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Smarandache groupoids

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Abstract

In this paper, we study the concept of Smarandache groupoids, subgroupoids, ideal of groupoids, semi-normal subgroupoids, Smarandache-Bol groupoids and Strong Bol groupoids and obtain many interesting results about them.

Keywords Smarandache groupoid; Smarandache subgroupoid; Smarandache ideal of a Smarandache groupoid; Smarandache semi-normal groupoid; Smarandache normal groupoid; Smarandache semi conjugate subgroupoid; Smarandache Bol groupoid; Smarandache Moufang groupoid.

Definition [1]: A groupoid $(G, *)$ is a non-empty set, closed with respect to an operation $*$ (in general $*$ need not to be associative).

Definition 1: A *Smarandache groupoid* G is a groupoid which has a proper subset $S \subset G$ which is a semigroup under the operation of G.

Example 1: Let $(G, *)$ be a groupoid on modulo 6 integers. $G = \{0, 1, 2, 3, 4, 5\}$ is given by the following table:

$*$	0	1	2	3	4	5
0	0	3	0	3	0	3
1	1	4	1	4	1	4
2	2	5	2	5	2	5
3	3	0	3	0	3	0
4	4	1	4	1	4	1
5	5	2	5	2	5	2

Clearly $S_1 = \{0, 3\}, S_2 = \{1, 4\}$ and $S_3 = \{2, 5\}$ are semigroups of G. So $(G, *)$ is a Smarandache groupoid.

Example 2: Let $G = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the set of integers modulo 10. Define an operation $*$ on G by choosing a pair $(1, 5)$ such that $a * b = 1a + 5b \pmod{10}$ for all $a, b \in G$.

The groupoid is given by the following table.

$*$	0	1	2	3	4	5	6	7	8	9
0	0	5	0	5	0	5	0	5	0	5
1	1	6	1	6	1	6	1	6	1	6
2	2	7	2	7	2	7	2	7	2	7
3	3	8	3	8	3	8	3	8	3	8
4	4	9	4	9	4	9	4	9	4	9
5	5	0	5	0	5	0	5	0	5	0
6	6	1	6	1	6	1	6	1	6	1
7	7	2	7	2	7	2	7	2	7	2
8	8	3	8	3	8	3	8	3	8	3
9	9	4	9	4	9	4	9	4	9	4

Clearly $S_1 = \{0, 5\}$, $S_2 = \{1, 6\}$, $S_3 = \{2, 7\}$, $S_4 = \{3, 8\}$ and $S_5 = \{4, 9\}$ are semigroupoids under the operation $*$. Thus $\{G, *, (1, 5)\}$ is a Smarandache groupoid.

Theorem 2. Let $Z_{2p} = \{0, 1, 2, \cdots, 2p-1\}$. Define $*$ on Z_{2p} for $a, b \in Z_{2p}$ by $a*b = 1a+pb($ mod $2p)$. $\{Z_{2p}, *, (1, p)\}$ is a Smarandache groupoid.

Proof. Under the operation $*$ defined on Z_{2p} we see $S_1 = \{0, p\}$, $S_2 = \{1, p + 1\}$, $S_3 = \{2, p + 2\}$,$\cdots$,$S_p = \{p - 1, 2p - 1\}$ are semigroup under the operation $*$. Hence $\{Z_{2p}, *, (1, p)\}$ is a Smarandache groupoid.

Example 3: Take $Z_6 = \{0, 1, 2, 3, 4, 5\}$. $(2, 5) = (m, n)$. For $a, b \in Z_6$ define $a * b = ma + nb(\text{mod } 6)$. The groupoid is given by the following table:

$*$	0	1	2	3	4	5
0	0	5	4	3	2	1
1	2	1	0	5	4	3
2	4	3	2	1	0	5
3	0	5	4	3	2	1
4	2	1	0	5	4	3
5	4	3	2	1	0	5

Every singleton is an idempotent semigroup of Z_6.

