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ORIGINAL PAPER

Algebraic and Einstein weighted operators of neutrosophic enthalpy values for multi-criteria decision making in neutrosophic multi-valued set settings

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Abstract

In fuzzy set theory, the aggregation is the process that combines input fuzzy sets into a single output fuzzy set. In this manner, an aggregation operator is an important tool in the fuzzy set theory and its applications. The purpose of this study is to present some algebraic operators among neutrosophic enthalpy values and to provide some aggregation operators with the help of general *t*-norms and *t*-conorms which produce a new theoretical base in the fuzzy environment. An enthalpy value is the information energy expressed by the complement of the Shannon's entropy and a neutrosophic enthalpy set is characterized with a truth, an indeterminacy and a falsity function defined on a universal set to $[0,1]^2$. The first component of each function is the average of the truth, the indeterminacy and the falsity sequence of a neutrosophic multi-valued set, respectively, and the second component of each function is the fuzzy complement of the normalized Shannon's entropy of the truth, the indeterminacy and the falsity of the same neutrosophic multi-valued set, respectively. Therefore, a neutrosophic enthalpy set contains both the level of the mean of the data and the degree of uncertainty of the data via enthalpy. Then, by using Algebraic and Einstein *t*-norms and *t*-conorms we give a multi-criteria decision making method based on these aggregation operators and a score function. This method is applied to a multi-criteria decision making problem with neutrosophic enthalpy set information and the comparison analysis is given with the existing methods to show the efficiency and sensitivity of the proposed method.

Keywords Neutrosophic enthalpy set · Shannon's entropy · Aggregation operator · Multi-criteria decision making

1 Introduction

Decision making is a process of choosing the optimal alternative based on conflicting criteria. In a multi-criteria decision making problem (MCDM), decision makers (DMs) evaluate each alternatives in terms of conflicting criteria and rank them from the best to the worst. However, DMs can obtain various incomplete and indefinite data during this process. To overcome the unclear information, Smarandache (1998) proposed the concept of neutrosophic set (NS) which is characterized by a truth membership function T, an indeterminacy membership function I and a falsity membership function F and each membership function take value in a real standard or non-standard subset of the non-standard unit interval $]^{-}0, 1^{+}[$. Besides, there is no restriction on the sum of the membership functions. Nonetheless, NSs are hard to be applied in practical problems since the values of the functions with

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respect to truth, indeterminacy and falsity lie in $]^{-}0, 1^{+}[$. Therefore, this concept has been extended to various kind of neutrosophic sets whose truth, falsity and indeterminacy membership functions take only one value from the closed interval [0, 1] (Wang et al. 2010, Ye 2014, Deli and Broumi 2015, Deli et al. 2015). All of these concepts are generalizations of the concept of fuzzy set introduced by Zadeh (1965) and they have been applied to various MCDM problems. However, DMs may face some difficulties while determining the truth, indeterminacy and falsity membership degrees of an element in some real life situations so they give rise of giving a few different values due to doubt. For such situations, Wang and Li (2015) developed multi-valued neutrosophic sets (MVNSs) based on NSs and Ye (2014a) proposed single-valued neutrosophic hesitant fuzzy sets (SVNHFSs) based on the hesitant sets. Recently, many researchers have also studied on decision making methods of probability versions of MVNSs (Liu and Cheng 2019, Liu et al. 2019, Peng et al. 2016). A MVNS is characterized by a truth-membership, an indeterminacy and a falsity functions that take values in [0, 1]. Therefore, the notion eliminates repetitive information. Yager (1986) introduced the concept of fuzzy multi-set to overcome this issue. This concept is given with a count function and so the membership degrees of elements are presented as a sequence having different sequence lengths. Therefore, more accurate results can be obtained by preventing the loss of repetitive information. The concept of fuzzy multi-set was extended to the concept of single valued neutrosophic multi-set (SVNMS) by Ye and Ye (2014). A SVNMS is characterized with a truth, an indeterminacy and a falsity membership function taking values in [0, 1], respectively, which are depicted by three sequences. Furthermore, more useful information can be obtained by using some statistical methods such as arithmetic mean and standard deviation. Ye et al. (2020) have used this idea in neutrosophic multi-valued environment which includes both MVNS and SVNS. The notions of neutrosophic enthalpy set (NES) and neutrosophic enthalpy value (NEV) are expressed with a truth, an indeterminacy and a falsity function that map a member of a universal set to a value in $[0, 1]^2$. The first component of each function is the average of the truth sequence, the indeterminacy sequence and the falsity sequence of a neutrosophic multi-valued set (NMVS), respectively, and the second component of each function is the fuzzy complement of a normalized Shannon's entropy, which measures the uncertainty in randomly distributed data, of the truth sequence, the indeterminacy sequence and the falsity sequence of the same NMVS, respectively.

