Smarandache Function Journal, Vol. 4-5, No. 1-2-3, 1994, pp. 3-7.

SOME REMARKS ON THE SMARANDACHE FUNCTION

by

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1. **On the method of calculus proposed by Florentin Smarandache.** In [6] is defined a numerical function $S: N^* \to N$, as follows:
S(n) is the smallest nonnegative integer such that S(n)! is divisible by n.

For example $S(1) = 0$, $S(2^{12}) = 16$.

This function characterizes the prime numbers in the sense that $p > 4$ is prime if and only if $S(p) = p$. As it is showed in [6] this function may be extended to all integers by defining $S(-n) = S(n)$. If a and b are relatively prime then $S(a \cdot b) = \max\{S(a), S(b)\}$ More general, if $[a, b]$ is the last common multiple of a and b then

$$S([a,b]) = \max\{S(a), S(b)\} \tag{1}$$

So, if $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_t^{\alpha_t}$ is the factorization of n into primes, then

$$S(n) = \max\{S(p_i^{\alpha_i}) \mid i = 1, \ldots, t\} \tag{2}$$

For the calculus of $S(p_i^{\alpha_i})$ in [6] it is used the fact that if $a = (p^n - 1)/(p - 1)$ then $S(p^a) = p^n$.
This equality results from the fact, if $\alpha_p(n)$ is the exponent of the prime p in the decomposition of n! into primes then

$$\alpha_p(n) = \sum_{i \geq 1} \left[\frac{n}{p^i} \right] \tag{3}$$

From (3) is results that $S(p^a) \leq p \cdot a$.

Now, if we note $a_n(p) = (p^n - 1)/(p - 1)$ then

$$S\left(p^{k_{m_1} a_{m_1}(p) + k_{m_2} a_{m_2}(p) + \ldots + k_{m_l} a_{m_l}(p)}\right) = k_{m_1} p^{m_1} + k_{m_2} p^{m_2} + \ldots + k_{m_l} p^{m_l} \tag{4}$$

for $k_{m_1}, k_{m_2}, \ldots, k_{m_{l-1}} \in \overline{1, p-1}$ and $k_{m_l} \in \{1, 2, \ldots, p\}$.

That is, if we consider the generalized scale

$$[p] : a_1(p), a_2(p), \ldots, a_n(p), \ldots$$

and the standard scale

$$(p) : 1 , p , p^2 , \ldots , p^n , \ldots$$

and we express the exponent a in the scale $[p]$, $a_{[p]} = \overline{k_{m_1} k_{m_2} \ldots k_{m_l}}$, then the left hand of the equality (4) is $S\left(p^{a_{[p]}}\right)$ and the right hand becomes $p\left(a_{[p]}\right)_{(p)}$. In other words, the right hand of (4) is the number obtained multiplying by p the exponent a writed in the scale $[p]$, readed it in the scale (p) .

So, (4) may be written as

$$S\left(p^{a_{[p]}}\right) = p\left(a_{[p]}\right)_{(p)} \qquad (5)$$

For example, to calculate $S(3^{89})$ we write the exponent a=89 in the scale

$$[3] : 1, 4 , 13 , 40 , 121 , \ldots$$

and so

$$a_{m_1}(p) \le a \Leftrightarrow (p^{m_1} - 1)/(p-1) \le a \Leftrightarrow p^{m_1} \le (p-1)\cdot a + 1 \Leftrightarrow m_1 \le \log_p((p-1)\cdot a + 1).$$

It results that m_1 is the integer part of $\log_p((p-1)\cdot a + 1)$.

For our example $m_1 = [\log_3(2a+1)] = \log_3 179 = 4$. Then first digit of $a_{[3]}$ is $k_4 = [a/a_4(3)] = 2$. So, $89 = 2a_4(3) + 9$.

For $a_1 = 9$ it results $m_2 = [\log_3(2a_1 + 1)] = 2$, $k_2 = [a_1/a_2(3)] = 2$ and so $a_1 = 2a_2(3) + 1$.

Then $89 = 2a_4(3) + 2a_2(3) + a_1(3) = 2021_{[3]}$, and $S(3^{89}) = 3(2021)_{(3)} = 183$.

