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Some Single-Valued Neutrosophic Power Heronian Aggregation Operators and Their Application to Multiple-Attribute Group Decision-Making

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Abstract: The power Heronian aggregation (PHA) operator can use the advantages of power average and the Heronian mean operator, which together could take into account the interrelationship of the aggregated arguments, and therefore alleviate the effects caused by unreasonable data through considering the support degree between input arguments. However, PHA operators cannot be used to process single-valued neutrosophic numbers (SVNNs), which is significant for extending it to SVNNs. We propose some new PHA operators for SVNNs and introduce a novel MAGDM method on the basis of the proposed operators. Firstly, the definition, properties, comparison method, and operational rules of SVNNs are introduced briefly. Then, some PHA operators are proposed, such as the single-valued neutrosophic power Heronian aggregation (SVNPHA) operator, the single-valued neutrosophic weighted power Heronian aggregation (SVNWPHA) operator, single-valued neutrosophic geometric power Heronian aggregation (SVNGPHA) operator, single-valued neutrosophic weighted geometric power Heronian aggregation (SVNWGPHA) operator. Furthermore, we discuss some properties of these new aggregation operators and several special cases. Moreover, the method to solve the MAGDM problems with SVNNs is proposed, based on the SVNWPHA and SVNWGPHA operators. Lastly, we verified the application and effectiveness of the proposed method by using an example for the MAGDM problem.

Keywords: MAGDM; Single-valued neutrosophic numbers; SVNWPHA operator; SVNWGPHA operator; Interaction

1. Introduction

In real decision-making, multiple-attribute group decision-making (MAGDM) methods are extensively used to rank alternatives for relevant attributes. The decision information involved in MAGDM problems is often fuzzy and easily expressed by fuzzy information owing to the limitations of human thinking and the complexity of decision-making problems. Accordingly, Zadeh [1] first proposed fuzzy sets (FSs) to handle fuzzy information. Then, Atanassov [2,3] proposed the intuitionistic fuzzy sets (IFSs) based on FS owing to fuzziness. IFSs can embody the degrees of satisfaction and dissatisfaction to express the judgment of alternative. IFS has immediately gained widespread attention and been extensively studied in the field of MAGDM [4–8].

Although the IFS theory has been generalized, it still cannot solve all uncertain problems in real decision-making, such as those that involve indeterminate and inconsistent information. Therefore, to

overcome these shortcomings, neutrosophic sets (NSs) was introduced by Smarandache [9], which can substantially express indeterminate or inconsistent information. The NSs is finer and smoother in describing fuzzy information than FS and IFS. However, the application of NSs is very difficult in practice due to the lack of specific description. Therefore, Wang et al. [10] defined single-valued neutrosophic sets (SVNSs), which is an instance of NSs. Meanwhile, SVNSs have attracted increasing attention from researchers, obtained numerous research achievements [11–15] and been extensively applied in many areas. The concept of distance of the two SVNSs and several similarity measures between SVNSs are defined and proposed by Majumdar and Samanta [16]. Ye [15] introduced the correlation coefficients and decision-making method of SVNSs. Şahin and Küçük [17] introduced a method to deal with SVNSs based on the neutrosophic subsethood measure. Based on the use of Hamming distance, Stanujkić [18] proposed a extend WASPAS method with single-valued intuitionistic fuzzy numbers, and application it in website evaluation. Liu, et al. [19] presented a single valued neutrosophic number Decision-making Trial and Evaluation Laboratory Method (SVNN-DEMATEL) model for evaluating and selecting a transport service provider. Pamučar et al. [20] proposed a Linguistic Neutrosophic Numbers Pairwise-CODAS model, which can eliminate subjective qualitative assessments and assumptions by decision makers in complex decision-making conditions. Pamučar et al. [21] presented a new Linguistic Neutrosophic Numbers Weighted Aggregated Sum Product Assessment (LNN WASPAS) model and further developed LNN CODAS (Combinative Distance-Based Assessment), LNN VIKOR (Multi-criteria Optimization and Compromise Solution) and LNN MABAC (Multi-Attributive Border Approximation Area Comparison) models.

Information aggregation generally became an important topic of MAGDM and received increasing attention from researchers [22–24]. A few new extended aggregation operators for NS and SVNS have been proposed previously [25–28]. Ji, et al. [29] proposed several Frank prioritized Bonferroni mean operators for SVNSs. Ren and Sugumaran [30] proposed some prioritized weighted geometric operators for SVNSs. Ye [31] extended several weighted arithmetic average operators for SNSs. Liu and Wang [32] extended a weighted Bonferroni mean operator for SVNSs. Li, et al. [33] extended Heronian mean (HM) operator for SVNSs. Yang and Li [34] extended the PA operator to SVNSs and developed some single-valued neutrosophic power weighted average (SVNPWA) operators. Wei and Zhang [35] proposed some Bonferroni power aggregation operators for SVNSs.

It is obvious that there are different functions for the different aggregation operators. Yager [36] firstly proposed power average (PA) operator, which can alleviate the effects caused by the unreasonable data through considering the support degree between input data. Xu [37] further defined a power geometric (PG) operator. Beliakov [38] first proposed Heronian mean operator, which can capture the interrelation of the input arguments. As an important generalization of Bonferroni mean, Yu [39] proposed the geometric Heronian mean. To consider the advantages of PA and HM operators together, P. Liu proposed some power Heronian aggregation (PHA) operators through combining the PA operator and HM operator to IVIFNs [40] and Linguistic Neutrosophic Sets (LNSs) [41].

The decision-making problems have increased in complexity, and in order to derive the best alternative for MAGDM problems, we need to consider both the influence of some unreasonable attribute values caused by the preference of decision maker, and the interrelationship and interaction among the attributes. Taking the advantages of the HM and PA operator, the PHA operator can achieve the following two functions: the interrelationship of the aggregated arguments could be taken into account, and the influence of unreasonable data could be eliminated by considering the support degree between input arguments. Besides, up to now, there has been no research on how to use PHA operator to aggregate the SVNNs. So, the goal and motivation of this study is to:

(1) Establish the single-valued neutrosophic PHA (SVNPHA) operator, single-valued neutrosophic geometric PHA (SVNGPHA) operators and the weighted form of these operators (the form of shorthand is SVNWPHA and SVNWGPHA).

(2) Discuss their properties and analyze special cases.

(3) Propose a novel MAGDM method based on the SVNWPHA and SVNWGPHA operators for SVNNs.

(4) Demonstrate the application and effectiveness of the developed methods.

The remainder of this paper is organized as follows. In Section 2, some definitions of SVNSs and some operational rules of SVNNs are introduced. In Section 3, we propose some new PHA operators, such as SVNPHA, SVNWPHA, SVNGPHA and SVNWGPHA operators. In Section 4, a MAGDM method is proposed based on the above operators in SVNNs environment. In Section 5, we demonstrate the application and effectiveness of the proposed method by using an example for the MAGDM problem. Section 6 presents the conclusion.

2. Preliminaries

2.1. The SVNNs

Definition 1. [42]. *Let X be a space of points with a generic element in X denoted by x. SVNS A in X is as follows:*

$$A = \{x(T_A(x), I_A(x), F_A(x)) | x \in X\} \tag{1}$$

where $T_A(x)$ is the truth-membership function, $I_A(x)$ is the indeterminacy-membership function, and $F_A(x)$ is the falsity-membership function. For each point x in X, we have $T_A(x)$, $I_A(x)$, $F_A(x) \in [0,1]$, and $0 \le T_A(x), I_A(x), F_A(x) \le 3$.

For convenience, we can use $x = (T_A, I_A, F_A)$ to represent an element in SVNS and call it an SVNN.

2.2. Operational Rules and Properties of SVNNs

Definition 2. [32]. *Let $x_i = (T_i, I_i, F_i)$ and $x_j = (T_j, I_j, F_j)$ be any two SVNNs and $\lambda > 0$, the operations are defined as follows:*

$$x_i \oplus x_j = (T_i + T_j - T_i T_j, I_i I_j, F_i F_j) \tag{2}$$

$$x_i \otimes x_j = (T_i T_j, I_i + I_j - I_i I_j, F_i + F_j - F_i F_j) \tag{3}$$

$$\lambda x_i = \left(1 - (1 - T_i)^\lambda, I_i^\lambda, F_i^\lambda\right) \tag{4}$$

$$x_i{}^\lambda = \left(T_i^\lambda, 1 - (1 - I_i)^\lambda 1 - (1 - F_i)^\lambda\right) \tag{5}$$

Theorem 1. *For any two SVNNs $x_i = (T_i, I_i, F_i)$ and $x_j = (T_j, I_j, F_j)$, and $\eta, \eta_1, \eta_2 > 0$, their operational rules have the following properties:*

$$x_i \otimes x_j = x_j \otimes x_i \tag{6}$$

$$x_i \oplus x_j = x_j \oplus x_i \tag{7}$$

$$\eta(x_i \oplus x_j) = \eta x_j \oplus \eta x_i \tag{8}$$

$$\eta_1 x_i \oplus \eta_2 x_i = (\eta_1 \oplus \eta_2) x_i \tag{9}$$

$$x_i{}^\eta \otimes x_j{}^\eta = (x_i \otimes x_j)^\eta \tag{10}$$

$$x_i{}^{\eta_1} \otimes x_i{}^{\eta_2} = x_i{}^{(\eta_1 \oplus \eta_2)} \tag{11}$$

2.3. Comparison of SVNNs

Definition 3. [43]. *Let $x = (T_i, I_i, F_i)$ be an SVNN. The score $s(x)$, accuracy $a(x)$, and certainty $c(x)$ functions of x can be defined as follows:*

$$s(x) = (T_i + 2 - I_i - F_i)/3 \tag{12}$$

$$a(x) = T_i - F_i \tag{13}$$

$$c(x) = T_i \tag{14}$$

Definition 4. [43]. *Suppose $x_i = (T_i, I_i, F_i)$ and $x_j = (T_j, I_j, F_j)$ be two SVNNs. The comparison method between x_i and x_j can be defined as follows:*

(1) *If $s(x_i) > s(x_j)$, then $x_i > x_j$;*

(2) *If $s(x_i) = s(x_j)$ and $a(x_i) > a(x_j)$, then $x_i > x_j$;*

(3) *If $s(x_i) = s(x_j)$, $a(x_i) = a(x_j)$, and $c(x_i) > c(x_j)$, then $x_i > x_j$;*

(4) *If $s(x_i) = s(x_j)$, $a(x_i) = a(x_j)$ and $c(x_i) = c(x_j)$, then $x_i \sim x_j$.*

Definition 5. [44,45]. *Let $x_i = (T_i, I_i, F_i)$ and $x_j = (T_j, I_j, F_j)$ be any two SVNNs, the normalized Euclidian distance between x_i and x_j are defined as follows:*

$$d(x_i, x_j) = \sqrt{\left[(T_i - T_j)^2 + (I_i - I_j)^2 + (F_i - F_j)^2 \right]/3} \tag{15}$$

3. Some Power Heronian Aggregation Operators with SVNNs

3.1. Single Valued Neutrosophic Power Heronian Aggregation Operators

As an important aggregation operator, the PA operator is first proposed byYager [36], which can overcome the influence of unreasonable arguments by considering the support degree between input arguments. The traditional PA operator is defined as below.

Definition 6. [36]. *Let $p, q \geq 0$, and $\phi_i (i = 1, 2, \cdots, n)$ be a group of nonnegative numbers. If*

$$PA(\phi_1, \phi_2, \cdots, \phi_n) = \frac{\sum\limits_{i=1}^{n} ((1 + T(\phi_i))\phi_i}{\sum\limits_{i=1}^{n} ((1 + T(\phi_i))} \tag{16}$$

where $T(\phi_i) = \sum\limits_{j=1, j\neq i}^{n} Sup(\phi_i, \phi_j)$. We denote $Sup(\phi_i, \phi_j)$ as the support degree for ϕ_i from ϕ_j. $Sup(\phi_i, \phi_j)$ satisfies the following axioms:

(1) *$Sup(\phi_i, \phi_j) = Sup(\phi_j, \phi_i)$;*

(2) *$Sup(\phi_i, \phi_j) \in [0, 1]$;*

(3) *$Sup(\phi_i, \phi_j) > Sup(\phi_l, \phi_k)$, if $|\phi_i - \phi_j| < |\phi_l - \phi_k|$.*

Then PA is called the power average (PA) operator.

For example, suppose $\phi_1 = 0.6$, $\phi_2 = 0.7$, $\phi_3 = 0.8$ are nonnegative numbers, the $PA(\phi_1, \phi_2, \phi_3)$ are calculated as follows:

Step 1. Calculate the $Sup(x_i, x_j)(i, j = 1, 2, 3)$ Thereafter, we have

$Sup(\phi_1, \phi_2) = 0.1$, $Sup(\phi_1, \phi_3) = 0.2$, $Sup(\phi_2, \phi_3) = 0.1$.

