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An algebraic approach to the variants of convexity for soft expert approximate function with intuitionistic fuzzy setting

Muhammad Ihsan[a], Muhammad Saeed[a], Khuram Ali Khan[b] and Ammara Nosheen[b]

[a]Department of Mathematics, University of Management and Technology, Lahore, Pakistan; [b]Department of Mathematics, University of Sargodha, Sargodha, Pakistan

ABSTRACT

A new area of research called intuitionistic fuzzy soft expert set is expected to overcome the drawbacks of an intuitionistic fuzzy soft set in terms of eligibility for soft expert-argument approximate function. This type of function views the power set of the universe as its co-domain and the cartesian product of attributes, experts, and their opinions as its domain. The domain of this function is larger as compared to the domain of a soft approximation function. It can manage a situation in which several expert opinions are taken into account by a single model. For the soft expert-argument approximate function with intuitionistic fuzzy setting, concepts such as set inclusion, (α, ν)-convexity(concave) sets, strongly (α, ν)-convexity (concave) sets, strictly (α, ν)-convexity (concave) sets, convex hull, and convex cone are conceived in this paper. Some set-theoretic inequalities are established with generalized properties and results on the basis of these specified notions. Additionally, by using a theoretic cum analytical approach, various elements of computational geometry, such as convex hull and convex cone, are theorized and some pertinent results are generalized.

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Fuzzy set; fuzzy soft expert set; (α, ν)-convexity; convex hull; convex cone

1. Introduction

In 1999, Molodtsov [1] used the idea of soft approximate function for constructing a novel model soft set in order to give fuzzy set like models [2,3] the parameterizations tool for dealing with uncertain data. This set makes use of the soft approximate function, which converts a single set of parameters into the starting universe's power set. There have been many discussions about the fundamentals of soft sets, but the contributions of Ali et al. [4], Babitha and Sunil [5,6], Ge and Yang [7], Li [8], Maji et al. [9], Pei and Miao [10], and Sezgin and Atagun [11] are thought to be particularly significant for characterizing the fundamental properties and set-theoretic operations of soft sets. Abbas et al. [12] examined the idea of soft points and addressed its drawbacks, similarities, and difficulties. Zadeh [13] introduced the concept of fuzzy set as a generalization of crisp set. In fuzzy set, each element has a membership degree. Xia [14] applied fuzzy in multi-criteria decision-making problems and named as EFMCDM-method. He applied this technique to deal with the problem that uncertainty is inevitably present in the MCDM process owing to human subjectivity. Intuitionistic fuzzy sets have a stronger capacity to represent and address the ambiguity of information than previous generations of fuzzy sets. Despite the development of numerous measuring techniques, there are still certain issues with the poor axioms of distance measurement or that lack discernment and lead to counterintuitive circumstances. To solve the aforementioned problems, Xiao [15] suggested a brand-new intuitionistic distance metric based on the Jensen-Shannon divergence. The Interval-Valued Intuitionistic Fuzzy Set has drawn a lot of interest because it is a powerful tool for modelling uncertainty. Using this structure, Wang et al. [16] introduced the interval-valued intuitionistic fuzzy Jenson-Shannon (IVIFJS) divergence, a new distance of interval-valued intuitionistic fuzzy set that can assess the similarity or difference between inter-valued intuitionistic fuzzy sets. The Pythagorean fuzzy set is a useful tool for addressing uncertainty in real-world applications, but it is still unclear how to gauge its level of uncertainty. By considering Pythagorean fuzziness entropy in terms of membership and non-membership degrees as well as Pythagorean hesitation entropy in terms of hesitation degree, Wang et al. [17] developed a novel entropy measure of Pythagorean fuzzy set. To address uncertainties with parameterizations tools, Maji et al. [18] advanced bonding impression of fuzzy soft set and debated its crucial things and outcomes. Numerous researchers [19,20] discussed the properties of the soft set, such as subset, absolute set, not set, etc., AND, OR, etc., and applied them to real-world issues. Rahman et al. [21] studies (m, n)convexity(concavity) using the structure of fuzzy soft set and discussed its certain properties. Many intriguing applications of soft set theory have been extended by certain researchers utilizing this idea of fuzzy soft

CONTACT Khuram Ali Khan khuram.ali@uos.edu.pk Department of Mathematics, University of Sargodha, Sargodha 40100, Pakistan

sets. Fuzzy soft sets have some applications, according to Roy and Maji [22]. On the basis of the theory of soft sets, Som [23] defined soft relation and fuzzy soft relation. Working on intuitionistic fuzzy soft relations were Mukherjee and Chakraborty [24]. Soft sets and the related ideas of fuzzy sets and rough sets were compared by Aktas and Çağman [25]. Operations on fuzzy soft sets were established by Yang et al. [26] and are based on the three fuzzy logic operators negation, triangular norm, and triangular co-norm. The soft set and fuzzy soft set were introduced into the incomplete environment by Zou and Xiao [27]. For their combined forecasting strategy based on fuzzy soft sets, Xiao et al. [28] employed predicting accuracy as the criterion of fuzzy membership function. The combination of an interval-valued fuzzy set and a soft set was introduced by Yang et al. [29]. In fuzzy soft-sets, Kong et al. [30] developed the typical parameter reduction and demonstrated that Roy and Maji's [22] technique is not practical in most situations. Çağman and Kartaş[31] introduced the concept of intuitionistic fuzzy soft set and successfully applied it in decision-making problems. Alkhazaleh et al. [32] advanced the structure of soft expert set, a mingling of soft set and expert set. They categorized its fundamentals results and strongly put in decision-making problems. Alkhazaleh et al. [33] instigated the model of fuzzy soft expert set by making a nice extension in soft expert set using fuzzy environment. They developed its operations and used in decision-making problems. He also introduced the concept of mappings on fuzzy soft expert set and described its properties. Broumi and Samarandache [34] introduced the concept of intuitionistic fuzzy soft expert set and applied in multi-criteria decision-making problems by giving examples. They [35] also introduced mappings on this structure and discussed certain theorems with examples. Smarandache [36] made extension in soft set by creating a new structure called hypersoft set. He made this work by dividing the parameters into sub-parameters.

1.1. Research gap, motivation and novelty

Convexity is very convenient in unalike arenas like optimization, recognition and classification of certain patterns, dualism difficulties and numerous extra-linked issues in operation research. The soft expert set is the mingling of soft and expert set. It is more flexible as it manages the restrictions of soft set for the respect of adept's thoughts. In order to make the existing convexity-like literature adequate with that situation, it is a literary necessity to get hold of a basic context for resolving such sort of issues under more flexible setting, i.e. soft expert set. convexity is very useful to solve and understand the optimization problems. Classically, the main focus of linear programming was in optimization area. At the beginning, it was considered that problems were majorly classified into two categories linear and non-linear optimized problems. Later on researchers found that the right division was between convex and non-convex problems because the some non-linear problems were found difficult to the others. But the use of convexity in uncertain environment was a big challenge. Various convex fuzzy and concave fuzzy set definitions exist, however, Zadeh [13] and Chaudhuri [37] are credited with introducing the first convex fuzzy set definition. Concavo-convex fuzzy sets were suggested by Sarkar [38] after Zadeh [13], with some features. Additionally, research on convex (concave) fuzzy sets has advanced quickly in both theory and application. A few examples include [39,40]. First, Deli [41] introduced the idea of convexity on soft set as well as fuzzy soft set. Deli proved some important results by using operations like union, intersection and compliment. He proved the following important results: (i)Intersection of convex soft sets is convex set but union is not necessarily a convex soft set. (ii) Union of concave soft sets is concave soft sets but intersection is not necessarily. (iii) The compliment of convex soft set is concave soft set and vice versa. (iv) Intersection of two strictly convex soft sets is convex but union is not necessarily. The same results have been proved for convex(concave) fuzzy soft sets. Later on, Majeed [42] instigated the geometrical futures like convex hull and cone s-set environment showing the use of convexity in geometry. Salleh and Sabir [43] talked about the certain features of convexity(concavity) on soft sets. Rahman et al. [44,45] introduced different structures of convexity(concavity) on soft and fuzzy soft sets. Ihsan et al. [46,47] conceptualized convexity on soft and fuzzy soft expert sets with certain properties. The research that is currently available on soft inclusions and inequalities for soft and fuzzy soft sets is only appropriate for managing soft expert argument approximate functions with intuitionistic fuzzy settings. In other words, it may be said that the literature now available on soft and fuzzy soft sets is unable to offer a mathematical model that might address all of the aforementioned real-world circumstances at once:

(1) When evaluating alternatives with an unclear nature (entities in a universal set), fuzzy membership grades must be applied to each entity that corresponds to each parameter.
(2) The situation where we can know the opinions of different experts in a single model without using any additional operations like union and intersection etc.
(3) When data is of two-dimensional type, i.e. membership and non-memberships values (intuitionistic fuzzy).

Therefore, motivated by the aforementioned gap in the literature, this study aims to create a new structure intuitionistic fuzzy expert sets for convexity(concavity) that is more adaptable than existing models because it can handle their constraints and is useful for making accurate and unbiased decisions because it places a strong emphasis on parameters as well as experts and their multi-decisive opinions. Consequently, an algebraic procedure is adapted to progress a core support of (α, ν)-convexity(concavity), (α, ν)-convexity(concavity) in 1st-sense and 2nd-senses, convex hull, convex cone on intuitionistic fuzzy soft expert sets (IFSEs) as well as some essential results have been proved. Numerical cases of these structures on this model are described too. The paper's primary contributions are summarized as follows:

(1) Under IFSEs's environment, the traditional concepts of (α, ν)-convex (concave) set, strictly (α, ν)-convex (concave) set, strongly (α, ν)-convex (concave) set, convex hull, and convex cone are generalized with the right to a soft expert approximate function.
(2) Set-theoretic inequalities are developed based on these set inclusions as well as other suggested notions when the concept of set inclusion for soft expert approximate function is defined.
(3) In order to evaluate the uniqueness of the proposed study, a detailed comparison is done with relevant, already published research works.

The paper is organized as Section 3 describes the definition of (α, ν)-convexity and concavity and related theorems. Section 4 describes the definition of (α, ν)-convexity and concavity in 1st and 2nd-senses with related theorems. Section 5 formulates the concept of strictly convex(concavity). Convex hull and cone are added to Section 6. In Sections 7 and 8, respectively, the comparison and conclusion with regard to future guidelines have been reached.

2. Preliminaries

This part shows the basic definitions from the literature. In this part, set of parameters will be denoted by $\mathcal{F}$ and $\mathcal{Z}$ as a universe of discourse and set of experts is by $\mathcal{Y}$ and $\mathbb{O}$ will be a set of opinions, $\mathcal{T} = \mathcal{F} \times \mathcal{Y} \times \mathbb{O}$. $P(\mathcal{Z})$ will be used as a power set.

Definition 2.1 ([13]): A set "F_z" is named as a *fuzzy set* shown by $F_z = \{(\hat{r}, N(\hat{r})) \mid \hat{r} \in \mathcal{Z}\}$ with $N : \mathcal{Z} \to [0, 1]$ and $N(\hat{r})$ shows the membership value of $\hat{r} \in \mathcal{Z}$.

Definition 2.2 ([1]): A soft set is a pair $(h_M, \mathcal{F})$ where h_M is defined by mapping $h_M : \mathcal{F} \to P(\mathcal{Z})$.

Definition 2.3 ([18]): A pair $(\Lambda_L, \ltimes)$ is will be a fs-set on $\mathcal{Z}$, with $\Lambda_L : \ltimes \to FP(\mathcal{Z})$ and $FP(\mathcal{Z})$ is being used as a collection of fuzzy subsets of $\mathcal{Z}$, $\ltimes \subseteq \mathcal{F}$.

Definition 2.4 ([32]): A pair $(\Theta, \mathcal{P})$ is known as a soft expert set with Θ is a mapping $\Theta : \mathcal{P} \to P(\mathcal{Z})$ and $\mathcal{P} \subseteq \mathcal{T} = \mathcal{F} \times \mathcal{Y} \times \mathbb{O}$.

Definition 2.5 ([34]): A pair $(\mathcal{E}, \mathcal{W})$ is called an IFSEs, with $\mathcal{E}$ is given by $\mathcal{E} : \mathcal{W} \to I^{\mathcal{Z}}$ while $I^{\mathcal{Z}}$ is being used as a collection of intuitionistic fuzzy subsets of $\mathcal{Z}$ and $\mathcal{W} \subseteq \mathcal{T}$.

Definition 2.6 ([34]): For two IFSEs $(\mathcal{J}_1, \mathcal{C})$ and $(\mathcal{J}_2, \mathcal{D})$ on $\mathcal{S}$, then $(\mathcal{J}_1, \mathcal{C}) \subseteq (\mathcal{J}_2, \mathcal{D})$ with $\mathcal{C} \subseteq \mathcal{D}$ and $\mathcal{J}_1(\lambda) \subseteq \mathcal{J}_2(\lambda)$, for all $\lambda \in \mathcal{C}$.

Definition 2.7 ([34]): Let $(\mathcal{G}, \mathcal{V})$ be an IFSEs on $\mathcal{S} \subseteq \mathcal{T}$, then its complement $(\mathcal{G}, \mathcal{V})^c$ is characterized by $(\mathcal{G}, \mathcal{V})^c = (\mathcal{G}^c, \rceil\mathcal{V})$ with $\mathcal{G}^c : \mathcal{S} \to I^{\mathcal{Z}}$ is a function and $\mathcal{G}^c(o) = c(\mathcal{G}(o)) = 1 - \mathcal{G}(o)$ for each $o \in \mathcal{S}$, here c represents a intuitionistic fuzzy complement.

Definition 2.8 ([34]): Let $(\mathcal{J}_1, \mathcal{C})$ and $(\mathcal{J}_2, \mathcal{D})$ be two IFSESs, then these are equal if $(\mathcal{J}_1, \mathcal{C}) \subseteq (\mathcal{J}_2, \mathcal{D})$ and $(\mathcal{J}_2, \mathcal{D}) \subseteq (\mathcal{J}_1, \mathcal{C})$.

Definition 2.9 ([34]): The union between IFSESs (η_1, Υ_1) and (η_2, Υ_2) is again an IFSEs (η_3, Υ_3) ; $\Upsilon_3 \doteq \Upsilon_1 \cup \Upsilon_2$, and $\forall \hat{e} \in \Upsilon_3$,

$$\eta_3(\hat{e}) = \begin{cases} \eta_1(\hat{e}) & ; \hat{e} \in \Upsilon_1 - \Upsilon_2 \\ \eta_2(\hat{e}) & ; \hat{e} \in \Upsilon_2 - \Upsilon_1 \\ s(\eta_1(\hat{e}), \eta_2(\hat{e})) & ; \hat{e} \in \Upsilon_1 \cap \Upsilon_2 \end{cases}$$

while s is a s-norm.

Definition 2.10 ([34]): The intersection of IFSEs $(\mho_1, \Upsilon_1)$ and $(\mho_2, \Upsilon_2)$ on $\mathcal{Z}$ is an IFSEs $(\mho_3, \Upsilon_3)$ while $\Upsilon_3 \doteq \Upsilon_1 \cap \Upsilon_2$, ; for all $\hat{e} \in \Upsilon_3$,

$$\mho_3(\hat{e}) = \begin{cases} \mho_1(\hat{e}) & ; \hat{e} \in \Upsilon_1 - \Upsilon_2 \\ \mho_2(\hat{e}) & ; \hat{e} \in \Upsilon_2 - \Upsilon_1 \\ t(\mho_1(\hat{e}), \mho_2(\hat{e})) & ; \hat{e} \in \Upsilon_1 \cap \Upsilon_2 \end{cases}$$

where t is a t-norm.

