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Some Results on the Relationship Between Smarandache Semigroups and Their Smarandache Subsemigroups

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ABSTRACT

In a journal titled "Smarandache Semigroups" discussed Validity of Some Classical Theorems of Group Theory in case of Smarandache Semigroup and Smarandache notions in group. Padilla Raul, introduced the notion of smarandache semigroups, titled "Smarandache Algebraic Structures". A smarandache semigroup is a semigroup A which has a proper subset B contained in A that is a group (with respect to the same binary operation on A). The nature of the structure is extremely interesting and attractive, since it handles a weak and a strong structure. Here we shall be establishing some basic properties of smarandache semigroups. We studied some aspects of smarandache semigroups and used those facts to establish some basic properties of smarandache semigroup and smarandache notion in group. Let $Z_n = \{1, \; 2, \; 3, \; . \; . \; . \; . \; , n-1\}$. In this paper, we established some results about relationship that exists between Smarandache semigroups and Smarandache Subsemigroups. And some results on uniqueness of related pair of a Smarandache inverse pair and Smarandache weakly Lagrange semigroup on Z_{2n}. Next, we have newly defined concepts: Smarandache Ideal of Smarandache semigroup, Smarandache weakly Cauchy semigroup.

1. Introduction

The classical theorems: Lagrange's theorem, Cayley's theorem and Cauchy's theorem play very important role in group theory and play a vital role. In this work, we shall study validity of the above-mentioned theorems for smarandache semigroups. Padilla Raul (1998), introduced the notion of smarandache semigroups in a paper titled "Smarandache Algebraic Structures", but in 1973, there is a publication by Smarandache F., titled "Special Algebraic Structures". A smarandache semigroup is a semigroup A which has proper subset B contained in A that is a group (with respect to the same binary operation on A). So, smarandache notions in groups, specifically: Smarandache inverse of an element in a group, Smarandache conjugate elements in a group, Smarandache double coset and Smarandache coset, is part of our work and this shall lead us to study some basic aspects of group theory.

Further, smarandache semigroups exhibit both properties of groups and semigroups simultaneously. It is interesting, in fact, it attracts researchers and algebraic. Since, smarandache handles two structures; we shall see how the mixture of a group and a semigroup behaves. In this work, we are interested only about smarandache semigroup of finite order. We have newly defined concepts and results in Smarandache semigroups which are important in this area of study, because they have exhibited some elementary properties of Smarandache semigroup and related pair of a Smarandache inverse pair. In fact, they have tackled one or two challenges in this area of study. Let us revisit a journal and papers on Smarandache semigroups: in 2003, there is journal publication on smarandache semigroups, by Kandasamy V. W. B, in 2013, a paper by Mohammed S. K. which handles smarandache semigroups, smarandache cyclic semigroups and Smarandache Lagrange Semigroups and in 2020, a paper by Boris T. which deals with some basic definitions about semigroups, groups and smarandache semigroups. And the end we give the characterisation of smarandache semigroups and so on. We have considered additional ones on groupoids; a paper published by Kandasamy V. W. B. in 2005: The concepts of smarandache groupoids, ideal of groupoid, semi normal subgroupoids, smarandache-BOI groupoids and strong BOI groupoids, next, a paper published by Siamwalla H. J. and Muktibod A. S. in 2012: Results toward classifying smarandache groupoids which are in $Z^*(\text{n})$ and not in $Z(n)$ when n is even and n is odd.

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Note: Throughout this work T_n is a set of all mapping from X to X, $X = \{1, 2, ... , n\}$ and S_n is a symmetric group of degree n and order $n!$.

Definition 1.1: A semigroup is a pair $(S, *)$ consisting of a non-empty set S endowed with an operation $*$ such that

$$a * b \in S \;\; \forall\; a, b \in S$$
$$(a * b) * c = a * (b * c) \;\; \forall\; a, b, c \in S$$

Definition 1.2: The Smarandache semigroup (S-semigroup) is defined to be a semigroup A such that a proper subset of A is a group with respect to the same induced operation.