Step 2. Calculate the power weighting vector through Expression (19). Thereafter, we have

$T(\phi_1) = Sup(\phi_1, \phi_2) + Sup(\phi_1, \phi_3) = 0.3$,

$T(\phi_2) = Sup(\phi_2, \phi_1) + Sup(\phi_2, \phi_3) = 0.2,$

$T(\phi_3) = Sup(\phi_3, \phi_1) + Sup(\phi_3, \phi_2) = Sup(\phi_1, \phi_3) + Sup(\phi_2, \phi_3) = 0.3.$

Step 3. Calculate the $PA(\phi_1, \phi_2, \phi_3)$ using the equation 15 (suppose p = q = 1). Thereafter, we can have

$$PA(\phi_1, \phi_2, \phi_3) = \frac{\sum\limits_{i=1}^{3} ((1+T(\phi_i))\phi_i)}{\sum\limits_{i=1}^{3} ((1+T(\phi_i))}$$

$$= \frac{((1+T(\phi_1))\phi_1 + ((1+T(\phi_2))\phi_2 + ((1+T(\phi_3))\phi_3}{(1+T(\phi_1)) + (1+T(\phi_2)) + (1+T(\phi_3))}$$

$$= \frac{(1+0.3)0.6 + (1+0.2)0.7 + (1+0.3)0.8}{(1+0.3)(1+0.2)(1+0.3)} = 0.7$$

Beliakov [38] first proposed the Heronian mean operator, which can determine the interrelation of the input arguments [46–48]. HM is defined as follows.

Definition 7. [48]. *Let $p, q \geq 0$, and $\phi_i(i = 1, 2, \cdots, n)$ be a group of nonnegative numbers. If*

$$HM^{p,q}(\phi_1, \phi_2, \cdots, \phi_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} \phi_i^p \phi_j^q \right)^{\frac{1}{p+q}} \tag{17}$$

Then $HM^{p,q}$ is called the Heronian mean (HM) operator.

For example, suppose $\phi_1 = 0.6$, $\phi_2 = 0.7$, $\phi_3 = 0.8$ are nonnegative numbers, the $HM^{p,q}(\phi_1, \phi_2, \phi_3)$ are calculated as follow (suppose $p = q = 1$):

$$HM^{1,1}(\phi_1, \phi_2, \cdots, \phi_n) = \left(\frac{2}{3(3+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} \phi_i^1 \phi_j^1 \right)^{\frac{1}{2}}$$

$$= \left(\frac{2}{3(3+1)} \phi_1\phi_1 + \phi_1\phi_2 + \phi_1\phi_3 + \phi_2\phi_2 + \phi_2\phi_3 + \phi_3\phi_3 \right)^{\frac{1}{2}}$$

$$= \left(\frac{1}{6}(0.6 * 0.6 + 0.6 * 0.7 + 0.6 * 0.8 + 0.7 * 0.7 + 0.7 * 0.8 + 0.8 * 0.8) \right)^{\frac{1}{2}}$$

$$= \left(\frac{1}{6} \times 2.95 \right)^{\frac{1}{2}} \approx 0.7012$$

Next, we shall develop the SVNPHA and SVNWPHA operators based on the operation laws of SVNNs.

Definition 8. *Let $p, q \geq 0$, and $x_i = (T_i, I_i, F_i)(i = 1, 2, \cdots, n)$ be a collection of SVNNs. If*

$$SVNPHA^{p,q}(x_1, x_2, \cdots, x_n)$$

$$= \left(\frac{2}{n(n+1)} \overset{n}{\underset{i=1, j=i}{\oplus}} \left(\left(\frac{n(1+T(x_i))}{\sum_{t=1}^{n}(1+T(x_t))} x_i \right)^p \otimes \left(\frac{n(1+T(x_j))}{\sum_{t=1}^{n}(1+T(x_t))} x_j \right)^q \right) \right)^{\frac{1}{p+q}} \tag{18}$$

where $T(x_i) = \sum\limits_{j=1, j\neq i}^{n} Sup(x_i, x_j)$. We shall denote $Sup(x_i, x_j)$ as the support degree for x_i from x_j. $Sup(x_i, x_j)$ satisfies the following three properties:

(1) $Sup(x_i, x_j) \in [0, 1]$;

(2) $Sup(x_i, x_j) = Sup(x_j, x_i)$;

(3) $Sup(x_i, x_j) > Sup(x_l, x_k)$, if $d(x_i, x_j) < d(x_l, x_k)$ in which $d(x_i, x_j)$ is the distance between SVNNs x_i and x_j.

Then, $SVNGPHA^{p,q}$ is called the single-valued neutrosophic power Heronian aggregation (SVNPHA) operator.

In order to simply this expression (18). We can define:

$$\widetilde{w}_i = \frac{(1 + T(x_i))}{\sum_{t=1}^n (1 + T(x_t))}. \tag{19}$$

and call $(\widetilde{w}_1, \widetilde{w}_2, \cdots, \widetilde{w}_n)$ as the power weighting vector with $\widetilde{w}_i \geq 0$, $\sum_{i=1}^n \widetilde{w}_i = 1$

Thereafter, Expression (18) can be expressed as follows:

$$SVNPHA^{p,q}(x_1, x_2, \cdots, x_n)$$
$$= \left(\frac{2}{n(n+1)} \overset{n}{\underset{i=1, j=i}{\oplus}} \left((n\widetilde{w}_i x_i)^p \otimes \left(n\widetilde{w}_j x_j \right)^q \right) \right)^{\frac{1}{p+q}} \tag{20}$$

Theorem 2. *Let* $x_i = (T_i, I_i, F_i)(i = 1, 2, \cdots, n)$ *be a group of SVNNs and* $p, q \geq 0$. *Then, the result aggregated from SVNPHA operator is still a SVNN, and even*

$$SVNPHA^{p,q}(x_1, x_2, \cdots, x_n)$$
$$= \left(\left(1 - \left(\prod_{i=1, j=i}^n \left(1 - \left(1 - (1 - T_i)^{n\widetilde{w}_i} \right)^p \left(1 - \left(1 - T_j \right)^{n\widetilde{w}_j} \right)^q \right) \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}}, \tag{21}$$

$$\left(1 - \left(\prod_{i=1, j=i}^n \left(1 - \left(1 - I_i^{n\widetilde{w}_i} \right)^p \left(1 - I_i^{n\widetilde{w}_j} \right)^q \right) \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}},$$

$$\left(1 - \left(1 - \left(\prod_{i=1, j=i}^n \left(1 - \left(1 - F_i^{n\widetilde{w}_i} \right)^p \left(1 - F_i^{n\widetilde{w}_j} \right)^q \right) \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right)$$

Proof. To prove Equation (21), we first prove that the following equation is right.

$$\overset{n}{\underset{i=1, j=i}{\oplus}} \left((n\widetilde{w}_i x_i)^p \otimes \left(n\widetilde{w}_j x_j \right)^q \right)$$
$$= \left(1 - \prod_{i=1, j=i}^n \left(1 - \left(1 - (1 - T_i)^{n\widetilde{w}_i} \right)^p \left(1 - \left(1 - T_j \right)^{n\widetilde{w}_j} \right)^q \right), \tag{22}$$
$$\prod_{i=1, j=i}^n \left(1 - \left(1 - I_i^{n\widetilde{w}_i} \right)^p \left(1 - I_j^{n\widetilde{w}_j} \right)^q \right), \prod_{i=1, j=i}^n \left(1 - \left(1 - F_i^{n\widetilde{w}_i} \right)^p \left(1 - F_j^{n\widetilde{w}_j} \right)^q \right) \right)$$

By the operational rules of SVNNs defined in (2–5), we have

$$(n\widetilde{w}_i x_i)^p = \left(\left(1 - (1 - T_i)^{n\widetilde{w}_i} \right)^p, 1 - \left(1 - I_i^{n\widetilde{w}_i} \right)^p, 1 - \left(1 - F_i^{n\widetilde{w}_i} \right)^p \right)$$
$$\left(n\widetilde{w}_j x_j \right)^q = \left(\left(1 - \left(1 - T_j \right)^{n\widetilde{w}_j} \right)^q, 1 - \left(1 - I_j^{n\widetilde{w}_j} \right)^q, 1 - \left(1 - F_j^{n\widetilde{w}_j} \right)^q \right)$$
$$(n\widetilde{w}_i x_i)^p \otimes \left(n\widetilde{w}_j x_j \right)^q \tag{23}$$
$$= \left(\left(1 - (1 - T_i)^{n\widetilde{w}_i} \right)^p \left(1 - \left(1 - T_j \right)^{n\widetilde{w}_j} \right)^q, 1 - \left(1 - I_i^{n\widetilde{w}_i} \right)^p \left(1 - I_j^{n\widetilde{w}_j} \right)^q, 1 - \left(1 - F_i^{n\widetilde{w}_i} \right)^p \left(1 - F_j^{n\widetilde{w}_j} \right)^q \right)$$

When $n = 2$, we have

$$
\overset{2}{\underset{i=1,j=i}{\oplus}} \left((2\widetilde{w}_i x_i)^p \otimes \left(2\widetilde{w}_j x_j\right)^q \right)
$$
$$
= \left((2\widetilde{w}_1 x_1)^p \otimes (2\widetilde{w}_1 x_1)^q \right) \oplus \left((2\widetilde{w}_1 x_1)^p \otimes (2\widetilde{w}_2 x_2)^q \right) \oplus \left((2\widetilde{w}_2 x_2)^p \otimes (2\widetilde{w}_2 x_2)^q \right)
$$
$$
= \left(\left(1 - (1-T_1)^{2\widetilde{w}_1}\right)^p \left(1 - (1-T_1)^{2\widetilde{w}_1}\right)^q, 1 - \left(1 - I_1^{2\widetilde{w}_1}\right)^p \left(1 - I_1^{2\widetilde{w}_1}\right)^q, 1 - \left(1 - F_1^{2\widetilde{w}_1}\right)^p \left(1 - F_1^{2\widetilde{w}_1}\right)^q \right)
$$
$$
\oplus \left(\left(1 - (1-T_1)^{2\widetilde{w}_1}\right)^p \left(1 - (1-T_2)^{2\widetilde{w}_2}\right)^q, 1 - \left(1 - I_1^{2\widetilde{w}_1}\right)^p \left(1 - I_2^{2\widetilde{w}_2}\right)^q, 1 - \left(1 - F_1^{2\widetilde{w}_1}\right)^p \left(1 - F_2^{2\widetilde{w}_2}\right)^q \right)
$$
$$
\oplus \left(\left(1 - (1-T_2)^{2\widetilde{w}_2}\right)^p \left(1 - (1-T_2)^{2\widetilde{w}_2}\right)^q, 1 - \left(1 - I_2^{2\widetilde{w}_2}\right)^p \left(1 - I_2^{2\widetilde{w}_2}\right)^q, 1 - \left(1 - F_2^{2\widetilde{w}_2}\right)^p \left(1 - F_2^{2\widetilde{w}_2}\right)^q \right)
$$

By using Equation (2), we get

$$
= \left(1 - \overset{2}{\underset{i=1,j=i}{\prod}} \left(1 - \left(1 - (1-T_i)^{2\widetilde{w}_i}\right)^p \left(1 - \left(1-T_j\right)^{2\widetilde{w}_j}\right)^q \right), \right.
$$
$$
\left. \overset{2}{\underset{i=1,j=i}{\prod}} \left(1 - \left(1 - I_i^{2\widetilde{w}_i}\right)^p \left(1 - I_j^{2\widetilde{w}_j}\right)^q \right), \overset{2}{\underset{i=1,j=i}{\prod}} \left(1 - \left(1 - F_i^{2\widetilde{w}_i}\right)^p \left(1 - F_j^{2\widetilde{w}_j}\right)^q \right) \right) \tag{24}
$$

That is, when $n = 2$, the Equation (22) is right.
Assume $n = m$, Equation (22) is right:

$$
= \left(1 - \overset{k}{\underset{i=1,j=i}{\prod}} \left(1 - \left(1 - (1-T_i)^{k\widetilde{w}_i}\right)^p \left(1 - \left(1-T_j\right)^{k\widetilde{w}_j}\right)^q \right), \right.
$$
$$
\left. \overset{k}{\underset{i=1,j=i}{\prod}} \left(1 - \left(1 - I_i^{k\widetilde{w}_i}\right)^p \left(1 - I_j^{k\widetilde{w}_j}\right)^q \right), \overset{k}{\underset{i=1,j=i}{\prod}} \left(1 - \left(1 - F_i^{k\widetilde{w}_i}\right)^p \left(1 - F_j^{k\widetilde{w}_j}\right)^q \right) \right) \tag{25}
$$

Furthermore, when $n = k + 1$, we have

$$
\overset{k+1}{\underset{i=1,j=i}{\oplus}} \left(((k+1)\widetilde{w}_i x_i)^p \otimes \left((k+1)\widetilde{w}_j x_j\right)^q \right)
$$
$$
= \overset{k}{\underset{i=1,j=i}{\oplus}} \left(((k+1)\widetilde{w}_i x_i)^p \otimes \left((k+1)\widetilde{w}_j x_j\right)^q \right) \overset{k}{\underset{i=1}{\oplus}} \left(((k+1)\widetilde{w}_i x_i)^p \otimes ((k+1)\widetilde{w}_{k+1} x_{k+1})^q \right) \tag{26}
$$
$$
\oplus \left(((k+1)\widetilde{w}_{k+1} x_{k+1})^p \otimes ((k+1)\widetilde{w}_{k+1} x_{k+1})^q \right)
$$

Firstly, we prove that

$$
\overset{k}{\underset{i=1}{\oplus}} \left(((k+1)\widetilde{w}_i x_i)^p \otimes ((k+1)\widetilde{w}_{k+1} x_{k+1})^q \right)
$$
$$
= \left(1 - \overset{k}{\underset{i=1}{\prod}} \left(1 - \left(1 - (1-T_i)^{(k+1)\widetilde{w}_i}\right)^p \left(1 - \left(1-T_{(k+1)}\right)^{(k+1)\widetilde{w}_{(k+1)}}\right)^q \right), \right.
$$
$$
\left. \overset{k}{\underset{i=1}{\prod}} \left(1 - \left(1 - I_i^{(k+1)\widetilde{w}_i}\right)^p \left(1 - I_{(k+1)}^{(k+1)\widetilde{w}_{(k+1)}}\right)^q \right), \overset{k}{\underset{i=1}{\prod}} \left(1 - \left(1 - F_i^{(k+1)\widetilde{w}_i}\right)^p \left(1 - F_{(k+1)}^{(k+1)\widetilde{w}_{(k+1)}}\right)^q \right) \right) \tag{27}
$$