The concept of entropy is one of the most important notions of the information theory. Entropy quantifies the amount of uncertainty involved in the value of a random variable or the outcome of a random process. Shannon's entropy that is one of the most important entropy types is frequently (Shannon 1948). Shannon's entropy is a useful measurement method and it is used to measure the uncertainty in randomly distributed data in information analysis. Therefore, this concept is closely related to the fuzzy set theory. Since each element of sequences (truth, indeterminacy and falsity sequences) in multi-valued sets corresponds to one piece of information, there is an uncertainty in the amount of information given by each sequence depending on the length of the sequences. This shows that the uncertainty of the amount of the information of multi-valued sequences in a NVMS can be measured via Shannon's entropy. An enthalpy value is the information energy expressed by the complement of the Shannon's entropy. The concepts of NES and NEV are defined by considering the enthalpy.

In this paper, we give some algebraic operations for NEVs by using general t-norms and t-conorms. We also provide some weighted aggregation operators via this algebraic operations. The proposed weighted aggregation operators are useful mathematical tools for multi-criteria MCDM problems in the NMVS environment. With a proper score function they can be used in the process of ranking of the alternatives in the MCDM. So, we develop a MCDM approach that improves the decision making reliability and supplies a new influential way for MCDM in the NMVS environment. In this manner, this paper produces a new theoretical base for MCDM.

The rest of the paper is organized as follows: In Section 2, we recall the concepts of Shannon's entropy, NMVS, NEV, t-norm and t-conorm. Then, we develop weighted aggregation operators for NEVs with the help of t-norms and t-conorms. In section 3, we present a multi-criteria decision making method based on the provided weighted aggregation operators by considering Algebraic and Einstein t-norms and t-conorms and we use a score function to rank the alternatives. Later, the proposed method is applied to a multi-criteria decision making problem and the results are compared with the results of some existing methods. In the last section, we make a comparative analysis and we conclude the paper.

2 Fundamental concepts

In this section, we recall some concepts related to Shannon's entropy and NMVSs. Then we recall the concepts of NES and NEV.

Definition 1 (Shannon 1948) Let $P = \{r_1, ..., r_n\}$ be a probability distribution on a universal set, then the Shannon's entropy of P is defined by

$$H(P) = -\sum_{i=1}^{n} r_i \ln(r_i). \tag{1}$$

Note that Shannon's entropy calculates the amount of the uncertainty of the information of an event.

Definition 2 (Ye et al. 2020) Let $X = \{x_1, ..., x_n\}$ be a finite set. A NMVS on X is given with

$$A = \{\langle x_i, MT_A(x_i), MI_A(x_i), MF_A(x_i)\rangle : i = 1, ..., n\}$$

where MT_A, MI_A and MF_A are the truth, the indeterminacy and the falsity sequence, respectively, i.e.,

$$\begin{aligned} MT_A(x_i) &= (\tau_A^1(x_i), \tau_A^2(x_i), ..., \tau_A^{p_i}(x_i)),\\ MI_A(x_i) &= (\upsilon_A^1(x_i), \upsilon_A^2(x_i), ..., \upsilon_A^{p_i}(x_i)), \end{aligned}$$

and

$$MF_A(x_i) = (\kappa_A^1(x_i), \kappa_A^2(x_i), ..., \kappa_A^{p_i}(x_i))$$

where p_i is the length of the sequences with respect to $x_i \in X$, $\tau_A^j(x_i), \upsilon_A^j(x_i), \kappa_A^j(x_i) \in [0, 1]$, $(j = 1, ..., p_i;\ i = 1, ..., n)$, and $0 \leq \sup MT_A(x_i) + \sup MI_A(x_i) + \sup MF_A(x_i) \leq 3$ for any $i = 1, ..., n$, If X has only one element x, then a NMVS $\beta_A = \langle MT_A(x), MI_A(x), MF_A(x)\rangle$, is called a neutrosophic multi-valued value (NMVV).

Let, $X = \{x_1, ...x_n\}$ be a finite set and let A be a NMVS on X. To calculate Shannon's entropy of the sequences the values should be normalized so that their sum is equal to 1. For this purpose let us define $SMT_A = \sum_{j=1}^{p_i} \tau_A^j(x_i)$, $SMI_A = \sum_{j=1}^{p_i} \upsilon_A^j(x_i)$ and $SMF_A = \sum_{j=1}^{p_i} \kappa_A^j(x_i)$. Now it is clear that

$$\sum\nolimits_{j=1}^{p_i} \frac{\tau_A^j(x_i)}{SMT_A} = 1, \sum\nolimits_{j=1}^{p_i} \frac{\upsilon_A^j(x_i)}{SMI_A} = 1,\ and \sum\nolimits_{j=1}^{p_i} \frac{\kappa_A^j(x_i)}{SMF_A} = 1$$

for $i = 1, 2, ...n$. Now, the values

$$\begin{aligned} H\left(\frac{1}{SMT_A} MTA(x_i)\right) &= -\frac{1}{SMT_A} \sum_{j=1}^{p_i} \tau_A^j(x_i) \ln\left(\frac{\tau_A^j(x_i)}{SMT_A}\right),\\ H\left(\frac{1}{SMI_A} MTI(x_i)\right) &= -\frac{1}{SMI_A} \sum_{j=1}^{p_i} \upsilon_A^j(x_i) \ln\left(\frac{\upsilon_A^j(x_i)}{SMI_A}\right),\\ H\left(\frac{1}{SMF_A} MFA(x_i)\right) &= -\frac{1}{SMF_A} \sum_{j=1}^{p_i} \kappa_A^j(x_i) \ln\left(\frac{\kappa_A^j(x_i)}{SMF_A}\right). \end{aligned}$$

are the Shannon's entropies of (normalized) $MT_A(x_i), MI_A(x_i), MF_A(x_i)$.