Definition 2.11 ([34]): Suppose $\{(H_i, \mathcal{S}) : i \in I\}$ be a finite set of IFSESs. Then

(i) The operation union of finite collection can be characterized as $(\sqcup_{i \in I} H_i, \mathcal{S}) = \cup_{i \in I} H_i(\alpha), \forall \alpha \in \mathcal{Z}$.
(ii) The operation intersection of finite collection can be characterized as $(\sqcap_{i \in I} H_i, \mathcal{S}) = \cap_{i \in I} H_i(\alpha), \forall \alpha \in \mathcal{Z}$.

Definition 2.12 ([41]): The fuzzy soft set on $\mathcal{F}$ is named as a convex fuzzy soft set if $h_S(u\epsilon_1 + (1 - u)\epsilon_2) \supseteq h_S(\epsilon_1) \cap h_S(\epsilon_2)$ for each $\epsilon_1, \epsilon_2 \in \mathcal{F}$ and $u \in \Gamma^{\bullet} = [0, 1]$.

Definition 2.13 ([41]): The fuzzy soft set on $\mathcal{F}$ is named as a convex fuzzy soft set if $h_S(u\epsilon_1 + (1 - u)\epsilon_2) \subseteq h_S(\epsilon_1) \cup h_S(\epsilon_2)$ for each $\epsilon_1, \epsilon_2 \in \mathcal{F}$ and $u \in \Gamma^{\bullet}$.

3. Convex and concave intuitionistic fuzzy soft expert set

This part of the paper describes the definitions of convex(concave) intuitionistic fuzzy soft expert set and various results have been proved.

Deli [41] is the first person who introduced the concept of convexity(concavity) in s-set and f-set environment. But this work is not suitable when some experts opinions is required. To deal with situation, Ihsan et al. [46] made extension in the work of Deli by introducing convexity in se-set. Then Ihsan et al. [47] put forward this idea in fuzzy soft expert environment by conceptualizing convexity in fse-set. Fuzz soft set is useful for handling membership values but not for non-membership environment. In order to handle these limitations, idea of convexity(concavity) is extended to intuitionistic fuzzy soft environment under multi-decisive opinion. Some dominant and valuable characteristics have been extended.

Definition 3.1: The IFSEs on $\mathcal{S}$ is called a convex IFSEs if $q_{ifses}(\alpha\epsilon_1 + (1-\alpha)\epsilon_2) \supseteq q_{ifses}(\epsilon_1) \cap q_{ifses}(\epsilon_2)$ for each $\epsilon_1, \epsilon_2 \in \mathcal{S}$ and $\alpha \in \Gamma^{\bullet}$.

Example 3.1: Assume that an organization delivers new sorts of items and needs to take the assessment of certain specialists about these items. Suppose $\mathcal{Z} = \{w_1, w_2, w_3, w_4\}$ is used as s set of products, $\mathcal{E} = \{b_1, b_2, b_3\} = \{1, 2, 3\}$ represents a collection of decision parameters where $b_i (i = 1, 2, 3)$ are the parameters; easy to handle, quality and moderate. Let $\mathcal{X} = \{f, g, h\} = \{1, 2, 3\}$ represents an experts set. Suppose that

$$\partial_1 = \partial(b_1, f, 1) = \left\{ \frac{w_1}{<0.5, 0.5>}, \frac{w_2}{<0.2, 0.6>}, \frac{w_3}{<0.4, 0.6>}, \frac{w_4}{<0.1, 0.8>} \right\},$$

$$\partial_2 = \partial(b_1, g, 1) = \left\{ \frac{w_1}{<0.4, 0.5>}, \frac{w_2}{<0.2, 0.1>}, \frac{w_3}{<0.4, 0.5>}, \frac{w_4}{<0.2, 0.6>} \right\}$$

$$\partial_3 = \partial(b_1, h, 1) = \left\{ \frac{w_1}{<0.7, 0.4>}, \frac{w_2}{<0.5, 0.6>}, \frac{w_3}{<0.6, 0.2>}, \frac{w_4}{<0.3, 0.5>} \right\},$$

$$\partial_4 = \partial(b_2, f, 1) = \left\{ \frac{w_1}{<0.9, 0.1>}, \frac{w_2}{<0.4, 0.3>}, \frac{w_3}{<0.7, 0.2>}, \frac{w_4}{<0.3, 0.2>} \right\},$$

$$\partial_5 = \partial(b_2, g, 1) = \left\{ \frac{w_1}{<0.4, 0.2>}, \frac{w_2}{<0.8, 0.2>}, \frac{w_3}{<0.3, 0.4>}, \frac{w_4}{<0.2, 0.3>} \right\},$$

$$\partial_6 = \partial(b_2, h, 1) = \left\{ \frac{w_1}{<0.5, 0.3>}, \frac{w_2}{<0.3, 0.4>}, \frac{w_3}{<0.2, 0.1>}, \frac{w_4}{<0.8, 0.1>} \right\},$$

$$\partial_7 = \partial(b_3, f, 1) = \left\{ \frac{w_1}{<0.2, 0.4>}, \frac{w_2}{<0.1, 0.9>}, \frac{w_3}{<0.4, 0.2>}, \frac{w_4}{<0.5, 0.3>} \right\},$$

$$\partial_8 = \partial(b_3, g, 1) = \left\{ \frac{w_1}{<0.4, 0.2>}, \frac{w_2}{<0.6, 0.3>}, \frac{w_3}{<0.2, 0.7>}, \frac{w_4}{<0.1, 0.9>} \right\},$$

$$\partial_9 = \partial(b_3, h, 1) = \left\{ \frac{w_1}{<0.2, 0.7>}, \frac{w_2}{<0.2, 0.3>}, \frac{w_3}{<0.3, 0.5>}, \frac{w_4}{<0.1, 0.2>} \right\},$$

$$\partial_{10} = \partial(b_1, f, 0) = \left\{ \frac{w_1}{<0.5, 0.5>}, \frac{w_2}{<0.3, 0.7>}, \frac{w_3}{<0.4, 0.6>}, \frac{w_4}{<0.2, 0.8>} \right\},$$

$$\partial_{11} = \partial(b_1, g, 0) = \left\{ \frac{w_1}{<0.1, 0.2>}, \frac{w_2}{<0.1, 0.9>}, \frac{w_3}{<0.3, 0.6>}, \frac{w_4}{<0.5, 0.2>} \right\},$$

$$\partial_{12} = \partial(b_1, h, 0) = \left\{ \frac{w_1}{<0.1, 0.2>}, \frac{w_2}{<0.6, 0.1>}, \frac{w_3}{<0.2, 0.3>}, \frac{w_4}{<0.3, 0.5>} \right\},$$

$$\partial_{13} = \partial(b_2, f, 0) = \left\{ \frac{w_1}{<0.2, 0.8>}, \frac{w_2}{<0.1, 0.3>}, \frac{w_3}{<0.3, 0.5>}, \frac{w_4}{<0.2, 0.7>} \right\},$$

$$\partial_{14} = \partial(b_2, g, 0) = \left\{ \frac{w_1}{<0.2, 0.7>}, \frac{w_2}{<0.2, 0.5>}, \frac{w_3}{<0.2, 0.7>}, \frac{w_4}{<0.3, 0.4>} \right\},$$

$$\partial_{15} = \partial(b_2, h, 0) = \left\{ \frac{w_1}{<0.3, 0.6>}, \frac{w_2}{<0.2, 0.7>}, \frac{w_3}{<0.2, 0.3>}, \frac{w_4}{<0.1, 0.2>} \right\},$$

$$\partial_{16} = \partial(b_3, f, 0) = \left\{ \frac{w_1}{<0.1, 0.4>}, \frac{w_2}{<0.2, 0.4>}, \frac{w_3}{<0.2, 0.7>}, \frac{w_4}{<0.2, 0.8>} \right\},$$

$$\partial_{17} = \partial(b_3, g, 0) = \left\{ \frac{w_1}{<0.1, 0.2>}, \frac{w_2}{<0.1, 0.6>}, \frac{w_3}{<0.2, 0.8>}, \frac{w_4}{<0.2, 0.3>} \right\},$$

$$\partial_{18} = \partial(b_3, h, 0) = \left\{ \frac{w_1}{<0.3, 0.5>}, \frac{w_2}{<0.2, 0.3>}, \right.$$

$$\frac{w_3}{<0.4, 0.6>}, \frac{w_4}{<0.1, 0.4>}\Bigg\},$$

The IFSEs can be described as

$$(\mathrm{H}, \mathrm{S}) = \left\{\begin{array}{l} \partial_1, \partial_2, \partial_3, \partial_4, \partial_5, \partial_6, \partial_7, \partial_8, \partial_9, \\ \partial_{10}, \partial_{11}, \partial_{12}, \partial_{13}, \partial_{14}, \partial_{15}, \partial_{16}, \partial_{17}, \partial_{18} \end{array}\right\}.$$

Example 3.2: Using Example (3.1), consider $\mathcal{C} = \{\partial(b_1, f, 1), \partial(b_2, f, 0), \partial(b_3, f, 1), \partial(b_1, g, 1), \partial(b_2, g, 1), \partial(b_3, g, 0), \partial(b_1, h, 0), \partial(b_2, h, 1), \partial(b_3, h, 1)\}$ and $\mathcal{D} = \{\partial(b_1, f, 1), \partial(b_2, f, 0), \partial(b_3, f, 1), \partial(b_1, g, 1), \partial(b_2, g, 1), \partial(b_3, g, 1), \partial(b_1, h, 0), \partial(b_2, h, 1)\}$. Let $(\mathcal{J}_1, \mathcal{C})$ and $(\mathcal{J}_2, \mathcal{D})$ be two IFSESs such that

$$(\mathcal{J}_1, \mathcal{C}) = \left\{\begin{array}{l} \partial_1 = \partial(b_1, f, 1) = \left\{\frac{w_1}{<0.5, 0.2>}, \right. \\ \frac{w_2}{<0.7, 0.3>}, \frac{w_3}{<0.4, 0.5>}, \\ \left. \frac{w_4}{<0.1, 0.4>}\right\}, \\ \partial_2 = \partial(b_1, g, 1) = \left\{\frac{w_1}{<0.4, 0.5>}, \right. \\ \frac{w_2}{<0.2, 0.8>}, \frac{w_3}{<0.4, 0.6>}, \\ \left. \frac{w_4}{<0.2, 0.3>}\right\}, \\ \partial_5 = \partial(b_2, g, 1) = \left\{\frac{w_1}{<0.4, 0.6>}, \right. \\ \frac{w_2}{<0.2, 0.3>}, \frac{w_3}{<0.3, 0.4>}, \\ \left. \frac{w_4}{<0.2, 0.3>}\right\}, \\ \partial_6 = \partial(b_2, h, 1) = \left\{\frac{w_1}{<0.5, 0.4>}, \right. \\ \frac{w_2}{<0.3, 0.4>}, \frac{w_3}{<0.6, 0.4>}, \\ \left. \frac{w_4}{<0.8, 0.2>}\right\}, \end{array}\right\}.$$

$$(\mathcal{J}_2, \mathcal{D}) = \left\{\begin{array}{l} \partial_1 = \partial(b_1, f, 1) = \left\{\frac{w_1}{<0.1, 0.3>}, \right. \\ \frac{w_2}{<0.2, 0.3>}, \frac{w_3}{<0.4, 0.5>}, \\ \left. \frac{w_4}{<0.2, 0.4>}\right\}, \\ \partial_2 = \partial(b_1, g, 1) = \left\{\frac{w_1}{<0.5, 0.4>}, \right. \\ \frac{w_2}{<0.2, 0.3>}, \frac{w_3}{<0.2, 0.1>}, \\ \left. \frac{w_4}{<0.3, 0.5>}\right\}, \\ \partial_5 = \partial(b_2, g, 1) = \left\{\frac{w_1}{<0.6, 0.3>}, \right. \\ \frac{w_2}{<0.3, 0.4>}, \frac{w_3}{<0.2, 0.7>}, \\ \left. \frac{w_4}{<0.4, 0.5>}\right\}, \\ \partial_6 = \partial(b_2, h, 1) = \left\{\frac{w_1}{<0.3, 0.5>}, \right. \\ \frac{w_2}{<0.4, 0.6>}, \frac{w_3}{<0.7, 0.3>}, \\ \left. \frac{w_4}{<0.9, 0.1>}\right\}, \end{array}\right\}.$$

Using union operation(max), we get

$$(\mathcal{H}, \mathcal{L}) = \left\{\begin{array}{l} \partial_1 = \partial(b_1, f, 1) = \left\{\frac{w_1}{<0.5, 0.3>}, \right. \\ \frac{w_2}{<0.7, 0.3>}, \frac{w_3}{<0.4, 0.5>}, \\ \left. \frac{w_4}{<0.2, 0.4>}\right\}, \\ \partial_2 = \partial(b_1, g, 1) = \left\{\frac{w_1}{<0.5, 0.5>}, \right. \\ \frac{w_2}{<0.2, 0.8>}, \frac{w_3}{<0.4, 0.6>}, \\ \left. \frac{w_4}{<0.3, 0.5>}\right\}, \\ \partial_5 = \partial(b_2, g, 1) = \left\{\frac{w_1}{<0.6, 0.3>}, \right. \\ \frac{w_2}{<0.3, 0.4>}, \frac{w_3}{<0.2, 0.4>}, \\ \left. \frac{w_4}{<0.4, 0.5>}\right\}, \\ \partial_6 = \partial(b_2, h, 1) = \left\{\frac{w_1}{<0.5, 0.4>}, \right. \\ \frac{w_2}{<0.4, 0.4>}, \frac{w_3}{<0.7, 0.3>}, \\ \left. \frac{w_4}{<0.9, 0.1>}\right\}, \end{array}\right\}.$$

Example 3.3: Referring to Example (3.1), let $\mathcal{C} = \{\partial(b_1, f, 1), \partial(b_2, f, 0), \partial(b_3, f, 1), \partial(b_1, g, 1), \partial(b_2, g, 1), \partial(b_3, g, 0), \partial(b_1, h, 0), \partial(b_2, h, 1), \partial(b_3, h, 1)\}$ and $\mathcal{D} = \{\partial(b_1, f, 1), \partial(b_2, f, 0), \partial(b_3, f, 1), \partial(b_1, g, 1), \partial(b_2, g, 1), \partial(b_3, g, 1), \partial(b_1, h, 0), \partial(b_2, h, 1)\}$.