We shall prove Equation (27) on mathematical induction on k

For $k = 2$, we have

$$
\begin{aligned}
&\overset{2}{\underset{i=1}{\oplus}}\left(((k+1)\widetilde{w}_i x_i)^p \otimes ((k+1)\widetilde{w}_{k+1} x_{k+1})^q\right) \\
&= \left((3\widetilde{w}_1 x_1)^p \otimes (3\widetilde{w}_3 x_3)^q\right) \oplus \left((3\widetilde{w}_2 x_2)^p \otimes (3\widetilde{w}_3 x_3)^q\right) \\
&= \left(\left(1-(1-T_1)^{3\widetilde{w}_1}\right)^p\left(1-(1-T_3)^{3\widetilde{w}_3}\right)^q, 1-\left(1-I_1^{3\widetilde{w}_1}\right)^p\left(1-I_3^{3\widetilde{w}_3}\right)^q, 1-\left(1-F_1^{3\widetilde{w}_1}\right)^p\left(1-F_3^{3\widetilde{w}_3}\right)^q\right) \\
&\oplus\left(\left(1-(1-T_2)^{3\widetilde{w}_2}\right)^p\left(1-(1-T_3)^{3\widetilde{w}_3}\right)^q, 1-\left(1-I_2^{3\widetilde{w}_2}\right)^p\left(1-I_3^{3\widetilde{w}_3}\right)^q, 1-\left(1-F_2^{3\widetilde{w}_2}\right)^p\left(1-F_3^{3\widetilde{w}_3}\right)^q\right) \\
&= \left(1-\overset{2}{\underset{i=1}{\prod}}\left(1-\left(1-(1-T_i)^{3\widetilde{w}_i}\right)^p\left(1-(1-T_3)^{3\widetilde{w}_3}\right)^q\right),\right. \\
&\qquad \overset{2}{\underset{i=1}{\prod}}\left(1-\left(1-I_i^{3\widetilde{w}_i}\right)^p\left(1-I_3^{3\widetilde{w}_3}\right)^q\right), \\
&\qquad \left.\overset{2}{\underset{i=1}{\prod}}\left(1-\left(1-F_i^{3\widetilde{w}_i}\right)^p\left(1-F_3^{3\widetilde{w}_3}\right)^q\right)\right)
\end{aligned} \tag{28}
$$

Suppose $k = a$, the Equation (27) is right, that is

$$
\begin{aligned}
&\overset{a}{\underset{i=1}{\oplus}}\left(((a+1)\widetilde{w}_i x_i)^p \otimes ((a+1)\widetilde{w}_{a+1} x_{a+1})^q\right) \\
&= \left(1-\overset{a}{\underset{i=1}{\prod}}\left(1-\left(1-(1-T_i)^{(a+1)\widetilde{w}_i}\right)^p\left(1-\left(1-T_{(a+1)}\right)^{(a+1)\widetilde{w}_{(a+1)}}\right)^q\right),\right. \\
&\qquad \left.\overset{a}{\underset{i=1}{\prod}}\left(1-\left(1-I_i^{(a+1)\widetilde{w}_i}\right)^p\left(1-I_{(a+1)}^{(a+1)\widetilde{w}_{(a+1)}}\right)^q\right), \overset{a}{\underset{i=1}{\prod}}\left(1-\left(1-F_i^{(a+1)\widetilde{w}_i}\right)^p\left(1-F_{(a+1)}^{(a+1)\widetilde{w}_{(a+1)}}\right)^q\right)\right)
\end{aligned}
$$

Then, when $k = a + 1$, we have

$$
\begin{aligned}
&\overset{a+1}{\underset{i=1}{\oplus}}\left(((a+2)\widetilde{w}_i x_i)^p \otimes ((a+2)\widetilde{w}_{a+2} x_{a+2})^q\right) \\
&= \overset{a}{\underset{i=1}{\oplus}}\left(((a+2)\widetilde{w}_i x_i)^p \otimes ((a+2)\widetilde{w}_{a+2} x_{a+2})^q\right) \oplus \left(((a+2)\widetilde{w}_{a+1} x_{a+1})^p \otimes ((a+2)\widetilde{w}_{a+2} x_{a+2})^q\right) \\
&= \left(1-\overset{a}{\underset{i=1}{\prod}}\left(1-\left(1-(1-T_i)^{(a+2)\widetilde{w}_i}\right)^p\left(1-\left(1-T_{(a+2)}\right)^{(a+2)\widetilde{w}_{(a+2)}}\right)^q\right),\right. \\
&\qquad \left.\overset{a}{\underset{i=1}{\prod}}\left(1-\left(1-I_i^{(a+2)\widetilde{w}_i}\right)^p\left(1-I_{(a+2)}^{(a+2)\widetilde{w}_{(a+2)}}\right)^q\right), \overset{a}{\underset{i=1}{\prod}}\left(1-\left(1-F_i^{(a+2)\widetilde{w}_i}\right)^p\left(1-F_{(a+2)}^{(a+2)\widetilde{w}_{(a+2)}}\right)^q\right)\right) \\
&\oplus\left(1-\left(1-\left(1-T_{(a+1)}\right)^{(a+2)\widetilde{w}_i}\right)^p\left(1-\left(1-T_{(a+2)}\right)^{(a+2)\widetilde{w}_{(a+2)}}\right)^q,\right. \\
&\qquad \left.\left(1-\left(1-I_{(a+1)}^{(a+2)\widetilde{w}_{(a+1)}}\right)^p\left(1-I_{(a+2)}^{(a+2)\widetilde{w}_{(a+2)}}\right)^q\right), \left(1-\left(1-F_{(a+1)}^{(a+2)\widetilde{w}_{(a+1)}}\right)^p\left(1-F_{(a+2)}^{(a+2)\widetilde{w}_{(a+2)}}\right)^q\right)\right) \\
&= \left(1-\overset{a+1}{\underset{i=1}{\prod}}\left(1-\left(1-(1-T_i)^{(a+2)\widetilde{w}_i}\right)^p\left(1-\left(1-T_{(a+2)}\right)^{(a+2)\widetilde{w}_{(a+2)}}\right)^q\right),\right. \\
&\qquad \left.\overset{a+1}{\underset{i=1}{\prod}}\left(1-\left(1-I_i^{(a+2)\widetilde{w}_i}\right)^p\left(1-I_{(a+2)}^{(a+2)\widetilde{w}_{(a+2)}}\right)^q\right), \overset{a+1}{\underset{i=1}{\prod}}\left(1-\left(1-F_i^{(a+2)\widetilde{w}_i}\right)^p\left(1-F_{(a+2)}^{(a+2)\widetilde{w}_{(a+2)}}\right)^q\right)\right)
\end{aligned}
$$

Therefore, when $k = a + 1$, the Equation (27) is true. Hence, Equation (27) is established for any k. Similarly, we can prove the other parts of Equation (26).
So, Equation (26) becomes

$$
\begin{aligned}
&\overset{k+1}{\underset{i=1,j=i}{\oplus}}\left(((k+1)\widetilde{w}_i x_i)^p \otimes ((k+1)\widetilde{w}_j x_j)^q\right) \\
&= \left(1-\overset{(k+1)}{\underset{i=1,j=i}{\prod}}\left(1-\left(1-(1-T_i)^{(k+1)\widetilde{w}_i}\right)^p\left(1-\left(1-T_j\right)^{(k+1)\widetilde{w}_j}\right)^q\right),\right. \\
&\qquad \left.\overset{(k+1)}{\underset{i=1,j=i}{\prod}}\left(1-\left(1-I_i^{(k+1)\widetilde{w}_i}\right)^p\left(1-I_j^{(k+1)\widetilde{w}_j}\right)^q\right), \overset{(k+1)}{\underset{i=1,j=i}{\prod}}\left(1-\left(1-F_i^{(k+1)\widetilde{w}_i}\right)^p\left(1-F_j^{(k+1)\widetilde{w}_j}\right)^q\right)\right)
\end{aligned}
$$

Therefore, when $n = k + 1$, the Equation (22) is true. Hence, Equation (22) is established for any n. □

From Equation (22) and the operational rules of SVNNs defined in (2–5), we have

$$\frac{2}{n(n+1)} \overset{n}{\underset{i=1,j=i}{\oplus}} \left((n\widetilde{w}_i x_i)^p \otimes (n\widetilde{w}_j x_j)^q \right)$$

$$= \left(\left(1 - \left(\prod_{i=1,j=i}^{n} \left(1 - \left(1 - (1-T_i)^{n\widetilde{w}_i} \right)^p \left(1 - \left(1 - T_j \right)^{n\widetilde{w}_j} \right)^q \right) \right)^{\frac{2}{n(n+1)}} \right. \right.$$

$$\left. \left(\prod_{i=1,j=i}^{n} \left(1 - \left(1 - I_i^{n\widetilde{w}_i} \right)^p \left(1 - I_j^{n\widetilde{w}_j} \right)^q \right) \right)^{\frac{2}{n(n+1)}}, \left(\prod_{i=1,j=i}^{n} \left(1 - \left(1 - F_i^{n\widetilde{w}_i} \right)^p \left(1 - F_j^{n\widetilde{w}_j} \right)^q \right) \right)^{\frac{2}{n(n+1)}} \right),$$

So

$$\left(\frac{2}{n(n+1)} \overset{n}{\underset{i=1,j=i}{\oplus}} \left((n\widetilde{w}_i x_i)^p \otimes (n\widetilde{w}_j x_j)^q \right) \right)^{\frac{1}{p+q}}$$

$$= \left(\left(1 - \left(\prod_{i=1,j=i}^{n} \left(1 - \left(1 - (1-T_i)^{n\widetilde{w}_i} \right)^p \left(1 - \left(1 - T_j \right)^{n\widetilde{w}_j} \right)^q \right) \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}}, \right.$$

$$1 - \left(1 - \left(\prod_{i=1.j=i}^{n} \left(1 - \left(1 - I_i^{n\widetilde{w}_i} \right)^p \left(1 - I_i^{n\widetilde{w}_j} \right)^q \right) \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}},$$

$$\left. 1 - \left(1 - \left(\prod_{i=1.j=i}^{n} \left(1 - \left(1 - F_i^{n\widetilde{w}_i} \right)^p \left(1 - F_i^{n\widetilde{w}_j} \right)^q \right) \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right)$$

Therefore, Equation (21) is right and we complete the proof of the Theorem 2.

To compute the power weight vector $\widetilde{w}_i$, the support degree between SVNNs should be calculate firstly. Under normal circumstances, we can use the similarity degree between SVNNs to replace the support degree and that is,

$$Sup(x_i, x_j) = 1 - d(x_i, x_j) \tag{29}$$

Example 1. *Suppose three SVNNs exist:* $x_1 = (0.260, 0.425, 0.315)$, $x_2 = (0.220, 0.450, 0.330)$, *and* $x_3 = (0.255, 0.500, 0.245)$. *Accordingly, we can use SVNPHA to generate a comprehensive value. In the following, the steps are given.*

Step 1. Calculate the $Sup(x_i, x_j)(i, j = 1, 2, 3)$ *by using Expressions (15) and (22). Thereafter, we have*

$Sup(x_1, x_2) = 0.97142$ $Sup(x_1, x_3) = 0.94070$

$Sup(x_2, x_1) = 0.97142$ $Sup(x_3, x_1) = 0.93958$

$Sup(x_3, x_1) = 0.94070$ $Sup(x_3, x_2) = 0.93958$

Step 2. Calculate the power weighting vector through Expression (19). Thereafter, we have

$T(x_1) = Sup(x_1, x_2) + Sup(x_1, x_3) = 1.91212$

$T(x_2) = Sup(x_2, x_1) + Sup(x_2, x_3) = 1.91100$

$T(x_3) = Sup(x_3, x_1) + Sup(x_3, x_2) = 1.88028$

$\widetilde{w}_1 = \frac{1+T(x_1)}{3+T(x_1)+T(x_2)+T(x_3)} = \frac{1+1.91212}{3+1.91212+1.91100+1.88028} = 0.33460$

$\widetilde{w}_2 = \frac{1+T(x_2)}{3+T(x_1)+T(x_2)+T(x_3)} = \frac{1+1.91100}{3+1.91212+1.91100+1.88028} = 0.33447$

$\widetilde{w}_3 = \frac{1+T(x_3)}{3+T(x_1)+T(x_2)+T(x_3)} = \frac{1+1.88028}{3+1.91212+1.91100+1.88028} = 0.33093$

