On the Concepts of Two-Fold Fuzzy Vector Spaces and Algebraic Modules

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On the Concepts of Two- Fold Fuzzy Vector Spaces and Algebraic Modules

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Abstract

The concept of two-fold algebras is generated by merging some special sets such as fuzzy sets, neutrosophic sets or plithogenic sets with various algebraic structures. The main objective of this research paper is to provide a strict mathematical definition and a general study of two-fold fuzzy vector spaces defined over the real numbers and also two-fold fuzzy vector spaces defined over the complex numbers, as well as two-fold fuzzy algebraic modules defined over commutative rings with unity. The elementary properties and algebraic operations of those structures will be explained through many theorems and mathematical proofs.

Keywords: two-fold fuzzy algebra; two-fold fuzzy vector space; two-fold fuzzy module

1. Introduction

The study of mathematical algebras defined by fuzzy sets and their various generalizations is one of the most recent and important diverse studies in both commutative and non-commutative algebras alike [1-2, 6-8, 15].
We notice in previous studies the application of fuzzy, neutrosophic, and plithogenic sets in generalizing many classical algebraic structures with wide applications, such as matrices, algebraic rings, vector spaces, and also modules [3-5, 9-12].
Smarandache presented the concept of two-fold algebras [14], where he combined the neutrosophic sets with the three parts that symbolize truth, falsity, and indeterminacy with some traditional algebraic structures to form new algebraic structures, as his distinguished study opened a new research door of algebraic nature, both theoretical and applied.
Abobala in [16] studied two-fold fuzzy algebras by combining it with the standard fuzzy number theoretical system, where many results related to this new type were proven through its algebraic operations and its connection to number theory.
These previous studies have prompted us to use vector spaces and combine them with fuzzy sets, in order to build new algebraic structures that we called two-fold fuzzy vector spaces and two-fold fuzzy modules and then study the basic properties related to these algebraic structures and the partial structures related to them.
On the other hand, we presented many theories and related mathematical proofs to describe the desired basic algebraic properties.

Main Discussion

Real Two-Fold Fuzzy Vector Spaces:

Definition:

Let S be a vector space over the field $\mathbb{R}$, and

$$\begin{cases} g: S \to [0.1] \\ f: \mathbb{R} \to [0.1] \end{cases} \text{ such that: } \begin{cases} g(x+y) = max\big(g(x).g(y)\big) \\ g(x \cdot \mathrm{r}) = \min\big(g(x).f(r)\big) \\ g(0) = 0 \,.\, f(1) = 1 \end{cases}$$

For $x.y \in S \text{ and } r \in \mathbb{R}$

We define the two- fold fuzzy vector space $S_{(f.g)}$ as follows:

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$S_{(f.g)} = \{x_{g(x)} ; x \in S\}$

Definition:

We define the following binary operations on $S_{(f.g)}$:

$$\begin{cases} *: S_{(f.g)} \times S_{(f.g)} \to S_{(f.g)} \\ \circ: S_{(f.g)} \times \mathbb{R}_f \to S_{(f.g)} \end{cases}$$

$$\begin{cases} x_{g(x)} * y_{g(y)} = (x+y)_{g(x+y)} = (x+y)_{\max(g(x).g(y))} \\ x_{g(x)} \circ r_{f(r)} = (xr)_{g(xr)} = (xr)_{\min(g(x).f(r))} \end{cases}$$

Theorem:

Let $S_{(f.g)}$ be the two-fold vector space defined above, then:

1] $*,\circ$ are abelian.

2] $*$ is associative.

3] $o_\circ$ is the identity of $*$, 1_1 is the identity of $\circ$.

4] $r \circ (x * y) = (r \circ x) * (r \circ y)$. $r \circ (t \circ x) = (rt) \circ x : x \in S . r.t \in \mathbb{R}$

5] $x * (-x) = 0$

Proof:

1] Let $x_{g(x)}.y_{g(y)} \in S_{(f.g)}.r \in \mathbb{R}.then$:

$x_{g(x)} * y_{g(y)} = (x+y)_{\max(g(x).g(y))} = (y+x)_{\max(g(x).g(y))} = y_{g(y)} * x_{g(x)}$

$x_{g(x)} \circ r_{f(r)} = (xr)_{g(x \cdot r)} = (r \cdot x)_{g(r \cdot x)} = r_{f(r)} \circ x_{g(x)}$