Theorem 3. Let $Z_{2p} = \{0, 1, 2, \cdots, p-1\}$. Define $*$ on Z_{2p} by $a * b = 2a + (2p-1)b(\text{mod } 2p)$ for $a, b \in Z_{2p}$. Then $\{Z_{2p}, *, (2, 2p - 1)\}$ is a Smarandache groupoid.

Proof. Under the operation $*$ defined on Z_{2p} we see that every element of Z_{2p} is idempotent, therefore every element forms a singleton semigroup. Hence the claim.

Example 4: Consider $Z_6 = \{Z_6, *, (4, 5)\}$ given by the following table:

*	0	1	2	3	4	5
0	0	5	4	3	2	1
1	4	3	2	1	0	5
2	2	1	0	5	4	3
3	0	5	4	3	2	1
4	4	3	2	1	0	5
5	2	1	0	5	4	3

$\{3\}$ is a semigroup. Hence $*$ is a Smarandache groupoid. It is easily verified that Z_6 is a Smarandache groupoid as $\{Z_6, *, (4,5)\}$ has an idempotent semigroup $\{3\}$ under $*$.

Theorem 4. Let $Z_{2p} = \{0, 1, 2, \cdots, 2p-1\}$ be the set of integers modulo $2p$. Define $*$ on $a, b \in Z_{2p}$ by $a(2p-2) + b(2p-1)(\text{mod } 2p)$. Then $\{Z_{2p}, *, (2p-2, 2p-1)\}$ is a Smarandache groupoid.

Proof. $Z_{2p} = \{0, 1, 2, \cdots, 2p-1\}$. Take $(2p-2, 2p-1) = 1$ from Z_{2p}. For $a, b \in Z_p$ define $a*b = a(2p-2)+b(2p-1)(\text{mod } 2p)$. Clearly for $a = b = p$ we have $(2p-2)p+(2p-1)p = p(\text{mod } 2p)$. Hence $\{p\}$ is an idempotent semigroup of Z_{2p}. So $\{Z_{2p}, *, (2p-2, 2p-1)\}$ is a Smarandache groupoid.

Definition 5: Let $(G, *)$ be a Smarandache groupoid. A non-empty subset H of G is said to be a Smarandache groupoid if H contains a proper subset $K \subset H$ such that K is a semigroup under the operation $*$.

Theorem 6. Not every subgroupoid of a Smarandache groupoid S is in general a Smarandache subgroupoid of S.

Proof. By an example.

Let $Z_6 = \{0, 1, 2, 3, 4, 5\}(\text{mod } 6)$. Take $(t, u) = (4, 5) = 1$. For $a, b \in Z_6$ define $*$ on Z_6 by $a * b = at + bu(\text{mod } 6)$ given by the following table:

*	0	1	2	3	4	5
0	0	5	4	3	2	1
1	4	3	2	1	0	5
2	2	1	0	5	4	3
3	0	5	4	3	2	1
4	4	3	2	1	0	5
5	2	1	0	5	4	3

Clearly $\{Z_6, *, (4,5)\}$ is a Smarandache groupoid for it contains $\{0, 3\}$ as a semigroup. But this groupoid has the following subgroupoids: $A_1 = \{0, 2, 4\}$ and $A_2 = \{1, 3, 5\}$. A_1 has no non-trivial semigroup($\{0\}$ is a trivial semigroup). But A_2 has a non-trivial semigroup, viz.$\{3\}$. Hence the claim.

Theorem 7. If a groupoid contains a Smarandache subgroupoid, then the groupoid is a Smarandache groupoid.

Proof. Let G be a groupoid and $H \subset G$ be a Smarandache subgroupoid, that is H contains a proper subset $P \subset H$ such that P is a semigroup. So $P \subset G$ and P is a semigroup. Hence G is a Smarandache groupoid.

Definition 8:

i) A Smarandache Left Ideal A of the Smarandache Groupoid G satisfies the following conditions:

1. A is a Smarandache subgroupoid. 2. For all $x \in G$, and $x \in A$, $\quad xa \in A$.

ii) Similarly, one defines a Smarandache Right Ideal.

iii) If A is both a Smarandache right and left ideals then A is a Smarandache Ideal. We take $\{0\}$ as a trivial Smarandache ideal.