Considering the above two IFSESs in (3.2) and using intersection operation(min), we get

$$(\mathcal{M}, \mathcal{L}) = \left\{\begin{array}{l} \partial_1 = \partial(b_1, f, 1) = \left\{\frac{w_1}{<0.1, 0.3>}, \right. \\ \frac{w_2}{<0.2, 0.3>}, \frac{w_3}{<0.4, 0.5>}, \\ \left. \frac{w_4}{<0.1, 0.4>}\right\}, \\ \partial_2 = \partial(b_1, g, 1) = \left\{\frac{w_1}{<0.4, 0.4>}, \right. \\ \frac{w_2}{<0.2, 0.8>}, \frac{w_3}{<0.2, 0.6>}, \\ \left. \frac{w_4}{<0.2, 0.5>}\right\}, \\ \partial_5 = \partial(b_2, g, 1) = \left\{\frac{w_1}{<0.4, 0.6>}, \right. \\ \frac{w_2}{<0.2, 0.4>}, \frac{w_3}{<0.2, 0.7>}, \\ \left. \frac{w_4}{<0.2, 0.5>}\right\}, \\ \partial_6 = \partial(b_2, h, 1) = \left\{\frac{w_1}{<0.3, 0.4>}, \right. \\ \frac{w_2}{<0.3, 0.6>}, \frac{w_3}{<0.6, 0.4>}, \\ \left. \frac{w_4}{<0.8, 0.2>}\right\}, \end{array}\right\}.$$

Example 3.4: Considering Example (3.1) with $\mathcal{E} = \{b_1, b_2, b_3\} = \{1, 2, 3\}$, and $\mathcal{X} = \{f, g, h\} = \{1, 2, 3\}$, $\partial_1 = q_{ifses}(1, 1, 1) = \{\frac{w_1}{<0.5,0.5>}, \frac{w_2}{<0.2,0.6>}, \frac{w_3}{<0.4,0.6>}, \frac{w_4}{<0.1,0.8>}\}$, $\partial_2 = q_{ifses}(1, 2, 1) = \{\frac{w_1}{<0.4,0.5>}, \frac{w_2}{<0.2,0.1>}, \frac{w_3}{<0.4,0.5>}, \frac{w_4}{<0.2,0.6>}\}$

Take $\alpha = 0.6$, and $\epsilon_1 = (1, 1, 1), \epsilon_2 = (1, 2, 1)$ then

$$\begin{aligned} &q_{ifses}(\alpha\epsilon_1 + (1-\alpha)\epsilon_2) \\ &\quad = q_{ifses}(0.6(1, 1, 1) + (1 - 0.6)(1, 2, 1)) \\ &\quad = q_{ifses}((0.6, 0.6, 0.6) + (0.4, 0.8, 0.4)) \\ &\quad = q_{ifses}(1, 1.4, 1). \end{aligned}$$

By applying the decimal round off property, we get $\{1, 1.4, 1\} = \{1, 1, 1\}$ $q_{ifses}(1, 1, 1) = \{\frac{w_1}{<0.5,0.5>}, \frac{w_2}{<0.2,0.6>}, \frac{w_3}{<0.4,0.6>}, \frac{w_4}{<0.1,0.8>}\}$ and $q_{ifses}(\epsilon_1) \cap q_{ifses}(\epsilon_2) = q_{ifses}(1, 1, 1) \cap q_{ifses}(1, 2, 1 = \{\frac{w_1}{<0.5,0.5>}, \frac{w_2}{<0.2,0.6>}, \frac{w_3}{<0.4,0.6>}, \frac{w_4}{<0.1,0.8>}\} \cap \{\frac{w_1}{<0.4,0.5>}, \frac{w_2}{<0.2,0.1>}, \frac{w_3}{<0.4,0.5>}, \frac{w_4}{<0.2,0.6>}\} = \{\frac{w_1}{<0.4,0.5>}, \frac{w_2}{<0.2,0.6>}, \frac{w_3}{<0.4,0.5>}, \frac{w_4}{<0.1,0.8>}\}$.

It is clear that

$$q_{ifses}\left(\alpha\epsilon_1 + (1-\alpha)\epsilon_2\right) \supseteq q_{ifses}\left(\epsilon_1\right) \cap q_{ifses}\left(\epsilon_2\right).$$

Ihsan et al. [47] have defined concavity in fuzzy soft expert environment. Now following definition is the extension of fuzzy soft expert set.

Definition 3.2: The IFSEs on $\mathcal{S}$ is called concave IFSEs if

$$q_{ifses}\left(\alpha\epsilon_1 + (1-\alpha)\epsilon_2\right) \subseteq q_{ifses}\left(\epsilon_1\right) \cup q_{ifses}\left(\epsilon_2\right)$$

for each $\epsilon_1, \epsilon_2 \in \mathcal{S}$ and $\alpha \in \Upsilon^{\bullet}$.

Example 3.5: Considering Example (3.1) with $\mathcal{E} = \{b_1, b_2, b_3\} = \{1, 2, 3\}$, and $\mathcal{X} = \{f, g, h\} = \{1, 2, 3\}$, $\partial_6 = q_{ifses}(1, 1, 1) = \{\frac{w_1}{<0.5,0.3>}, \frac{w_2}{<0.3,0.4>}, \frac{w_3}{<0.2,0.1>}, \frac{w_4}{<0.8,0.1>}\}$, $\partial_7 = q_{ifses}(1, 2, 1) = \{\frac{w_1}{<0.2,0.4>}, \frac{w_2}{<0.1,0.9>}, \frac{w_3}{<0.4,0.2>}, \frac{w_4}{<0.5,0.3>}\}$.

Take $\alpha = 0.6$, and $\epsilon_1 = (1, 1, 1), \epsilon_2 = (1, 2, 1)$ then

$$\begin{aligned} &q_{ifses}(\alpha\epsilon_1 + (1-\alpha)\epsilon_2) \\ &\quad = q_{ifses}(0.6(1, 1, 1) + (1 - 0.6)(1, 2, 1)) \\ &\quad = q_{ifses}((0.6, 0.6, 0.6) + (0.4, 0.8, 0.4)) \\ &\quad = q_{ifses}(1, 1.4, 1). \end{aligned}$$

By applying the decimal round off property, we get $\{1, 1.4, 1\} = \{1, 1, 1\}$ $q_{ifses}(1, 1, 1) = \{\frac{w_1}{<0.5,0.3>}, \frac{w_2}{<0.3,0.4>}, \frac{w_3}{<0.2,0.1>}, \frac{w_4}{<0.8,0.1>}\}$ and $q_{ifses}(\epsilon_1) \cap q_{ifses}(\epsilon_2) = q_{ifses}(1, 1, 1) \cup q_{ifses}(1, 2, 1) = \{\frac{w_1}{<0.5,0.3>}, \frac{w_2}{<0.3,0.4>}, \frac{w_3}{<0.2,0.1>}, \frac{w_4}{<0.8,0.1>}\} \cup \{\frac{w_1}{<0.2,0.4>}, \frac{w_2}{<0.1,0.9>}, \frac{w_3}{<0.4,0.2>}, \frac{w_4}{<0.5,0.3>}\} = \{\frac{w_1}{<0.5,0.3>}, \frac{w_2}{<0.3,0.4>}, \frac{w_3}{<0.4,0.1>}, \frac{w_4}{<0.8,0.1>}\}$.

It is clear that

$$q_{IFSEs}\left(\alpha\epsilon_1 + (1-\alpha)\epsilon_2\right) \subseteq q_{IFSEs}\left(\epsilon_1\right) \cup q_{IFSEs}\left(\epsilon_2\right).$$

Set-inclusion has already been introduced in Ihsan et al. [47] for fuzzy environment. Now this work is for its extension.

Definition 3.3: Suppose $\mathcal{W}$ be an IFSEs on $\mathcal{Z}$ and $\rho \subseteq \mathcal{Z}$. Then ρ-inclusion of $\mathcal{W}$ is characterized as

$$\mathcal{W}^{\rho} = \{\epsilon \in \mathcal{F} : g_{\mathcal{W}}(\epsilon) \supseteq \rho\}.$$

Theorem 3.1: *$P_1 \cap P_2$ is a (α, v)-convex IFSESs while P_1 and P_2 are (α, v)-convex IFSESs.*

Proof: Assume $\exists \epsilon_1, \epsilon_2 \in \mathcal{S}$ and $\alpha \in (0, 1]$, and $P_3 = P_1 \cap P_2$.

$$\begin{aligned} &q_{ifses(P_3)}\left(\alpha\epsilon_1 + v\left(1-\alpha\right)\epsilon_2\right) \\ &\quad = q_{ifses(P_1)}\left(\alpha\epsilon_1 + v\left(1-\alpha\right)\epsilon_2\right) \\ &\quad\quad \cap q_{ifses(P_2)}\left(\alpha\epsilon_1 + v\left(1-\alpha\right)\epsilon_2\right) \end{aligned}$$

$\because P_1$ and P_2 are (α, v)-convex IFSESs, so

$$\begin{aligned} &q_{ifses(P_1)}\left(\alpha\epsilon_1 + v\left(1-\alpha\right)\epsilon_2\right) \\ &\quad \supseteq h_{ifses(P_1)}\left(\epsilon_1\right) \cap q_{ifses(P_1)}\left(\epsilon_2\right) \quad (1) \\ &q_{ifses(P_2)}\left(\alpha\epsilon_1 + v\left(1-\alpha\right)\epsilon_2\right) \\ &\quad \supseteq q_{ifses(P_2)}\left(\epsilon_1\right) \cap q_{ifses(P_2)}\left(\epsilon_2\right) \quad (2) \end{aligned}$$

By applying the intersection operation, we have

$$\begin{aligned} &q_{ifses(P_3)}(\alpha\epsilon_1 + v(1-\alpha)\epsilon_2) \\ &\quad \supseteq \{(q_{ifses(P_1)}(\epsilon_1) \cap q_{ifses(P_1)}(\epsilon_2)) \\ &\quad\quad \cap (q_{ifses(P_2)}(\epsilon_1) \cap q_{ifses(P_2)}(\epsilon_2))\} \\ &q_{ifses(P_3)}(\alpha\epsilon_1 + v(1-\alpha)\epsilon_2) \\ &\quad \supseteq q_{ifses(P_3)}(\epsilon_1) \cap q_{ifses(P_3)}(\epsilon_2 \end{aligned}$$

hence the result is proved. ■

Remark 3.1: If $\{P_k : k \in \{1, 2, 3, \ldots\}\}$ is any collection of (α, v)-convex IFSESs, then the $\bigcap_{k\in I} P_k$ is a (α, v)-convex IFSEs.

Theorem 3.2: *Y is a (α, v)-convex IFSEs on $\mathcal{S}$ iff for each $\alpha \in (0, 1]$ and $\rho \in P(\mathcal{Z})$, Y^{ρ} is (α, v)-convex IFSEs on $\mathcal{S}$.*

Proof: Suppose that Y is (α, v)-convex IFSEs. If $\epsilon_1, \epsilon_2 \in \mathcal{S}$ and $\rho \in P(Z)$, then $q_{ifses(Y)}(\epsilon_1) \supseteq \rho$ and $q_{ifses(Y)}(\epsilon_2) \supseteq \rho$.

By the convexity of Y, we have

$$\begin{aligned} &q_{ifses(Y)}\left(\alpha\epsilon_1 + v\left(1-\alpha\right)\epsilon_2\right) \\ &\quad \supseteq q_{ifses(Y)}\left(\epsilon_1\right) \cap q_{ifses(Y)}\left(\epsilon_2\right) \end{aligned}$$

so Y^{ρ} is (α, v)-convex IFSEs $\mathcal{S}$.

Conversely, suppose that Y^{ρ} is (α, v)-convex IFSEs for each $\alpha \in (0, 1]$.

Then, for $\epsilon_1, \epsilon_2 \in \mathcal{S}$, Y^{ρ} is (α, v)-convex IFSEs for $\rho = q_{ifses(Y)}(\epsilon_1) \cap q_{ifses(Y)}(\epsilon_2)$.

$\because q_{ifses(Y)}(\epsilon_1) \supseteq \rho$ and $q_{ifses(Y)}(\epsilon_2) \supseteq \rho$, we have $\epsilon_1 \in Y^{\rho}$ and $\epsilon_2 \in Y^{\rho}$, hence $\alpha\epsilon_1 + v(1-\alpha)\epsilon_2 \in Y^{\rho}$.

$\therefore q_{ifses(Y)}(\alpha\epsilon_1 + v(1-\alpha)\epsilon_2) \supseteq q_{ifses(Y)}(\epsilon_1) \cap q_{ifses(Y)}(\epsilon_2)$,
$\Rightarrow Y$ is (α, v)-convex IFSEs. ■

Theorem 3.3: *L' is (α, v)-concave IFSEs while L is (α, v)-convex IFSEs.*

Proof: Assume that $\exists \epsilon_1, \epsilon_2 \in \mathcal{S}$, $\alpha \in (0, 1]$.
$\because L$ is (α, v)-convex IFSEs,

$$q_{ifses(L)}(\alpha\epsilon_1 + v(1-\alpha)\epsilon_2) \supseteq q_{fses(L)}(\epsilon_1) \cap q_{ifses(L)}(\epsilon_2)$$

or

$$\mathcal{Z}\setminus q_{ifses(L)}(\alpha\epsilon_1 + v(1-\alpha)\epsilon_2) \subseteq \mathcal{Z}\setminus \left\{q_{ifses(L)}(\epsilon_1) \cap q_{ifses(L)}(\epsilon_2)\right\}$$

If $q_{ifses(L)}(\epsilon_1) \supseteq q_{ifses(L)}(\epsilon_2)$, then we may write

$$\mathcal{Z}\setminus q_{ifses(L)}(\alpha\epsilon_1 + v(1-\alpha)\epsilon_2) \subseteq \mathcal{Z}\setminus q_{ifses(L)}(\epsilon_2) \quad (3)$$

If $q_{ifses(L)}(\epsilon_1) \subseteq q_{ifses(L)}(\epsilon_2)$, then we may write

$$\mathcal{Z}\setminus q_{ifses(L)}(\alpha\epsilon_1 + v(1-\alpha)\epsilon_2) \subseteq \mathcal{Z}\setminus q_{ifses(L)}(\epsilon_1) \quad (4)$$

From Equations (1) and (2), we have

$$\mathcal{Z}\setminus q_{ifses(L)}(\alpha\epsilon_1 + v(1-\alpha)\epsilon_2) \subseteq \left\{\mathcal{Z}\setminus q_{ifses(L)}(\epsilon_1) \cup \mathcal{Z}\setminus q_{ifses(L)}(\epsilon_2)\right\}. \quad (5)$$

So, L' is (α, v)-concave IFSEs. ■

Theorem 3.4: *P' is (α, v)-convex IFSEs while P is (α, v)-concave IFSEs.*

Proof: Assume that $\exists \epsilon_1, \epsilon_2 \in \mathcal{S}$, $\alpha \in (0, 1]$.
$\because P$ is (α, v)-concave IFSEs,

$$q_{ifses(P)}(\alpha\epsilon_1 + v(1-\alpha)\epsilon_2) \subseteq q_{ifses(P)}(\epsilon_1) \cup q_{ifses(P)}(\epsilon_2)$$

or

$$\mathcal{Z}\setminus q_{ifses(P)}(\alpha\epsilon_1 + v(1-\alpha\epsilon)_2) \supseteq \mathcal{Z}\setminus \left\{q_{ifses(P)}(\epsilon_1) \cup q_{ifses(P)}(\epsilon_2)\right\}$$

If $q_{ifses(P)}(\epsilon_1) \supseteq q_{ifses(P)}(\epsilon_2)$ then we may write

$$\mathcal{Z}\setminus q_{ifses(P)}(\alpha\epsilon_1 + (1-\alpha)\epsilon_2) \supseteq \mathcal{Z}\setminus q_{ifses(P)}(\epsilon_1) \quad (6)$$

If $q_{ifses(P)}(\epsilon_1) \subseteq q_{ifses(P)}(\epsilon_2)$ $\therefore$

$$\mathcal{Z}\setminus q_{ifses(P)}(\alpha\epsilon_1 + v(1-\alpha)\epsilon_2) \supseteq \mathcal{Z}\setminus q_{ifses(P)}(\epsilon_2) \quad (7)$$

from (4) and (5), we have

$$\mathcal{Z}\setminus q_{ifses(P)}(\alpha\epsilon_1 + v(1-\alpha)\epsilon_2) \supseteq \left\{\mathcal{Z}\setminus q_{ifses(P)}(\epsilon_1) \cap \mathcal{Z}\setminus q_{ifses(P)}(\epsilon_2)\right\} \quad (8)$$

which shows that P' is (α, v)-convex IFSEs. ■

Theorem 3.5: *M is (α, v)-concave IFSEs on $\mathcal{S}$ iff for each $\alpha \in (0, 1]$ and $\rho \in P(\mathcal{Z})$, M^ρ is (α, v)-concave IFSEs on $\mathcal{S}$.*

Proof: Suppose that M is (α, v)-concave IFSEs. If $\epsilon_1, \epsilon_2 \in \mathcal{S}$ and $\rho \in P(\mathcal{Z})$, then $q_{ifses(M)}(\epsilon_1) \supseteq \rho$ and $q_{ifses(M)}(\epsilon_2) \supseteq \rho$.

By the concavity of M, we have

$$q_{ifses(M)}(\alpha\epsilon_1 + v(1-\alpha)\epsilon_2) \subseteq q_{ifses(M)}(\epsilon_1) \cup q_{ifses(M)}(\epsilon_2)$$

hence M^ρ is a (α, v)-concave IFSEs.