Example 1.1: Let $Z_{12} = \{0, 1, 2, . . . 11\}$ be the semigroup under multiplication modulo 12. The subset A = {1, 11} is a subgroup with respect to the same operation. So Z_{12} is a Smarandache semigroup.

Definition 1.3: Let S be a S-semigroup. A proper subset A of S is said to be a Smarandache subsemigroup of S if A itself is a S-semigroup, that is A is a semigroup of S containing a proper subset B such that B is a group under the operations of S. Note we do not accept A to be a group. A must only be a semigroup.

Example 1.2: Let $Z_{12} = \{0, 1, . . . ,11\}$ be the semigroup under multiplication modulo 12. The subset A = {0, 1, 11} is a smarandache semigroup, so A is a smarandache subsemi- group of Z_{12} with respect to the same operation.

Definition 1.4: Let G be a group. An element $x \in G \setminus \{1\}$ is said to have a Smarandache inverse y in G if $xy = 1$, $a, b \in G \setminus \{1, x, y\}$ we have $xa = y\,(or\;\; ax = y), yb = x\,(or\, by = x)$ with $ab = 1$.

Example 1.3: Let G = < g / g^6 = 1> be the cyclic group of order 6. Now g in G has $g^5 \in$ G such that g • g^5 = 1 further g^2, $g^4 \in$ G \ {1, g, g^5} is such that g^5 • g^2 = g and g • g^4 = g^5 with g^2 • g^4 = 1. Clearly g^3 • g^3 = 1 but $g^3 \in$G has no Smarandache inverse, for we cannot find $g^i \in$ G with g^3 • g^i = g^3. So for (g, g^5) the Smarandache inverse pair, the related pair is (g^2, g^4) and $g^3 \in$ G has no Smarandache inverse.

Definition 1.5: Let S be a finite S-semigroup. If the order of every subgroup of S divides the order of the S-semigroup S then we say S is a Smarandache Lagrange semigroup.

Example 1.4: Let Z_4 = {0, 1, 2, 3} be the semigroup under multiplication modulo 4. Clearly, Z_4 is a S-semigroup. Further the only subgroup in Z_4 is A = {1, 3} and o (A) | 4, so Z_4 is a Smarandache Lagrange semigroup.

Definition 1.6: Let S be a finite S-semigroup. If there exist at least one subgroup A that is a proper subset ($A \subset S$) having the same operations of S whose order divides the order of S then we say that S is a Smarandache weakly Lagrange semigroup.

Example 1.5: Let Z_{10} = {0, 1, 2, ... 9} is a S-semigroup under multiplication modulo 10. Clearly A = {1, 9} is a subset of Z_{10} which is a subgroup such that o (A) | 10. Therefore, Z_{10} is a Smarandache weakly Lagrange semigroup.

2. Methodology

We have studied some aspects of Smarandache Semigroups and used those facts to establish some basic properties of Smarandache semigroup and Smarandache notion in group.

2.1 Newly Defined Concepts with Examples

Definition 2.1: Let S be a S-semigroup and let A be a proper subset of S, then A is said to be a Smarandache ideal of S-semigroup, if A is both Smarandache left and right ideal. A is Smarandache left ideal, if

i. A is a Smarandache subsemigroup.
ii. $SA \subseteq A, \forall x \in S, a \in A$

Dually, we can define Smarandache right ideal.

Example 2.1: Let Z_6 = {0, 1, 2, 3, 4, 5} is a S- semigroup under multiplication modulo 6. Then A = {0, 2, 4} is a proper subset of Z_6 which is smarandache ideal of Z_6.

Definition 2.2: Let S be a S-semigroup, if every element of at least one subgroup of S is a Smarandache Cauchy element, then S is a Smarandache weakly Cauchy semigroup.

Example 2.2: The Smarandache symmetric semigroup T_3 of order 3^3 is a Smarandache weakly Cauchy semigroup.