Step 3. Calculate the comprehensive value $x = (T, I, F)$ *using the SVNPHA operator (suppose p = q = 1). Thereafter, we can have*

$$T = \left(1 - \left(\prod_{i=1,j=i}^{3}\left(1 - \left(1 - (1-T_i)^{3\widetilde{w}_i}\right)\left(1 - \left(1-T_j\right)^{3\widetilde{w}_j}\right)\right)\right)^{\frac{2}{3(3+1)}}\right)^{\frac{1}{2}}$$

$$= \left(1 - \left(\begin{array}{c}\left(1 - \left(1-(1-T_1)^{3\widetilde{w}_1}\right)\left(1-(1-T_1)^{3\widetilde{w}_1}\right)\right)*\left(1 - \left(1-(1-T_1)^{3\widetilde{w}_1}\right)\left(1-(1-T_2)^{3\widetilde{w}_2}\right)\right)* \\ \left(1 - \left(1-(1-T_1)^{3\widetilde{w}_1}\right)\left(1-(1-T_3)^{3\widetilde{w}_3}\right)\right)*\left(1 - \left(1-(1-T_2)^{3\widetilde{w}_2}\right)\left(1-(1-T_2)^{3\widetilde{w}_2}\right)\right)* \\ \left(1 - \left(1-(1-T_2)^{3\widetilde{w}_2}\right)\left(1-(1-T_3)^{3\widetilde{w}_3}\right)\right)*\left(1 - \left(1-(1-T_3)^{3\widetilde{w}_3}\right)\left(1-(1-T_3)^{3\widetilde{w}_3}\right)\right)\end{array}\right)^{\frac{2}{3(3+1)}}\right)^{\frac{1}{2}}$$

$$= 0.24518$$

and

$$I = 1 - \left(1 - \left(\prod_{i=1.j=i}^{3}\left(1 - \left(1 - I_i^{3\widetilde{w}_i}\right)^p\left(1 - I_i^{3\widetilde{w}_j}\right)^q\right)\right)^{\frac{2}{3(3+1)}}\right)^{\frac{1}{2}}$$

$$= 1 - \left(1 - \left(\begin{array}{c}\left(1 - \left(1-I_1^{3\widetilde{w}_1}\right)^p\left(1-I_1^{3\widetilde{w}_1}\right)^q\right)*\left(1 - \left(1-I_1^{3\widetilde{w}_1}\right)^p\left(1-I_2^{3\widetilde{w}_2}\right)^q\right)* \\ \left(1 - \left(1-I_1^{3\widetilde{w}_1}\right)^p\left(1-I_3^{3\widetilde{w}_3}\right)^q\right)*\left(1 - \left(1-I_2^{3\widetilde{w}_2}\right)^p\left(1-I_2^{3\widetilde{w}_2}\right)^q\right)* \\ \left(1 - \left(1-I_2^{3\widetilde{w}_2}\right)^p\left(1-I_3^{3\widetilde{w}_3}\right)^q\right)*\left(1 - \left(1-I_3^{3\widetilde{w}_3}\right)^p\left(1-I_3^{3\widetilde{w}_3}\right)^q\right)\end{array}\right)^{\frac{2}{3(3+1)}}\right)^{\frac{1}{2}}$$

$$= 0.45754$$

and

$$F = 1 - \left(1 - \left(\prod_{i=1.j=i}^{3}\left(1 - \left(1 - F_i^{3\widetilde{w}_i}\right)^p\left(1 - F_i^{3\widetilde{w}_j}\right)^q\right)\right)^{\frac{2}{3(3+1)}}\right)^{\frac{1}{2}}$$

$$I = 1 - \left(1 - \left(\begin{array}{c}\left(1 - \left(1-F_1^{3\widetilde{w}_1}\right)^p\left(1-F_1^{3\widetilde{w}_1}\right)^q\right)*\left(1 - \left(1-F_1^{3\widetilde{w}_1}\right)^p\left(1-F_2^{3\widetilde{w}_2}\right)^q\right)* \\ \left(1 - \left(1-F_1^{3\widetilde{w}_1}\right)^p\left(1-F_3^{3\widetilde{w}_3}\right)^q\right)*\left(1 - \left(1-F_2^{3\widetilde{w}_2}\right)^p\left(1-F_2^{3\widetilde{w}_2}\right)^q\right)* \\ \left(1 - \left(1-F_2^{3\widetilde{w}_2}\right)^p\left(1-F_3^{3\widetilde{w}_3}\right)^q\right)*\left(1 - \left(1-F_3^{3\widetilde{w}_3}\right)^p\left(1-F_3^{3\widetilde{w}_3}\right)^q\right)\end{array}\right)^{\frac{2}{3(3+1)}}\right)^{\frac{1}{2}}$$

$$= 0.29532$$

Thus, we can obtain the comprehensive value $x = (0.24519, 0.45746, 0.29532)$.

Theorem 3. *(Idempotency). Let* $x_i(i = 1, 2, \cdots, n)$ *be a collection of SVNNs and* $x_1 = x_2 = \cdots = x_n = x$. *Hence,*

$$SVNPHA^{p,q}(x_1, x_2, \cdots, x_n) = x.$$

Proof.

$$SVNPHA^{p,q}(x_1, x_2, \cdots, x_n) = \left(\frac{2}{n(n+1)} \bigoplus_{i=1,j=i}^{n}\left((n\widetilde{w}_i x_i)^p \otimes \left(n\widetilde{w}_j x_j\right)^q\right)\right)^{\frac{1}{p+q}}$$

$$= \left(\frac{2}{n(n+1)} \bigoplus_{i=1,j=i}^{n}\left((n\widetilde{w}_i x)^p \otimes \left(n\widetilde{w}_j x\right)^q\right)\right)^{\frac{1}{p+q}} = \left(\frac{2}{n(n+1)} \bigoplus_{i=1,j=i}^{n}\left((n\frac{1}{n}x)^p \otimes \left(n\frac{1}{n}x\right)^q\right)\right)^{\frac{1}{p+q}}$$

$$= \left(\frac{2}{n(n+1)} \bigoplus_{i=1,j=i}^{n}\left(x^{p+q}\right)\right)^{\frac{1}{p+q}} = x$$

thereby completing the proof of Theorem 3. □

Theorem 4. *(Commutativity).* Let $(x'_1, x'_2, \cdots x'_n)$ *be any permutation of* $(x_1, x_2, \cdots, x_n)$, *then* $SVNPHA^{p,q}(x'_1, x'_2, \cdots x'_n) = SVNPHA^{p,q}(x_1, x_2, \cdots, x_n)$

Proof. Since $(x'_1, x'_2, \cdots x'_n)$ be any permutation of $(x_1, x_2, \cdots, x_n)$, then

$$SVNPHA^{p,q}(x_1, x_2, \cdots, x_n) = \left(\frac{2}{n(n+1)} \overset{n}{\underset{i=1,j=i}{\oplus}} \left((n\widetilde{w}_i x_i)^p \otimes (n\widetilde{w}_j x_j)^q \right) \right)^{\frac{1}{p+q}}$$

$$= \left(\frac{2}{n(n+1)} \overset{n}{\underset{i=1,j=i}{\oplus}} \left((n\widetilde{w}'_i x'_i)^p \otimes (n\widetilde{w}'_j x'_j)^q \right) \right)^{\frac{1}{p+q}} = SVNPHA^{p,q}(x'_1, x'_2, \cdots, x'_n)$$

thereby completing the proof of Theorem 4.□

Theorem 5. *(Boundedness).* Let $x_i = (T_i, I_i, F_i)(i = 1, 2, \cdots, n)$ *be a collection of SVNNs,* $x^- = (min_i\{T_i\}, max_i\{I_i\}, max_i\{F_i\})$, *and* $x^+ = (max_i\{T_i\}, min_i\{I_i\}, min_i\{F_i\})$. *Hence,*

$$x^- \leq SVNPHA^{p,q}(x_1, x_2, \cdots, x_n) \leq x^+$$

Proof. By the comparison method in Definition 3, we have $x_i \geq x^-$, then based on the theorem 2 and 3, we have

$$SVNPHA^{p,q}(x_1, x_2, \cdots, x_n) \geq SVNPHA^{p,q}(x^-, x^-, \cdots, x^-) = x^-$$

Similarly, we can obtain

$$SVNPHA^{p,q}(x_1, x_2, \cdots, x_n) \leq SVNPHA^{p,q}(x^+, x^+, \cdots, x^+) = x^+, \text{ then}$$

$$SVNPHA^{p,q}(x_1, x_2, \cdots, x_n) \leq SVNPHA^{p,q}(x^+, x^+, \cdots, x^+) = x^+, \text{ then} (30) x^- \leq SVNPHA^{p,q}(x_1, x_2, \cdots, x_n) \leq x^+$$

thereby completing the proof of Theorem 5.□

In the follow, we can discuss some special cases about $SVNNPHA^{p,q}$ operator.

(1) If $q = 0$, then the Expression (21) operator is reduced to the single-valued neutrosophic power generalized linear descending weight operator as follows:

$$SVNPHA^{p,0}(x_1, x_2, \cdots, x_n)$$

$$= \left(\left(1 - \left(\prod_{i=1,}^{n} \left(1 - \left(1 - (1-T_i)^{n\widetilde{w}_i} \right)^p \right)^{n+1-i} \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p}}, \right.$$

$$\left. \left(1 - \left(\prod_{i=1}^{n} \left(1 - \left(1 - I_i^{n\widetilde{w}_i} \right)^p \right)^{n+1-i} \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p}}, 1 - \left(1 - \left(\prod_{i=1}^{n} \left(1 - \left(1 - F_i^{n\widetilde{w}_i} \right)^p \right)^{n+1-i} \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p}} \right)$$

(2) If $p = 0$ then the Expression (21) operator is reduced to the single-valued neutrosophic power generalized linear ascending weight operator as follows:

$$SVNPHA^{q}(x_1, x_2, \cdots, x_n)$$

$$= \left(\left(1 - \left(\prod_{i=1,j=i}^{n} \left(1 - \left(1 - (1-T_j)^{n\widetilde{w}_j} \right)^q \right)^i \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{q}}, \right.$$

$$\left. \left(1 - \left(\prod_{i=1,j=i}^{n} \left(1 - \left(1 - I_j^{n\widetilde{w}_j} \right)^q \right)^i \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{q}}, 1 - \left(1 - \left(\prod_{i=1,j=i}^{n} \left(1 - \left(1 - F_j^{n\widetilde{w}_j} \right)^q \right)^i \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{q}} \right)$$

(3) If $p = q = \frac{1}{2}$, then the Expression (21) operator is reduced to the single-valued neutrosophic power basic Heronian operator as follows:

$$SVNPHA^{\frac{1}{2},\frac{1}{2}}(x_1, x_2, \cdots, x_n)$$

$$= \left(\left(1 - \left(\prod_{i=1,j=i}^{n} \left(1 - \left(1 - (1 - T_i)^{\frac{n(1+T(x_i))}{\sum_{t=1}^{n}(1+T(x_t))}} \right)^{\frac{1}{2}} \left(1 - \left(1 - T_j \right)^{\frac{n(1+T(x_j))}{\sum_{t=1}^{n}(1+T(x_t))}} \right)^{\frac{1}{2}} \right) \right)^{\frac{2}{n(n+1)}} \right), \right.$$

$$\left(1 - \left(\prod_{i=1,j=i}^{n} \left(1 - \left(1 - I_i^{\frac{n(1+T(x_i))}{\sum_{t=1}^{n}(1+T(x_t))}} \right)^{\frac{1}{2}} \left(1 - I_i^{\frac{n(1+T(x_i))}{\sum_{t=1}^{n}(1+T(x_t))}} \right)^{\frac{1}{2}} \right) \right)^{\frac{2}{n(n+1)}} \right),$$

$$\left. 1 - \left(1 - \left(\prod_{i=1,j=i}^{n} \left(1 - \left(1 - F_i^{\frac{n(1+T(x_i))}{\sum_{t=1}^{n}(1+T(x_t))}} \right)^{\frac{1}{2}} \left(1 - F_i^{\frac{n(1+T(x_i))}{\sum_{t=1}^{n}(1+T(x_t))}} \right)^{\frac{1}{2}} \right) \right)^{\frac{2}{n(n+1)}} \right) \right)$$

(4) If $p = q = 1$, then the operator of Equation (21) is reduced to the single-valued neutrosophic number power line Heronian operator as follows:

$$SVNPHA^{1,1}(x_1, x_2, \cdots, x_n)$$

$$= \left(\left(1 - \left(\prod_{i=1,j=i}^{n} \left(1 - \left(1 - (1 - T_i)^{\frac{n(1+T(x_i))}{\sum_{t=1}^{n}(1+T(x_t))}} \right) \left(1 - \left(1 - T_j \right)^{\frac{n(1+T(x_j))}{\sum_{t=1}^{n}(1+T(x_t))}} \right) \right) \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{2}}, \right.$$

$$1 - \left(1 - \left(\prod_{i=1,j=i}^{n} \left(1 - \left(1 - I_i^{\frac{n(1+T(x_i))}{\sum_{t=1}^{n}(1+T(x_t))}} \right) \left(1 - I_i^{\frac{n(1+T(x_i))}{\sum_{t=1}^{n}(1+T(x_t))}} \right) \right) \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{2}},$$

$$\left. 1 - \left(1 - \left(\prod_{i=1,j=i}^{n} \left(1 - \left(1 - F_i^{\frac{n(1+T(x_i))}{\sum_{t=1}^{n}(1+T(x_t))}} \right) \left(1 - F_i^{\frac{n(1+T(x_i))}{\sum_{t=1}^{n}(1+T(x_t))}} \right) \right) \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{2}} \right)$$

In the SVNPHA operators, we only take into account the power weight vector and interrelationship among SVNNs but not the weight of every SVNN. However, in many realistic decision-making, the weights of attributes are also an important parameter. Thus, we propose the single-valued neutrosophic weight PHA (SVNWPHA) operator as follows.