2] $x_{g(x)} * \left(y_{g(y)} * z_{g(z)}\right) = x * \left((y+z)_{g(y+z)}\right) = (x+y+z)_{g(x+y+z)} = (x+y)_{g(x+y)} * z_{g(z)} = \left(x_{g(x)} * y_{g(y)}\right) * z_{g(z)}$

3] $x_{g(x)} * \circ_{g(\circ)} = (x + \circ)_{max(g(x).\circ)} = x_{g(x)}$

$x_{g(x)} \circ 1_{f(1)} = (x \cdot 1)_{min(g(x).1)} = x_{g(x)}$

4] $r_{f(r)} \circ \left(x_{g(x)} * y_{g(y)}\right) = r_{f(r)} \circ (x+y)_{g(x+y)} = (rx+ry)_{g(rx+ry)} = (rx)_{g(rx)} * (ry)_{g(ry)} = \left(r_{f(r)} \circ x_{g(x)}\right) * \left(r_{f(r)} \circ y_{g(y)}\right)$

$r_{f(r)} \circ \left(t_{f(t)} \circ x_{g(x)}\right) = r_{f(r)}\left((tx)_{g(tx)}\right) = (rtx)_{g(rtx)} = (rt)_{f(rt)} \circ x_{g(x)}$

5] $x_{g(x)} * (-x)_{g(-x)} = (x-x)_{g(x-x)} = \circ_\circ$

Definition:

Let T be a subspace of S, then:

$T_{(f.g)} = \{x_{g(x)} ; x \in T\}$ is called a two-fold fuzzy subspace of $S_{(f.g)}$

Theorem:

Let T be a non-empty subset of S, then $T_{(f.g)}$ is a two-fold fuzzy subspace of $S_{(f.g)}$ if and only if:

$$\begin{cases} x * y \in T_{(f.g)} \\ r \circ x \in T_{(f.g)} \end{cases} \text{for all } r \in \mathbb{R}. \quad x.y \in T$$

Proof:

$T_{(f.g)}$ is a two-fold fuzzy subspace of $S_{(f.g)}$ if and only if T is a subspace of S, which is equivalent to:

$$\begin{cases} x + y \in \text{T} \\ rx \in \text{T} \end{cases} \text{for all } x.y \in T. \quad r \in \mathbb{R}$$

So that, $\begin{cases} x_{g(x)} * y_{g(y)} = (x+y)_{g(x+y)} \in T_{(f.g)} \\ r_{f(r)} \circ x_{g(x)} = (rx)_{g(rx)} \in T_{(f.g)} \end{cases}$

Definition:

Let $S_{(f.g)}.W_{(f.h)}$ be two. two-fold fuzzy vector spaces over $\mathbb{R}$, and $H: S \to W$ be a classical linear transformation, then:

$H_F: S_{(f.g)} \to W_{(f.h)} : H_F\left(x_{g(x)}\right) = (H(x))_{h(H(x))}$ is called the two-fold fuzzy linear transformation.

We define:

$k_{er}(H_F) = \{x_{g(x)} \in S_{(f.g)} ; \ H_F\left(x_{g(x)}\right) = \circ_\circ\}$

$I_m(H_F) = \{Z_{h(z)} \in W_{(f.h)} ; \exists\, x_{g(x)} \in S_{(f.g)} ; H_F\left(x_{g(x)}\right) = Z_{h(z)}\}$

Theorem:

Let $H_F: S_{(f.g)} \to W_{(f.h)}$ be a two-fold fuzzy linear transformation, then we have:

1] $H_F(x * y) = H_F(x) * H_F(y)$, $H_F(r \circ x) = r \circ H_F(x)$.

2] $k_{er}(H_F) = (k_{er}(H))_{(f,g)}$, $I_m(H_F) = (I_m(H))_{(f,h)}$.

3] $k_{er}(H_F)$ is a two-fold fuzzy subspace of $S_{(f.g)}$.

4] $I_m(H_F)$ is a two-fold fuzzy subspace of $W_{(f.h)}$.