Now we are ready to define the concepts of enthalpy, NES and NEV. Let $P = \{r_1, ..., r_n\}$ be a probability distribution on universal set, then the enthalpy of P is given with

$$C(P) = 1 - \frac{1}{n} H(P).$$

Note that $C(P) \in [0, 1]$.

Definition 3 Let $X = \{x_1, ..., x_n\}$ be a finite set.

A NES E on X is given with

$$E = \{\langle x_i, (m_{T_i}, c_{T_i}), (m_{I_i}, c_{I_i}), (m_{F_i}, c_{F_i})\rangle : i = 1, ..., n\} \tag{2}$$

where for $i = 1, 2, ..., n$,

$$m_{Ti} := \frac{SMT_A}{p_i}, m_{I_i} := \frac{SMI_A}{p_i}, m_{F_i} := \frac{SMF_A}{p_i}$$

and

$$c_{T_i} := C\left(\frac{1}{SMT_A} MTA(x_i)\right), c_{I_i} := C\left(\frac{1}{SMI_A} MTI(x_i)\right), c_{F_i} := C\left(\frac{1}{SMF_A} MFA(x_i)\right).$$

Definition 4 If X has only one element x, then the NES

$$a(x) = \langle (m_T(x), c_T(x)), (m_I(x), c_I(x)), (m_F(x), c_F(x))\rangle \tag{3}$$

is called a NEV, which is simply denoted by

$$a = \langle (m_T, c_T), (m_I, c_I), (m_F, c_F)\rangle.$$

To define some algebraic operators between NEVs we need t-norms and t-conorms. Now let us recall these concepts.

Definition 5 (Klir and Yuan 1996; Nguyen and Walker 1997) A function $T:[0,1]^2 \rightarrow [0,1]$ is called a t-norm if it satisfies the following conditions:

1. $T(1,x)=x$, for all x.
2. $T(x,y)=T(y,x)$, for all x and y.
3. $T(x,T(y,z))=T(T(x,y),z)$, for all x, y and z.
4. If $x \leq x'$ and $y \leq y'$ then $T(x,y) \leq T(x',y')$.

Definition 6 (Klir and Yuan 1996; Nguyen and Walker 1997) A function $S:[0,1]^2 \rightarrow [0,1]$ is called a t-conorm if it satisfies the following conditions:

1. $S(0,x)=x$, for all x.
2. $S(x,y)=S(y,x)$, for all x and y.
3. $S(x,S(y,z))=S(S(x,y),z)$, for all x, y and z.
4. If $x \leq x'$ and $y \leq y'$ then $S(x,y) \leq S(x',y')$.

Let T be a t-norm. The function $S:[0,1]^2 \rightarrow [0,1]$ given by $S(x,y)=1-T(1-x,1-y)$ is a t-conorm which is called the dual t-conorm of T. Furthermore, these functions can be expressed by $T(x,y)=g^{-1}(g(x)+g(y))$ and $S(x,y)=h^{-1}(h(x)+h(y))$ where $g:[0,1]\rightarrow[0,\infty]$ is a decreasing function such that $g(1)=0$ that is called the additive generator of T and $h:[0,1]\rightarrow[0,\infty]$ is the function defined by $h(x)=g(1-x)$ (see e.g., Klir and Yuan 1996; Nguyen and Walker 1997). Here, infinity is included in the ranges of g and h in the meaning of the limit as $x\rightarrow 0^+$ and $x \rightarrow 1^-$, respectively.

3 Arithmetic and geometric aggregation operators

In this section, we define some algebraic operations among NEVs. Then, we define an arithmetic and a geometric aggregation operator based on t-norms and t-conorms.

Definition 7 Let T be a t-norm and S be the dual t-conorm of T. Let g be the additive generator of T, $h(x)=g(1-x)$ and S be the dual t-conorm of T. For NEVs