Definition 9. *Let $x_i = (T_i, I_i, F_i)(i = 1, 2, \cdots, n)$ be a group of SVNNs and $p, q \geq 0$, $W = (w_1, w_2, \cdots, w_n)^T$ is the weight vector of $x_i(i = 1, 2, \cdots, n)$, satisfying $w_i \in [0, 1]$ and $\sum_{i}^{n} w_i = 1$. If:*

$$SVNWPHA^{p,q}(x_1, x_2, \cdots, x_n)$$

$$= \left(\frac{1}{n(n+1)} \bigoplus_{i=1,j=i}^{n} \left(\left(\frac{n\widetilde{w}_i w_i}{\sum_{t=1}^{n} \widetilde{w}_t w_t} x_i \right)^p \otimes \left(\frac{n\widetilde{w}_j w_j}{\sum_{t=1}^{n} \widetilde{w}_t w_t} x_j \right)^q \right) \right)^{\frac{1}{p+q}} \tag{30}$$

where $\widetilde{w}_i = \frac{(1+T(x_i))}{\sum_{t=1}^{n}(1+T(x_t))}$ and $\sum_{i}^{n} \widetilde{w}_i = 1$. $T(x_i) = \sum_{j=1,j\neq i}^{n} Sup(x_i, x_j)$ we shall denote $Sup(x_i, x_j)$ as the support degree for x_i from x_j. $Sup(x_i, x_j)$ satisfies the following three properties:
(1) $Sup(x_i, x_j) \in [0, 1]$;
(2) $Sup(x_i, x_j) = Sup(x_j, x_i)$;
(3) $Sup(x_i, x_j) > Sup(x_l, x_k)$, if $d(x_i, x_j) < d(x_l, x_k)$ in which $d(x_i, x_j)$ is the distance between SVNNs x_i and x_j.

Then, $SVNWPHA^{p,q}$ is called the single-valued neutrosophic weight power Heronian aggregation (SVNPHA) operator.

Theorem 6. *Let $p, q \geq 0$ and $x_i = (T_i, I_i, F_i)(i = 1, 2, \cdots, n)$ be a collection of SVNNs, $W = (w_1, w_2, \cdots, w_n)^T$, satisfying $w_i \in [0,1]$ and $\sum_i^n w_i = 1$. Then, the result aggregated from SVNWPHA is still a SVNNs, and even*

$$SVNWPHA^{p,q}(x_1, x_2, \cdots, x_n)$$

$$= \left(\left(1 - \left(\prod_{i=1, j=i}^{n} \left(1 - \left(1 - (1-T_i)^{\frac{n\widetilde{w}_i w_i}{\sum_{t=1}^n \widetilde{w}_t w_t}} \right)^p \left(1 - \left(1 - T_j \right)^{\frac{n\widetilde{w}_j w_j}{\sum_{t=1}^n \widetilde{w}_t w_t}} \right)^q \right) \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right.,$$

$$\left(1 - \left(\prod_{i=1.j=i}^{n} \left(1 - \left(1 - I_i^{\frac{n\widetilde{w}_i w_i}{\sum_{t=1}^n \widetilde{w}_t w_t}} \right)^p \left(1 - I_i^{\frac{n\widetilde{w}_j w_j}{\sum_{t=1}^n \widetilde{w}_t w_t}} \right)^q \right) \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} ,$$

$$\left. 1 - \left(1 - \left(\prod_{i=1.j=i}^{n} \left(1 - \left(1 - F_i^{\frac{n\widetilde{w}_i w_i}{\sum_{t=1}^n \widetilde{w}_t w_t}} \right)^p \left(1 - F_i^{\frac{n\widetilde{w}_j w_j}{\sum_{t=1}^n \widetilde{w}_t w_t}} \right)^q \right) \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right) \tag{31}$$

As with the proof of Theorem 2, it is omitted from this paper.

Obviously, when $W = (1/n, 1/n, \cdots, 1/n)^T$, the SVNWPHA operator is reduced to the SVNPHA operator.

Similar to the above SVNPHA operator, the SVNWPHA operator also has the same properties.

Theorem 7. *(Idempotency). Let $x_i(i = 1, 2, \cdots, n)$ be a collection of SVNNs, $W = (w_1, w_2, \cdots, w_n)^T$, satisfying $w_i \in [0,1]$ and $\sum_i^n w_i = 1$, and $x_1 = x_2 = \cdots = x_n = x$. Hence,*

$$SVNWPHA^{p,q}(x_1, x_2, \cdots, x_n) = x.$$

Theorem 8. *(Commutativity). Let $(x'_1, x'_2, \cdots x'_n)$ be any permutation of $(x_1, x_2, \cdots, x_n)$, $W = (w_1, w_2, \cdots, w_n)^T$, satisfying $w_i \in [0,1]$ and $\sum_i^n w_i = 1$. Hence,*

$$SVNWPHA^{p,q}(x'_1, x'_2, \cdots x'_n) = SVNWPHA^{p,q}(x_1, x_2, \cdots, x_n).$$

Theorem 9. *(Boundedness). Let $x_i = (T_i, I_i, F_i)(i = 1, 2, \cdots, n)$ be a collection of SVNNs $W = (w_1, w_2, \cdots, w_n)^T$, satisfying $w_i \in [0,1]$ and $\sum_i^n w_i = 1$, $x^- = (\min_i\{T_i\}, \max_i\{I_i\}, \max_i\{F_i\})$, and $x^+ = (\max_i\{T_i\}, \min_i\{I_i\}, \min_i\{F_i\})$. Hence,*

$$x^- \leq SVNWPHA^{p,q}(x_1, x_2, \cdots, x_n) \leq x^+$$

3.2. Single Valued Neutrosophic Geometric Power Heronian Aggregation Operators

Based on the PA operator [36] and geometric mean [49], Xu [37] further defined a power geometric (PG) operator:

Definition 10. [37]. *Let $p, q \geq 0$, and $\phi_i (i = 1, 2, \cdots, n)$ be a collection of nonnegative numbers. if:*

$$PG(\phi_1, \phi_2, \cdots, \phi_n) = \prod_{i=1}^{n} \phi_i^{\frac{1+T(\phi_i)}{\sum_{i=1}^{n}((1+T(\phi_i((}}$$

(32)

where $T(\phi_i) = \sum_{j=1, j \neq i}^{n} Sup(\phi_i, \phi_j)$. We denote $Sup(\phi_i, \phi_j)$ as the support degree for ϕ_i from ϕ_j. $Sup(\phi_i, \phi_j) >$ $Sup(\phi_l, \phi_k)$ satisfies the following axioms:

(1) $Sup(\phi_i, \phi_j) \in [0, 1]$;
(2) $Sup(\phi_i, \phi_j) Sup(\phi_j, \phi_i)$
(3) $Sup(\phi_i, \phi_j) > Sup(\phi_l, \phi_k)$, if $|\phi_i - \phi_j| < |\phi_l - \phi_k|$

Definition 11. [39]. *Let $p, q \geq 0$, and $\phi_i (i = 1, 2, \cdots, n)$ be a collection of nonnegative numbers. If:*

$$GHM^{p,q}(\phi_1, \phi_2, \cdots, \phi_n) = \frac{1}{p+q} \left(\prod_{i=1, j=i}^{n} (p\phi_i + q\phi_j)^{\frac{2}{n(n+1)}} \right)$$

(33)

Then $GHM^{p,q}$ is called the geometric Heronian mean (GHM) operator.

Next, we shall develop the SVNGPHA and SVNWGPHA operators based on the operation laws of SVNNs.

Definition 12. *Let $p, q \geq 0$, and $x_i = (T_i, I_i, F_i)(i = 1, 2, \cdots, n)$ be a collection of single-valued neutrosophic numbers. if:*

$$SVNGPHA^{p,q}(x_1, x_2, \cdots, x_n) = \frac{1}{p+q} \mathop{\otimes}_{\substack{i=1, \\ j=i}}^{n} \left(px_i a_i^{\frac{1+T(x_i)}{\sum_{i=1}^{n}((1+T(x_i((}} \oplus qx_j^{\frac{1+T(x_j)}{\sum_{i=1}^{n}((1+T(x_j((}} \right)^{\frac{2}{n(n+1)}}$$

(34)

where $T(x_i) = \sum_{j=1, j \neq i}^{n} Sup(x_i, x_j)$. we shall denote $Sup(x_i, x_j)$ as the support degree for x_i from x_j. $Sup(x_i, x_j)$ satisfies the following three properties:

(1) $Sup(x_i, x_j) \in [0, 1]$;
(2) $Sup(x_i, x_j) = Sup(x_j, x_i)$
(3) $Sup(x_i, x_j) > Sup(x_j, x_i)$, if $d(x_i, x_j) < d(x_l, x_k)$ in which $d(x_i, x_j)$ is the distance between SVNNs x_i and x_j

Then, $SVNGPHA^{p,q}$ is called the single-valued neutrosophic geometric power Heronian aggregation (SVNGPHA) operator.

In order to simply this expression X. We can define

$$\widetilde{w}_i = \frac{(1 + T(x_i))}{\sum_{t=1}^{n} (1 + T(x_t))}$$

(35)

and call $(\widetilde{w}_1, \widetilde{w}_2, \cdots, \widetilde{w}_n)$ as the power weighting vector with $w_i \geq 0, \sum_{i=1}^{n} w_i = 1$.

Then, Expression (18) can be shown as follows:

$$SVNGPHA^{p,q}(x_1, x_2, \cdots, x_n) = \frac{1}{p+q} \mathop{\otimes}_{\substack{i=1, \\ j=i}}^{n} \left(px_i^{n\widetilde{w}_i} \oplus qx_j^{n\widetilde{w}_j} \right)^{\frac{2}{n(n+1)}}$$

(36)

Theorem 10. *Let $p, q \geq 0$, and $x_i = (T_i, I_i, F_i)(i = 1, 2, \cdots, n)$ be a collection of SVNNs. Then, the result aggregated from SVNGPHA is still a SVNN, and even*

$$
SVNNPGHA^{p,q}(x_1, x_2, \cdots, x_n)
$$
$$
= \left(1 - \left(1 - \prod_{i=1, j=i}^{n} \left(1 - \left(1 - T_i^{n\widetilde{w}_i} \right)^p \left(1 - T_j^{n\widetilde{w}_j} \right)^q \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right),
$$
$$
\left(1 - \prod_{i=1, j=i}^{n} \left(1 - \left(1 - (1 - I_i)^{n\widetilde{w}_i} \right)^p \left(1 - (1 - I_j)^{n\widetilde{w}_j} \right)^q \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}},
$$
$$
\left(1 - \prod_{i=1, j=i}^{n} \left(1 - \left(1 - (1 - F_i)^{n\widetilde{w}_i} \right)^p \left(1 - (1 - F_j)^{n\widetilde{w}_j} \right)^q \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right) \tag{37}
$$

Similar to the above SVNPHA operator, the SVNGPHA operator also has the same properties.

Theorem 11. *(Idempotency). Let $x_i(i = 1, 2, \cdots, n)$ be a collection of SVNNs, and $x_1 = x_2 = \cdots = x_n = x$, then*

$$
SVNGPHA^{p,q}(x_1, x_2, \cdots, x_n) = x. \tag{38}
$$

Theorem 12. *(Commutativity). Let $(x'_1, x'_2, \cdots x'_n)$ be any permutation of $(x_1, x_2, \cdots, x_n)$, then*

$$
SVNGPHA^{p,q}(x'_1, x'_2, \cdots x'_n) = SVNGPHA^{p,q}(x_1, x_2, \cdots, x_n) \tag{39}
$$

Theorem 13. *(Boundedness). Let $x_i = (T_i, I_i, F_i)(i = 1, 2, \cdots, n)$ be a collection of SVNNs, and $x^- = (min_i\{T_i\}, max_i\{I_i\}, max_i\{F_i\})$, $x^+ = (max_i\{T_i\}, min_i\{I_i\}, min_i\{F_i\})$ then*

$$
x^- \leq SVNGPHA^{p,q}(x_1, x_2, \cdots, x_n) \leq x^+ \tag{40}
$$

In the SVNGPHA operators, we only take into account the power weight vector and interrelationship among SVNNs but not the weight of every SVNN. However, in many realistic decision-making, the weights of attributes are also an important parameter. Thus, we propose the single-valued neutrosophic numbers weight geometric power Heronian aggregation (SVNWGPHA) operator as follows.

Definition 13. *Let $p, q \geq 0$, and $x_i = (T_i, I_i, F_i)(i = 1, 2, \cdots, n)$ be a collection of single-valued neutrosophic numbers, $W = (w_1, w_2, \cdots, w_n)^T$ is the weight vector of $x_i(i = 1, 2, \cdots, n)$, satisfying $w_i \in [0, 1]$ and $\sum_{i}^{n} w_i = 1$. If:*

$$
SVNWGPHA^{p,q}(x_1, x_2, \cdots, x_n) = \frac{1}{p+q} \underset{\substack{i = 1, \\ j = i}}{\overset{n}{\otimes}} \left(px_i^{\frac{n\widetilde{w}_i w_i}{\sum_{t=1}^{n} \widetilde{w}_t w_t}} \oplus qx_j^{\frac{n\widetilde{w}_j w_j}{\sum_{t=1}^{n} \widetilde{w}_t w_t}} \right)^{\frac{2}{n(n+1)}} \tag{41}
$$

where $\widetilde{w}_i = \frac{(1 + T(x_i))}{\sum_{t=1}^{n}(1 + T(x_t))}$ and $\sum_{i}^{n} \widetilde{w}_i = 1$. $T(x_i) = \sum_{j=1, j \neq i}^{n} Sup(x_i, x_j)$. We shall denote $Sup(x_i, x_j)$ as the support degree for x_i from x_j. $Sup(x_i, x_j)$ satisfies the following three properties:
(1) $Sup(x_i, x_j) \in [0, 1]$;
(2) $Sup(x_i, x_j) = Sup(x_j, x_i)$;

(3) $Sup(x_i, x_j) > Sup(x_l, x_k)$, if $d(x_i, x_j) < d(x_l, x_k)$ in which $d(x_i, x_j)$ is the distance between SVNNs x_i and x_j.