Conversely suppose that M^ρ is (α, v)-concave IFSEs for each $\alpha \in (0, 1]$.

Subsequently, for $\epsilon_1 \epsilon_2 \in \mathcal{S}$, M^ρ is (α, v)-concave IFSEs for $\rho = q_{ifses(M)}(\epsilon_1) \cup q_{ifses(M)}(\epsilon_2)$.
$\therefore q_{ifses(M)}(\epsilon_1) \supseteq \rho$ and $q_{ifses(M)}(\epsilon_2) \supseteq \rho$,
we have $\epsilon_1 \in M^\rho$ and $\epsilon_2 \in M^\rho$,
hence $\alpha\epsilon_1 + v(1-\alpha)\epsilon_2 \in M^\rho$.
$\therefore, q_{ifses(M)}(\alpha\epsilon_1 + v(1-\alpha)\epsilon_2) \subseteq q_{ifses(M)}(\epsilon_1) \cup q_{ifses(M)}(\epsilon_2)$,
it is clear that M is (α, v)-concave IFSEs. ■

Theorem 3.6: *$P_1 \cup P_2$ is a (α, v)-concave IFSEs while P_1 and P_2 are (α, v)-concave IFSESs.*

Proof: Assume $\exists \epsilon_1, \epsilon_2 \in \mathcal{S}$ and $\alpha \in (0, 1]$, and $P_3 = P_1 \cup P_2$.

$$\begin{aligned} q_{ifses(P_3)}(\alpha\epsilon_1 + v(1-\alpha)\epsilon_2) &= q_{ifses(P_1)}(\alpha\epsilon_1 + v(1-\alpha)\epsilon_2) \\ &\cup q_{ifses(P_2)}(\alpha\epsilon_1 + v(1-\alpha)\epsilon_2) \end{aligned}$$

$\because P_1$ and P_2 are (α, v)-concave ifse-sets, so

$$q_{ifses(P_1)}(\alpha\epsilon_1 + v(1-\alpha)\epsilon_2) \subseteq q_{ifses(P_1)}(\epsilon_1) \cup q_{ifses(P_1)}(\epsilon_2) \quad (9)$$

$$q_{ifses(P_2)}(\alpha\epsilon_1 + v(1-\alpha)\epsilon_2) \subseteq q_{ifses(P_2)}(\epsilon_1) \cup q_{ifses(P_2)}(\epsilon_2) \quad (10)$$

By applying union operation, we have

$$\begin{aligned} q_{ifses(P_3)}(\alpha\epsilon_1 + v(1-\alpha)\epsilon_2) &\subseteq \{(q_{ifses(P_1)}(\epsilon_1) \cup q_{ifses(P_1)}(\epsilon_2)) \\ &\cup (q_{ifses(P_2)}(\epsilon_1) \cup q_{ifses(P_2)}(\epsilon_2))\}. \\ q_{ifses(P_3)}(\alpha\epsilon_1 + v(1-\alpha)\epsilon_2) &\subseteq q_{ifses(P_3)}(\epsilon_1) \cup q_{ifses(P_3)}(\epsilon_2). \end{aligned}$$

Here we have proved the union of two concave IFSESs, it can be proved for more than two concave ifse-sets. Hence we can generalized this result up to any countable number of concave IFSESs. ■

4. (α, v)-Convex and (α, v)-Concave intuitionistic fuzzy soft expert sets in 1st and 2^{st}-Sense

This portion will present the concept of (α, v)-convex and (α, v)-concave IFSESs in 1nd and 2nd-sense with some proved results.

Rahman et al. [44] have discussed (m; n) convexity with 1st sense for a fuzzy environment and now an extension is made by introducing this work in an intuitionistic fuzzy environment.

Definition 4.1: An IFSEs is named as a (α, v)-convex IFSEs in 1st-sense if

$$q_{ifses}\left(\alpha\epsilon_1 + v\left(1-\alpha^{\delta}\right)\epsilon_2\right) \supseteq q_{fses}(\epsilon_1) \cap q_{ifses}(\epsilon_1)$$

for $\alpha, \delta \in (0, 1], v \in [0, 1]$ and $\epsilon_1, \epsilon_2 \in \mathcal{S}$.

Example 4.1: Referring to Example (3.1) with $E = \{b_1, b_2, b_3\} = \{1, 2, 3\}$, and $X = \{f, g, h\} = \{1, 2, 3\}$, $\partial_1 = q_{ifses}(1, 1, 1) = \{\frac{w_1}{<0.5,0.5>}, \frac{w_2}{<0.2,0.6>}, \frac{w_3}{<0.4,0.6>}, \frac{w_4}{<0.1,0.8>}\}$, $\partial_2 = q_{ifses}(1, 2, 1) = \{\frac{w_1}{<0.4,0.5>}, \frac{w_2}{<0.2,0.1>}, \frac{w_3}{<0.4,0.5>}, \frac{w_4}{<0.2,0.6>}\}$

Take $\alpha = 0.6$, $v = 1$, $\delta = 0.5$ and $\epsilon_1 = (1, 1, 1)$, $\epsilon_2 = (1, 2, 1)$ then $q_{ifses}(\alpha\epsilon_1 + 1(1-\alpha^{\delta})\epsilon_2) = q_{ifses}(0.6(1, 1, 1) + 1(1-0.6^{0.5})(1, 2, 1) = q_{ifses}((0.6, 0.6, 0.6) + 0.22(1, 2, 1)) = q_{ifses}((0.6, 0.6, 0.6) + (0.22, 0.44, 0.22)) = q_{ifses}(0.28, 0.50, 0.28)$. By applying the decimal round off property, we get $\{0.28, 0.50, 0.28\} = \{0, 1, 0\}$ and also $(0, 1, 0) = (b_1, f, 0)$ $q_{ifses}(0, 1, 0) = \{\frac{w_1}{<0.5,0.5>}, \frac{w_2}{<0.3,0.7>}, \frac{w_3}{<0.4,0.6>}, \frac{w_4}{<0.2,0.8>}\}$ and $q_{ifses}(\epsilon_1) \cap q_{ifses}(\epsilon_2) = q_{ifses}(1, 1, 1) \cap q_{ifses}(1, 2, 1) = \{\frac{w_1}{<0.5,0.5>}, \frac{w_2}{<0.2,0.6>}, \frac{w_3}{<0.4,0.6>}, \frac{w_4}{<0.1,0.8>}\} \cap \{\frac{w_1}{<0.4,0.5>}, \frac{w_2}{<0.2,0.1>}, \frac{w_3}{<0.4,0.5>}, \frac{w_4}{<0.2,0.6>}\} = \{\frac{w_1}{0.4,0.5}, \frac{w_2}{<0.2,0.6>}, \frac{w_3}{<0.4,0.6>}, \frac{w_4}{<0.1,0.8>}\}$.

It is clear that

$$q_{ifses}(\alpha\epsilon_1 + (1-\alpha)\epsilon_2) \supseteq q_{ifses}(\epsilon_1) \cap q_{ifses}(\epsilon_2).$$

Rahman et al. [44] have discussed (m; n) concavity with 1st sense for fuzzy environment and now extension is made by introducing this work in intuitionistic fuzzy environment.

Definition 4.2: An IFSEs is named as a (α, v)-concave IFSEs in 1st-sense if

$$q_{ifses}\left(\alpha\epsilon_1 + v\left(1-\alpha^{\delta}\right)\epsilon_2\right) \subseteq q_{ifses}(\epsilon_1) \cap q_{ifses}(\epsilon_2)$$

for $\alpha, \delta \in (0, 1], v \in [0, 1]$ and $\epsilon_1, \epsilon_2 \in \mathcal{S}$.

Rahman et al. [44] have discussed (m; n) convexity with 2^{st} sense for fuzzy environment and now extension is made by introducing this work in intuitionistic fuzzy environment.

Definition 4.3: An IFSEs is named as a (α, v)-convex IFSEs in 2nd-sense if

$$q_{ifses}\left(\alpha\epsilon_1 + v(1-\alpha)^{\delta}\epsilon_2\right) \supseteq q_{ifses}(\epsilon_1) \cap q_{fses}(\epsilon_2)$$

for $\alpha, \delta \in (0, 1], v \in [0, 1]$ and $\epsilon_1, \epsilon_2 \in \mathcal{S}$.

Example 4.2: Referring to Example (3.1), Referring to Example (3.1) with $E = \{b_1, b_2, b_3\} = \{1, 2, 3\}$, and $X = \{f, g, h\} = \{1, 2, 3\}$, $\partial_1 = q_{ifses}(1, 1, 1) = \{\frac{w_1}{<0.5,0.5>}, \frac{w_2}{<0.2,0.6>}, \frac{w_3}{<0.4,0.6>}, \frac{w_4}{<0.1,0.8>}\}$, $\partial_2 = q_{ifses}(1, 2, 1) = \{\frac{w_1}{<0.4,0.5>}, \frac{w_2}{<0.2,0.1>}, \frac{w_3}{<0.4,0.5>}, \frac{w_4}{<0.2,0.6>}\}$.

Take $\alpha = 0.6$, $v = 1$, $\delta = 0.5$, and $\epsilon_1 = (1, 1, 1)$, $\epsilon_2 = (1, 2, 1)$ then $q_{ifses}(\alpha\epsilon_1 + 1(1-\alpha)^{\delta}\epsilon_2) = q_{ifses}(0.6(1, 1, 1) + 1(1-0.6)^{0.5}(1, 2, 1)) = q_{ifses}((0.6, 0.6, 0.6) + 0.63(1, 2, 1)) = q_{ifses}((0.6, 0.6, 0.6) + (0.63, 1.26, 0.63)) = q_{fses}(0.69, 1.32, 0.69)$.

By applying the decimal round off property, we get $\{0.69, 1.32, 0.69\} = \{1, 1, 1\}$ and also $(1, 1, 1) = (b_1, f, 1)$ $q_{ifses}(1, 1, 1) = \{\frac{w_1}{<0.5,0.5>}, \frac{w_2}{<0.2,0.6>}, \frac{w_3}{<0.4,0.6>}, \frac{w_4}{<0.1,0.8>}\}$ and $q_{ifses}(\epsilon_1) \cap q_{ifses}(\epsilon_2) = q_{ifses}(1, 1, 1) \cap q_{ifses}(1, 2, 1) = \{\frac{w_1}{<0.5,0.5>}, \frac{w_2}{<0.2,0.6>}, \frac{w_3}{<0.4,0.6>}, \frac{w_4}{<0.1,0.8>}\} \cap \{\frac{w_1}{<0.4,0.5>}, \frac{w_2}{<0.2,0.1>}, \frac{w_3}{<0.4,0.5>}, \frac{w_4}{<0.2,0.6>}\} = \{\frac{w_1}{<0.4,0.5>}, \frac{w_2}{<0.2,0.6>}, \frac{w_3}{<0.4,0.6}, \frac{w_4}{<0.1,0.8>}\}$.

It is clear that

$$q_{ifses}(\alpha\epsilon_1 + (1-\alpha)\epsilon_2) \supseteq q_{ifses}(\epsilon_1) \cap q_{ifses}(\epsilon_2).$$

Rahman et al. [44] have discussed (m; n) concavity with 1st sense for fuzzy environment and now extension is made by introducing this work in intuitionistic fuzzy environment.

Definition 4.4: An IFSEs is named as a (α, v)-concave IFSEs in 2nd-sense if

$$q_{ifses}\left(\alpha\epsilon_1 + v(1-\alpha)^{\delta}\epsilon_2\right) \subseteq q_{ifses}(\epsilon_1) \cap q_{ifses}(\epsilon_1)$$

for $\alpha, \delta \in (0, 1], v \in [0, 1]$ and $\epsilon_1, \epsilon_2 \in \mathcal{S}$.

Theorem 4.1: *$P_1 \cap P_2$ is a (α, v)-convex IFSEs in 1st-sense while P_1 and P_2 are (α, v)-convex IFSESs in 1st-sense.*

Proof: Assume $\exists\epsilon_1, \epsilon_2 \in \mathcal{S}$ and $\alpha \in (0, 1]$, and $G_3 = P_1 \cap P_2$.

$$q_{ifses(P_3)}\left(\alpha\epsilon_1 + v\left(1-\alpha^{\delta}\right)\epsilon_2\right) = q_{ifses(P_1)}\left(\alpha\epsilon_1 + v\left(1-\alpha^{\delta}\right)\epsilon_2\right) \cap q_{ifses(P_2)}\left(\alpha\epsilon_1 + v\left(1-\alpha^{\delta}\right)\epsilon_2\right)$$

$\because P_1$ and P_2 are (α, v)- convex IFSESs in 1st-sense, so

$$q_{ifses(P_1)}\left(\alpha\epsilon_1 + v\left(1-\alpha^{\delta}\right)\epsilon_2\right) \supseteq q_{ifses(P_1)}(\epsilon_1) \cap q_{ifses(P_1)}(\epsilon_1) \quad (11)$$

$$q_{ifses(P_2)}\left(\alpha\epsilon_1 + v\left(1-\alpha^{\delta}\right)\epsilon_2\right) \supseteq q_{ifses(P_2)}(\epsilon_1) \cap q_{ifses(P_2)}(\epsilon_1) \quad (12)$$

By applying the intersection operation, we get

$$q_{ifses(P_3)}\left(\alpha\epsilon_1 + v\left(1-\alpha^{\delta}\right)\epsilon_2\right) \supseteq \{(q_{ifses(P_1)}(\epsilon_1) \cap q_{ifses(P_1)}(\epsilon_1)) \cap (q_{ifses(P_2)}(\epsilon_1) \cap q_{ifses(P_2)}(\epsilon_1))\} \quad (13)$$

$q_{ifses(P_3)}(\alpha\epsilon_1 + v(1-\alpha^\delta)\epsilon_2) \supseteq q_{fses(P_3)}(\epsilon_1) \cap q_{ifses(P_3)}(\epsilon_1)$

hence the result is proved. ■

Remark 4.1: If $\{P_k : k \in \{1,2,3,\ldots\}\}$ is any collection of (α, v)-convex IFSESs in 1st-sense, then the $\bigcap_{k\in I} P_k$ is a (α, v)-convex IFSEs in 1st-sense.

Theorem 4.2: *$P_1 \cap P_2$ is a (α, v)-convex IFSEs in 2^{st}-sense while P_1 and P_2 are (α, v)-convex IFSESs in 2^{st}-sense.*

Proof: Assume $\exists\epsilon_1, \epsilon_2 \in \mathcal{S}$ and $\alpha \in (0,1]$, and $P_3 = P_1 \cap P_2$.

$$q_{ifses(P_3)}\left(\alpha\epsilon_1 + v(1-\alpha)^\delta \epsilon_2\right) = q_{IFSEs(P_1)}\left(\alpha\epsilon_1 + v(1-\alpha)^\delta \epsilon_2\right) \cap q_{IFSEs(P_2)}\left(\alpha\epsilon_1 + v(1-\alpha)^\delta \epsilon_2\right)$$

$\because P_1$ and P_2 are (α, v)-convex IFSESs in 2^{st}-sense, so

$$q_{IFSEs(P_1)}\left(\alpha\epsilon_1 + v(1-\alpha)^\delta \epsilon_2\right) \supseteq q_{IFSEs(P_1)}(\epsilon_1) \cap q_{IFSEs(P_1)}(\epsilon_1) \quad (14)$$

$$q_{IFSEs(P_2)}\left(\alpha\epsilon_1 + v(1-\alpha)^\delta \epsilon_2\right) \supseteq q_{IFSEs(P_2)}(\epsilon_1) \cap q_{IFSEs(P_2)}(\epsilon_1) \quad (15)$$

By applying the union operation, we get

$$q_{IFSEs(P_3)}\left(\alpha\epsilon_1 + v(1-\alpha)^\delta\epsilon_2\right) \subseteq \left\{\left(q_{IFSEs(P_1)}(\epsilon_1) \cup q_{fses(P_1)}(\epsilon_2)\right) \cup \left(q_{IFSEs(P_2)}(\epsilon_1) \cup q_{IFSEs(P_2)}(\epsilon_2)\right)\right\}$$

$$q_{IFSEs(P_3)}\left(\alpha\epsilon_1 + v(1-\alpha)^\delta \epsilon_2\right) \supseteq q_{IFSEs(P_3)}(\epsilon_1) \cap q_{IFSEs(P_3)}(\epsilon_1) \quad (16)$$

hence the result is proved. ■

Remark 4.2: If $\{P_k : k \in \{1,2,3,\ldots\}\}$ is any collection of (α, v)-convex IFSESs in 2^{st}-sense sense, then the $\bigcap_{k\in I} P_k$ is a (α, v)-convex IFSEs in 2^{st}-sense sense.