$$a_1=\langle(m_{T_1},c_{T_1}),(m_{I_1},c_{I_1}),(m_{F_1},c_{F_1})\rangle$$

and

$$a_2=\langle(m_{T_2},c_{T_2}),(m_{I_2},c_{I_2}),(m_{F_2},c_{F_2})\rangle$$

and $\lambda>0$, we define

1. $a_1 \oplus a_2=\langle(S(m_{T_1},m_{T_2}),S(c_{T_1},c_{T_2})),(T(m_{I_1},m_{I_2}),T(c_{I_1},c_{I_2})),(T(m_{F_1},m_{F_2}),T(c_{F_1},c_{F_2}))\rangle$, 2. $a_1 \otimes a_2=\langle(T(m_{T_1},m_{T_2}),T(c_{T_1},c_{T_2})),(S(m_{I_1},m_{I_2}), S(c_{I_1},c_{I_2})),(S(m_{F_1},m_{F_2}),S(c_{F_1},c_{F_2}))\rangle$, 3. $\lambda a_1=\langle(h^{-1}(\lambda h(m_{T_1}),h^{-1}(\lambda h(c_{T_1})),(g^{-1}(\lambda g(m_{I_1}),g^{-1}(\lambda g(c_{I_1})),(g^{-1}(\lambda g(m_{F_1}),g^{-1}(\lambda g(c_{F_1}))\rangle$, 4. $a_1^{\lambda}=\langle(g^{-1}(\lambda g(m_{T_1}),g^{-1}(\lambda g(c_{T_1})),(h^{-1}(\lambda h(m_{I_1}),h^{-1}(\lambda h(c_{I_1})),(h^{-1}(\lambda h(m_{F_1}),h^{-1}(\lambda h(c_{F_1}))\rangle$.

In the following, we define a weighted arithmetic aggregation operator with the help of additive generators of t-norms.

Definition 8 Let g be the additive generator of a t-norm and $h(x)=g(1-x)$. A weighted aggregation operator $TS-NEVWAA$ based on g and h is given by

$$TS-NEVWAA(a_1,\ldots,a_n): = \begin{pmatrix} \left(h^{-1}\left(\sum_{i=1}^{n}\omega_i h(m_{T_i})\right),h^{-1}\left(\sum_{i=1}^{n}\omega_i h(c_{T_i})\right)\right), \\ \left(g^{-1}\left(\sum_{i=1}^{n}\omega_i g(m_{I_i})\right),g^{-1}\left(\sum_{i=1}^{n}\omega_i g(c_{I_i})\right)\right), \\ \left(g^{-1}\left(\sum_{i=1}^{n}\omega_i g(m_{F_i})\right),g^{-1}\left(\sum_{i=1}^{n}\omega_i g(c_{F_i})\right)\right) \end{pmatrix} \tag{4}$$

where $\{a_1,\ldots,a_n\}$ is a collection of NEVs and $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$ is a weight vector with $\omega_i \in [0,1]$ such that $\sum_{i=1}^{n}\omega_i=1$.

Theorem 1 *Components of* $TS-NEVWAA(a_1,\ldots,a_n)$ *belong* [0, 1].

Proof The range of g^{-1},h^{-1} is [0, 1]. Thus the proof is trivial.

Definition 9 Let g be the additive generator of a t-norm and $h(x)=g(1-x)$. A weighted aggregation operator $TS-NEVWGA$ based on g and h is given by

$$TS-NEVWGA(a_1,\ldots,a_n): = \begin{pmatrix} \left(g^{-1}\left(\sum_{i=1}^{n}\omega_i g(m_{T_i})\right),g^{-1}\left(\sum_{i=1}^{n}\omega_i g(c_{T_i})\right)\right), \\ \left(h^{-1}\left(\sum_{i=1}^{n}\omega_i h(m_{I_i})\right),h^{-1}\left(\sum_{i=1}^{n}\omega_i h(c_{I_i})\right)\right), \\ \left(h^{-1}\left(\sum_{i=1}^{n}\omega_i h(m_{F_i})\right),h^{-1}\left(\sum_{i=1}^{n}\omega_i h(c_{F_i})\right)\right) \end{pmatrix} \tag{5}$$

where $\{a_1,\ldots,a_n\}$ is a collection of NEVs and $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$ is a weight vector with $\omega_i \in [0,1]$ such that $\sum_{i=1}^{n}\omega_i=1$.

Theorem 2 *Components of* $TS-NEVWGA(a_1,\ldots,a_n)$ *belong* [0, 1].

Proof Since the range of g^{-1},h^{-1} is [0, 1] the proof is trivial.

In the following two remarks we consider some particular additive generators to obtain some particular weighted aggregation operators from Definitions 8 and 9.

Remark 1 Let $g(t) = -\log t$. In this case, we obtain $h(t) = -\log(1-t)$. Then, T and S are called the Algebraic t-norm and the Algebraic t-conorm, respectively (see Beliakov et al. 2007). In this case, the weighted aggregation operators $TS-NEVWAA$ and $TS-NEVWGA$ transform into

$$NEVWAA(a_1,\ldots,a_n): = \left(\begin{array}{c} \left(1-\prod_{i=1}^{n}(1-m_{T_i})^{\omega_i}, 1-\prod_{i=1}^{n}(1-c_{T_i})^{\omega_i}\right), \\ \left(\prod_{i=1}^{n}(m_{I_i})^{\omega_i}, \prod_{i=1}^{n}(c_{I_i})^{\omega_i}\right), \left(\prod_{i=1}^{n}(m_{F_i})^{\omega_i}, \prod_{i=1}^{n}(c_{F_i})^{\omega_i}\right) \end{array} \right) \tag{6}$$

and

$$NEVWGA(a_1,\ldots,a_n) := \left(\begin{array}{c} \left(\prod_{i=1}^{n}(m_{T_i})^{\omega_i}, \prod_{i=1}^{n}(c_{T_i})^{\omega_i}\right), \\ \left(1-\prod_{i=1}^{n}(1-m_{I_i})^{\omega_i}, 1-\prod_{i=1}^{n}(1-c_{I_i})^{\omega_i}\right), \\ \left(1-\prod_{i=1}^{n}(1-m_{F_i})^{\omega_i}, 1-\prod_{i=1}^{n}(1-c_{F_i})^{\omega_i}\right) \end{array} \right), \tag{7}$$

respectively.