Then, SVNWGPHA^{p,q} is called the single-valued neutrosophic weight geometric power Heronian aggregation (SVNWGPHA) operator.

Theorem 14. *Let $p, q \geq 0$, and $x_i = (T_i, I_i, F_i)(i = 1, 2, \cdots, n)$ be a collection of SVNNs, $W = (w_1, w_2, \cdots, w_n)^T$, satisfying $w_i \in [0, 1]$ and $\sum\limits_{i}^{n} w_i = 1$. Then, the result aggregated from SVNWGPHA is still a SVNNs, and even*

$$SVNWGPHA^{p,q}(x_1, x_2, \cdots, x_n)$$

$$= \left(1 - \left(1 - \prod_{i=1,j=i}^{n} \left(1 - \left(1 - T_i^{w_i \widetilde{w}_i} \right)^p \left(1 - T_j^{w_j \widetilde{w}_j} \right)^q \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right. ,$$

$$\left(1 - \prod_{i=1,j=i}^{n} \left(1 - \left(1 - (1 - I_i)^{w_i \widetilde{w}_i} \right)^p \left(1 - \left(1 - I_j \right)^{w_j \widetilde{w}_j} \right)^q \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} , \qquad (42)$$

$$\left. \left(1 - \prod_{i=1,j=i}^{n} \left(1 - \left(1 - (1 - F_i)^{w_i \widetilde{w}_i} \right)^p \left(1 - \left(1 - F_j \right)^{w_j \widetilde{w}_j} \right)^q \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right)$$

Similar to the proof of Theorem 2 it is omitted in this study.

Obviously, when $W = (1/n, 1/n, \cdots, 1/n)^T$, the SVNWGPHA operator is reduced to the SVNGPHA operator.

Similar to the SVNWPHA operator, the SVNWGPHA operator has the same properties.

Theorem 15. *(Idempotency). Let $x_i(i = 1, 2, \cdots, n)$ be a collection of SVNNs, $W = (w_1, w_2, \cdots, w_n)^T$, satisfying $w_i \in [0, 1]$ and $\sum\limits_{i}^{n} w_i = 1$, and $x_1 = x_2 = \cdots = x_n = x$, then SVNWGPHA$^{p,q}(x_1, x_2, \cdots, x_n) = x$.*

Theorem 16. *(Commutativity). Let $(x'_1, x'_2, \cdots x'_n)$ be any permutation of $(x_1, x_2, \cdots, x_n)$ $W = (w_1, w_2, \cdots, w_n)^T$, satisfying $w_i \in [0, 1]$ and $\sum\limits_{i}^{n} w_i = 1$ then*

$$SVNWGPHA^{p,q}(x'_1, x'_2, \cdots x'_n) = SVNWGPHA^{p,q}(x_1, x_2, \cdots, x_n)$$

Theorem 17. *(Boundedness). Let $x_i = (T_i, I_i, F_i)(i = 1, 2, \cdots, n)$ be a collection of SVNNs, $W = (w_1, w_2, \cdots, w_n)^T$, satisfying $w_i \in [0, 1]$ And $\sum\limits_{i}^{n} w_i = 1$ and $x^- = (min_i\{T_i\}, max_i\{I_i\}, max_i\{F_i\})$, $x^+ = (max_i\{T_i\}, min_i\{I_i\}, min_i\{F_i\})$ then*

$$x^- \leq SVNWGPHA^{p,q}(x_1, x_2, \cdots, x_n) \leq x^+$$

4. MAGDM Method Based on the SVNWPHA or SVNWGPHA Operator

In this part, we introduce the application of the SVNWPHA or SVNWGPHA operator in MAGDM. Given the MAGDM problems based on SVNNs, let $E = \{E_1, E_2, \cdots, E_m\}$ be the set of alternatives and $G = \{G_1, G_2, \cdots, G_n\}$ be the set of attributes respectively. The is the weight of the attributes $G_j(j = 1, 2, \cdots, n)$, where $0 \leq w_j \leq 1(j = 1, 2, \cdots, n)$ and $\sum_{k=1}^{t} \gamma_k = 1$. Suppose that the set of decision makers is $D = \{D_1, D_2, \cdots, D_t\}$ and $\gamma_k(k = 1, 2, \cdots, t)$ represents a weight of decision maker D_k with $0 \leq \gamma_k \leq 1(k = 1, 2, \cdots, t)$, $\sum_{k=1}^{t} \gamma_k = 1$. Suppose that $H^k = \left[\varphi_{ij}^k \right]_{m \times n}$ is the decision matrix, where

$\varphi_{ij}^k = \left(T_{ij}^k, I_{ij}^k, F_{ij}^k\right)$ takes the form of SVNN, $T_{ij}^k, I_{ij}^k, F_{ij}^k \in [0,1]$, and $0 \le T_{ij}^k + I_{ij}^k + F_{ij}^k \le 3$, which describes the decision-making information of the attributes G_j in terms of the alternative E_i provided by the decision maker D_k. Accordingly, the rank of the alternatives based on the decision information given by decision makers could be attained.

The method includes the following steps:

Step 1: Calculate the supports $Sup(\varphi_{ij}^k, \varphi_{ij}^h)(k, h = 1, 2, 3, \cdots, t)$ by

$$Sup(\varphi_{ij}^k, \varphi_{ij}^h) = 1 - d(\varphi_{ij}^k, \varphi_{ij}^h) \tag{43}$$

where $d(\varphi_{ij}^k, \varphi_{ij}^h)$ is the normalized Euclidian distance between two SVNNs is φ_{ij}^k and φ_{ij}^h, which is given in Definition 5.

Step 2: Calculate

$$T(\varphi_{ij}^k) = \sum_{j=1, j \neq i}^{n} Sup(\varphi_{ij}^k, \varphi_{ij}^h)(k, h = 1, 2, 3, \cdots, t) \tag{44}$$

Step 3: Calculate the weight vector $\widetilde{w}_i^k$ of power operator associated with the SVNNs φ_{ij}^k

$$\widetilde{w}_{ij}^k = \frac{(1 + T(\varphi_{ij}^k))}{\sum_{k=1}^{t}(1 + T(\varphi_{ij}^k))}(k = 1, 2, \cdots, t) \tag{45}$$

Step 4: Aggregate and fuse the decision information given by each decision maker $D_k(k = 1, 2, \cdots, t)$ by

$$\overline{\varphi}_{ij} = \left(T_{ij}, I_{ij}, F_{ij}\right) = SVNWPHA^{p,q}\left(\varphi_{ij}^1, \varphi_{ij}^2, \cdots, \varphi_{ij}^t\right) \tag{46}$$

$$\text{Or} \overline{\varphi}_{ij} = \left(T_{ij}, I_{ij}, F_{ij}\right) = SVNWGPHA^{p,q}\left(\varphi_{ij}^1, \varphi_{ij}^2, \cdots, \varphi_{ij}^t\right) \tag{47}$$

in order to get the collective decision matrix $H = \left[\overline{\varphi}_{ij}\right]_{m \times n}(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$.

Step 5: Calculate the supports $Sup(\varphi_{ik}, \varphi_{ij})(i = 1, 2, \cdots, m; h, j = 1, 2, \cdots, n)$ by

$$Sup(\varphi_{ik}, \varphi_{ij}) = 1 - d(\varphi_{ik}, \varphi_{ij}) \tag{48}$$

where $d(\varphi_{ik}, \varphi_{ij})$ is the normalized Euclidian distance between two SVNNs is φ_{ik} and φ_{ij}, which is given in Definition 5.

Step 6: Calculate

$$T(\widetilde{\varphi}_{ij}) = \sum_{k=1, k \neq j}^{n} Sup(\varphi_{ij}, \varphi_{ik})(k, j = 1, 2, 3, \cdots, n). \tag{49}$$

Step 7: Calculate the weight vector $\widetilde{w}_i$ of power operator associated with the SVNNs φ_{ij}^k

$$\widetilde{w}_{ij} = \frac{(1 + T(\varphi_{ij}))}{\sum_{j=1}^{n}(1 + T(\varphi_{ij}))}(j = 1, 2, \cdots, n) \tag{50}$$

Step 8: Compute the comprehensive value of each alternative by

$$\varphi_i = SVNWPHA^{p,q}(\overline{\varphi}_{i1}, \overline{\varphi}_{i2}, \cdots, \overline{\varphi}_{in}) \tag{51}$$

$$\text{or} \varphi_i = SVNWGPHA^{p,q}(\overline{\varphi}_{i1}, \overline{\varphi}_{i2}, \cdots, \overline{\varphi}_{in}) \tag{52}$$

where $(i = 1, 2, 3, \cdots, m)$

Step 9: Obtain the score $S(\varphi_i)$, accuracy $H(\varphi_i)$, and certainty $c(\varphi_i)$ functions based on Definition 4.

Step 10: Ranking $\varphi_i(i = 1, 2, \cdots, n)$ by the comparison method in Definition 5.

Step 11: End.

5. Illustrative Example

This section verified the application and effectiveness of the proposed method by using an example for the MAGDM problem.

Example 2. *Assuming that the air quality in Guangzhou needs to be assessed, and the air quality data in Guangzhou for 2006–2009 was collected as a series of alternatives that is* $\{E_1, E_2, E_3, E_4\} = \{2006, 2007, 2008, 2009\}$ *[32]. Accordingly, three attributes are considered: (1) $SO_2(G_1)$ (2) $NO_2(G_2)$ (3) $PH_{10}(G_3)$. Importance degree of the measured attributes is $W = (0.40, 0.20, 0.40)^T$. Three air quality monitoring stations assessed as experts are expressed by $\{D_1, D_2, D_3\}$ and the importance of these experts is $\gamma = (0.314, 0.355, 0.331)^T$, respectively. Suppose that $H^k = \left[\varphi_{ij}^k\right]_{m \times n}$ is the decision matrix, where $\varphi_{ij}^k = \left(T_{ij}^k, I_{ij}^k, F_{ij}^k\right)$ takes the form of SVNN, $T_{ij}^k, I_{ij}^k, F_{ij}^k \in [0, 1]$, and $0 \le T_{ij}^k + I_{ij}^k + F_{ij}^k \le 3$, which describes the decision-making information of the attributes G_j in terms of the alternative E_i provided by the decision maker D_k. Table 1, Table 2, and Table 3 show the three normalized standardized decision matrices from three experts by the SVNNs, respectively.*

Table 1. Air quality data from air quality monitoring station D_1.

	G_1	G_2	G_3
E_1	(0.265,0.350,0.385)	(0.330,0.390,0.280)	(0.245,0.275,0.480)
E_2	(0.345,0.245,0.410)	(0.430,0.290,0.280)	(0.245,0.275,0.480)
E_3	(0.365,0.300,0.335)	(0.480,0.315,0.205)	(0.340,0.370,0.290)
E_4	(0.430,0.300,0.270)	(0.460,0.245,0.295)	(0.310,0.520,0.170)

Table 2. Air quality data from air quality monitoring station D_2.

	G_1	G_2	G_3
E_1	(0.125, 0.470, 0.405)	(0.220, 0.420, 0.360)	(0.345, 0.490, 0.165)
E_2	(0.335, 0.310, 0.330)	(0.300, 0.370, 0.330)	(0.205, 0.630, 0.165)
E_3	(0.315, 0.380, 0.305)	(0.330, 0.565, 0.105)	(0.280, 0.520, 0.200)
E_4	(0.365, 0.365, 0.270)	(0.355, 0.320, 0.325)	(0.425, 0.485, 0.090)

Table 3. Air quality data from air quality monitoring station D_3.