Example 5: Let $\{Z_6, *, (4, 5)\}$ be a Smarandache groupoid. $A = \{1, 3, 5\}$ is a Smarandache subgroupoid and A is Smarandache left ideal and not a Smarandache right ideal. Easy to verify.

Theorem 9. Let G be a groupoid. An ideal of G in general is not a Smarandache ideal of G even if G is a Smarandache groupoid.

Proof. By an example. Consider the groupoid $G = \{Z_6, *, (2, 4)\}$ given by the following table.

$*$	0	1	2	3	4	5
0	0	4	2	0	4	2
1	2	0	4	2	0	4
2	4	2	0	4	2	0
3	0	4	2	0	4	2
4	2	0	4	2	0	4
5	4	2	0	4	2	0

Clearly G is a Smarandache groupoid for $\{0, 3\}$ is a semigroup of G. Now, $\{0, 4, 2\}$ is an ideal of G but is not a Smarandache ideal as $\{0, 4, 2\}$ is not a Smarandache subgroupoid.

Definition 10: Let G be a Smarandache groupoid and V be a Smarandache subgroupoid of G. We say V is a Smarandache semi-normal subgroupoid if:

1. $aV = X$ for all $a \in G$; 2. $Va = Y$ for all $a \in G$, where either X or Y is a Smarandache subgroupoid of G but X and Y are both subgroupoids.

Example 6: Consider the groupoid $G = \{Z_6, *, (4, 5)\}$ given by the table.

$*$	0	1	2	3	4	5
0	0	5	4	3	2	1
1	4	3	2	1	0	5
2	2	1	0	5	4	3
3	0	5	4	3	2	1
4	4	3	2	1	0	5
5	2	1	0	5	4	3

Clearly G is a Smarandache groupoid as $\{3\}$ is a semigroup. Take $A = \{1, 3, 5\}$. A is also a Smarandache subgroupoid. Now $aA = A$ is a Smarandache groupoid. $Aa = \{0, 2, 4\}$. $\{0, 2, 4\}$ is not a Smarandache subgroupoid of G. Hence A is a Smarandache semi-normal subgroupoid.

Definition 11: Let A be a Smarandache groupoid and V be a Smarandache subgroupoid. V is said to be *Smarandache normal subgroupoid* if $aV = X$ and $Va = Y$ where both X and Y are Smarandache subgroupoids of G.

Theorem 12. Every Smarandache normal subgroupoid is a Smarandache semi-normal subgroupoid, and not conversely.

Proof. By the definition 10 and 11, we see every Smarandache normal subgroupoid is Smarandache semi-normal subgroupoid. We prove the converse by an example. In example 6 we see A is a Smarandache semi-normal subgroupoid but not a normal subgroupoid as $Aa = \{0, 2, 4\}$ is only a subgroupoid and not a Smarandache subgroupoid.

Example 7: Let $G = \{Z_8, *, (2, 6)\}$ be a groupoid given by the following table:

$*$	0	1	2	3	4	5	6	7
0	0	6	4	2	0	6	4	2
1	2	0	6	4	2	0	6	4
2	4	2	0	6	4	2	0	6
3	6	4	2	0	6	4	2	0
4	0	6	4	2	0	6	4	2
5	2	0	6	4	2	0	6	4
6	4	2	0	6	4	2	0	6
7	6	4	2	0	6	4	2	0

Clearly G is a Smarandache groupoid for $\{0, 4\}$ is a semigroupoid G. $A = \{0, 2, 4, 6\}$ is a Smarandache subgroupoid. Clearly $Aa = A$ for all $a \in G$. So A is a Smarandache normal subgroupoid of G.