Remark 2 Let $g(t) = \log\left(\frac{2-t}{t}\right)$. Then, we have $h(t) = \log\left(\frac{1+t}{1-t}\right)$. In this case, T and S are called the Einstein t-norm and the Einstein t-conorm, respectively (see Beliakov et al. 2007). In this case, the weighted aggregation operators $TS-NEVWAA$ and $TS-NEVWGA$ transform into

$$NEVWAA^1(a_1,\ldots,a_n): = \left(\begin{array}{c} \left(\frac{\prod_{i=1}^{n}(1+m_{T_i})^{\omega_i}-\prod_{i=1}^{n}(1-m_{T_i})^{\omega_i}}{\prod_{i=1}^{n}(1+m_{T_i})^{\omega_i}+\prod_{i=1}^{n}(1-m_{T_i})^{\omega_i}}, \frac{\prod_{i=1}^{n}(1+c_{T_i})^{\omega_i}-\prod_{i=1}^{n}(1-c_{T_i})^{\omega_i}}{\prod_{i=1}^{n}(1+c_{T_i})^{\omega_i}+\prod_{i=1}^{n}(1-c_{T_i})^{\omega_i}} \right), \\ \left(\frac{2\prod_{i=1}^{n}(m_{I_i})^{\omega_i}}{\prod_{i=1}^{n}(2-m_{I_i})^{\omega_i}+\prod_{i=1}^{n}(m_{I_i})^{\omega_i}}, \frac{2\prod_{i=1}^{n}(c_{I_i})^{\omega_i}}{\prod_{i=1}^{n}(2-c_{I_i})^{\omega_i}+\prod_{i=1}^{n}(c_{I_i})^{\omega_i}} \right), \\ \left(\frac{2\prod_{i=1}^{n}(m_{F_i})^{\omega_i}}{\prod_{i=1}^{n}(2-m_{F_i})^{\omega_i}+\prod_{i=1}^{n}(m_{F_i})^{\omega_i}}, \frac{2\prod_{i=1}^{n}(c_{F_i})^{\omega_i}}{\prod_{i=1}^{n}(2-c_{F_i})^{\omega_i}+\prod_{i=1}^{n}(c_{F_i})^{\omega_i}} \right) \end{array} \right) \tag{8}$$

and

$$NEVWGA^1(a_1,\ldots,a_n): = \left(\begin{array}{c} \left(\frac{2\prod_{i=1}^{n}(m_{T_i})^{\omega_i}}{\prod_{i=1}^{n}(2-m_{T_i})^{\omega_i}+\prod_{i=1}^{n}(m_{T_i})^{\omega_i}}, \frac{2\prod_{i=1}^{n}(c_{T_i})^{\omega_i}}{\prod_{i=1}^{n}(2-c_{T_i})^{\omega_i}+\prod_{i=1}^{n}(c_{T_i})^{\omega_i}} \right), \\ \left(\frac{\prod_{i=1}^{n}(1+m_{I_i})^{\omega_i}-\prod_{i=1}^{n}(1-m_{I_i})^{\omega_i}}{\prod_{i=1}^{n}(1+m_{I_i})^{\omega_i}+\prod_{i=1}^{n}(1-m_{I_i})^{\omega_i}}, \frac{\prod_{i=1}^{n}(1+c_{I_i})^{\omega_i}-\prod_{i=1}^{n}(1-c_{I_i})^{\omega_i}}{\prod_{i=1}^{n}(1+c_{I_i})^{\omega_i}+\prod_{i=1}^{n}(1-c_{I_i})^{\omega_i}} \right), \\ \left(\frac{\prod_{i=1}^{n}(1+m_{F_i})^{\omega_i}-\prod_{i=1}^{n}(1-m_{F_i})^{\omega_i}}{\prod_{i=1}^{n}(1+m_{F_i})^{\omega_i}+\prod_{i=1}^{n}(1-m_{F_i})^{\omega_i}}, \frac{\prod_{i=1}^{n}(1+c_{F_i})^{\omega_i}-\prod_{i=1}^{n}(1-c_{F_i})^{\omega_i}}{\prod_{i=1}^{n}(1+c_{F_i})^{\omega_i}+\prod_{i=1}^{n}(1-c_{F_i})^{\omega_i}} \right) \end{array} \right), \tag{9}$$

respectively.

Du et al. (2021) have provided a score function for neutrosophic Z-numbers that also consist of triples of pairs of components lying in [0, 1]. Due the similar structure of NEVs we can use the score function S defined for neutrosophic Z-numbers by Du et al. (2021) to rank the results of proposed weighted aggregation operators.