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Proof:

1] $H_F(x_{g(x)} * y_{g(y)}) = H_F((x+y)_{g(x+y)}) = (H(x+y))_{h(H(x+y))} = (H(x)+H(y))_{h(H(x)+H(y))} = H_F(x_{g(x)}) * H_F(y_{g(y)})$

Also, $H_F(r_{f(r)} \circ x_{g(x)}) = H_F((rx)_{g(rx)}) = (H(rx))_{h(g(rx))} = (r \cdot H(x))_{h`(g(rx))} = r_{f(r)} \circ H_F(x_{g(x)})$

2] $k_{er}(H_F) = \{x_{g(x)} \in S_{(f.g)};\ H(x) = 0\} = \{x_{g(x)} \in S_{(f.g)}; x \in k_{er}(H)\} = (k_{er}(H))_{(f.g)}$

$I_m(H_F) = \{Z_{h(Z)} \in W_{(f.h)}; \exists\, x_{g(x)} \in S_{(f.g)}; H(x) = Z\} = \{Z_{h(Z)} \in W_{(f.h)}; Z \in I_m(H)\} = I_m(H))_{(f.h)}$

3] Since $k_{er}(H)$ is a subspace of S, then $k_{er}(H_F) = k_{er}(H))_{(f.g)}$ is two-fold fuzzy subspace of $S_{(f.g)}$.

4] The proof is similar to 3.

Definition:

Let $E = \{s_1, \dots, s_n\}$ be a basic of S over $\mathbb{R}$, then:

$E_F = \{s_{1_{g(s_1)}}, \dots, s_{n_{g(s_n)}}\}$ is called a two-fold basis of $S_{(f.g)}$over $\mathbb{R}_f$

Theorem:

Let $E = \{s_1, \dots, s_n\}$ be a basic of S, and E_F be the corresponding two-fold fuzzy basis of $S_{(f.g)}$.

Assume that: $x_{g(x)} \in S_{(f.g)}$. then there exists $r_{1_{f(r_1)}}. \dots .r_{n_{f(r_n)}} \in \mathbb{R}_f$

Such as: $x_{g(x)} = r_{1_{f(r_1)}} \circ s_{1_{g(s_1)}} + \dots + r_{n_{f(r_n)}} \circ s_{n_{g(s_n)}} = \sum_{i=1}^{n} r_{i_{f(r_i)}} \circ s_{i_{g(s_i)}}$

Proof:

$x \in S$, then $x = \sum_{i=1}^{n} r_i\, s_i$; $s_i \in E . r_i \in \mathbb{R}$ so that:

$$x_{g(x)} = (\sum_{i=1}^{n} r_i\, s_i)_{g(\sum_{i=1}^{n} r_i s_i)} = \sum_{i=1}^{n} (r_i)_{f(r_i)} \circ (s_i)_{g(s_i)}$$

Remark:

If dim S= n , then $\dim(S_{(f.g)}) = n$

Remark:

If $\sum_{i=1}^{n} (r_i)_{f(r_i)} \circ (s_i)_{g(s_i)} = \circ_{g(\circ)}$. then $r_i = 0$

Complex Two-Fold Fuzzy Vector Spaces:

Definition:

Let S be a vector space over the field C, and

$$\begin{cases} g: S \to [0.1] \\ f: C \to [0.1] \end{cases} \text{ such that: } \begin{cases} g(x+y) = max(g(x).g(y)) \\ g(x \cdot r) = \min(g(x).f(r)) \\ g(0) = 0 . f(1) = 1 \end{cases}$$

For $x.y \in S$ and $r \in C$

We define the two- fold fuzzy vector space $S_{(f.g)}$ as follows:

$S_{(f.g)} = \{x_{g(x)} ; x \in S\}$

Definition:

We define the following binary operations on $S_{(f.g)}$:

$$\begin{cases} *: S_{(f.g)} \times S_{(f.g)} \to S_{(f.g)} \\ \circ: S_{(f.g)} \times C_f \to S_{(f.g)} \end{cases}$$

$$\begin{cases} x_{g(x)} * y_{g(y)} = (x+y)_{g(x+y)} = (x+y)_{\max(g(x).g(y))} \\ x_{g(x)} \circ r_{f(r)} = (xr)_{g(xr)} = (xr)_{\min(g(x).f(r))} \end{cases}$$

Theorem:

Let $S_{(f.g)}$ be the two-fold vector space defined above, then:

1] $*$,$\circ$ are abelian.