Indeed, $\sum_{i\ge1}\left[\frac{183}{3^i}\right] = 61 + 20 + 6 + 2 = 89$.

Let us observe that the calculus in the generalized scale $[p]$ is essentially different from the calculus in the standard scale (p) . That because if we note $b_n(p) = p^n$ then it results

$$a_{n+1}(p) = pa_n(p) + 1 \quad \text{and} \quad a_{n+1}(p) = pa_n(p) + 1 \qquad (6)$$

For this, to add some numbers in the scale $[p]$ we do as follows. We start to add from the digits of "decimals", that is from the column of $a_2(p)$. If adding some digits it is obtained $pa_2(p)$ then it is utilized a unit from the class of units (coefficients of $a_1(p)$) to obtain $pa_2(p) + 1 = a_3(p)$. Continuing to add, if agains it is obtained $pa_2(p)$, then a new unit must be used, from the class of units, etc.

For example if $m_{[5]} = 442$, $n_{[5]} = 412$ and $r_{[5]} = 44$ then

$$m+n+r = 442 + \\ 412 \\ \underline{44} \\ \text{dcba}$$

We start to add from the column corresponding to $a_2(5)$:

$$4a_2(5) + a_2(5) + 4a_2(5) = 5a_2(5) + 4a_2(5) .$$

Now utilizing a unit from the first column we obtain

$$5a_2(5) + 4a_2(5) = a_3(5) + 4a_2(5) \text{ , so } b = 4 .$$

Continuing, $4a_3(5) + 4a_3(5) + a_3(5) = 5a_3(5) + 4a_3(5)$ and using a new unit it results $4a_3(5) + 4a_3(5) + a_3(5) = a_4(5) + 4a_3(5)$, so $c = 4$ and $d = 1$. Finally, adding the remained units $4a_1(5) + 2a_1(5) = 5a_1(5) + a_1(5) = 5a_1(5) + 1 = a_2(5)$ it results that b must be modified and $a = 0$. So $m+n+r = 1450_5$.

We have applied the formula (5) to the calculus of the values of S for any integer between $N_1 = 31,000,000$ and $N_2 = 31,001,000$. A program has been designed to generate the factorization of every integer $n \in [N_1, N_2]$ (TIME (minutes) : START : 40:8:93, STOP : 56:38:85, more than 16 minutes) .

Afterwards, the Smarandache function has been calculated for every $n = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_t^{a_t}$ as follows :

1) max $p_i \cdot a_i$ is determined
2) $S_0 = S(p_i^{a_i})$, for i determined above
3) Because $S(p_j^{a_j}) \le p_j \cdot a_j$, we ignore the factors for which $p_j \cdot a_j \le S_0$.
4) Are calculated $S(p_j^{a_j})$ for $p_j \cdot a_j > S_0$ and is determined the greatest of these values.

(TIME (minutes): START: 25:52:75, STOP: 25:55:27, leas than 3 seconds)

2. Some diofantine equations concerning the function S.

In this section we shall apply the formula (5) for the study of the solutions of some diofantine equations proposed in (6).

a) Using (5) it can be proved that the diofantine equation

$$S(x \cdot y) = S(x) + S(y) \tag{7}$$

has infinitely many solutions. Indeed, let us observe that from (2) every relatively prime integers x_0 and y_0 can' t be a solution from (7). Let now $x = p^a \cdot A$, $y = p^b \cdot B$ be such that $S(x) = S(p^a)$ and $S(y) = S(p^b)$.

Then $S(x \cdot y) = S(p^{a+b})$ and (7) becomes

$$p\left((a+b)_{[p]}\right)_{(p)} = p\left(a_{[p]}\right)_{(p)} + p\left(b_{[p]}\right)_{(p)}$$

or

$$\left((a+b)_{[p]}\right)_{(p)} = \left(a_{[p]}\right)_{(p)} + \left(b_{[p]}\right)_{(p)} \qquad (8)$$

There exists infinitely many values for a and b satisfying yhis equality. For example $a = a_3(p) = 100_{[p]}$, $b = a_2(p) = 10_{[p]}$ and (8) becomes $\left(110_{[p]}\right)_{(p)} = \left(100_{[p]}\right)_{(p)} + \left(10_{[p]}\right)_{(p)}$.

b) We shall prove now that the equation

$$S(x\cdot y) = S(x)\cdot S(y)$$

has no solution $x, y > 1$.