Let $A = \langle (m_T, c_T), (m_I, c_I), (m_F, c_F) \rangle$ be the aggregated version of a collection of NEVs via provided weighted aggregation operators. The score function S is defined by

$$S(A) = \frac{(2 + m_T c_T - m_I c_I - m_F c_F)}{3}. \tag{10}$$

If $S(A_1) \geq S(A_2)$, then the precedence of A_1 is higher than A_2 (Du et al. 2021). In this case we write $A_1 \succeq A_2$.

4 Applications to MCDM

In this section, a MCDM problem that is adapted from Ye et al. (2020) is considered in order to highlight the applicability of the proposed arithmetic and geometric average operators. Moreover, the availability and the power of the method are approved through comparative analysis with other existing methods.

4.1 Steps of the proposed method

Consider a MCDM problem with unknown criteria weights, the set of alternatives $G = \{G_1, G_2, \ldots, G_m\}$ and the set of criteria $A = \{a_1, a_2, \ldots, a_n\}$. Assume that the suitable evaluations of each alternative is represented by the following NMVS:

$G_j = \left\{ \left\langle a_i, (\tau^1_{G_j}, \tau^2_{G_j}, \ldots, \tau^{p_i}_{G_j}), (\upsilon^1_{G_j}, \upsilon^2_{G_j}, \ldots, \upsilon^{p_i}_{G_j}), (\kappa^1_{G_j}, \kappa^2_{G_j}, \ldots, \kappa^{p_i}_{G_j}) \right\rangle : i = 1, \ldots, n \right\}.$

Step 1: Each NMVS G_j is represented as a NES:

$$E_j = \left\{ \left\langle a_i, (m_{T_{E_j}}, e_{T_{E_j}}), (m_{I_{E_j}}, e_{I_{E_j}}), (m_{F_{E_j}}, e_{F_{E_j}}) \right\rangle : i = 1, \ldots, n \right\}$$

for $j = 1, 2, \ldots, m$ by using Definition 3.

Step 2: Each NES $E_j = \{a_{j1}, \ldots, a_{jn}\}$ can be considered as a collection of NEVs for $j = 1, \ldots, m$. Then, the decision matrix of NESs is constructed as follows:

$$E = \begin{bmatrix} E_1 \\ E_2 \\ . \\ . \\ . \\ E_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ . & . & \ldots & . \\ . & . & \ldots & . \\ . & . & \ldots & . \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}_{m \times n}$$

Step 3: For a fixed $j = 1, \ldots, m$ the NEVs $\{a_{j1}, a_{j2}, \ldots, a_{jn}\}$ can be aggregated to the NEV a'_j by using one of the weighted aggregation operators NEVWAA, NEVWGA, NEVWAA^1 or NEVWGA^1.

Step4: The scores of a'_j for $j = 1, .., m$ is calculated by using the score function given by (10).

Step5: The alternative that has the highest score can be selected.

4.2 A Multi-Criteria Decision Making Problem

Ye et al. (2020) has studied a MCDM problem under NMVS setting with consistency fuzzy sets. Now, we consider the same problem in terms of NESs via proposed weighted aggregation operators. Let a client want to buy an appropriate car. Consider the set of criteria as follow:

$$X = \{x_1(\text{Fuel economy}), x_2(\text{Price}), x_3(\text{Amenity}), x_4(\text{Safety})\}$$

Moreover, assume that each car G_j for $j = 1, 2, 3, 4$ is represented as a NMVSs (see, Ye et al. 2020) with respect to the criteria as follow:

$$G_1 = \left\{ \begin{array}{c} \langle x_1, (0.5, 0.7), (0.3, 0.7), (0.2, 0.6) \rangle, \langle x_2, (0.4, 0.4, 0.5) \rangle, \\ \langle x_3, (0.7, 0.8), (0.7, 0.7), (0.5, 0.6) \rangle, \langle x_4, (0.1, 0.5), (0.2, 0.5), (0.7, 0.8) \rangle \end{array} \right\},$$
$$G_2 = \left\{ \begin{array}{c} \langle x_1, (0.7, 0.9), (0.7, 0.7), (0.1, 0.5) \rangle, \langle x_2, (0.7, 0.6, 0.8) \rangle, \\ \langle x_3, (0.9, 0.4, 0.6) \rangle, \langle x_4, (0.5, 0.5), (0.1, 0.2), (0.7, 0.9) \rangle \end{array} \right\},$$
$$G_3 = \left\{ \begin{array}{c} \langle x_1, (0.3, 0.6), (0.3, 0.4), (0.2, 0.7) \rangle, \langle x_2, (0.2, 0.2, 0.2) \rangle, \\ \langle x_3, (0.6, 0.9), (0.5, 0.5), (0.2, 0.5) \rangle, \langle x_4, (0.4, 0.7), (0.2, 0.5), (0.2, 0.3) \rangle \end{array} \right\},$$
$$G_4 = \left\{ \begin{array}{c} \langle x_1, (0.8, 0.9), (0.6, 0.7), (0.1, 0.2) \rangle, \langle x_2, (0.3, 0.5, 0.2) \rangle, \\ \langle x_3, (0.1, 0.5), (0.4, 0.7), (0.2, 0.5) \rangle, \langle x_4, (0.4, 0.4), (0.2, 0.2), (0.8, 0.8) \rangle \end{array} \right\}.$$

