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Smarandache Curves According to the Extended Darboux Frame in Euclidean 4-Space

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Abstract: In this paper, considering the extended Darboux frame in Euclidean 4-space, we define some special Smarandache curves. We calculate the Frenet apparatus of these curves depending on the invariants of the extended Darboux frame of second kind.

Keywords: Smarandache curve, extended Darboux frame field, curvatures.

1. Introduction

In differential geometry, relations between the curves are wide and important field of study for many researchers. Parallel to this, new studies have been done about curves and special curves. The most familiar special curves are involute-evolute curves, Bertrand curves, Mannheim curves and Smarandache curves. A regular curve, whose position vector is obtained by Frenet frame vectors on another regular curve is called Smarandache curve in Minkowski space-time [1]. Special Smarandache curves have been studied at many researches in both Euclidean space and Minkowski space [1-5]. There are also some studies on special Smarandache curves in Galilean and pseudo-Galilean spaces [6-8].

Ali [2] has introduced TN, NB and TNB-Smarandache curves according to Frenet frame $\{\mathsf{T},\mathsf{N},\mathsf{B}\}$ in Euclidean 3-space and obtained Frenet-Serret invariants of TN-Smarandache curve. In [4]; Tg, Tn, gn and Tgn-Smarandache curves according to Darboux frame $\{\mathsf{T},\mathsf{g},\mathsf{n}\}$ have been given in Euclidean 3-space. In this study, the authors have calculated the Frenet apparatus of these special Smarandache curves and found some properties of these curves. Also, Smarandache curves in 4-dimensional Euclidean space according to the Frenet frame and parallel transport frame have been studied in [5] and obtained the Frenet-Serret and Bishop invariants for the Smarandache curves.

In the light of the existing studies in this area, we define some special Smarandache curves such as TE, TD and TN-Smarandache curves in Euclidean 4-space according to the extended Darboux frame (or shortly ED-frame) defined in [9]. Then considering the extended Darboux frame of second kind, we obtain the Frenet apparatus of these special Smarandache curves depending on the extended Darboux frame invariants.

2. Preliminaries

Definition 1. Let $\mathbf{x}=\sum_{i=1}^{4} x_i\mathbf{e_i}$, $\mathbf{y}=\sum_{i=1}^{4} y_i\mathbf{e_i}$ and $\mathbf{z}=\sum_{i=1}^{4} z_i\mathbf{e_i}$ be three vectors in $\mathbb{R}^4$, where $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4\}$ be the standard basis of $\mathbb{R}^4$. Then the ternary product or vector product of these vectors is defined by [10]

$$\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}=\begin{vmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4\\ x_1 & x_2 & x_3 & x_4\\ y_1 & y_2 & y_3 & y_4\\ z_1 & z_2 & z_3 & z_4\end{vmatrix}.$$

Let $\mathcal{M}$ be a regular hypersurface in $\mathbb{E}^4$, α be a Frenet curve with arc-length parametrization on $\mathcal{M}$ and $\{\mathsf{t},\mathsf{n},\mathsf{b}_1,\mathsf{b}_2\}$ be the moving Frenet frame along α. Then we have [11]

$$\mathsf{t}=\frac{\alpha'}{\|\alpha'\|},\qquad \mathsf{b}_2=\frac{\alpha'\otimes\alpha''\otimes\alpha'''}{\|\alpha'\otimes\alpha''\otimes\alpha'''\|},\tag{1}$$

$$\mathsf{b}_1=\frac{\mathsf{b}_2\otimes\alpha'\otimes\alpha''}{\|\mathsf{b}_2\otimes\alpha'\otimes\alpha''\|},\qquad \mathsf{n}=\frac{\mathsf{b}_1\otimes\mathsf{b}_2\otimes\alpha'}{\|\mathsf{b}_1\otimes\mathsf{b}_2\otimes\alpha'\|},$$

$$k_1=\frac{\langle\mathsf{n},\alpha''\rangle}{\|\alpha'\|^2},\quad k_2=\frac{\langle\mathsf{b}_1,\alpha'''\rangle}{\|\alpha'\|^3 k_1},\quad k_3=\frac{\langle\mathsf{b}_2,\alpha^{(4)}\rangle}{\|\alpha'\|^4 k_1k_2}.\tag{2}$$

Since the curve α lies on $\mathcal{M}$, we have another frame field such as the ED-frame field $\{\mathsf{T},\mathsf{E},\mathsf{D},\mathsf{N}\}$ along α, where

$\mathsf{T}=\alpha'$, $\quad\mathsf{N}=\mathcal{N}(\alpha)$,

$\mathsf{E}=\dfrac{\alpha''-\langle\alpha'',\mathsf{N}\rangle\mathsf{N}}{\|\alpha''-\langle\alpha'',\mathsf{N}\rangle\mathsf{N}\|}$ if $\{\mathsf{N},\mathsf{T},\alpha''\}$ is linearly independent (Case1),