2] $*$ is associative.

3] $o_\circ$ is the identity of $*$, 1_1 is the identity of $\circ$.

4] $r \circ (x * y) = (r \circ x) * (r \circ y)$. $r \circ (t \circ x) = (rt) \circ x : x \in S . r.t \in \mathbb{R}$

5] $x * (-x) = 0$

Proof:

1] Let $x_{g(x)}.y_{g(y)} \in S_{(f.g)}.r \in C$. then:

$x_{g(x)} * y_{g(y)} = (x+y)_{\max(g(x).g(y))} = (y+x)_{\max(g(x).g(y))} = y_{g(y)} * x_{g(x)}$

$x_{g(x)} \circ r_{f(r)} = (xr)_{g(x \cdot r)} = (r \cdot x)_{g(r \cdot x)} = r_{f(r)} \circ x_{g(x)}$

2] $x_{g(x)} * (y_{g(y)} * z_{g(z)}) = x * ((y+z)_{g(y+z)}) = (x+y+z)_{g(x+y+z)} = (x+y)_{g(x+y)} * z_{g(z)} = (x_{g(x)} * y_{g(y)}) * z_{g(z)}$

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3] $x_{g(x)} * \circ_{g(\circ)} = (x + \circ)_{max(g(x).\circ)} = x_{g(x)}$

$x_{g(x)} \circ 1_{f(1)} = (x \cdot 1)_{min(g(x).1)} = x_{g(x)}$

4] $r_{f(r)} \circ \left(x_{g(x)} * y_{g(y)}\right) = r_{f(r)} \circ (x + y)_{g(x+y)} = (rx + ry)_{g(rx+ry)} = (rx)_{g(rx)} * (ry)_{g(ry)} = \left(r_{f(r)} \circ x_{g(x)}\right) * \left(r_{f(r)} \circ y_{g(y)}\right)$

$r_{f(r)} \circ \left(t_{f(t)} \circ x_{g(x)}\right) = r_{f(r)}\left((tx)_{g(tx)}\right) = (rtx)_{g(rtx)} = (rt)_{f(rt)} \circ x_{g(x)}$

5] $x_{g(x)} * (-x)_{g(-x)} = (x - x)_{g(x-x)} = \circ_{\circ}$

Definition:

Let T be a subspace of S, then:

$T_{(f.g)} = \{x_{g(x)} ;\ x \in T\}$ is called a two-fold fuzzy subspace of $S_{(f.g)}$

Theorem:

Let T be a non-empty subset of S, then $T_{(f.g)}$ is a two-fold fuzzy subspace of $S_{(f.g)}$ if and only if:

$\begin{cases} x * y \in T_{(f.g)} \\ r \circ x \in T_{(f.g)} \end{cases}$ for all $r \in$ C. $x.y \in T$

Proof:

$T_{(f.g)}$ is a two-fold fuzzy subspace of $S_{(f.g)}$ if and only if T is a subspace of S, which is equivalent to:

$\begin{cases} x + y \in \mathrm{T} \\ rx \in \mathrm{T} \end{cases}$ for all $x.y \in T$. $r \in$ C

So that, $\begin{cases} x_{g(x)} * y_{g(y)} = (x + y)_{g(x+y)} \in T_{(f.g)} \\ r_{f(r)} \circ x_{g(x)} = (rx)_{g(rx)} \in T_{(f.g)} \end{cases}$

Definition:

Let $S_{(f.g)}.W_{(f.h)}$ be two two-fold fuzzy vector spaces over C, and $H: S \to W$ be a classical linear transformation, then:

$H_F: S_{(f.g)} \to W_{(f.h)} : H_F\left(x_{g(x)}\right) = (H(x))_{h(H(x))}$is called the two-fold fuzzy linear transformation.

We define:

$k_{er}(H_F) = \{x_{g(x)} \in S_{(f.g)} ;\ H_F\left(x_{g(x)}\right) = \circ_{\circ}\}$

$I_m(H_F) = \{Z_{h(z)} \in W_{(f.h)} ; \exists\, x_{g(x)} \in S_{(f.g)} ;\ H_F\left(x_{g(x)}\right) = Z_{h(z)}\}$

Theorem:

Let $H_F: S_{(f.g)} \to W_{(f.h)}$ be a two-fold fuzzy linear transformation, then we have:

1] $H_F(x * y) = H_F(x) * H_F(y)$, $H_F(r \circ x) = r \circ H_F(x)$.