Theorem 4.3: *L is a (α, v)-convex IFSEs in* 1st-*sense on $\mathcal{S}$ iff for each $\alpha \in (0,1]$ and $\rho \in P(\mathcal{Z})$, L^ρ is (α, v)-convex IFSEs in* 1st-*sense on $\mathcal{S}$.*

Proof: Suppose that L is (α, v)-convex IFSEs in 1st-sense. If $\epsilon_1, \epsilon_2 \in \mathcal{S}$ and $\rho \in P(\mathcal{Z})$, then $q_{fses(L)}(\epsilon_1) \supseteq \rho$ and $q_{fses(L)}(\epsilon_2) \supseteq \rho$.

By the convexity of D, we have

$$q_{IFSEs(L)}\left(\alpha\epsilon_1 + v\left(1-\alpha^\delta\right)\epsilon_2\right) \supseteq q_{IFSEs(L)}(\epsilon_1) \cap q_{IFSEs(L)}(\epsilon_2)$$

so L^ρ is (α, v)-convex IFSEs in 1st-sense.

Conversely, suppose that L^ρ is (α, v)-convex IFSEs in 1st-sense for each $\alpha \in (0,1]$.

Then, for $\epsilon_1, \epsilon_2 \in \mathcal{S}$, L^ρ is (α, v)-convex IFSEs in 1st-sense for $\rho = q_{IFSEs(L)}(\epsilon_1) \cap q_{IFSEs(L)}(\epsilon_2)$.

$\because q_{IFSEs(L)}(\epsilon_1) \supseteq \rho$ and $q_{IFSEs(L)}(\epsilon_2) \supseteq \rho$, we have $\epsilon_1 \in L^\rho$ and $\epsilon_2 \in L^\rho$, hence $\alpha\epsilon_1 + v(1-\alpha^\delta)\epsilon_2 \in D^\rho$.

$\therefore q_{IFSEs(L)}(\alpha\epsilon_1 + v(1-\alpha^\delta)\epsilon_2) \supseteq q_{IFSEs(L)}(\epsilon_1) \cap q_{IFSEs(L)}(\epsilon_2)$, which implies that L is (α, v)-convex IFSEs in 1st-sense. ■

Theorem 4.4: *L is a (α, v)-convex IFSEs in 2^{st}-sense on $\mathcal{S}$ iff for each $\alpha \in (0,1]$ and $\rho \in P(\mathcal{Z})$, L^ρ is (α, v)-convex IFSEs in 2^{st}-sense on $\mathcal{S}$.*

Proof: Suppose that L is (α, v)-convex IFSEs in 2^{st}-sense. If $\epsilon_1, \epsilon_2 \in \mathcal{S}$ and $\rho \in P(Z)$, then $q_{IFSEs(L)}(\epsilon_1) \supseteq \rho$ and $q_{IFSEs(L)}(\epsilon_2) \supseteq \rho$.

By the convexity of L, we have

$$q_{IFSEs(L)}\left(\alpha\epsilon_1 + v(1-\alpha)^\delta \epsilon_2\right) \supseteq q_{IFSEs(L)}(\epsilon_1) \cap q_{IFSEs(L)}(\epsilon_2)$$

so L^ρ is (α, v)-convex IFSEs in 2^{st}-sense.

Conversely, suppose that L^ρ is (α, v)-convex ifse-set in 2^{st}-sense for each $\alpha \in (0,1]$.

Then, for $\epsilon_1, \epsilon_2 \in \mathcal{S}$, L^ρ is (α, v)-convex ifse-set in2^{st}-sense.

For $\rho = q_{IFSEs(L)}(\epsilon_1) \cap q_{IFSEs(L)}(\epsilon_2)$.

$\because q_{IFSEs(L)}(\epsilon_1) \supseteq \rho$ and $q_{IFSEs(L)}(\epsilon_2) \supseteq \rho$, we have $\epsilon_1 \in L^\rho$ and $\epsilon_2 \in L^\rho$, hence $\alpha\epsilon_1 + v(1-\alpha)^\delta\epsilon_2 \in L^\rho$.

$\therefore q_{IFSEs(L)}(\alpha\epsilon_1 + v(1-\alpha)^\delta\epsilon_2) \supseteq q_{IFSEs(L)}(\epsilon_1) \cap q_{IFSEs(L)}(\epsilon_2)$, which implies that L is (α, v)-convex IFSEs in 2^{st}-sense. ■

Theorem 4.5: *O' is (α, v)-concave IFSEs in* 1st-*sense while O is (α, v)-convex IFSEs in* 1st-*sense.*

Proof: Assume that $\exists\epsilon_1, \epsilon_2 \in \mathcal{S}$, $\alpha \in (0,1]$.

$\because O$ is (α, v)-convex IFSEs in 1st-sense,

$$q_{IFSEs(O)}\left(\alpha\epsilon_1 + v\left(1-\alpha^\delta\right)\epsilon_2\right) \supseteq q_{IFSEs(O)}(\epsilon_1) \cap q_{IFSEs(O)}(\epsilon_2)$$

or

$$\mathcal{Z}\setminus q_{IFSEs(O)}\left(\alpha\epsilon_1 + v\left(1-\alpha^\delta\right)\epsilon_2\right) \subseteq \mathcal{Z}\setminus\left\{q_{IFSEs(O)}(\epsilon_1) \cap q_{IFSEs(O)}(\epsilon_2)\right\}$$

If $q_{IFSEs(O)}(\epsilon_1) \supseteq q_{IFSEs(O)}(\epsilon_2)$, then we may write

$$\mathcal{Z}\setminus q_{IFSEs(O)}\left(\alpha\epsilon_1 + v\left(1-\alpha^\delta\right)\epsilon_2\right) \subseteq \mathcal{Z}\setminus q_{IFSEs(O)}(\epsilon_2) \quad (17)$$

If $q_{IFSEs(O)}(\epsilon_1) \subseteq q_{IFSEs(O)}(\epsilon_2)$, then we may write

$$\mathcal{Z}\setminus q_{IFSEs(O)}\left(\alpha\epsilon_1 + v\left(1-\alpha^\delta\right)\epsilon_2\right) \subseteq \mathcal{Z}\setminus q_{IFSEs(O)}(\epsilon_1) \quad (18)$$

From Equations (17) and (18), we have

$$\mathcal{Z}\setminus q_{IFSEs(O)}\left(\alpha\epsilon_1 + v\left(1-\alpha^\delta\right)\epsilon_2\right) \subseteq \left\{\mathcal{Z}\setminus q_{IFSEs(O)}(\epsilon_1) \cup \mathcal{Z}\setminus q_{IFSEs(O)}(\epsilon_2)\right\}. \quad (19)$$

So, O' is (α, v)-concave IFSEs in 1st-sense. ■

Theorem 4.6: *O' is (α, v)-concave IFSEs in 2^{st}-sense while O is (α, v)-convex IFSEs in 2^{st}-sense.*

Proof: Assume that $\exists \epsilon_1, \epsilon_2 \in \mathcal{S}$, $\alpha \in (0, 1]$.
$\because O$ is (α, v)-convex IFSEs in 2^{st}-sense,

$$q_{IFSEs(O)}\left(\alpha\epsilon_1 + v(1-\alpha)^{\delta}\epsilon_2\right) \supseteq q_{IFSEs(O)}(\epsilon_1) \cap q_{IFSEs(O)}(\epsilon_2)$$

or

$$\mathcal{Z}\setminus q_{IFSEs(O)}\left(\alpha\epsilon_1 + v(1-\alpha)^{\delta}\epsilon_2\right) \subseteq \mathcal{Z}\setminus\left\{q_{IFSEs(O)}(\epsilon_1) \cap q_{IFSEs(O)}(\epsilon_2)\right\}$$

If $q_{IFSEs(O)}(\epsilon_1) \supseteq q_{IFSEs(O)}(\epsilon_2)$, then we may write

$$\mathcal{Z}\setminus q_{IFSEs(O)}\left(\alpha\epsilon_1 + v(1-\alpha)^{\delta}\epsilon_2\right) \subseteq \mathcal{Z}\setminus q_{IFSEs(O)}(\epsilon_2) \tag{20}$$

If $q_{IFSEs(O)}(\epsilon_1) \subseteq q_{IFSEs(O)}(\epsilon_2)$, then we may write

$$\mathcal{Z}\setminus q_{IFSEs(O)}\left(\alpha\epsilon_1 + v(1-\alpha)^{\delta}\epsilon_2\right) \subseteq \mathcal{Z}\setminus q_{IFSEs(O)}(\epsilon_1) \tag{21}$$

From Equations (20) and (21), we have

$$\mathcal{Z}\setminus q_{IFSEs(O)}\left(\alpha\epsilon_1 + v(1-\alpha)^{\delta}\epsilon_2\right) \subseteq \left\{\mathcal{Z}\setminus q_{IFSEs(O)}(\epsilon_1) \cup \mathcal{Z}\setminus q_{IFSEs(O)}(\epsilon_2)\right\}. \tag{22}$$

So, O' is (α, v)-concave IFSEs in 2^{st}-sense. ■

Theorem 4.7: *Æ′ is (α, v)-convex IFSEs in 1st-sense while Æ is (α, v)-concave IFSEs in 1st-sense.*

Proof: Assume that $\exists \epsilon_1, \epsilon_2 \in \mathcal{S}$, $\alpha \in (0, 1]$.
$\because$Æ is (α, v)-concave IFSEs in 1st-sense,

$$q_{IFSEs(Æ)}\left(\alpha\epsilon_1 + v\left(1-\alpha^{\delta}\right)\epsilon_2\right) \subseteq q_{IFSEs(Æ)}(\epsilon_1) \cup q_{IFSEs(Æ)}(\epsilon_2)$$

or

$$\mathcal{Z}\setminus q_{IFSEs(Æ)}\left(\alpha\epsilon_1 + v\left(1-\alpha^{\delta}\right)\epsilon_2\right) \supseteq \mathcal{Z}\setminus\left\{q_{IFSEs(Æ)}(\epsilon_1) \cup q_{IFSEs(Æ)}(\epsilon_2)\right\}$$

If $q_{IFSEs(Æ)}(\epsilon_1) \supseteq q_{IFSEs(Æ)}(\epsilon_2)$ then we may write

$$\mathcal{Z}\setminus q_{IFSEs(Æ)}\left(\alpha\epsilon_1 + \left(1-\alpha^{\delta}\right)\epsilon_2\right) \supseteq \mathcal{Z}\setminus q_{IFSEs(Æ)}(\epsilon_1) \tag{23}$$

If $q_{IFSEs(Æ)}(\epsilon_1) \subseteq q_{IFSEs(Æ)}(\epsilon_2)$ $\therefore$

$$\mathcal{Z}\setminus q_{IFSEs(Æ)}\left(\alpha\epsilon_1 + v\left(1-\alpha^{\delta}\right)\epsilon_2\right) \supseteq \mathcal{Z}\setminus q_{IFSEs(Æ)}(\epsilon_2) \tag{24}$$

From (23) and (24), we have

$$\mathcal{Z}\setminus q_{IFSEs(Æ)}\left(\alpha\epsilon_1 + v\left(1-\alpha^{\delta}\right)\epsilon_2\right) \supseteq \left\{\mathcal{Z}\setminus q_{IFSEs(Æ)}(\epsilon_1) \cap \mathcal{Z}\setminus q_{IFSEs(Æ)}(\epsilon_2)\right\} \tag{25}$$

which shows that Æ′ is (α, v)-convex IFSEs in 1st-sense. ■

Theorem 4.8: *Æ′ is (α, v)-convex IFSEs in 2^{st}-sense while Æ is (α, v)-concave IFSEs in 2^{st}-sense.*

Proof: Assume that $\exists \epsilon_1, \epsilon_2 \in \mathcal{S}$, $\alpha \in (0, 1]$.
$\because$Æ is (α, v)-concave IFSEs in 2^{st}-sense,

$$q_{IFSEs(Æ)}\left(\alpha\epsilon_1 + v(1-\alpha)^{\delta}\epsilon_2\right) \subseteq q_{IFSEs(Æ)}(\epsilon_1) \cup q_{IFSEs(Æ)}(\epsilon_2)$$

or

$$\mathcal{Z}\setminus q_{IFSEs(Æ)}\left(\alpha\epsilon_1 + v(1-\alpha)^{\delta}\epsilon_2\right) \supseteq \mathcal{Z}\setminus\left\{q_{IFSEs(Æ)}(\epsilon_1) \cup q_{IFSEs(Æ)}(\epsilon_2)\right\}$$

If $q_{IFSEs(Æ)}(\epsilon_1) \supseteq q_{IFSEs(Æ)}(\epsilon_2)$ then we may write

$$\mathcal{Z}\setminus q_{IFSEs(Æ)}\left(\alpha\epsilon_1 + (1-\alpha)^{\delta}\epsilon_2\right) \supseteq \mathcal{Z}\setminus q_{IFSEs(Æ)}(\epsilon_1) \tag{26}$$

If $q_{IFSEs(Æ)}(\epsilon_1) \subseteq q_{IFSEs(Æ)}(\epsilon_2)$ $\therefore$

$$\mathcal{Z}\setminus q_{IFSEs(Æ)}\left(\alpha\epsilon_1 + v(1-\alpha)^{\delta}\epsilon_2\right) \supseteq \mathcal{Z}\setminus q_{IFSEs(Æ)}(\epsilon_2) \tag{27}$$

From (26) and (27), we have

$$\mathcal{Z}\setminus q_{IFSEs(Æ)}\left(\alpha\epsilon_1 + v(1-\alpha)^{\delta}\epsilon_2\right) \supseteq \left\{\mathcal{Z}\setminus q_{IFSEs(Æ)}(\epsilon_1) \cap \mathcal{Z}\setminus q_{IFSEs(Æ)}(\epsilon_2)\right\} \tag{28}$$

which shows that Æ′ is (α, v)-convex IFSEs in 2^{st}-sense. ■

Theorem 4.9: *Œ is (α, v)-concave IFSEs in 1st-sense on $\mathcal{S}$ iff for each $\alpha \in (0, 1]$ and $\rho \in P(\mathcal{Z})$, $Œ^{\rho}$ is (α, v)-concave IFSEs in 1st-sense on $\mathcal{S}$.*

Proof: Suppose that Œ is (α, v)-concave IFSEs in 1st-sense. If $\epsilon_1, \epsilon_2 \in \mathcal{S}$ and $\rho \in P(\mathcal{Z})$, then $q_{IFSEs(Œ)}(\epsilon_1) \supseteq \rho$ and $q_{IFSEs(Œ)}(\epsilon_2) \supseteq \rho$.

By the (α, v)-concavity of Œ in 1st-sense, we have

$$q_{IFSEs(Œ)}\left(\alpha\epsilon_1 + v\left(1-\alpha^{\delta}\right)\epsilon_2\right) \subseteq q_{IFSEs(Œ)}(\epsilon_1) \cup q_{IFSEs(Œ)}(\epsilon_2)$$

hence $Œ^{\rho}$ is a (α, v)-concave IFSEs in 1st-sense.

Conversely, suppose that $Œ^{\rho}$ is (α, v)-concave IFSEs in 1st-sense for each $\alpha \in (0, 1]$.