Definition 13: Let G be a Smarandache groupoid H and P be subgroupoids of G, we say H and P are Smarandache semi-conjugate subgroupoids of G if:

1. H and P are Smarandache subgroupoids.
2. $H = xP$ or Px, for some $x \in G$.
3. $P = xH$ or Hx, for some $x \in G$.

Example 8: Consider the groupoid $G = \{Z_{12}, *, (1, 3)\}$ which is given by the following table:

$*$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	3	6	9	0	3	6	9	0	3	6	9
1	1	4	7	10	1	4	7	10	1	4	7	10
2	2	5	8	11	2	5	8	11	2	5	8	11
3	3	6	9	0	3	6	9	0	3	6	9	0
4	4	7	10	1	4	7	10	1	4	7	10	1
5	5	8	11	2	5	8	11	2	5	8	11	2
6	6	9	0	3	6	9	0	3	6	9	0	3
7	7	10	1	4	7	10	1	4	7	10	1	4
8	8	11	2	5	8	11	2	5	8	11	2	5
9	9	0	3	6	9	0	3	6	9	0	3	6
10	10	1	4	7	10	1	4	7	10	1	4	7
11	11	2	5	8	11	2	5	8	11	2	5	8

Clearly G is a Smarandache groupoid for $\{0, 6\}$ is a semigroup of G. Let $A_1 = \{0, 3, 6, 9\}$ and $A_2 = \{2, 5, 8, 11\}$ be two subgroupoids. Clearly A_1 and A_2 are Smarandache subgroups of G as $\{0, 6\}$ and $\{2, 8\}$ are semigroups of A_1 and A_2 respectively.

Now:

$$\begin{aligned} A_1 &= 3\{2, 5, 8, 11\} = 3A_2 \\ &= \{0, 3, 6, 9\} \end{aligned}$$

and similarly:

$$A_2 = 2\{0, 3, 6, 9\} = 2A_1.$$

Hence A_1 and A_2 are conjugate Smarandache subgroupoids of G.

Definition 15: Let $G_1, G_2, G_3, \cdots, G_n$ be n groupoids. We say $G = G_1 \times G_2 \times \cdots \times G_n$ is a *Smarandache direct product of groupoids* if G has a proper subset H of G which is a semigroup under the operations of G. It is important to note that each G_i need not be a Smarandache groupoid for in that case G will be obviously a Smarandache groupoid. Hence we take any set of n groupoids and find the direct product.

Definition 16: Let $(G, *)$ and $(G^{'}, \circ)$ be any two Smarandache groupoids. A map ϕ from $(G, *)$ to $(G^{'}, \circ)$ is said to be a *Smarandache groupoid homomorphism* if $\phi(a * b) = \phi(a) \circ \phi(b)$ for all $a, b \in A$.

We say the *Smarandache groupoid* homomorphism is an *isomorphism* if ϕ is an isomorphism.

Definition 17: Let G be a Smarandache groupoid. We say G is a *Smarandache commutative groupoid* if there is a proper subset A of G which is a commutative semigroup under the operation of G.

Definition 18: Let G be Smarandache groupoid. We say G is *Smarandache inner commutative groupoid* if every semigroup contained in every Smarandache subgroupoid of G is commutative.

Theorem 19. Every Smarandache inner commutative groupoid G is a Smarandache commutative groupoid and not conversely.

Proof. By the very definition 18 and 19 we see if G is a Smarandache inner commutative groupoid then G is Smarandache commutative groupoid.

To prove the converse we prove it by an example. Let $Z_2 = \{0, 1\}$ be integers modulo 2. Consider set of all 2×2 matrices with entries from $Z_2 = (0, 1)$ denote it by $M_{2\times 2}$.

$$M_{2\times 2} = \left\{ \begin{array}{c} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{array} \right.$$

$$\left. \begin{array}{c} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{array} \right\}.$$

$M_{2\times 2}$ is made into a groupoid by for $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ in $M_{2\times 2}$.

$$\begin{aligned} A \circ B &= \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \circ \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \\ &= \begin{pmatrix} a_1b_3 + a_2b_1(\mathrm{mod}2) & a_1b_4 + a_2b_2(\mathrm{mod}2) \\ a_3b_3 + a_4b_1(\mathrm{mod}2) & a_3b_4 + a_4b_2(\mathrm{mod}2) \end{pmatrix} \end{aligned}$$

Clearly $(M_{2\times 2}, \circ)$ is a Smarandache groupoid for $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

So $\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \circ \right\}$ is a semigroup.