Let $m = S(x)$ and $n = S(y)$. It is sufficient to prove that $S(x\cdot y) = m\cdot n$. But it is said that $m!\cdot n!$ divide $(m+n)!$, so

$$(m\cdot n)! \,\vdots\, (m+n)! \,\vdots\, m!\cdot n! \,\vdots\, x\cdot y$$

c) If we note by (x, y) the greatest common divisor of x and y, then the ecuation

$$(x,y) = (S(x), S(y)) \qquad \textbf{(9)}$$

has infinitelly many solutions. Indeed, because $x \geq S(x)$, the equality holding if and only if x is a prime it results that (9) has as solution every pair x, y of prime numbers and also every pair of product of prime numbers.

Let now $S(x) = p\left(a_{[p]}\right)_{(p)}$, $S(y) = q\left(b_{[q]}\right)_{(q)}$ be such that $(x, y) = d > 1$. Then because $(p.q) = 1$, if

$$a_1 = \left(a_{[p]}\right)_{[p]}, \quad b_1 = \left(b_{[p]}\right)_{[p]} \quad \text{and} \quad (p, b_1) = (a_1, q) = 1.$$

it result that the equality (9) becomes

$$\left(\left(a_{[p]}\right)_{(p)}, \left(b_{[q]}\right)_{(q)}\right) = d$$

and it is satisfied for various positive integers a and b. For example if $x = 2\cdot 3^a$ and $y = 2\cdot 5^b$ it results $d = 2$ and the equality $\left(\left(a_{[3]}\right)_{(3)}, \left(b_{[5]}\right)_{(5)}\right) = 2$ is satisfied for many values of $a, b \in N$.

d) If $[x, y]$ is the least common multiple of x and y then the equation

$$[x,y] = [S(x), S(y)] \qquad (10)$$

has as solutions every pair of prime numbers. Now, if x and y are composite numbers such that $S(x) = S\left(p_i^{a_i}\right)$ and $S(y) = S\left(p_j^{a_j}\right)$ with $p_i \neq p_j$ then the pair x, y can't be solution of the equation because in this case we have

$$[x,y] > p_i^{a_i}\cdot p_j^{a_j} > S(x)\cdot S(y) \geq S(x), S(y)$$

and if $x = p^a \cdot A$ and $y = p^b \cdot B$ with $S(x) = S\left(p^a\right)$, $S(y) = S\left(p^b\right)$ then

$$[S(x), S(y)] = \left[p\left(a_{[p]}\right)_{(p)}, p\left(b_{[p]}\right)_{(p)}\right] = p\cdot\left(a_{[p]}\right)_{(p)}, \left(b_{[p]}\right)_{(p)}$$

and $[x,y] = p^{\min(a,b)} \cdot [A,B]$ so (10) is satified also for many values of non relatively prime integers.

e) Finaly we consider the equation

$$S(x) + y = x + S(y)$$

which has as solution every pair of prime numbers, but also the composit numbers $x = y$. It can be found other composit number as solutions. For example if p and q are consecutive prime numbers such that

$$q - p = h > 0 \qquad (11)$$

and $x = p \cdot A$, $y = q \cdot B$ then our equatic is equivalent to

$$y - x = S(y) - S(x) \qquad (12)$$

If we consider the diofantine equation $qB - pA = h$ it results from (11) that $A_0 = B_0 = 1$ is a particular solution, so the general solution is $A = 1 + rq$, $B = 1 + rp$,for arbitrary integer r. Then for $r = 1$ it results $x = p(1+q)$, $y = q(1+p)$ and $y - x = h$. In addition, because p and q are consecutive primes it results that $p+1$ and $q+1$ are composite and so

$$S(x) = p \quad , \quad S(y) = q \quad , \quad S(y) - S(x) = h$$

and (12) holds.

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