Subsequently, for $\epsilon_1, \epsilon_2 \in \mathcal{S}$, $Œ^{\rho}$ is (α, v)-concave IFSEs in 1st-sense for $\rho = q_{IFSEs(Œ)}(\epsilon_1) \cup q_{IFSEs(Œ)}(\epsilon_2)$.

$\therefore q_{IFSEs(Œ)}(\epsilon_1) \supseteq \rho$ and $q_{IFSEs(Œ)}(\epsilon_2) \supseteq \rho$, we have $\epsilon_1 \in Œ^{\rho}$ and $\epsilon_2 \in Œ^{\rho}$, hence $\alpha\epsilon_1 + v(1-\alpha^{\delta})\epsilon_2 \in Œ^{\rho}$ $\therefore q_{IFSEs(Œ)}(\alpha\epsilon_1 + v(1-\alpha^{\delta})\epsilon_2) \subseteq q_{IFSEs(Œ)}(\epsilon_1) \cup q_{IFSEs(Œ)}(\epsilon_2)$, it is clear that Œ is (α, v)-concave IFSEs in 1st-sense. ■

Theorem 4.10: *Œ is (α, v)-concave IFSEs in 2^{st}-sense on $\mathcal{S}$ iff for each $\alpha \in (0, 1]$ and $\rho \in P(\mathcal{Z})$,*

$Œ^\rho$ is (α, v)-concave IFSEs in 2^{st}-sense on $\mathcal{S}$.

Proof: Suppose that Œ is (α, v)-concave IFSEs in 2^{st}-sense. If $\epsilon_1, \epsilon_2 \in \mathcal{S}$ and $\rho \in P(\mathcal{Z})$, then $q_{IFSEs(Œ)}(\epsilon_1) \supseteq \rho$ and $q_{IFSEs(Œ)}(\epsilon_2) \supseteq \rho$.

By the (α, v)-concavity of Œ in 2^{st}-sense, we have

$$q_{IFSEs(Œ)}\left(\alpha\epsilon_1 + v(1-\alpha)^\delta \epsilon_2\right) \subseteq q_{IFSEs(Œ)}(\epsilon_1) \cup q_{IFSEs(Œ)}(\epsilon_2)$$

hence $Œ^\rho$ is a (α, v)-concave IFSEs in 2^{st}-sense.

Conversely suppose that $Œ^\rho$ is (α, v)-concave IFSEs in 2^{st}-sense for each $\alpha \in (0, 1]$.

Subsequently, for $\epsilon_1, \epsilon_2 \in \mathcal{S}$, $Œ^\rho$ is (α, v)-concave IFSEs in 2^{st}-sense for $\rho = q_{IFSEs(Œ)}(\epsilon_1) \cup q_{IFSEs(Œ)}(\epsilon_2)$.

$\therefore q_{IFSEs(Œ)}(\epsilon_1) \supseteq \rho$ and $q_{IFSEs(Œ)}(\epsilon_2) \supseteq \rho$, we have $\epsilon_1 \in Œ^\rho$ and $\epsilon_2 \in Œ^\rho$, hence $\alpha\epsilon_1 + v(1-\alpha)^\delta\epsilon_2 \in Œ^\rho$. $\therefore$, $q_{IFSEs(Œ)}(\alpha\epsilon_1 + v(1-\alpha)^\delta\epsilon_2) \subseteq q_{IFSEs(Œ)}(\epsilon_1) \cup q_{IFSEs(Œ)}(\epsilon_2)$, it is clear that Œ is (α, v)-concave IFSEs in 2^{st}- sense. ∎

Theorem 4.11: *$P_1 \cup P_2$ is a (α, v)-concave IFSEs in 1st-sense while P_1 and P_2 are (α, v)-concave IFSEss in 1st-sense.*

Proof: Assume $\exists\epsilon_1, \epsilon_2 \in \mathcal{S}$ and $\alpha \in (0, 1]$, and $P_3 = P_1 \cup P_2$.

$$\begin{aligned} q_{IFSEs(P_3)}\left(\alpha\epsilon_1 + v\left(1-\alpha^\delta\right)\epsilon_2\right) &= q_{IFSEs(P_1)}\left(\alpha\epsilon_1 + v\left(1-\alpha^\delta\right)\epsilon_2\right) \\ &\cup q_{IFSEs(P_2)}\left(\alpha\epsilon_1 + v\left(1-\alpha^\delta\right)\epsilon_2\right) \end{aligned}$$

$\because P_1$ and P_2 are (α, v)-concave IFSEss in 1st-sense, so

$$q_{IFSEs(P_1)}\left(\alpha\epsilon_1 + v\left(1-\alpha^\delta\right)\epsilon_2\right) \subseteq q_{IFSEs(P_1)}(\epsilon_1) \cup q_{IFSEs(P_1)}(\epsilon_2) \tag{29}$$

$$q_{IFSEs(P_2)}\left(\alpha\epsilon_1 + v\left(1-\alpha^\delta\right)\epsilon_2\right) \subseteq q_{IFSEs(P_2)}(\epsilon_1) \cup q_{IFSEs(P_2)}(\epsilon_2) \tag{30}$$

By applying the union operation, we get

$$\begin{aligned} q_{IFSEs(P_3)}\left(\alpha\epsilon_1 + v\left(1-\alpha^\delta\right)\epsilon_2\right) &\subseteq \{(q_{IFSEs(P_1)}(\epsilon_1) \cup q_{IFSEs(P_1)}(\epsilon_2)) \\ &\cup (q_{IFSEs(P_2)}(\epsilon_1) \cup q_{IFSEs(P_2)}(\epsilon_2))\} \end{aligned} \tag{31}$$

$\Rightarrow q_{IFSEs(P_3)}(\alpha\epsilon_1 + v(1-\alpha^\delta)\epsilon_2) \subseteq q_{IFSEs(P_3)}(\epsilon_1) \cup q_{IFSEs(P_3)}(\epsilon_2)$.

Here we have proved the union of two concave IFSEss in 1st-sense, it can be proved for more than two concave IFSEss in 1st-sense. Hence we can generalized this result up to any countable number of concave IFSEss in 1st-sense. ∎

5. Strongly and strictly convex (concave)

This section contains definitions of strongly convex IFSEs, strictly convex IFSEs, strongly concave IFSEs, strictly concave IFSEs and describes their characteristics.

Salleh and Sabir [43] conceptualized some characteristics of convexity(concavity) like strictly and strongly in s-set environment. Now these properties (Definitions 5.1, 5.2, 5.3, 5.4) have been generalized for intuitionistic fuzzy environment.

Definition 5.1: The IFSEs over $\mathcal{S}$ are called strongly convex IFSEs if

$$q_{IFSEs}(\alpha\epsilon_1 + (1-\alpha)\epsilon_2) \supset q_{IFSEs}(\epsilon_1) \cap q_{IFSEs}(\epsilon_2)$$

for each $\epsilon_1, \epsilon_2 \in \mathcal{S}, \epsilon_1 \neq \epsilon_2$ and $\alpha \in \Gamma^0 = (0, 1)$.

Definition 5.2: The IFSEs over $\mathcal{S}$ are called strongly concave IFSEs if

$$q_{IFSEs}(\alpha\epsilon_1 + (1-\alpha)\epsilon_2) \subset q_{IFSEs}(\epsilon_1) \cup q_{IFSEs}(\epsilon_2)$$

for each $\epsilon_1, \epsilon_2 \in \mathcal{S}, \epsilon_1 \neq \epsilon_2$ and $\alpha \in \Gamma^0$.

Definition 5.3: The IFSEs over $\mathcal{S}$ are called strictly convex IFSEs if

$$q_{IFSEs}(\alpha\epsilon_1 + (1-\alpha)\epsilon_2) \supset q_{IFSEs}(\epsilon_1) \cap q_{IFSEs}(\epsilon_2)$$

for each $\epsilon_1, \epsilon_2 \in \mathcal{S}, q_{IFSEs}(\epsilon_1) \neq q_{IFSEs}(\epsilon_2)$ and $\alpha \in \Gamma^0$.

Definition 5.4: The IFSEs on $\mathcal{S}$ are said to be strictly concave IFSEs if

$$q_{IFSEs}(\alpha\epsilon_1 + (1-\alpha)\epsilon_2) \subset q_{IFSEs}(\epsilon_1) \cup q_{IFSEs}(\epsilon_2)$$

for each $\epsilon_1, \epsilon_2 \in \mathcal{S}, q_{IFSEs}(\epsilon_1) \neq q_{IFSEs}(\epsilon_2)$ and $\alpha \in \Gamma^0$.

Theorem 5.1: *Suppose $(q_{IFSEs}, \mathcal{S})$ be a strictly convex IFSEs. If $\exists\alpha \in \Gamma^0 \forall\epsilon_1, \epsilon_2 \in \mathcal{S}$ such that*

$$q_{IFSEs}(\alpha\epsilon_1 + (1-\alpha)\epsilon_2) \supseteq q_{IFSEs}(\epsilon_1) \cap q_{IFSEs}(\epsilon_2)$$

then $(q_{IFSEs}, \mathcal{S})$ is a convex IFSEs.

Proof: Assume that $q_{IFSEs}(\epsilon_1) \subseteq q_{IFSEs}(\epsilon_2)$ and $\exists\epsilon_1, \epsilon_2 \in \mathcal{S}, \delta \in \Gamma^0$

$$\mathcal{Z} \backslash q_{IFSEs(T)}(\delta\epsilon_1 + (1-\delta)\epsilon_2) \supseteq \mathcal{Z} \backslash \left\{q_{IFSEs(T)}(\epsilon_1) \cap q_{IFSEs(T)}(\epsilon_2)\right\} \tag{32}$$

if $q_{IFSEs}(\epsilon_1) \subset q_{IFSEs}(\epsilon_2)$, then (30) contradicts that $(q_{IFSEs}, \mathcal{S})$ is a strictly convex IFSEs.

Now if we take $q_{IFSEs}(\epsilon_1) = q_{IFSEs}(\epsilon_2)$ and $\delta \in [0, \alpha]$, suppose $\epsilon_3 = \frac{\delta}{\alpha}\epsilon_1 + (1 - \frac{\delta}{\alpha})\epsilon_2$ and $\tau = (\frac{1}{\delta} - 1)(\frac{1}{\alpha} - 1)^{-1}$. So

$$q_{IFSEs(T)}(\delta\epsilon_1 + (1-\delta)\epsilon_2) = q_{IFSEs(T)}\left(\alpha\left(\frac{\delta}{\alpha}\epsilon_1 + \left(1 - \frac{\delta}{\alpha}\right)\epsilon_2\right) + (1-\delta)\epsilon_2\right)$$

$$= q_{IFSEs(T)}\left(\alpha\epsilon_3 + (1-\alpha)\epsilon_2\right) \supseteq \left\{q_{IFSEs(T)}(\epsilon_1) \cap q_{IFSEs(T)}(\epsilon_2)\right\}$$

$\Rightarrow$

$$q_{IFSEs(T)}\left(\delta\epsilon_1 + (1-\delta)\epsilon_2\right) \supseteq \left\{q_{IFSEs(T)}(\epsilon_1) \cap q_{IFSEs(T)}(\epsilon_2)\right\} \tag{33}$$

now

$$q_{IFSEs(T)}(\epsilon_3) = q_{IFSEs(T)}\left(\frac{\delta}{\alpha}\epsilon_1\right) + \left(1 - \frac{\delta}{\alpha}\epsilon_2\right) = q_{IFSEs(T)}\left(\tau\epsilon_1 + (1-\tau) \times \left(\delta\epsilon_1 + (1-\delta)\epsilon_2\right)\right) \tag{34}$$

From (32), (33) and $q_{IFSEs}(\epsilon_1) = q_{IFSEs}(\epsilon_2)$, we have

$$q_{IFSEs(T)}\left(\delta\epsilon_1 + (1-\delta)\epsilon_2\right) \supseteq q_{IFSEs}(\epsilon_3), \tag{35}$$

also from (31), (32) $q_{IFSEs}(\epsilon_1) = q_{IFSEs}(\epsilon_2)$ and strictly convex IFSEs condition, we have

$$q_{IFSEs(T)}\left(\delta\epsilon_1 + (1-\delta)\epsilon_2\right) \subset q_{IFSEs}(\epsilon_3), \tag{36}$$

or

$$\mathcal{Z}\setminus q_{IFSEs(T)}\left(\delta\epsilon_1 + (1-\delta)\epsilon_2\right) \supseteq \mathcal{Z}\setminus q_{IFSEs}(\epsilon_3), \tag{37}$$

thus (34) and (36) contradict the fact. ∎

If $q_{IFSEs}(\epsilon_1) = q_{IFSEs}(\epsilon_2)$ and $\delta \in [\alpha, 1]$, suppose $\epsilon_4 = \frac{\delta-\alpha}{1-\alpha}\epsilon_1 + \frac{1-\delta}{1-\alpha}\epsilon_2$ then $q_{IFSEs(T)}(\delta\varepsilon_1 + (1-\delta)\varepsilon_2) = \{q_{IFSEs(T)}(\alpha\varepsilon_1 + (1-\alpha)\varepsilon_4) \supseteq q_{IFSEs(T)}(\varepsilon_1) \cap q_{IFSEs(T)}(\varepsilon_4)\}$

$$q_{IFSEs(T)}\left(\delta\epsilon_1 + (1-\delta)\epsilon_2\right) \supseteq \left\{q_{IFSEs(T)}(\epsilon_1) \cap q_{IFSEs(T)}(\epsilon_4)\right\} \tag{38}$$

From (32), (37) and $q_{IFSEs}(\epsilon_1) = q_{IFSEs}(\epsilon_2)$, we have

$$q_{IFSEs(T)}\left(\delta\epsilon_1 + (1-\delta)\epsilon_2\right) \supseteq q_{IFSEs}(\epsilon_4), \tag{39}$$

Also, $(\delta\epsilon_1 + (1-\delta)\epsilon_2) = (\alpha\epsilon_1 + (1-\alpha)\epsilon_4)$ becomes

$$\begin{aligned}\epsilon_4 &= \frac{1}{1-\alpha}(\delta\epsilon_1 + (1-\delta)\epsilon_2) - \frac{\alpha}{1-\alpha}\epsilon_1 \\ &= \frac{1}{1-\alpha}(\delta\epsilon_1 + (1-\delta)\epsilon_2) - \frac{\alpha}{1-\alpha}\epsilon_1\left(\frac{1}{\delta}(\delta\epsilon_1 + (1-\delta)\epsilon_2) - \frac{1-\delta}{\delta}\epsilon_2\right) \\ &= \frac{\delta-\alpha}{(1-\alpha)\delta}(\delta\epsilon_1 + (1-\delta)\epsilon_2) + \left(1 - \frac{\delta-\alpha}{(1-\alpha)\delta}\right)\epsilon_2\end{aligned}$$

$$\epsilon_4 = \frac{\delta-\alpha}{(1-\alpha)\delta}\left(\delta\epsilon_1 + (1-\delta)\epsilon_2\right) + \left(1 - \frac{\delta-\alpha}{(1-\alpha)\delta}\right)\epsilon_2 \tag{40}$$

now from (32), (39) $q_{IFSEs}(\epsilon_1) = q_{IFSEs}(\epsilon_2)$ and strictly convex IFSEs condition, we have

$$\mathcal{Z}\setminus q_{IFSEs(T)}\left(\delta\epsilon_1 + (1-\delta)\epsilon_2\right) \supseteq \mathcal{Z}\setminus q_{IFSEs}(\epsilon_4) \tag{41}$$

Hence (39) and (40) contradict the fact.

Theorem 5.2: *Suppose $(q_{IFSEs}, \mathcal{S})$ be a strictly concave IFSEs. If $\exists\alpha \in \Gamma^0 \forall\epsilon_1, \epsilon_2 \in \mathcal{S}$ such that*

$$q_{IFSEs}\left(\alpha\epsilon_1 + (1-\alpha)\epsilon_2\right) \subseteq q_{IFSEs}(\epsilon_1) \cap q_{IFSEs}(\epsilon_2)$$

then $(q_{IFSEs}, \mathcal{S})$ is a concave IFSEs.