2.2 Some Results on S - Semigroups and S - Subsemigroups

Lemma 3.1: Let $Z_n = \{0,1,2,...,n-1\}$ be a S – semigroup, then Z_n has no proper S-sub-semigroup for $n = 3$, but for $n \geq 4$, it has at least one proper S-subsemigroup and the operation is multiplication modulo n.

Proof: For $n = 3$, we have $Z_3 = \{0,1,2\}$ with respect to multiplication modulo 3 is a S-semigroup. We can find sub-semigroup of Z_3, but not proper S - subsemigroup.

Therefore, $(Z_3, \times)$ has no proper S-sub-semigroup

Let $Z_n = \{0,1,2,...,n-1\}$ be a S-semigroup with respect to multiplication modulo n, for $n \geq 4$, we can get a subsemigroup which is S-subsemigroup of the form.

$A = \{0,1,n-1\}$ with respect to the same binary operation.

Hence, the claim.

Lemma 3.2: The Smarandache symmetric semigroup $\boldsymbol{S(n)}$ has no proper S-subsemigroup for $n = 2$, but for $n \geq 3$, it has proper S-subsemigroup, where $S(n)$ is a set of all mapping from X to X. $X = (1,\ 2,\ 3,\ .\ .\ .\ ,n)$.

Proof: For $n = 2$, we get

$$S(2) = \left\{ \begin{pmatrix} 12 \\ 12 \end{pmatrix}, \begin{pmatrix} 12 \\ 21 \end{pmatrix}, \begin{pmatrix} 12 \\ 11 \end{pmatrix}, \begin{pmatrix} 12 \\ 22 \end{pmatrix} \right\}$$

$$let\ A = \left\{ \begin{pmatrix} 12 \\ 12 \end{pmatrix}, \begin{pmatrix} 12 \\ 21 \end{pmatrix}, \begin{pmatrix} 12 \\ 11 \end{pmatrix} \right\}$$

$$take \begin{pmatrix} 12 \\ 11 \end{pmatrix} \begin{pmatrix} 12 \\ 21 \end{pmatrix} = \begin{pmatrix} 12 \\ 22 \end{pmatrix} \notin A, so$$

A is not close

$$let\ B = \left\{ \begin{pmatrix} 12 \\ 12 \end{pmatrix}, \begin{pmatrix} 12 \\ 21 \end{pmatrix}, \begin{pmatrix} 12 \\ 22 \end{pmatrix} \right\}$$

$$take \begin{pmatrix} 12 \\ 22 \end{pmatrix} \begin{pmatrix} 12 \\ 21 \end{pmatrix} = \begin{pmatrix} 12 \\ 11 \end{pmatrix} \notin B, so$$

B is not close
Therefore, we cannot get a proper S-subsemigroup of S (2).

For $n \geq 3$, we can have S-subsemigroup of this form

$$Let\ C = \left\langle \begin{pmatrix} 1\ 2\ 3\ \dots n \\ 1\ 1\ 1\ \dots 1 \end{pmatrix}, \begin{pmatrix} 1\ 2\ 3\ \dots n-1\ n \\ 2\ 3\ 4\ \dots\ \ \ n\ \ \ 1 \end{pmatrix} \right\rangle$$

In fact, it gives the S-subsemigroup.

Lemma 3.3: A semigroup S which has a Smarandache subsemigroup is a Smarandache semigroup.

Proof: Let S be a semigroup and A be a smarandache subsemigroup of S. Then by definition of smarandache semigroup, there is a subsemigroup of S which is a subgroup and so S is a smarandache semigroup.

3. Results

3.1 Smarandache Weakly Lagrange Semigroup

Lemma 4.1: Z_{2n} with respect to multiplication modulo $2n$, is a Smarandache weakly Lagrange semigroup, for $n \geq 2$.

Proof: Let $Z_{2n} = \{0, 1, 2, \ldots, 2n-1\}$ be a S-semigroup with respect to multiplication modulo $2n$, it has a subgroup $A = \{1, 2n-1\}$, for $n \geq 2$ whose order divide the order of Z_{2n}, i.e. $o(A)/o(Z_{2n})$.