2] $k_{er}(H_F) = (k_{er}(H))_{(f,g)}$, $I_m(H_F) = (I_m(H))_{(f,h)}$.

3] $k_{er}(H_F)$ is a two-fold fuzzy subspace of $S_{(f.g)}$.

4] $I_m(H_F)$ is a two-fold fuzzy subspace of $W_{(f.h)}$.

Proof:

1] $H_F\left(x_{g(x)} * y_{g(y)}\right) = H_F\left((x + y)_{g(x+y)}\right) = (H(x + y))_{h(H(x+y))} = (H(x) + H(y))_{h(H(x)+H(y))} = H_F\left(x_{g(x)}\right) * H_F\left(y_{g(y)}\right)$

Also, $H_F\left(r_{f(r)} \circ x_{g(x)}\right) = H_F\left((rx)_{g(rx)}\right) = (H(rx))_{h(g(rx))} = (r \cdot H(x))_{h`(g(rx))} = r_{f(r)} \circ H_F\left(x_{g(x)}\right)$

2] $k_{er}(H_F) = \{x_{g(x)} \in S_{(f.g)};\ H(x) = 0\} = \{x_{g(x)} \in S_{(f.g)}; x \in k_{er}(H)\} = (k_{er}(H))_{(f.g)}$

$I_m(H_F) = \{Z_{h(Z)} \in W_{(f.h)}; \exists\, x_{g(x)} \in S_{(f.g)}; H(x) = Z\} = \{Z_{h(Z)} \in W_{(f.h)}; Z \in I_m(H)\} = I_m(H))_{(f.h)}$

3] Since $k_{er}(H)$ is a subspace of S, then $k_{er}(H_F) = k_{er}(H))_{(f.g)}$ is two-fold fuzzy subspace of $S_{(f.g)}$.

4] The proof is similar to 3.

Definition:

Let $E = \{s_1, \ldots, s_n\}$ be a basic of S over C, then:

$E_F = \left\{s_{1_{g(s_1)}}, \ldots, s_{n_{g(s_n)}}\right\}$is called a two-fold basis of $S_{(f.g)}$over C_f

Theorem:

Let $E = \{s_1, \ldots, s_n\}$ be a basic of S, and E_F be the corresponding two-fold fuzzy basis of $S_{(f.g)}$.

Assume that: $x_{g(x)} \in S_{(f.g)}$. then there exists $r_{1_{f(r_1)}}. \ldots .r_{n_{f(r_n)}} \in C_f$

Such as: $x_{g(x)} = r_{1_{f(r_1)}} \circ s_{1_{g(s_1)}} + \cdots + r_{n_{f(r_n)}} \circ s_{n_{g(s_n)}} = \sum_{i=1}^{n} r_{i_{f(r_i)}} \circ s_{i_{g(s_i)}}$

Proof:

$x \in S$, then $x = \sum_{i=1}^{n} r_i\, s_i$; $s_i \in E.r_i \in$ C so that:

$$x_{g(x)} = \left(\sum_{i=1}^{n} r_i\, s_i\right)_{g\left(\sum_{i=1}^{n} r_i s_i\right)} = \sum_{i=1}^{n} (r_i)_{f(r_i)} \circ (s_i)_{g(s_i)}$$