	G_1	G_2	G_3
E_1	(0.260, 0.425, 0.315)	(0.220, 0.450, 0.330)	(0.255, 0.500, 0.245)
E_2	(0.270, 0.370, 0.360)	(0.320, 0.215, 0.465)	(0.135, 0.575, 0.290)
E_3	(0.245, 0.465, 0.290)	(0.250, 0.570, 0.180)	(0.175, 0.660, 0.165)
E_4	(0.390, 0.340, 0.270)	(0.305, 0.475, 0.220)	(0.465, 0.485, 0.050)

5.1. Decision-Making Steps

Use SVNWPHA operator to solve the problem involves the following steps:

Step 1: Calculate the supports $Sup(\varphi_{ij}^k, \varphi_{ij}^h)(k, h = 1, 2, 3, \cdots, t)$ by formulas (40) (for simplicity, we denote $Sup(\varphi_{ij}^k, \varphi_{ij}^h)$ with S_{ij}^{kh}), and we get

$$
\begin{aligned}
&S_{11}^{12} = S_{11}^{21} = 0.89292 \quad S_{11}^{13} = S_{11}^{31} = 0.94070 \quad S_{11}^{23} = S_{11}^{32} = 0.90279 \\
&S_{12}^{12} = S_{12}^{21} = 0.91958 \quad S_{12}^{13} = S_{12}^{31} = 0.92211 \quad S_{12}^{23} = S_{12}^{32} = 0.97551 \\
&S_{13}^{12} = S_{13}^{21} = 0.80315 \quad S_{13}^{13} = S_{13}^{31} = 0.93462 \quad S_{13}^{23} = S_{13}^{32} = 0.75818
\end{aligned}
$$

$$S_{21}^{12} = S_{21}^{21} = 0.94021 \quad S_{21}^{13} = S_{21}^{31} = 0.91102 \quad S_{21}^{23} = S_{21}^{32} = 0.94607$$
$$S_{22}^{12} = S_{22}^{21} = 0.90726 \quad S_{22}^{13} = S_{22}^{31} = 0.86841 \quad S_{22}^{23} = S_{22}^{32} = 0.88077$$
$$S_{23}^{12} = S_{23}^{21} = 0.80605 \quad S_{23}^{13} = S_{23}^{31} = 0.85834 \quad S_{23}^{23} = S_{23}^{32} = 0.91140$$
$$S_{31}^{12} = S_{31}^{21} = 0.94285 \quad S_{31}^{13} = S_{31}^{31} = 0.87938 \quad S_{31}^{23} = S_{31}^{32} = 0.93584$$
$$S_{32}^{12} = S_{32}^{21} = 0.82205 \quad S_{32}^{13} = S_{32}^{31} = 0.80121 \quad S_{32}^{23} = S_{32}^{32} = 0.93662$$
$$S_{33}^{12} = S_{33}^{21} = 0.89323 \quad S_{33}^{13} = S_{33}^{31} = 0.92897 \quad S_{33}^{23} = S_{33}^{32} = 0.89696$$
$$S_{41}^{12} = S_{41}^{21} = 0.94693 \quad S_{41}^{13} = S_{41}^{31} = 0.96734 \quad S_{41}^{23} = S_{41}^{32} = 0.97959$$
$$S_{42}^{12} = S_{42}^{21} = 0.92351 \quad S_{42}^{13} = S_{42}^{31} = 0.93946 \quad S_{42}^{23} = S_{42}^{32} = 0.88812$$
$$S_{43}^{12} = S_{43}^{21} = 0.91663 \quad S_{43}^{13} = S_{43}^{31} = 0.88504 \quad S_{43}^{23} = S_{43}^{32} = 0.96734$$

Step 2: Calculate the $T(\varphi_{ij}^k)$ by formulas (41), and we get

$$T(\varphi_{11}^1) = 1.83362 \quad T(\varphi_{11}^2) = 1.79571 \quad T(\varphi_{11}^3) = 1.84349$$
$$T(\varphi_{12}^1) = 1.84170 \quad T(\varphi_{12}^2) = 1.89509 \quad T(\varphi_{12}^3) = 1.89762$$
$$T(\varphi_{13}^1) = 1.73777 \quad T(\varphi_{13}^2) = 1.55367 \quad T(\varphi_{13}^3) = 1.68543$$
$$T(\varphi_{21}^1) = 1.85123 \quad T(\varphi_{21}^2) = 1.88628 \quad T(\varphi_{21}^3) = 1.85710$$
$$T(\varphi_{22}^1) = 1.77567 \quad T(\varphi_{22}^2) = 1.78803 \quad T(\varphi_{22}^3) = 1.74917$$
$$T(\varphi_{23}^1) = 1.66439 \quad T(\varphi_{23}^2) = 1.71745 \quad T(\varphi_{23}^3) = 1.76974$$
$$T(\varphi_{31}^1) = 1.82222 \quad T(\varphi_{31}^2) = 1.87868 \quad T(\varphi_{31}^3) = 1.81522$$
$$T(\varphi_{32}^1) = 1.62326 \quad T(\varphi_{32}^2) = 1.75867 \quad T(\varphi_{32}^3) = 1.73783$$
$$T(\varphi_{33}^1) = 1.68752 \quad T(\varphi_{33}^2) = 1.79019 \quad T(\varphi_{33}^3) = 1.69125$$
$$T(\varphi_{41}^1) = 1.91427 \quad T(\varphi_{41}^2) = 1.92652 \quad T(\varphi_{41}^3) = 1.94693$$
$$T(\varphi_{42}^1) = 1.75763 \quad T(\varphi_{42}^2) = 1.81164 \quad T(\varphi_{42}^3) = 1.72224$$
$$T(\varphi_{43}^1) = 1.80167 \quad T(\varphi_{43}^2) = 1.88397 \quad T(\varphi_{43}^3) = 1.85238$$

Step 3: Calculate the weight vector $\widetilde{w}_{ij}^k$ by formulas (42), and we can have

$$\widetilde{w}_{11}^1 = 0.33444 \quad \widetilde{w}_{11}^2 = 0.32996 \quad \widetilde{w}_{11}^3 = 0.33560$$
$$\widetilde{w}_{12}^1 = 0.32911 \quad \widetilde{w}_{12}^2 = 0.33530 \quad \widetilde{w}_{12}^3 = 0.33559$$
$$\widetilde{w}_{13}^1 = 0.32187 \quad \widetilde{w}_{13}^2 = 0.33659 \quad \widetilde{w}_{13}^3 = 0.34154$$
$$\widetilde{w}_{21}^1 = 0.33222 \quad \widetilde{w}_{21}^2 = 0.33548 \quad \widetilde{w}_{21}^3 = 0.33230$$
$$\widetilde{w}_{22}^1 = 0.33390 \quad \widetilde{w}_{22}^2 = 0.33539 \quad \widetilde{w}_{22}^3 = 0.33071$$
$$\widetilde{w}_{23}^1 = 0.32686 \quad \widetilde{w}_{23}^2 = 0.33336 \quad \widetilde{w}_{23}^3 = 0.33978$$
$$\widetilde{w}_{31}^1 = 0.33140 \quad \widetilde{w}_{31}^2 = 0.33803 \quad \widetilde{w}_{31}^3 = 0.33057$$
$$\widetilde{w}_{32}^1 = 0.32307 \quad \widetilde{w}_{32}^2 = 0.33975 \quad \widetilde{w}_{32}^3 = 0.33718$$
$$\widetilde{w}_{33}^1 = 0.32899 \quad \widetilde{w}_{33}^2 = 0.34156 \quad \widetilde{w}_{33}^3 = 0.32945$$
$$\widetilde{w}_{41}^1 = 0.33163 \quad \widetilde{w}_{41}^2 = 0.33302 \quad \widetilde{w}_{41}^3 = 0.33535$$
$$\widetilde{w}_{42}^1 = 0.33259 \quad \widetilde{w}_{42}^2 = 0.33910 \quad \widetilde{w}_{42}^3 = 0.32832$$
$$\widetilde{w}_{43}^1 = 0.32814 \quad \widetilde{w}_{43}^2 = 0.33778 \quad \widetilde{w}_{43}^3 = 0.33408$$

Step 4: Aggregate and fuse the decision information given by each decision maker $D_k(k = 1, 2, \cdots, t)$ by $\overline{\varphi}_{ij} = (T_{ij}, I_{ij}, F_{ij}) = SVNWPHA^{p,q}\left(\varphi_{ij}^1, \varphi_{ij}^2, \cdots, \varphi_{ij}^t\right)$ and get the collective decision matrix $H = \left[\overline{\varphi}_{ij}\right]_{m \times n}(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$.

$$H = \begin{bmatrix} (0.21711, 0.41468, 0, 36736) & (0.25588, 0.42022, 0.32522) & (0.28543, 0.41789, 0.27328) \\ (0.32463, 0.30823, 0.36485) & (0.34967, 0.28874, 0.35349) & (0.19520, 0.52345, 0.26649) \\ (0.30911, 0.37891, 0.30961) & (0.35477, 0.47827, 0.15918) & (0.26718, 0.50949, 0.21400) \\ (0.39419, 0.33533, 0.27024) & (0.37427, 0.33893, 0.27872) & (0.40485, 0.49598, 0.09536) \end{bmatrix}$$

Step 5: Calculate the supports $Sup(\varphi_{ik}, \varphi_{ij})(i = 1, 2, \cdots, m; h, j = 1, 2, \cdots, n)$ by formulas (45) (for simplicity, we denote $Sup(\varphi_{ik}, \varphi_{ij})$ with $S_{ik,ij}$), and we get

$$S_{11,12} = S_{12,11} = 0.96679 \quad S_{11,13} = S_{13,11} = 0.93285 \quad S_{12,13} = S_{13,12} = 0.96547$$
$$S_{21,22} = S_{22,21} = 0.98054 \quad S_{21,23} = S_{23,21} = 0.84428 \quad S_{22,23} = S_{23,22} = 0.83018$$

$S_{31,32} = S_{32,31} = 0.89263 \quad S_{31,33} = S_{33,31} = 0.90348 \quad S_{32,33} = S_{33,32} = 0.93768$

$S_{41,42} = S_{42,41} = 0.98733 \quad S_{41,43} = S_{43,41} = 0.86276 \quad S_{42,43} = S_{43,42} = 0.85950$

Step 6: Calculate the $T(\widetilde{\varphi}_{ij})$ by formulas (46), and we can have

$T(\widetilde{\varphi}_{11}) = 1.89963 \quad T(\widetilde{\varphi}_{12}) = 1.93226 \quad T(\widetilde{\varphi}_{13}) = 1.89832$

$T(\widetilde{\varphi}_{21}) = 1.82482 \quad T(\widetilde{\varphi}_{22}) = 1.81072 \quad T(\widetilde{\varphi}_{23}) = 1.67446$

$T(\widetilde{\varphi}_{31}) = 1.79610 \quad T(\widetilde{\varphi}_{32}) = 1.83031 \quad T(\widetilde{\varphi}_{33}) = 1.84115$

$T(\widetilde{\varphi}_{41}) = 1.85009 \quad T(\widetilde{\varphi}_{42}) = 1.84683 \quad T(\widetilde{\varphi}_{43}) = 1.72226$

Step 7: Calculate the weight vector $\widetilde{w}_{ij}$ by formulas (47), and we can get

$\widetilde{w}_{11} = 0.33200 \quad \widetilde{w}_{12} = 0.33575 \quad \widetilde{w}_{13} = 0.33225$

$\widetilde{w}_{21} = 0.33991 \quad \widetilde{w}_{22} = 0.33807 \quad \widetilde{w}_{23} = 0.32202$

$\widetilde{w}_{31} = 0.33036 \quad \widetilde{w}_{32} = 0.33413 \quad \widetilde{w}_{33} = 0.33551$

$\widetilde{w}_{41} = 0.33851 \quad \widetilde{w}_{42} = 0.33816 \quad \widetilde{w}_{43} = 0.32333$

Step 8: Compute the comprehensive value of each alternative by formulas (48), and we can have

$\varphi_1 = (0.25283, 0.42147, 0.32590),$

$\varphi_2 = (0.28233, 0.38086, 0.32904),$

$\varphi_3 = (0.30193, 0.45360, 0.23771),$

$\varphi_4 = (0.39140, 0.39712, 0.20151)$

$\varphi_4 = (0.39140, 0.39712, 0.20151).$

Step 9: We calculate the value $s(x_i)$ of $A_i(i = 1, 2, 3, 4)$

$s(\varphi_1) = 0.50182$

$s(\varphi_2) = 0.52414$

$s(\varphi_3) = 0.53687$

$s(\varphi_4) = 0.59759$

Step 10: Rank the alternatives.

We use the score function $s(\varphi_i)$ as basis to can rank the alternatives $\{E_1, E_2, E_3, E_4\}$ as follows: $E_4 > E_3 > E_2 > E_1$. Thus, the best alternative is E_4.

5.2. Sensitivity Analysis with Different Parameters.

Sensitivity analysis is an effective tool to verify the validity of decision-making methods [50,51]. Considering that different parameters may lead to different decision results. In this section, a sensitivity analysis is performed to see whether the parameters p and q have an effect on the ranking result. We assign the different values of p and q to steps (1) and (2) in Section 5.1 respectively. Tables 4 and 5 show the ranking results as following.

Tables 4 and 5 show that the ranking results have a few differences when we assign the different values of and. For example, regarding the ranking results by the proposed method based on the SVNWPHA operator, if $p = 0$ and $q = 1$, the ranking result is $E_4 > E_3 > E_1 > E_2$; if $p = 2$ and $q = 2$, the ranking result is $E_4 > E_3 > E_2 > E_1$; if $p = 10$ and $q = 10$ the ranking order is $E_4 > E_2 > E_3 > E_1$. Hence, the regularity of the ranking order is difficult to obtain under this situation. However, when the parameters are above 0 and below 10, the ranking result is relatively stable. Accordingly, the best is E_4.

In fact, the values of parameters p and q are considerably high, the calculation becomes substantially more complex, and the interactions between different attribute values become considerably prominent. In general, we recommend $p = q = 1$ for computational simplicity. This process is simple and straightforward and considers the interrelationship between input arguments. In addition, Tables 4 and 5 show that as the increase of p and q, the score functions based on the SVNWGPHA operator became smaller and smaller, while SVNWPHA operator is contrary, becoming greater and greater.