Steps 1 and 2: Each NMVS G_j is transformed into a NES and the decision matrix is constructed (see Table 1)

Table 1 The Decision Matrix of NEVs

	x_1	x_2
E_1	$\langle(0.6,0.6604),(0.5,0.6946),(0.4,0.7189)\rangle$	$\langle(0.4,1),(0.4,1),(0.5,1)\rangle$
E_2	$\langle(0.8,0.6574),(0.7,0.6536),(0.3,0.7747)\rangle$	$\langle(0.7,1),(0.6,1),(0.8,1)\rangle$
E_3	$\langle(0.45,0.6816),(0.35,0.6586),(0.45,0.7352)\rangle$	$\langle(0.2,1),(0.2,1),(0.2,1)\rangle$
E_4	$\langle(0.85,0.6543),(0.65,0.6549),(0.15,0.6818)\rangle$	$\langle(0.3,1),(0.5,1),(0.2,1)\rangle$

	x_3	x_4
E_1	$\langle(0.75,0.6546),(0.7,0.6535),(0.55,0.6555)\rangle$	$\langle(0.3,0.7747),(0.35,0.7009),(0.75,0.6546)\rangle$
E_2	$\langle(0.9,1),(0.4,1),(0.6,1)\rangle$	$\langle(0.5,0.6535),(0.15,0.6818),(0.8,0.6574)\rangle$
E_3	$\langle(0.75,0.6635),(0.5,0.6535),(0.35,0.7009)\rangle$	$\langle(0.55,0.6723),(0.35,0.7009),(0.25,0.6635)\rangle$
E_4	$\langle(0.3,0.7747),(0.55,0.6723),(0.35,0.7009)\rangle$	$\langle(0.4,0.6535),(0.2,0.6535),(0.8,0.6535)\rangle$

Table 2 Scores of alternatives

	$S(a_1^{'})$	$S(a_2^{'})$	$S(a_3^{'})$	$S(a_4^{'})$
NEVWAA (6)	0.6109	0.6595	0.6581	**0.7143**
NEVWGA (7)	0.4616	0.4593	**0.5345**	0.5008
NEVWAA^1 (8)	0.6048	0.6463	**0.6514**	0.6396
NEVWGA^1 (9)	0.4677	0.4723	**0.5403**	0.5184

Bold values indicate the maximum of the corresponding row

Table 3 Rankings of alternatives

NEVWAA (6)	$G_4 \succ G_2 \succ G_3 \succ G_1$
NEVWGA (7)	$G_3 \succ G_4 \succ G_1 \succ G_2$
NEVWAA^1 (8)	$G_3 \succ G_2 \succ G_4 \succ G_1$
NEVWGA^1 (9)	$G_3 \succ G_4 \succ G_2 \succ G_1$

Step 3: The aggregated value $a_j^{'}$ of E_j for $j = 1,2,3,4$ is obtained with respect to NEVWAA, NEVWGA, NEVWAA^1 or NEVWGA^1. We consider the same weight vector $\omega = (0.5, 0.25, 0.125, 0.125)$ used in Ye et al. (2020).

1. a_j' values obtained with NEVWAA (6):

$$a_1^{'} = \langle(0.5523,1),(0.4716,0.7559),(0.4761,0.7627)\rangle,$$
$$a_2^{'} = \langle(0.7724,1),(0.5180,0.7706),(0.4725,0.8352)\rangle,$$
$$a_3^{'} = \langle(0.4662,1),(0.3181,0.7360),(0.3308,0.7791)\rangle,$$
$$a_4^{'} = \langle(0.6821,1),(0.5144,0.7301),(0.2209,0.7409)\rangle.$$

2. a_j' values obtained with NEVWGA (7):

$$a_1^{'} = \langle(0.5112,0.7465),(0.4926,1),(0.5043,1)\rangle,$$
$$a_2^{'} = \langle(0.7404,0.7688),(0.5995,1),(0.5919,1)\rangle,$$
$$a_3^{'} = \langle(0.4015,0.7463),(0.3374,1),(0.3588,1)\rangle,$$
$$a_4^{'} = \langle(0.5234,0.7429),(0.5621,1),(0.3243,1)\rangle.$$

3. a_j' values obtained with NEVWAA^1 (8):

$$a_1^{'} = \langle(0.5461,1),(0.4748,0.7631),(0.4799,0.7698)\rangle,$$
$$a_2^{'} = \langle(0.7694,1),(0.5345,0.7817),(0.4901,0.8420)\rangle,$$
$$a_3^{'} = \langle(0.4558,1),(0.3207,0.7442),(0.3346,0.7854)\rangle,$$
$$a_4^{'} = \langle(0.6624,1),(0.5239,0.7386),(0.2281,0.7564)\rangle.$$