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Remark:
If $dim\ S = n$, then $\dim(S_{(f.g)}) = n$
Remark:
If $\sum_{i=1}^{n}(r_i)_{f(r_i)} \circ (s_i)_{g(s_i)} = \circ_{g(\circ)}\ .then\ r_i = 0$
Two-Fold Fuzzy Modules:
Definition:
Let S be a module over the ring R with unity 1, and
$\begin{cases} g: S \to [0.1] \\ f: \mathrm{R} \to [0.1] \end{cases}$ such that: $\begin{cases} g(x+y) = max\big(g(x).g(y)\big) \\ g(x \cdot \mathrm{r}) = \min\big(g(x).f(r)\big) \\ g(0) = 0\ .f(1) = 1 \end{cases}$
For $x.y \in S\ and\ r \in \mathrm{R}$
We define the two- fold fuzzy module $S_{(f.g)}$ as follows:
$S_{(f.g)} = \{x_{g(x)}\ ; x \in S\}$
Definition:
We define the following binary operations on $S_{(f.g)}$:
$\begin{cases} *: S_{(f.g)} \times S_{(f.g)} \to S_{(f.g)} \\ \circ: S_{(f.g)} \times \mathrm{R}_f \to S_{(f.g)} \end{cases}$
$\begin{cases} x_{g(x)} * y_{g(y)} = (x+y)_{g(x+y)} = (x+y)_{\max(g(x).g(y))} \\ x_{g(x)} \circ r_{f(r)} = (xr)_{g(xr)} = (xr)_{\min(g(x).f(r))} \end{cases}$
Theorem:
Let $S_{(f.g)}$ be the two-fold module defined above, then:
1] $*,\circ$ are abelian.
2] $*$ is associative.
3] $o_\circ$ is the identity of $*$, 1_1 is the identity of $\circ$.
4] $r \circ (x * y) = (r \circ x) * (r \circ y)\quad .\qquad r \circ (t \circ x) = (rt) \circ x : x \in S\ .r.t \in \mathrm{R}$
5] $x * (-x) = 0$
Proof:
1] Let $x_{g(x)}.y_{g(y)} \in S_{(f.g)}.r \in \mathrm{R}\ .then$:
$x_{g(x)} * y_{g(y)} = (x+y)_{\max(g(x).g(y))} = (y+x)_{\max(g(x).g(y))} = y_{g(y)} * x_{g(x)}$
$x_{g(x)} \circ r_{f(r)} = (xr)_{g(x \cdot r)} = (r \cdot x)_{g(r \cdot x)} = r_{f(r)} \circ x_{g(x)}$
2] $x_{g(x)} * \big(y_{g(y)} * z_{g(z)}\big) = x * \big((y+z)_{g(y+z)}\big) = (x+y+z)_{g(x+y+z)} = (x+y)_{g(x+y)} * z_{g(z)} = \big(x_{g(x)} * y_{g(y)}\big) * z_{g(z)}$
3] $x_{g(x)} * \circ_{g(\circ)} = (x + \circ)_{max(g(x).\circ)} = x_{g(x)}$
$x_{g(x)} \circ 1_{f(1)} = (x \cdot 1)_{min(g(x).1)} = x_{g(x)}$
4] $r_{f(r)} \circ \big(x_{g(x)} * y_{g(y)}\big) = r_{f(r)} \circ (x+y)_{g(x+y)} = (rx+ry)_{g(rx+ry)} = (rx)_{g(rx)} * (ry)_{g(ry)} = \big(r_{f(r)} \circ x_{g(x)}\big) * \big(r_{f(r)} \circ y_{g(y)}\big)$
$r_{f(r)} \circ \big(t_{f(t)} \circ x_{g(x)}\big) = r_{f(r)}\big((tx)_{g(tx)}\big) = (rtx)_{g(rtx)} = (rt)_{f(rt)} \circ x_{g(x)}$
5] $x_{g(x)} * (-x)_{g(-x)} = (x-x)_{g(x-x)} = \circ_\circ$
Definition:
Let T be a submodule of S, then:
$T_{(f.g)} = \{x_{g(x)}\ ;\ x \in T\}$ is called a two-fold fuzzy submodule of $S_{(f.g)}$
Theorem:
Let T be a non-empty subset of S, then $T_{(f.g)}$ is a two-fold fuzzy submodule of $S_{(f.g)}$ if and only if:
$\begin{cases} x * y \in T_{(f.g)} \\ r \circ x \in T_{(f.g)} \end{cases}$ for all $r \in \mathrm{R}.\quad x.y \in T$
Proof:
$T_{(f.g)}$ is a two-fold fuzzy submodule of $S_{(f.g)}$ if and only if T is a submodule of S, which is equivalent to:
$\begin{cases} x+y \in \mathrm{T} \\ rx \in \mathrm{T} \end{cases}$ for all $x.y \in T.\quad r \in \mathrm{R}$
So that, $\begin{cases} x_{g(x)} * y_{g(y)} = (x+y)_{g(x+y)} \in T_{(f.g)} \\ r_{f(r)} \circ x_{g(x)} = (rx)_{g(rx)} \in T_{(f.g)} \end{cases}$
Definition:
Let $S_{(f.g)}.W_{(f.h)}$ be two two-fold fuzzy modules over C, and $H: S \to W$ be a classical homomorphisms.

