

TWO TRIANGLES WITH THE SAME ORTHOCENTER AND A VECTORIAL PROOF OF STEVANOVIC'S THEOREM

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Abstract. In this article we'll emphasize on two triangles and provide a vectorial proof of the fact that these triangles have the same orthocenter. This proof will, further allow us to develop a vectorial proof of the Stevanovic's theorem relative to the orthocenter of the Fuhrmann's triangle.

Lemma 1. Let ABC an acute angle triangle, H its orthocenter, and A', B', C' the symmetrical points of H in rapport to the sides BC, CA, AB . We denote by X, Y, Z the symmetrical points of A, B, C in rapport to $B'C', C'A', A'B'$. The orthocenter of the triangle XYZ is H .

Proof. We will prove that $XH \perp YZ$, by showing that $\overrightarrow{XH} \cdot \overrightarrow{YZ} = 0$. We have (see Figure1) $\overrightarrow{VH} = \overrightarrow{AH} - \overrightarrow{AX}$, $\overrightarrow{BC} = \overrightarrow{BY} + \overrightarrow{YZ} + \overrightarrow{ZC}$, from here $\overrightarrow{YZ} = \overrightarrow{BC} - \overrightarrow{BY} - \overrightarrow{ZC}$.

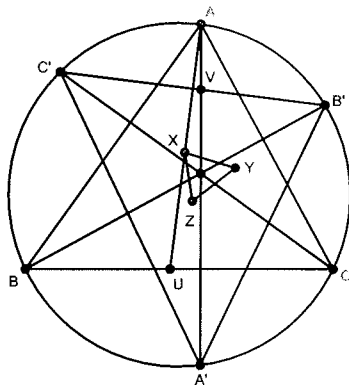


Figure 1

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Because Y is the symmetric of B in rapport to $A'C'$ and Z is the symmetric of C in rapport to $A'B'$, the parallelogram's rule gives us that: $\overrightarrow{BY} = \overrightarrow{BC'} + \overrightarrow{BA'}$, $\overrightarrow{CZ} = \overrightarrow{CB'} + \overrightarrow{CA'}$. Therefore

$$\overrightarrow{YZ} = \overrightarrow{BC} - (\overrightarrow{BC'} + \overrightarrow{BA'}) + \overrightarrow{B'C} + \overrightarrow{A'C}$$

But, $\overrightarrow{BC'} = \overrightarrow{BH} + \overrightarrow{HC'}$, $\overrightarrow{BA'} = \overrightarrow{BH} + \overrightarrow{HA'}$, $\overrightarrow{CB'} = \overrightarrow{CH} + \overrightarrow{HB'}$, $\overrightarrow{CA'} = \overrightarrow{CH} + \overrightarrow{HA'}$. By substituting these relations in the \overrightarrow{YZ} , we find:

$$\overrightarrow{YZ} = \overrightarrow{BC} + \overrightarrow{C'B'}$$

We compute

$$\begin{aligned} \overrightarrow{XH} \cdot \overrightarrow{YZ} &= (\overrightarrow{AH} - \overrightarrow{AX}) \cdot (\overrightarrow{BC} + \overrightarrow{C'B'}) = \\ &= \overrightarrow{AX} \cdot \overrightarrow{BC} + \overrightarrow{AH} \cdot \overrightarrow{C'B'} - \overrightarrow{AX} \cdot \overrightarrow{BC} - \overrightarrow{AX} \cdot \overrightarrow{C'B'} \end{aligned}$$

Because $AH \perp BC$ we have $\overrightarrow{AH} \cdot \overrightarrow{BC} = 0$, also $AX \perp B'C'$ and therefore $\overrightarrow{AX} \cdot \overrightarrow{B'C'} = 0$. We need to prove also that $\overrightarrow{XH} \cdot \overrightarrow{YZ} = \overrightarrow{AH} \cdot \overrightarrow{C'B'} - \overrightarrow{AX} \cdot \overrightarrow{BC}$. We note: $\{U\} = AX \cap BC$ and $\{V\} = AH \cap B'C'$. Then

$$\begin{aligned} \overrightarrow{AX} \cdot \overrightarrow{BC} &= AX \cdot BC \cdot \cos(\angle AUC) \\ \overrightarrow{AH} \cdot \overrightarrow{C'B'} &= AH \cdot C'A' \cdot \cos(\angle AVC') \end{aligned}$$

We observe that $\angle AUC \equiv \angle AVC'$ (angles with the sides respectively perpendicular). The point B' is the symmetric of H in rapport to AC , consequently $\angle HAC \equiv \angle CAB'$, also the point C' is the symmetric of the point H in rapport to AB , and therefore $\angle HAB \equiv \angle BAC'$.