Proof: Assume that $q_{IFSEs}(\epsilon_1) \supseteq q_{IFSEs}(\epsilon_2)$ and $\exists\epsilon_1, \epsilon_2 \in \mathcal{S}, \delta \in \Gamma^0$

$$\mathcal{Z}\setminus q_{IFSEs(T)}\left(\delta\epsilon_1 + (1-\delta)\epsilon_2\right) \subseteq \mathcal{Z}\setminus \left\{q_{IFSEs(T)}(\epsilon_1) \cap q_{IFSEs(T)}(\epsilon_2)\right\} \tag{42}$$

if $q_{IFSEs}(\epsilon_1) \supset q_{IFSEs}(\epsilon_2)$, then (42) contradicts that $(q_{IFSEs}, \mathcal{S})$ is a strictly concave IFSEs.

now if we take $q_{IFSEs}(\epsilon_1) = q_{IFSEs}(\epsilon_2)$ and $\delta \in [0, \alpha]$, suppose $\epsilon_3 = \frac{\delta}{\alpha}\epsilon_1 + (1 - \frac{\delta}{\alpha})\epsilon_2$ and $\tau = (\frac{1}{\delta} - 1)(\frac{1}{\alpha} - 1)^{-1}$. so $q_{IFSEs(T)}\left(\delta\epsilon_1 + (1-\delta)\epsilon_2\right) = q_{IFSEs(T)}\left(\alpha(\frac{\delta}{\alpha}\epsilon_1 + \left(1 - \frac{\delta}{\alpha}\right)\epsilon_2) + (1-\delta)\epsilon_2\right) = q_{IFSEs(T)}\left(\alpha\epsilon_3 + (1-\alpha)\epsilon_2\right) \subseteq \left\{q_{IFSEs(T)}(\epsilon_1) \cap q_{IFSEs(T)}(\epsilon_2)\right\}$

$\Rightarrow$

$$q_{IFSEs(T)}\left(\delta\epsilon_1 + (1-\delta)\epsilon_2\right) \subseteq \left\{q_{IFSEs(T)}(\epsilon_1) \cap q_{IFSEs(T)}(\epsilon_2)\right\} \tag{43}$$

now $q_{IFSEs(T)}(\epsilon_3) = q_{IFSEs(T)}(\frac{\delta}{\alpha}\epsilon_1) + (1 - \frac{\delta}{\alpha}\epsilon_2)$

$$= q_{IFSEs(T)}\left(\tau\epsilon_1 + (1-\tau)\left(\delta\epsilon_1 + (1-\delta)\epsilon_2\right)\right) \tag{44}$$

From (41), (43) and $q_{IFSEs}(\epsilon_1) = q_{IFSEs}(\epsilon_2)$, we have

$$q_{IFSEs(T)}\left(\delta\epsilon_1 + (1-\delta)\epsilon_2\right) \subseteq q_{IFSEs}(\epsilon_3), \tag{45}$$

now from (42), (44) $q_{IFSEs}(\epsilon_1) = q_{IFSEs}(\epsilon_2)$ and strictly concave se-set condition, we have

$$q_{IFSEs(T)}\left(\delta\epsilon_1 + (1-\delta)\epsilon_2\right) \supset q_{IFSEs}(\epsilon_3), \tag{46}$$

or

$$\mathcal{Z}\setminus q_{IFSEs(T)}\left(\delta\epsilon_1 + (1-\delta)\epsilon_2\right) \subseteq \mathcal{Z}\setminus q_{IFSEs}(\epsilon_3), \tag{47}$$

thus (46) and (47) contradicts the fact.

If $q_{IFSEs}(\epsilon_1) = q_{IFSEs}(\epsilon_2)$ and $\delta \in [\alpha, 1]$, suppose $\epsilon_4 = \frac{\delta-\alpha}{1-\alpha}\epsilon_1 + \frac{1-\delta}{1-\alpha}\epsilon_2$ then $q_{IFSEs(T)}(\delta\epsilon_1 + (1-\delta)\epsilon_2) = q_{IFSEs(T)}(\alpha\epsilon_1 + (1-\alpha)\epsilon_4) \subseteq \{q_{IFSEs(T)}(\epsilon_1) \cap q_{IFSEs(T)}(\epsilon_4)\}$

$$q_{IFSEs(T)}\left(\delta\epsilon_1 + (1-\delta)\epsilon_2\right) \subseteq \left\{q_{IFSEs(T)}(\epsilon_1) \cap q_{IFSEs(T)}(\epsilon_4)\right\} \tag{48}$$

From (42), (48) and $q_{IFSEs}(\epsilon_1) = q_{IFSEs}(\epsilon_2)$, we have

$$q_{IFSEs(T)}\left(\delta\epsilon_1 + (1-\delta)\epsilon_2\right) \subseteq q_{IFSEs}(\epsilon_4), \tag{49}$$

Also, $(\delta\epsilon_1 + (1-\delta)\epsilon_2) = (\alpha\epsilon_1 + (1-\alpha)\epsilon_4)$ becomes

$$\begin{aligned}\epsilon_4 &= \frac{1}{1-\alpha}(\delta\epsilon_1 + (1-\delta)\epsilon_2) - \frac{\alpha}{1-\alpha}\epsilon_1 \\ &= \frac{1}{1-\alpha}(\delta\epsilon_1 + (1-\delta)\epsilon_2)\end{aligned}$$

$$-\frac{\alpha}{1-\alpha}\epsilon_1\left(\frac{1}{\delta}(\delta\epsilon_1+(1-\delta)\epsilon_2)-\frac{1-\delta}{\delta}\epsilon_2\right)$$
$$=\frac{\delta-\alpha}{(1-\alpha)\delta}(\delta\epsilon_1+(1-\delta)\epsilon_2)+\left(1-\frac{\delta-\alpha}{(1-\alpha)\delta}\right)\epsilon_2$$

$$\epsilon_4=\frac{\delta-\alpha}{(1-\alpha)\,\delta}\,(\delta\epsilon_1+(1-\delta)\,\epsilon_2)+\left(1-\frac{\delta-\alpha}{(1-\alpha)\,\delta}\right)\epsilon_2 \quad (50)$$

Now from (42), (49), $q_{IFSEs}(\epsilon_1)=q_{IFSEs}(\epsilon_2)$ and strictly concave IFSEs condition, we have

$$\mathcal{Z}\backslash\, q_{IFSEs(T)}\,(\delta\epsilon_1+(1-\delta)\,\epsilon_2)\subseteq\mathcal{Z}\backslash\, q_{IFSEs}\,(\epsilon_4)\,, \quad (51)$$

Hence (49) and (5) contradict the fact. ■

Theorem 5.3: *Suppose $(q_{IFSEs},\mathcal{S})$ be a convex IFSEs. If $\exists\alpha\in\Gamma^0\forall\epsilon_1,\epsilon_2\in\mathcal{S}$ with $q_{IFSEs}(\epsilon_1)\neq q_{IFSEs}(\epsilon_2)$*

$$q_{IFSEs}\,(\alpha\epsilon_1+(1-\alpha)\,\epsilon_2)\supset q_{IFSEs}\,(\epsilon_1)\cap q_{IFSEs}\,(\epsilon_2)$$

then $(q_{IFSEs},\mathcal{S})$ is a strictly convex IFSEs.

Proof: Assume $\exists\epsilon_1,\epsilon_2\in S,\delta\in\Gamma^0$ such that

$$\mathcal{Z}\backslash\, q_{IFSEs}\,(\delta\epsilon_1+(1-\delta)\,\epsilon_2)\supset\mathcal{Z}\backslash\,\{q_{IFSEs}\,(\epsilon_1)\cap q_{IFSEs}\,(\epsilon_2)\,\} \quad (52)$$

If $q_{IFSEs}(\epsilon_1)\supset q_{IFSEs}(\epsilon_2)$, then above equation becomes

$$\mathcal{Z}\backslash\, q_{IFSEs}\,(\delta\epsilon_1+(1-\delta)\,\epsilon_2)\supset\mathcal{Z}\backslash\, q_{IFSEs}\,(\epsilon_1)\,, \quad (53)$$

But by the convexity condition, we have

$$q_{IFSEs}\,(\delta\epsilon_1+(1-\delta)\,\epsilon_2)\supseteq\{q_{IFSEs}\,(\epsilon_1)\cap q_{IFSEs}\,(\epsilon_2)\,\} \quad (54)$$

By Equations (52) and (53), we get

$$q_{IFSEs}\,(\delta\epsilon_1+(1-\delta)\,\epsilon_2)=\{q_{IFSEs}\,(\epsilon_1)\cap q_{IFSEs}\,(\epsilon_2)\,\}\,. \quad (55)$$

Continuing with $q_{IFSEs}(\epsilon_1)\supset q_{IFSEs}(\epsilon_2)$, we get

$$q_{IFSEs}\,(\delta\epsilon_1+(1-\delta)\,\epsilon_2)=q_{IFSEs}\,(\epsilon_2) \quad (56)$$

or

$$q_{IFSEs}\,(\delta\epsilon_1+(1-\delta)\,\epsilon_2)\subset q_{IFSEs}\,(\epsilon_1) \quad (57)$$

By supposition and (56), we have

$$q_{IFSEs}\,(\alpha\epsilon_1+(1-\alpha)\,(\delta\epsilon_1+(1-\delta)\,\epsilon_2))\supset q_{IFSEs}\,((\delta\epsilon_1+(1-\delta)\,\epsilon_2)) \quad (58)$$

We generalize it for taking $n\in\{1,2,3,\ldots,\}$

$$q_{IFSEs}\,\left(\alpha^n\epsilon_1+\left(1-\alpha^n\right)(\delta\epsilon_1+(1-\delta)\,\epsilon_2)\right)\supset q_{IFSEs}\,((\delta\epsilon_1+(1-\delta)\,\epsilon_2)) \quad (59)$$

Suppose $\epsilon_3=(\tau\epsilon_1+(1-\tau)\epsilon_2)$ with $\tau=\delta-\alpha^n\delta+\alpha^n\in\Gamma^0$ for any value of n.

Then we see from above equation

$$q_{IFSEs}(\epsilon_3)=q_{IFSEs}(\tau\epsilon_1+(1-\tau)\epsilon_2)$$
$$=q_{IFSEs}(\alpha^n\epsilon_1+(1-\alpha^n)(\delta\epsilon_1+(1-\delta)\epsilon_2))$$
$$\supset q_{IFSEs}(\delta\epsilon_1+(1-\delta)\epsilon_2)$$

$$q_{IFSEs}(\epsilon_3)\supset q_{IFSEs}\,(\delta\epsilon_1+(1-\delta)\,\epsilon_2) \quad (60)$$

Also, suppose $\epsilon_4=(\gamma\epsilon_1+(1-\gamma)\epsilon_2)$ with $\gamma=\delta-\alpha^n+\frac{1}{1-\alpha}+\frac{\delta\alpha^n}{1-\alpha}\in\Gamma^0$ for any n. we have

$$q_{IFSEs}\,(\delta\epsilon_1+(1-\delta)\,\epsilon_2)=q_{IFSEs}\,(\alpha\epsilon_3+(1-\alpha)\,\epsilon_4) \quad (61)$$

Also when, $q_{IFSEs}(\epsilon_3)\subseteq q_{IFSEs}(\epsilon_4)$, subsequently by using the convexity of IFSEs and Equation (61) we have $\mathcal{Z}\backslash q_{IFSEs}(\epsilon_3)\supset\mathcal{Z}\backslash q_{IFSEs}(\delta\epsilon_1+(1-\delta)\epsilon_2)$ which contradicts Equation (59).

Now when $\mathcal{Z}\backslash q_{IFSEs}(\epsilon_3)\subseteq\mathcal{Z}\backslash q_{IFSEs}(\epsilon_4)$, subsequently by using the supposition of theorem and Equation (53), we have $q_{IFSEs}(\delta\epsilon_1+(1-\delta)\epsilon_2)\supset\{q_{IFSEs}(\epsilon_3)\cap q_{IFSEs}(\epsilon_4)\}\supseteq\{q_{IFSEs}(\epsilon_1)\cap q_{IFSEs}(\epsilon_2)\,\}\cap\{q_{IFSEs}(\epsilon_1)\cap q_{IFSEs}(\epsilon_2)\}=q_{IFSEs}(\epsilon_1)$ $q_{IFSEs}(\delta\epsilon_1+(1-\delta)\epsilon_2)\supset q_{IFSEs}(\epsilon_1)$ which contradicts Equation (61). ■

Corollary 5.1: *Suppose $(q_{IFSEs},\mathcal{S})$ be a concave IFSEs. If $\exists\alpha\in\Gamma^0\forall\epsilon_1,\epsilon_2\in\mathcal{S}$ with $q_{IFSEs}(\epsilon_1)\neq q_{IFSEs}(\epsilon_2)$*

$$q_{IFSEs}\,(\alpha\epsilon_1+(1-\alpha)\,\epsilon_2)\subset q_{IFSEs}\,(\epsilon_1)\cap q_{IFSEs}\,(\epsilon_2)$$

then $(q_{IFSEs},\mathcal{S})$ is a strictly concave IFSEs.

Proof: This is simple by taking the complement of the above equations. ■

Theorem 5.4: *Suppose $(q_{IFSEs},\mathcal{S})$ be a strongly convex soft fuzzy expert set. If $\exists\alpha\in\Gamma^0\forall\epsilon_1,\epsilon_2\in\mathcal{S}$ with*

$$q_{IFSEs}\,(\alpha\epsilon_1+(1-\alpha)\,\epsilon_2)\supseteq q_{IFSEs}\,(\epsilon_1)\cap q_{IFSEs}\,(\epsilon_2) \quad (62)$$

then $(q_{IFSEs},\mathcal{S})$ is a convex IFSEs.

Proof: Assume $\exists\epsilon_1,\epsilon_2\in\mathcal{S},\delta\in\Gamma^0$ such that

$$\mathcal{Z}\backslash\, q_{IFSEs}\,(\delta\epsilon_1+(1-\delta)\,\epsilon_2)\supseteq\mathcal{Z}\backslash\,\{q_{IFSEs}\,(\epsilon_1)\cap q_{IFSEs}\,(\epsilon_2)\} \quad (63)$$

If $\epsilon_1\neq\epsilon_2$ then Equation (62) contradicts the fact that $(q_{IFSEs},\mathcal{S})$ is a convex IFSEs.

If $\epsilon_1=\epsilon_2$, select $\delta\neq\delta_1\in\Gamma^0$ with $\delta=\alpha\delta_1+(1-\alpha)\delta_1$.

Suppose $\epsilon_1=\delta_1\epsilon_1+(1-\delta_1)\epsilon_2$, $\epsilon_2=\delta_2\epsilon_1+(1-\delta_2)\epsilon_2$

Equation (62) becomes

$$\mathcal{Z}\backslash\, q_{IFSEs}\,(\epsilon_1)\supseteq\mathcal{Z}\backslash\,\{q_{IFSEs}\,(\epsilon_1)\cap q_{IFSEs}\,(\epsilon_2)\} \quad (64)$$

$$\mathcal{Z}\backslash\, q_{IFSEs}\,(\epsilon_2)\supseteq\mathcal{Z}\backslash\,\{q_{IFSEs}\,(\epsilon_1)\cap q_{IFSEs}\,(\epsilon_2)\} \quad (65)$$

By Equations (62), (63) and (64), we have

$$q_{IFSEs}\,(\alpha\epsilon_1+(1-\alpha)\,\epsilon_2)\supseteq\{q_{IFSEs}\,(\epsilon_1)\cap q_{IFSEs}\,(\epsilon_2)\}$$

$$\subset \{q_{IFSEs}(\epsilon_1) \cap q_{IFSEs}(\epsilon_2)\}$$
$$\cap \{q_{IFSEs}(\epsilon_1) \cap q_{IFSEs}(\epsilon_2)\}$$
$$= q_{IFSEs}(\epsilon_1) \cap q_{IFSEs}(\epsilon_2)$$

And this contradicts that $(q_{IFSEs}, \mathcal{S})$ is a strongly convex IFSEs. ■

Corollary 5.2: *Suppose $(q_{IFSEs}, \mathcal{S})$ be a strongly concave IFSEs. If $\exists \alpha \in \Gamma^0 \forall \epsilon_1, \epsilon_2 \in \mathcal{S}$ with*

$$q_{IFSEs}(\alpha\epsilon_1 + (1-\alpha)\epsilon_2) \subseteq q_{IFSEs}(\epsilon_1) \cup q_{IFSEs}(\epsilon_2) \quad (66)$$

then $(q_{IFSEs}, \mathcal{S})$ is a concave IFSEs.