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, then:
$H_F : S_{(f.g)} \to W_{(f.h)} : H_F\left(x_{g(x)}\right) = (H(x))_{h(H(x))}$is called the two-fold fuzzy homomorphisms.
We define:
$k_{er}(H_F) = \{x_{g(x)} \in S_{(f.g)} ;\ \ H_F\left(x_{g(x)}\right) = \circ_\circ\}$
$I_m(H_F) = \{Z_{h(z)} \in W_{(f.h)} ; \exists\, x_{g(x)} \in S_{(f.g)} ;\ H_F\left(x_{g(x)}\right) = Z_{h(z)}\}$

Theorem:
Let $H_F : S_{(f.g)} \to W_{(f.h)}$ be a two-fold fuzzy homomorphisms, then we have:
1] $H_F(x * y) = H_F(x) * H_F(y), \quad H_F(r \circ x) = r \circ H_F(x)$.
2] $k_{er}(H_F) = (k_{er}(H))_{(f,g)},\ I_m(H_F) = (I_m(H))_{(f,h)}$.
3] $k_{er}(H_F)$ is a two-fold fuzzy submodule of $S_{(f.g)}$.
4] $I_m(H_F)$ is a two-fold fuzzy submodule of $W_{(f.h)}$.

Proof:
1] $H_F\left(x_{g(x)} * y_{g(y)}\right) = H_F\left((x+y)_{g(x+y)}\right) = (H(x+y))_{h(H(x+y))} = (H(x) + H(y))_{h(H(x)+H(y))} = H_F\left(x_{g(x)}\right) * H_F\left(y_{g(y)}\right)$
Also, $H_F\left(r_{f(r)} \circ x_{g(x)}\right) = H_F\left((rx)_{g(rx)}\right) = (H(rx))_{h(g(rx))} = (r \cdot H(x))_{h`(g(rx))} = r_{f(r)} \circ H_F\left(x_{g(x)}\right)$
2] $k_{er}(H_F) = \{x_{g(x)} \in S_{(f.g)};\ H(x) = 0\} = \{x_{g(x)} \in S_{(f.g)}; x \in k_{er}(H)\} = (k_{er}(H))_{(f.g)}$
$I_m(H_F) = \{Z_{h(Z)} \in W_{(f.h)}; \exists\, x_{g(x)} \in S_{(f.g)}; H(x) = Z\} = \{Z_{h(Z)} \in W_{(f.h)}; Z \in I_m(H)\} = I_m(H))_{(f.h)}$
3] Since $k_{er}(H)$ is a subspace of S, then $k_{er}(H_F) = k_{er}(H))_{(f.g)}$ is two-fold fuzzy submodule of $S_{(f.g)}$.
4] The proof is similar to 3.

Definition:
Let $E = \{s_1, \dots, s_n\}$ be a basic of S over R, then:
$E_F = \left\{s_{1_{g(s_1)}}, \dots, s_{n_{g(s_n)}}\right\}$ is called a two-fold basis of $S_{(f.g)}$over R_f

Theorem:
Let $E = \{s_1, \dots, s_n\}$ be a basic of S, and E_F be the corresponding two-fold fuzzy basis of $S_{(f.g)}$.
Assume that: $x_{g(x)} \in S_{(f.g)}$. then there exists $r_{1_{f(r_1)}}. \dots .r_{n_{f(r_n)}} \in R_f$
Such as: $x_{g(x)} = r_{1_{f(r_1)}} \circ s_{1_{g(s_1)}} + \dots + r_{n_{f(r_n)}} \circ s_{n_{g(s_n)}} = \sum_{i=1}^{n} r_{i_{f(r_i)}} \circ s_{i_{g(s_i)}}$

Proof:
$x \in S$, then $x = \sum_{i=1}^{n} r_i\, s_i$; $s_i \in E . r_i \in R$ so that:
$x_{g(x)} = (\sum_{i=1}^{n} r_i\, s_i)_{g(\sum_{i=1}^{n} r_i s_i)} = \sum_{i=1}^{n} (r_i)_{f(r_i)} \circ (s_i)_{g(s_i)}$.