From these last two relations we find that $\angle B'AC' = 2\angle A$. The sinus theorem applied in the triangles $AB'C'$ and ABC gives:

$$B'C' = 2R \cdot \sin 2A$$

$$BC = 2R \sin A$$

We'll show that

$$AX \cdot BC = AH \cdot C'B',$$

and from here

$$AX \cdot 2R \sin A = AH \cdot 2R \cdot \sin 2A$$

which is equivalent to

$$AX = 2AH \cos A$$

We noticed that $\angle B'AC' = 2A$. Because $AX \perp B'C'$, it results that $\angle TAB \equiv \angle A$, we noted $\{T\} = AX \cap B'C'$. On the other side

$$AC' = AH, AT = \frac{1}{2}AY, \text{ and } AT = AC' \cos A = AH \cos A,$$

therefore $\overrightarrow{XH} \cdot \overrightarrow{YZ} = 0$.

Similarly, we prove that $YH \perp XZ$, and therefore H is the orthocenter of triangle XYZ .

Lemma 2. Let ABC a triangle inscribed in a circle, I the intersection of its bisector lines, and A', B', C' the intersections of the circumscribed circle with the bisectors AI, BI, CI respectively. The orthocenter of the triangle $A'B'C'$ is I .

Proof.

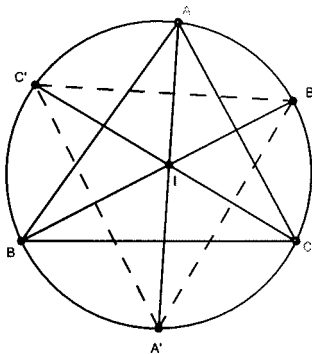


Figure 2

We'll prove that $A'I \perp B'C'$ (see Figure 2). Let $\alpha = m(\widehat{A'C}) = m(\widehat{A'B})$, $\beta = m(\widehat{B'C}) = m(\widehat{B'A})$, $\gamma = m(\widehat{C'A}) = m(\widehat{C'B})$. Then $m(\angle A'IC') = \frac{1}{2}(\alpha + \beta + \gamma)$. Because $2(\alpha + \beta + \gamma) = 360^\circ$ it results $m(\angle A'IC') = 90^\circ$, therefore $A'I \perp B'C'$.

Similarly, we prove that $B'I \perp A'C'$, and consequently the orthocenter of the triangle $A'B'C'$ is I .

Definition. Let ABC a triangle inscribed in a circle with the center in O and A', B', C' the middle of the arcs \widehat{BC} , \widehat{CA} , \widehat{AB} respectively. The triangle XYZ formed by the symmetric of the points A', B', C' respectively in rapport to BC, CA, AB is called the *Fuhrmann triangle* of the triangle ABC .

Note. In 2002 the mathematician Milorad Stevanovic proved the following theorem:

Theorem (M. Stevanovic). *In an acute angle triangle the orthocenter of the Fuhrmann's triangle coincides with the center of the circle inscribed in the given triangle.*

Proof. We note $A'B'C'$ the given triangle and let A, B, C respectively the middle of the arcs $\widehat{B'C'}$, $\widehat{C'A'}$, $\widehat{A'B'}$ (see Figure 1). The lines AA', BB', CC' being bisectors in the triangle $A'B'C'$ are concurrent in the center of the circle inscribed in this triangle, which will note H , and which, in conformity with Lemma 2 is the orthocenter of the triangle ABC . Let XYZ the Fuhrmann triangle of the triangle $A'B'C'$, in conformity with Lemma 1, the orthocenter of XYZ coincides with H the orthocenter of ABC , therefore with the center of the inscribed circle in the given triangle $A'B'C'$.

References

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