Now consider $A_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \circ \right\}$ is a Smarandache groupoid but A_1 is non-commutative Smarandache groupoid for A_1 contains a non-commutative semigroupoid S. $S = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \circ \right\}$ such that $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \circ$

$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. So $(M_{2\times 2}, \circ)$ is a Smarandache commutative groupoid but not Smarandache inner commutative groupoid.

Definition 20: A groupoid G is said to be a *Moufang groupoid* if for every x, y, z in G we have $(xy)(zx) = (x(yz))x$.

Definition 21: A Smarandache groupoid $(G, *)$ is said to be *Smarandache Moufang groupoid* if there exists $H \subset G$ such that H is a Smarandache groupoid satisfying the Moufang identity: $(xy)(zx) = (x(yz)x)$ for all x, y, z in H.

Definition 22: Let S be a Smarandache groupoid. If every Smarandache subgroupoid H of S satisfies the Moufang identity for all x, y, z in H then S is a *Smarandache Strong Moufang groupoid*.

Theorem 23. Every Smarandache Strong Moufang groupoid is a Smarandache Moufang groupoidand not conversely.

Proof. Every Strong Smarandache Moufang groupoid is a Smarandache Moufang groupoid. The proof of the converse can be proved by constructing examples.

Definition 24: A groupoid G is said to be a *Bol groupoid* if $((xy)z)y = x((yz))y$ for all $x, y, z \in G$.

Definition 25: Let G be a groupoid. G is said to be *Smarandache − Bol groupoid* if G has a subgroupoid H of G such that H is a Smarandache subgroupoid and satisfies the identity $((xy)z)y = x((yz))y$ for all x, y, z in H.

Definition 26: Let G be a groupoid. We say G is *Smarandache Strong Bol groupoid* if every Smarandache subgroupoid of G is a Bol groupoid.

Theorem 27. Every Smarandache Strong Bol groupoid is a Smarandache Bol groupoid and the converse is not true.

Proof. Obvious.

Theorem 28. Let $Z_n = \{0, 1, 2, \cdots, n-1\}$ be the set of integers modulo n. Let $G = \{Z_n, *, (t, u)\}$ be a Smarandache groupoid. G is a Smarandache Bol groupoid if $t^3 = t(mod \quad n)$ and $u^2 = u(mod \quad n)$.

Proof. Easy to verify.

Example 9: Let $G = \{Z_6, *, (2, 3)\}$ defined by the following table:

*	0	1	2	3	4	5
0	0	3	0	3	0	3
1	2	5	2	5	2	5
2	4	1	4	1	4	1
3	0	3	0	3	0	3
4	2	5	2	5	2	5
5	4	1	4	1	4	1

$\{0, 3\}$ is a Smarandache subgroupoid and since $2^3 = 2(mod \quad 6)$ and $3^2 = 3(mod \quad 6)$ we see G is a Smarandache Bol groupoid.

Problem 2: Let $\{0, 1, 2, \cdots, n-1\}$ be the ring of integers modulo n. $G = \{Z_n, *, (t, u)\}$ be a groupoid. Find conditions on n, t and u so that G:

1. is a Smarandache groupoid.
2. has Smarandache semi-normal subgroupoids.
3. has Smarandache normal subgroupoids.
4. is Smarandache commutative.
5. is Smarandache inner commutative.
6. is a Smarandache-Bol groupoid.
7. is a Smarandache Strong Bol groupoid.
8. is a Smarandache-Moufang groupoid.
9. is a Smarandache-Strong-Moufang groupoid.
10. has always a pair of Smarandache conjugate subgroupoid.

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