4. Conclusion

In this paper is we provided a strict mathematical definition and a general study of two-fold fuzzy vector spaces defined over the real numbers and also two-fold fuzzy vector spaces defined over the complex numbers, as well as two-fold fuzzy algebraic modules defined over commutative rings with unity.
The elementary properties and algebraic operations of those structures are explained through many theorems and mathematical proofs.

References

[1] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971), 512-517.
[2] P. Sivaramakrishna Das, Fuzzy groups and level subgroups, J. Math. Anal. Appl. 84 (1981), 264-269.
[3] Mohammad Abobala. (2020). n-Cyclic Refined Neutrosophic Algebraic Systems of Sub-Indeterminacies, An Application to Rings and Modules. International Journal of Neutrosophic Science, 12 (2), 81-95 (**Doi :** https://doi.org/10.54216/IJNS.120203)
[4] Malath Fared Alaswad, Rasha Dallah. (2023). Neutrosophic Divisor Point of A Straight Line Segment With A Given Ratio. Pure Mathematics for Theoretical Computer Science, 2 (1), 18-23 (**Doi :** https://doi.org/10.54216/PMTCS.020102)

[5] Mohammad Abobala, Mikail Bal, Ahmed Hatip. (2021). A Review on Recent Advantages in Algebraic Theory of Neutrosophic Matrices. International Journal of Neutrosophic Science, 17 (1), 68-86 (**Doi :** https://doi.org/10.54216/IJNS.170105)
[6] T. M. Anthony, H. Sherwood, A characterization of fuzzy subgroups, Fuzzy Sets and Systems 7 (1982), 297-305.
[7] W. M.Wu, "Normal fuzzy subgroups," Fuzzy Mathematics, vol.1, no. 1, pp. 21–30, 1981.

Doi: https://doi.org/10.54216/JNFS.070205
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[8] Palaniappan, N, Naganathan,S and Arjunan, K " A study on Intuitionistic L-Fuzzy Subgroups", Applied Mathematical Sciences, vol. 3 , 2009, no. 53 , 2619-2624.
[9] Noor Edin Rabeh, Othman Al-Basheer, Sara Sawalmeh, Rozina Ali. (2023). An Algebraic Approach to n-Plithogenic Square Matrices For $\mathbf{18} \leq \boldsymbol{n} \leq \mathbf{19}$. Journal of Neutrosophic and Fuzzy Systems, 7 (2), 08-23 (**Doi :** https://doi.org/10.54216/JNFS.070201)
[10] M.A. Ibrahim, A.A.A. Agboola , E.O. Adeleke, S.A. Akinleye. (2020). On Neutrosophic Quadruple Hypervector Spaces. International Journal of Neutrosophic Science, 4 (1), 20-35 (**Doi :** https://doi.org/10.54216/IJNS.040103)
[11] Mohammad Abobala. (2020). Classical Homomorphisms Between n-Refined Neutrosophic Rings. International Journal of Neutrosophic Science, 7 (2), 74-78 (**Doi :** https://doi.org/10.54216/IJNS.070204)
[12] Mohammad Abobala. (2020). Classical Homomorphisms Between Refined Neutrosophic Rings and Neutrosophic Rings. International Journal of Neutrosophic Science, 5 (2), 72-75 (**Doi :** https://doi.org/10.54216/IJNS.050202)
[13] P.K. Sharma , "(α , β) – Cut of Intuitionistic fuzzy Groups" International Mathematical Forum ,Vol. 6, 2011 , no. 53 , 2605-2614.
[14] Florentine Smarandache. "Neutrosophic Two-Fold Algebra", Plithogenic Logic and Computation, Vol.1, No.1 2024. PP.11-15.
[15] Mohammad Abobala. "On The Foundations of Fuzzy Number Theory and Fuzzy Diophantine Equations." Galoitica: Journal of Mathematical Structures and Applications, Vol. 10, No. 1, 2023 ,PP. 17-25 (**Doi :** https://doi.org/10.54216/GJMSA.0100102).
[16] Mohammad Abobala. (2023). On a Two-Fold Algebra Based on the Standard Fuzzy Number Theoretical System. Journal of Neutrosophic and Fuzzy Systems, 7 (2), 24-29 (**Doi :** https://doi.org/10.54216/JNFS.070202)

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