Hence, Z_{2n} is a S-weakly Lagrange semigroup.

3.2 Related pair of a Smarandache inverse pair

Lemma 5.1: A related pair (a, b) of a Smarandache inverse pair (x, y) is unique, but Smarandache inverse pair (x, y) of a related pair (a, b) is not unique in general.

Proof: Let (a_1, b_1) and (a_2, b_2) be two related pairs of Smarandache inverse pair (x, y), then

$$xa_1 = y\left(or\ a_1 x = y\right) \quad (1)$$

$$yb_1 = x\left(or\ b_1 y = x\right) \quad (2)$$

$$xa_2 = y\left(or\ a_2 x = y\right) \quad (3)$$

$$yb_2 = y\left(or\ b_2 y = x\right) \quad (4)$$

From (1) and (3), we have
$xa_1 = xa_2$
Since, $x, y, a_1, b_1, a_2\ and\ b_2 \in G,$ where G is a group, then by applying cancellation law, we get $a_1 = a_2$
From (2) and (4), we have

$yb_1 = yb_2$

$\Rightarrow b_1 = b_2$

Hence, the related pair is unique.

To prove the second part, consider the following example.
Let $G = < g/g^8 = 1 >$ be a cyclic group generated by g and then we can form the pairs as follows:

$$(g, g^7), (g^2, g^6), (g^5, g^3) \; and \; (g^4, g^4)$$

Let (g, g^7) be a Smarandache inverse pair, to see this we have
$g \cdot g^7 = 1, for \; g^2, g^6 \in G \setminus \{1, g, g^7\}, g \cdot g^6 = g^7 \; and \; g^7 \cdot g^2 = g \; with \; g^2 \cdot g^6 = 1.$
So, (g^2, g^6) is the related pair of the Smarandache inverse pair (g, g^7).
For the second pair, let (g^3, g^5) be a Smarandache inverse pair, to show this, we have $g^3, g^5 = 1, for \; g^2, g^6 \in G \setminus \{1, g^3, g^5\}, g^3 \cdot g^2 = g^5 \; and \; g^5 \cdot g^6 = g^3 \; with \; g^2 \cdot g^6 = 1.$
So, (g^2, g^6) is the related pair of the Smarandache inverse pair (g^3, g^5).

Therefore, $(g, g^7) and \; (g^3, g^5)$ have the same related pair(g^2, g^6), hence Smarandache inverse pair is not unique in general.

Next an interesting example on Smarandache inverse-free group;
Klein-4-group, a group of this form
$G^* = \{e, a, b, c: a^2 = b^2 = c^2 = e, ab = c, bc = a, ac = b\}$ is a Smarandache inverse-free group, we can prove this by applying a theorem that says;
Let G be a group, if $x \in G$ is such that $x^2 = 1$, then x has no Smarandache inverse.
Therefore, every element in G^* has no Smarandache inverse, hence G^* is a Smarandache inverse-free group.

4. Conclusion

This study deals with mixture of two structures (Semigroups and Groups) called Smarandache Semigroups. We have studied some aspects of smarandache semigroups and established some results, which help by exhibiting basic properties of Smarandache Semigroup and Smarandache notion in group.

Let us have some statements of the results:

Let $Z_n = \{0,1,2,...,n-1\}$ be a S – semigroup, then Z_n has no proper S-sub-semigroup for $n = 3$, but for $n \geq 4$, it has at least one proper S-subsemigroup and the operation is multiplication modulo n.

The Smarandache symmetric semigroup $\boldsymbol{S(n)}$ has no proper S-subsemigroup for $n = 2$, but for $n \geq 3$, it has proper S-subsemigroup, where $S(n)$ is a set of all mappings from X to X. $X = (1, 2, 3, . . . , n)$.

A related pair (a, b) of a Smarandache inverse pair (x, y) is unique, but Smarandache inverse pair (x, y) of a related pair (a, b) is not unique in general.

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