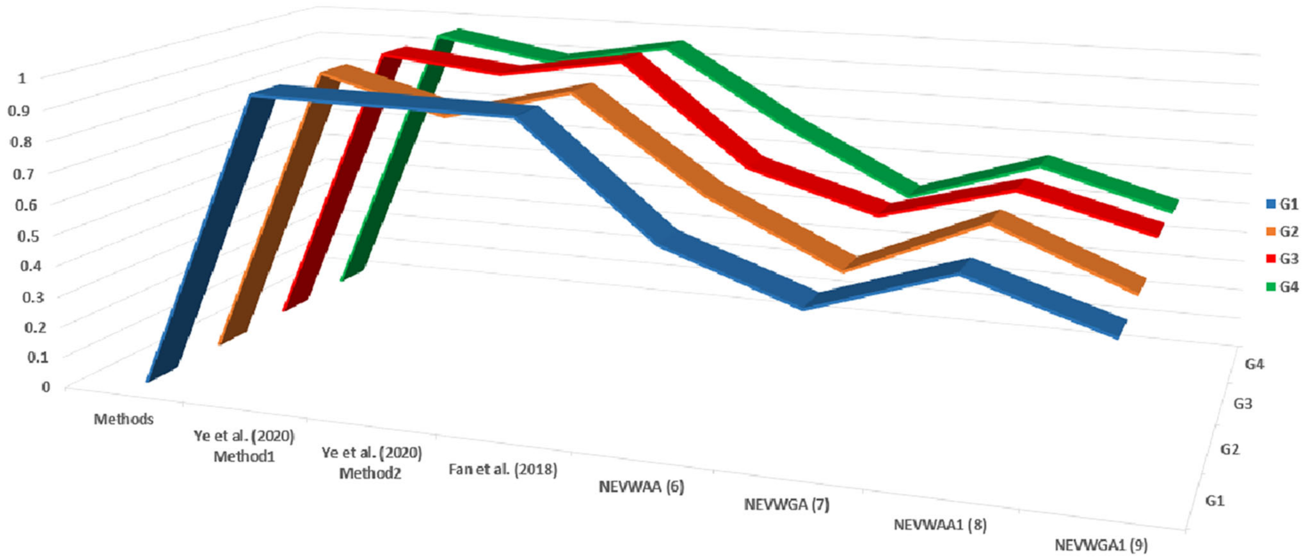
4. a_j' values obtained with NEVWGA^1 (9):

$$a_1^{'} = \langle(0.5181,0.7546),(0.4889,1),(0.4991,1)\rangle,$$
$$a_2^{'} = \langle(0.7458,0.7799),(0.5907,1),(0.5739,1)\rangle,$$
$$a_3^{'} = \langle(0.4109,0.7539),(0.3344,1),(0.3546,1)\rangle,$$
$$a_4^{'} = \langle(0.5478,0.7512),(0.5564,1),(0.2998,1)\rangle.$$

Steps 4 and 5: The score of a_j' with respect to the score function S given with (10) is provided in Table 2) for each of the weighted aggregation operators. Also the rankings are given in Table 3.

5 Comparative analysis

In this section, we compare the results of the proposed methods with results of some existing methods in literature. In order to solve the MCDM problem considered in Subsection 4.2, several different methods have been developed under different fuzzy environments. First, Deli et al. (2015) have proposed a weighted arithmetic and a weighted geometric operator for bipolar neutrosophic sets and applied them to the same car selection problem. Then, Fan et al.



	Ye et al. (2020) Method1	Ye et al. (2020) Method2	Fan et al. (2018)	NEVWAA (6)	NEVWGA (7)	NEVWAA1 (8)	NEVWGA1 (9)
G1	0.964	0.9586	0.9535	0.6109	0.4616	0.6048	0.4677
G2	0.9624	0.8485	0.9511	0.6595	0.4593	0.6463	0.4723
G3	0.9578	0.9138	0.9813	0.6581	0.5345	0.6514	0.5403
G4	0.9601	0.8881	0.9616	0.7143	0.5008	0.6396	0.5184

Fig. 1 The comparison of different methods

(2018) have studied on this problem under SVNMS settings with a cosine similarity measure. Ye et al. (2020) have introduced a new decision making method and correlation coefficients that transform NMVSs into consistency neutrosophic sets and have applied it to same problem. In this paper, we consider the same problem by using NEVs. For this same MCDM problem, the results obtained with different methods are given in Figure 1.

The results in Figure 1 show that as the proposed method changes, the choice of the best alternative changes. In this study, our goal is to obtain a more sensitive decision making approach with the average and the enthalpy that are transformed from multi valued sequences from NMVSs. The reason of the difference of the best selection with some other studies may be caused by the sensitivity of the provided method.

6 Conclusion

A NEV contains both the average of the data that can be expressed with different sequence lengths and the degree of uncertainty of the data via enthalpy.In this paper, we provide some algebraic operations among NEVs and we give some weighted aggregation operators for classes of NEVs. The proposed weighted aggregation operators provide useful ranking method when they are considered with a proper score function. We also develope a decision making approach which not only improves the decision making reliability but also supplies a new influential way for MCDM problems in the NMVS environment.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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