Proof: This can be done by using the proof of the above Theorem 5.4. In Theorem 5.4, definition of strongly convex IFSEs has been used and to prove this corollary just make use of strongly concave IFSEs. ■

Theorem 5.5: *Suppose $(q_{IFSEs}, \mathcal{S})$ be a convex IFSEs. If $\exists \alpha \in \Gamma^0 \forall \epsilon_1, \epsilon_2$ such that $\epsilon_1 \neq \epsilon_2 \in \mathcal{S}$ then*

$$q_{IFSEs}(\alpha\epsilon_1 + (1-\alpha)\epsilon_2) \supset q_{IFSEs}(\epsilon_1) \cap q_{IFSEs}(\epsilon_2)$$

then $(q_{IFSEs}, \mathcal{S})$ is a strongly convex IFSEs.

Proof: Assume $\exists\, (\epsilon_1 \neq \epsilon_2)\, \epsilon_1, \epsilon_2 \in, \delta \in \Gamma^0$ such that

$$Z \backslash q_{IFSEs}(\delta\epsilon_1 + (1-\delta)\epsilon_2) \supset Z \backslash \{q_{IFSEs}(\epsilon_1) \cap q_{IFSEs}(\epsilon_2)\} \quad (67)$$

By the condition of convex IFSEs and from above equation, we have

$$q_{IFSEs}(\delta\epsilon_1 + (1-\delta)\epsilon_2) = \{q_{IFSEs}(\epsilon_1) \cap q_{IFSEs}(\epsilon_2)\} \quad (68)$$

Moreover, it can be written as

$$(\alpha\epsilon_1 + (1-\alpha)\epsilon_2) = (\delta\epsilon_1 + (1-\delta)\epsilon_2) \quad (69)$$

While $\epsilon_1 = (\delta\epsilon_1 + (1-\delta)\epsilon_2)$, $\epsilon_2 = (\delta\epsilon_1 + (1-\delta)\epsilon_2)$ with selecting $\delta \in \Gamma^0$

Also by the definition of convexity of IFSEs and above defining ϵ_1, ϵ_2, we get

$$q_{IFSEs}(\epsilon_1) \supseteq \{q_{IFSEs}(\epsilon_1) \cap q_{IFSEs}(\epsilon_2)\} \quad (70)$$
$$q_{IFSEs}(\epsilon_2) \supseteq \{q_{IFSEs}(\epsilon_1) \cap q_{IFSEs}(\epsilon_2)\} \quad (71)$$

Using Equations (68), (69), (70) and given statement, we have

$$q_{IFSEs}(\delta\epsilon_1 + (1-\delta)\epsilon_2) = q_{IFSEs}(\alpha\epsilon_1 + (1-\alpha)\epsilon_2)$$
$$\supset \{q_{IFSEs}(\epsilon_1) \cap q_{IFSEs}(\epsilon_2)\}$$
$$\supseteq \{q_{IFSEs}(\epsilon_1) \cap q_{IFSEs}(\epsilon_2)\}$$
$$\cap \{q_{IFSEs}(\epsilon_1) \cap q_{IFSEs}(\epsilon_2)\}$$
$$= q_{IFSEs}(\epsilon_1) \cap q_{IFSEs}(\epsilon_2)$$

which contradicts to Equation (67). ■

Corollary 5.3: *Suppose $(q_{IFSEs}, \mathcal{S})$ be a concave IFSEs. If $\exists \alpha \in \Gamma^0 \forall \epsilon_1, \epsilon_2$ such that $\epsilon_1 \neq \epsilon_2 \in \mathcal{S}$ then*

$$q_{IFSEs}(\alpha\epsilon_1 + (1-\alpha)\epsilon_2) \subset q_{IFSEs}(\epsilon_1) \cap q_{IFSEs}(\epsilon_2)$$

then $(q_{IFSEs}, \mathcal{S})$ is a strongly concave IFSEs.

Proof: This can be done by using the proof of the above Theorem 5.5. In Theorem 5.5, definition of strongly convex IFSEs with certain conditions has been used and to prove this corollary just make use of strongly concave IFSEs with the same certain conditions. ■

6. Convex hull and convex cone

In this section, a new concept of convex hull and cone for IFSEs (6.1 and 6.2) has been developed with the help of existing concept of convex hull and cone on s-set by Majeed [42]. The notation X(Z) will be used for collection of IFSEss.

Definition 6.1: The convex hull of an IFSEs $(C, \mathcal{S})$ can be defined as

$\widetilde{Conh}\ (C, \mathcal{S}) = \ \overline{\cap}_{(T,)\supseteq(C,\mathcal{S})}\{(T, \mathcal{S}) : (T, \mathcal{S}) \in$ X(Z) are IFSEs }.

Theorem 6.1: *Let $(T, \mathcal{S}) \in X(Z)$ then its convex hull is given by as*

$$ConhC(\epsilon) = \hat{\cup}_{m \in N} \hat{\cup}_{P \in K(\epsilon,m)} \overline{\cap}\{C(v) : v \in P\}$$

where

$$K(\epsilon, m) = \left\{ \{\epsilon_1, \epsilon_2, \ldots, \epsilon_m\} \subset S : \exists\, \varpi \in [0,\, 1] \text{ with } \sum_{i=1}^{m} \varpi = 1,\ \sum_{i=1}^{m} \varpi_i \epsilon_i = 1 \right\}.$$

Proof: Using definition of convex hull of $(H, \mathcal{S})$, we have $convhC(\epsilon) = \ \overline{\cap} T(\epsilon) : C(\epsilon) \supseteq T(\epsilon)$ such that $(T, \mathcal{S}) \in X(Z)$. Now define the IFSEs as

$$convh\tilde{C}(\epsilon) = \hat{\cup}_{m \in N} \hat{\cup}_{P \in K(\epsilon,m)} \overline{\cap}\{C(v) : v \in P\}$$

now we have to show that $convhC(\epsilon) = convh\tilde{C}(\epsilon)$. To prove this, we will show

$$convhC(\epsilon) \subseteq convh\tilde{C}(\epsilon) \quad \text{andthen } convhC(\epsilon) \supseteq \tilde{C}(\epsilon).$$

As we know that every IFSEs is a convex IFSEs. Then by using the definition of convex IFSEs, we have

$$T(\epsilon) \supseteq \hat{\cup}_{m \in N} \hat{\cup}_{P \in K(\epsilon,m)} \overline{\cap}\{C(v) :\ v \in P\} \supseteq convh\tilde{C}(\epsilon)$$

Using the intersection to the left side of the above relation, we get

$$\overline{\cap}_{C(\epsilon)\supseteq T(\epsilon)} T(\epsilon) \supseteq convh\tilde{C}(\epsilon)$$
$$\Rightarrow convhC(\epsilon) \supseteq convh\tilde{C}(\epsilon).$$

Now we will prove that $convhC(\epsilon) \subseteq convh\widetilde{C}(\epsilon)$. It is enough to show that $(\widetilde{C}, S)$ is a convex IFSEs. Actually, since $convh(C, S)$ is the smallest convex IFSEs containing (C, S), $convh(C, S) \subseteq (\widetilde{C}, S)$. Suppose $\epsilon = \sum_{i=1}^{q} v_i\epsilon_i$ and $\epsilon' = \sum_{i=1}^{l} \gamma_i\epsilon_i'$ such that $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_q\} \in K(\epsilon, q)$ and $\{\epsilon_1', \epsilon_2', \ldots, \epsilon_l'\} \in K(\epsilon', l)$, with $\sum_{i=1}^{q} \epsilon_i = 1$ and $\sum_{i=1}^{l} \epsilon_i' = 1$. Therefore,

$$\widetilde{C}\left(\alpha\epsilon + (1-\alpha)\,\epsilon'\right) = \widetilde{C}\left(\alpha\sum_{i=1}^{q} v_i\epsilon_i + (1-\alpha)\sum_{i=1}^{l}\gamma_i\epsilon_i'\right)$$

such that $\alpha\sum_{i=1}^{q} v_i\epsilon_i + (1-\alpha)\sum_{i=1}^{l}\gamma_i\epsilon_i' = 1$ and $\{\epsilon_1, \epsilon_2, \epsilon_3, \ldots, \epsilon_q, \epsilon_1', \epsilon_2', \epsilon_3', \ldots, \epsilon_l'\} \in C(\alpha\epsilon + (1-\alpha)\epsilon', l+q)$. As $S = R^n$, so $\alpha\epsilon + (1-\alpha)\epsilon' \in S = R^n$. Using the definition of IFSEs described above to ϵ, ϵ' and to $x = \alpha\epsilon + (1-\alpha)\epsilon'$, we have

$$\begin{aligned}
&\hat{\cup}_{l+q\in N}\hat{\cup}_{D\cup V\in K(x,l+q)}\bar{\cap}_{i=1}^{l+q}\{C(x_i) : x_i \in D\cup V\}\\
&\supseteq \hat{\cup}_{q\in N}\hat{\cup}_{D\in K(\epsilon,q)}\bar{\cap}_{i=1}^{q}\{C(\epsilon_i) : \epsilon_i \in D\}\\
&\cap \hat{\cup}_{l\in N}\hat{\cup}_{V\in K(\epsilon,l)}\bar{\cap}_{i=1}^{l}\{C(\epsilon_i) : \epsilon_i \in V\},
\end{aligned}$$

i.e.

$$\widetilde{C}\left(\alpha\epsilon + (1-\alpha)\,\epsilon'\right) \supseteq \widetilde{C}(\epsilon) \cap \widetilde{C}\left(\epsilon'\right).$$

hence the required result is proved. ■

Definition 6.2: Let $(C, \mathcal{S})$ be IFSEs, then it is called a cone if

$$C(\alpha\epsilon) \supseteq C(\epsilon)$$

for all $\epsilon \in \mathcal{S}$ and $\alpha > 0$.

If the IFSEs is convex, then it is named as a convex cone.

Theorem 6.2: *A IFSEs $(C, \mathcal{S})$ is said to be convex cone iff for every $\epsilon, \epsilon' \in S$ and $\alpha > 0$*

(1) $C(\alpha\epsilon) \supseteq C(\epsilon)$
(2) $C(\epsilon + \epsilon') \supseteq C(\epsilon) \cap C(\epsilon')$.

Proof: Suppose $(C, \mathcal{S})$ be an IFSEs, then by definition statement (1) is proved.

Now to prove 2nd statement take $\alpha = \frac{1}{2}$ and for any $\epsilon, \epsilon' \in \mathcal{S}$, then by definition of convex cone

$$C\left(\frac{1}{2}\epsilon + \frac{1}{2}\epsilon'\right) \supseteq C(\epsilon) \cap C\left(\epsilon'\right)$$

and

$$C\left(2\left(\frac{1}{2}\epsilon + \frac{1}{2}\epsilon'\right)\right) \supseteq C\left(\frac{1}{2}\epsilon + \frac{1}{2}\epsilon'\right)$$

From the above equations, we have

$$\begin{aligned}
C\left(\epsilon + \epsilon'\right) &= C\left(2\left(\frac{1}{2}\epsilon + \frac{1}{2}\epsilon'\right)\right)\\
&\supseteq C\left(\frac{1}{2}\epsilon + \frac{1}{2}\epsilon'\right) \supseteq C(\epsilon) \cap C\left(\epsilon'\right)
\end{aligned}$$

$$C\left(\epsilon + \epsilon'\right) \supseteq C(\epsilon) \cap C\left(\epsilon'\right)$$

hence the 2nd result is proved. Conversely, we have to show that $(C, \mathcal{S})$ is convex IFSEs.

By first condition $(C, \mathcal{S})$ is cone. By making use of 1st and 2nd conditions with $\alpha \in [0, 1]$, it is clear that $(C, \mathcal{S})$ is convex IFSEs.

i.e.

$$\begin{aligned}
C\left(\alpha\epsilon + (1-\alpha)\epsilon'\right) &\supseteq C(\alpha\epsilon) \cap C\left((1-\alpha)\epsilon'\right)\\
&\supseteq H(\epsilon) \cap H\left(\epsilon'\right)\\
\Rightarrow C\left(\alpha\epsilon + (1-\alpha)\epsilon'\right) &\supseteq C(\epsilon) \cap C\left(\epsilon'\right).
\end{aligned}$$

Which shows that $(C, \mathcal{S})$ is convex IFSEs. ■

Corollary 6.1: *If $(C, \mathcal{S})$ is IFSEs then it is a convex cone iff*

$$C\left(\sum_{i=1}^{m}\alpha_i\epsilon_i\right) \supseteq \cap_{i=1}^{m} C(\epsilon_i)$$

for all $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_i\} \in \mathcal{S}$ *and* $\alpha_i > 0$.

7. Comparison

A comparison analysis of the model has been shown in Table 1. In this table, some prominent characteristics of existing models have been compared with proposed structure. These characteristics include membership function ($\dot{M}\check{V}$), non-membership ($\dot{N}\dot{M}\check{V}$), single argument approximate function ($\hat{S}\underset{.}{A}\underset{.}{A}\dot{F}$) and multi-decisive opinion ($\dot{M}\bar{D}\ddot{O}$). The Yes and No will be denoted by $\check{Y}$ and $\breve{N}$ in the following Table 1. From Table 1, it is clear that our proposed model is more generalized than the above described models.

8. Conclusions

In this article, (α, v)-convexity cum concavity, (α, v)-convexity cum concavity in 1st-sense and 2nd-sense, convex hull and convex cone are developed for IFSEs. Several classical axiomatic properties and operational results have been generalized under IFSEs. Although the proposed framework is an intelligent approach for convex optimization and parameterization with entitlement of fuzzy membership grades and intuitionistic fuzzy non-membership grades yet it has limitations

Table 1. Comparison with different models.

Authors	Models	$\dot{M}\check{V}$	$\dot{N}\dot{M}\check{V}$	$(\hat{S}\underset{.}{A}\underset{.}{A}\dot{F})$	$\dot{M}\bar{D}\ddot{O}$
Deli [41]	Soft set	$\breve{N}$	$\breve{N}$	$\check{Y}$	$\breve{N}$
Majeed [42]	Soft set	$\breve{N}$	$\breve{N}$	$\check{Y}$	$\breve{N}$
Salleh and Sabir [43]	Soft set	$\breve{N}$	$\breve{N}$	$\check{Y}$	$\breve{N}$
Rahman et al. [45]	Soft set	$\breve{N}$	$\breve{N}$	$\check{Y}$	$\check{Y}$
Ihsan et al. [46]	Soft expert set	$\breve{N}$	$\breve{N}$	$\check{Y}$	$\breve{N}$
Rahman et al. [44]	Fuzzy soft set	$\breve{N}$	$\breve{N}$	$\check{Y}$	$\breve{N}$
Ihsan et al. [47]	Fuzzy soft expert set	$\check{Y}$	$\breve{N}$	$\check{Y}$	$\check{Y}$
Proposed Model	IFSEs	$\check{Y}$	$\check{Y}$	$\check{Y}$	$\check{Y}$

regarding the consideration of neutral grades; therefore, the future work includes the extension of this study for such environments to address its limitations. Moreover, the generalization of various other variants of classical convexity may also a part of its future work.

Disclosure statement

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