Table 4. Ranking results by the single-valued neutrosophic weighted power Heronian aggregation (SVNWPHA) operator under different values of p and q.

p,q	$s(x_i)(i=1,2,3,4)$	Rank
$p = q = \frac{1}{2}$	$s(\varphi_1) = 0.49506, s(\varphi_2) = 0.52439$ $s(\varphi_3) = 0.53074, s(\varphi_4) = 0.59130$	$E_4 \succ E_3 \succ E_2 \succ E_1$
$p = \frac{1}{2}, q = 0$	$s(\varphi_1) = 0.49015, s(\varphi_2) = 0.54500$ $s(\varphi_3) = 0.55601, s(\varphi_4) = 0.60136$	$E_4 \succ E_3 \succ E_2 \succ E_1$
$p = 0, q = \frac{1}{2}$	$s(\varphi_1) = 0.51363, s(\varphi_2) = 0.48627$ $s(\varphi_3) = 0.51719, s(\varphi_4) = 0.61253$	$E_4 \succ E_3 \succ E_1 \succ E_2$
$p = 1, q = 0$	$s(\varphi_1) = 0.49524, s(\varphi_2) = 0.54099$ $s(\varphi_3) = 0.55930, s(\varphi_4) = 0.60495$	$E_4 \succ E_3 \succ E_2 \succ E_1$
$p = 0, q = 1$	$s(\varphi_1) = 0.51856, s(\varphi_2) = 0.47906$ $s(\varphi_3) = 0.52142, s(\varphi_4) = 0.61686$	$E_4 \succ E_3 \succ E_1 \succ E_2$
$p = q = 1$	$s(\varphi_1) = 0.50181, s(\varphi_2) = 0.52414$ $s(\varphi_3) = 0.53687, s(\varphi_4) = 0.59759$	$E_4 \succ E_3 \succ E_2 \succ E_1$
$p = 2, q = 1$	$s(\varphi_1) = 0.50761, s(\varphi_2) = 0.51412$ $s(\varphi_3) = 0.55005, s(\varphi_4) = 0.60531$	$E_4 \succ E_3 \succ E_2 \succ E_1$
$p = 1, q = 2$	$s(\varphi_1) = 0.51402, s(\varphi_2) = 0.49559$ $s(\varphi_3) = 0.53910, s(\varphi_4) = 0.60724$	$E_4 \succ E_3 \succ E_2 \succ E_1$
$p = q = 2$	$s(\varphi_1) = 0.51657, s(\varphi_2) = 0.49761$ $s(\varphi_3) = 0.54928, s(\varphi_4) = 0.61044$	$E_4 \succ E_3 \succ E_2 \succ E_1$
$p = 1, q = 5$	$s(\varphi_1) = 0.54700, s(\varphi_2) = 0.45546$ $s(\varphi_3) = 0.55995, s(\varphi_4) = 0.63528$	$E_4 \succ E_3 \succ E_2 \succ E_1$
$p = 5, q = 1$	$s(\varphi_1) = 0.53347, s(\varphi_2) = 0.49032$ $s(\varphi_3) = 0.57562, s(\varphi_4) = 0.62919$	$E_4 \succ E_3 \succ E_2 \succ E_1$
$p = q = 5$	$s(\varphi_1) = 0.55657, s(\varphi_2) = 0.57806$ $s(\varphi_3) = 0.57887, s(\varphi_4) = 0.64230$	$E_4 \succ E_3 \succ E_2 \succ E_1$
$p = q = 10$	$s(\varphi_1) = 0.59847, s(\varphi_2) = 0.62051$ $s(\varphi_3) = 0.60646, s(\varphi_4) = 0.67307$	$E_4 \succ E_3 \succ E_2 \succ E_1$

Table 5. Ranking results by the SVNWGPHA operator under different values of the p and q

p,q	$s(x_i)(i=1,2,3,4)$	Rank
$p = q = \frac{1}{2}$	$1s(\varphi_1) = 0.50276, s(\varphi_2) = 0.51780$ $s(\varphi_3) = 0.53938, s(\varphi_4) = 0.59863$	$E_4 \succ E_3 \succ E_2 \succ E_1$
$p = \frac{1}{2}, q = 0$	$s(\varphi_1) = 0.49486, s(\varphi_2) = 0.54943$ $s(\varphi_3) = 0.56159, s(\varphi_4) = 0.59670$	$E_4 \succ E_3 \succ E_2 \succ E_1$
$p = 0, q = \frac{1}{2}$	$s(\varphi_1) = 0.50025, s(\varphi_2) = 0.50114$ $s(\varphi_3) = 0.50270, s(\varphi_4) = 0.60233$	$E_4 \succ E_3 \succ E_2 \succ E_1$
$p = 1, q = 0$	$s(\varphi_1) = 0.49240, s(\varphi_2) = 0.55310$ $s(\varphi_3) = 0.55714, s(\varphi_4) = 0.59381$	$E_4 \succ E_3 \succ E_2 \succ E_1$
$p = 0, q = 1$	$s(\varphi_1) = 0.49698, s(\varphi_2) = 0.50524$ $s(\varphi_3) = 0.49852, s(\varphi_4) = 0.59670$	$E_4 \succ E_2 \succ E_3 \succ E_1$
$p = q = 1$	$s(\varphi_1) = 0.49811, s(\varphi_2) = 0.52414$ $s(\varphi_3) = 0.53251, s(\varphi_4) = 0.59198$	$E_4 \succ E_3 \succ E_2 \succ E_1$
$p = 2, q = 1$	$s(\varphi_1) = 0.49175, s(\varphi_2) = 0.54053$ $s(\varphi_3) = 0.53188, s(\varphi_4) = 0.58625$	$E_4 \succ E_2 \succ E_3 \succ E_1$
$p = 1, q = 2$	$s(\varphi_1) = 0.49235, s(\varphi_2) = 0.52494$ $s(\varphi_3) = 0.\ 51525, s(\varphi_4) = 0.58324$	$E_4 \succ E_2 \succ E_3 \succ E_1$
$p = q = 2$	$s(\varphi_1) = 0.48828, s(\varphi_2) = 0.53810$ $s(\varphi_3) = 0.51795, s(\varphi_4) = 0.57959$	$E_4 \succ E_2 \succ E_3 \succ E_1$
$p = 1, q = 5$	$s(\varphi_1) = 0.47267, s(\varphi_2) = 0.45546$ $s(\varphi_3) = 0.48019, s(\varphi_4) = 0.56004$	$E_4 \succ E_3 \succ E_2 \succ E_1$
$p = 5, q = 1$	$s(\varphi_1) = 0.47392, s(\varphi_2) = 0.49032$ $s(\varphi_3) = 0.51153, s(\varphi_4) = 0.57101$	$E_4 \succ E_3 \succ E_2 \succ E_1$
$p = q = 5$	$s(\varphi_1) = 0.46401, s(\varphi_2) = 0.45504$ $s(\varphi_3) = 0.48164, s(\varphi_4) = 0.55471$	$E_4 \succ E_3 \succ E_1 \succ E_2$
$p = q = 10$	$s(\varphi_1) = 0.43931, s(\varphi_2) = 0.41810$ $s(\varphi_3) = 0.44617, s(\varphi_4) = 0.5331$	$E_4 \succ E_3 \succ E_1 \succ E_2$

5.3. Comparison with the Existing Methods

To verify the effectiveness and explain the advantage of the proposed method relating to the SVNWPHA and SVNWGPHA operators, we can plan to compare it with the method proposed by Li, Liu and Chen [33], which is based on the NNIGWHM operator; the method proposed by Yang and Li [34], which is based on the SVNPWA operator; and the method proposed by Ye [31], which is based on the SNSWAA operator. Table 6 shows the ranking results for these four methods.

Table 6. The ranking results produced by the different methods.

Method	Scorefunctions$(x_i)(i=1,2,3,4)$	Ranking
Method proposed by Li, Liu and Chen [33] $(p=q=1)$	$s(\varphi_1) = 0.417, s(\varphi_2) = 0.468$ $s(\varphi_3) = 0.496, s(\varphi_4) = 0.665$	$E_4 > E_3 > E_2 > E_1$
Method proposed by Yang and Li [34]	$s(\varphi_1) = 0.43769, s(\varphi_2) = 0.50444$ $s(\varphi_3) = 0.51456, s(\varphi_4) = 0.67933$	$E_4 > E_3 > E_2 > E_1$
Method proposed by Ye [31]	$s(\varphi_1) = 0.42237, s(\varphi_2) = 0.46408$ $s(\varphi_3) = 0.49441, s(\varphi_4) = 0.66230$	$E_4 > E_3 > E_2 > E_1$
The proposed method in this paper $(p=q=1)$		
SVNWPHA operator	$s(\varphi_1) = 0.50181, s(\varphi_2) = 0.52414$ $s(\varphi_3) = 0.53687, s(\varphi_4) = 0.59759$	$E_4 > E_3 > E_2 > E_1$
SVNWGPHA operator	$s(\varphi_1) = 0.49811, s(\varphi_2) = 0.52414$ $s(\varphi_3) = 0.53251, s(\varphi_4) = 0.59198$	$E_4 > E_3 > E_2 > E_1$

In Table 6, we found that the ranking results by these different methods are the same, and that this can evidently explain and demonstrate the effectiveness of the proposed methods.

In the following, we likewise compare the desirable properties of our proposed SVNWPHA and SVNWGPHA operator with the NNIGWHM operator proposed by Li, Liu and Chen [33], the SVNPWA operator proposed by Yang and Li [34], and the SNSWAA operator proposed by Ye [31] to determine the advantages of the proposed operator, which are listed in Table 7.

Table 7. A comparison of the properties for different operators.

Operators Properties	SNSWAA	SVNPWA	NNIGWHM	SVNWPHA SVNWGPHA
Consider the interrelationship of the aggregated arguments	No	No	Yes	Yes
Consider the suppose degree between the input arguments	No	Yes	No	Yes
parameters	No	No	Yes	Yes

Table 7 shows that only the proposed operators satisfy all of the properties. Our proposed operator and the NNIGWHM operator are related to the HM operators, which can consider the relationships between the aggregated arguments. Moreover, our proposed operator can also consider the support degree between aggregated arguments while NNIGWHM operator cannot consider it. To show this advantage of the proposed method, we give the following example.

Example 3. *We only change a little data from Example 2 in this example. After careful observation, we can find that there are lesser differences among the attribute values of alternation A_3 given by the air quality monitoring station D_1, which are (0.365,0.300,0.335), (0.480,0.315,0.205), (0.340,0.370,0.290), the only difference between the old value and the new value is that we change the attribute value φ_{31}^1 from (0.365,0.300,0.335) to a small value (0.005,0.300,0.335), then we can observe the changes of ranking results for the method proposed by Li, Liu and Chen [33] and the proposed method with SVNWPHA operator. The ranking results are shown in Table 8.*

Table 8. The ranking results compared with Li, Liu and Chen [33] method ($p = q = 1$).

Method	Scorefunctions(x_i)(i=1,2,3,4)	Ranking
Method proposed by Li, Liu and Chen [33]	$s(\varphi_1) = 0.417, s(\varphi_2) = 0.468$ $s(\varphi_3) = 0.456, s(\varphi_4) = 0.665$	$E_4 \succ E_2 \succ E_3 \succ E_1$
The proposed method with SVNWPHA operator	$s(\varphi_1) = 0.50181, s(\varphi_2) = 0.52414$ $s(\varphi_3) = 0.53540, s(\varphi_4) = 0.59759$	$E_4 \succ E_3 \succ E_2 \succ E_1$

From Table 8, we can find that the ranking results of the proposed method with SVNWPHA operator are different from the method proposed by Li, Liu and Chen [33] when changing the attribute value to a small value (0.005, 0.300, 0.335). The reason for resulting in this condition is that the proposed method considers the support degree between aggregated arguments and it can reduce the influence of unreasonable data. So, in this example, although the score function of alternation A_3 for two methods are becoming smaller, the ranking result by the proposed method is kept while it is changed by Li, Liu and Chen [33]'s method. This can verify the advantage of the proposed method, which can relieve the influence of the too big or too small data. It also can explain the result of the proposed method in this paper being more reasonable than the method of Li, Liu and Chen [40].

In addition, our proposed operator and the SVNPWA operator are based on the PA operators, which can alleviate the effects caused by unreasonable data through considering the support degree between input arguments. However, our proposed operator also considers the interrelationship of aggregated arguments while the SVNPWA operator cannot consider it. That is, the advantages of our proposed operator are that its combination with the PA and HM operators can acquire the advantages of the NNIGWHM and SVNPWA operators. The proposed method can consider the interrelationship of the aggregated arguments and consider the support degree between input arguments, which alleviates the effects caused by the unreasonable data. Furthermore, the operator has two parameters, thereby rendering it extremely flexible in the process of information aggregation. However, any type of operator has advantages and disadvantages because of the simultaneous consideration of the PA and HM operators. The proposed operator calculation is slightly more complicated than the other three.

6. Conclusions

Owing to the limitations of human thinking and the complexity of decision-making problems, the decision information involved in MAGDM problems is often incomplete, indeterminate, and inconsistent. SVNNs can easily and considerably depict this type of information. The power Heronianaggregation (PHA) operator can take the advantages of power average and HM operator. Based on the PHA and SVNNs, this study proposes the SVNPHA operator, SVNWPHA operator, SVNGPHA operator and SVNWGPHA operator. Then, we likewise investigated their properties in detail. Furthermore, we used the SVNWPHA and SVNWGPHA operators as bases to develop a method for MAGDM under the environment where the information is expressed by SVNNs. Lastly, we verified the application and effectiveness of the proposed method by using an example for the air quality evaluation. The main advantages of this study are as follows. Our method could consider the interrelationship of the aggregated arguments and alleviate the effects caused by the unreasonable data through taking into account the support degree between input arguments. Moreover, decision makers can choose the SVNWPHA or SVNWGPHA operators and changing the values of p and q in accordance with their subject preference and practical need. This condition renders the proposed method substantially flexible and reliable. However, because of the simultaneous consideration of the PA and HM operators, the proposed operator calculation is slightly complicated. In future research, other problems such as medical diagnosis and pattern recognition will be handled with these operators.

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