$\mathsf{E}=\dfrac{\alpha'''-\langle\alpha''',\mathsf{N}\rangle\mathsf{N}-\langle\alpha''',\mathsf{T}\rangle\mathsf{T}}{\|\alpha'''-\langle\alpha''',\mathsf{N}\rangle\mathsf{N}-\langle\alpha''',\mathsf{T}\rangle\mathsf{T}\|}$ if $\{\mathsf{N},\mathsf{T},\alpha''\}$ is linearly dependent (Case2),

$\mathsf{D}=\mathsf{N}\otimes\mathsf{T}\otimes\mathsf{E}$,

and $\mathcal{N}$ is the unit normal vector field of $\mathcal{M}$. Then we have the following differential equations for the ED-frame field of first kind

$$\begin{bmatrix} \mathsf{T}' \\ \mathsf{E}' \\ \mathsf{D}' \\ \mathsf{N}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g^1 & 0 & \kappa_n \\ -\kappa_g^1 & 0 & \kappa_g^2 & \tau_g^1 \\ 0 & -\kappa_g^2 & 0 & \tau_g^2 \\ -\kappa_n & -\tau_g^1 & -\tau_g^2 & 0 \end{bmatrix} \begin{bmatrix} \mathsf{T} \\ \mathsf{E} \\ \mathsf{D} \\ \mathsf{N} \end{bmatrix} \quad \text{(Case 1)},$$

and the ED-frame field of second kind

$$\begin{bmatrix} \mathsf{T}' \\ \mathsf{E}' \\ \mathsf{D}' \\ \mathsf{N}' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \kappa_n \\ 0 & 0 & \kappa_g^2 & \tau_g^1 \\ 0 & -\kappa_g^2 & 0 & 0 \\ -\kappa_n & -\tau_g^1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathsf{T} \\ \mathsf{E} \\ \mathsf{D} \\ \mathsf{N} \end{bmatrix} \quad \text{(Case 2)}, \tag{3}$$

where κ_g^i and τ_g^i are the geodesic curvature and the geodesic torsion of order i, $(i=1,2)$, respectively, [9].

3. Smarandache Curves According to The Extended Darboux Frame in $\mathbb{E}^4$

In this part, we deal with some special Smarandache curves according to the ED-frame of second kind in Euclidean 4-space. We find the Frenet apparatus of TE-Smarandache curve, TD-Smarandache curve and TN-Smarandache curve depending on the ED-frame invariants.

3.1 TE-Smarandache curve in $\mathbb{E}^4$

Definition 2. Let $\mathcal{M} \subset \mathbb{E}^4$ be an oriented hypersurface and $\alpha : I \subset \mathbb{R} \to \mathcal{M}$ be a Frenet curve with arc-length parameter s in $\mathbb{E}^4$. Let us denote the ED-frame field of $\alpha(s)$ with $\{\mathsf{T}(s), \mathsf{E}(s), \mathsf{D}(s), \mathsf{N}(s)\}$. TE-Smarandache curve β is defined by

$$\beta(s) = \frac{1}{\sqrt{2}}\Big(\mathsf{T}(s) + \mathsf{E}(s)\Big). \tag{4}$$

Considering the ED-frame of second kind, let us now calculate the Frenet apparatus $\{\mathsf{T}^*, \mathsf{n}^*, \mathsf{b}_1^*, \mathsf{b}_2^*, k_1^*, k_2^*, k_3^*\}$ of TE-Smarandache curve β depending on the ED-frame invariants. Let s^* be the arc-length parameter of β. If we differentiate (4) with respect to s and use (3), we have

$$\beta' = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\Big(\kappa_g^2 \mathsf{D} + (\kappa_n + \tau_g^1)\mathsf{N}\Big). \tag{5}$$

Since

$$\| \beta' \| = \frac{ds^*}{ds} = \sqrt{\frac{(\kappa_g^2)^2 + (\kappa_n + \tau_g^1)^2}{2}}, \tag{6}$$

we find the unit tangent vector of TE-Smarandache curve β as

$$\mathsf{T}^* = \frac{1}{\sqrt{(\kappa_g^2)^2 + (\kappa_n + \tau_g^1)^2}}\Big(\kappa_g^2 \mathsf{D} + (\kappa_n + \tau_g^1)\mathsf{N}\Big). \tag{7}$$

From (5), we have

$$\beta'' = \frac{1}{\sqrt{2}}\Big(-\kappa_n(\kappa_n + \tau_g^1)\mathsf{T} - \Big((\kappa_g^2)^2 + \tau_g^1(\kappa_n + \tau_g^1)\Big)\mathsf{E} + (\kappa_g^2)'\mathsf{D} + \Big(\kappa_n' + (\tau_g^1)'\Big)\mathsf{N}\Big) \tag{8}$$

and

$$\beta''' = \frac{1}{\sqrt{2}}\Big(\mu_1 \mathsf{T} + \mu_2 \mathsf{E} + \mu_3 \mathsf{D} + \mu_4 \mathsf{N}\Big), \tag{9}$$

where

$$\mu_1 = -\kappa_n'(\kappa_n + \tau_g^1) - 2\kappa_n\Big(\kappa_n' + (\tau_g^1)'\Big),$$
$$\mu_2 = -(\tau_g^1)'(\kappa_n + \tau_g^1) - 2\tau_g^1\Big(\kappa_n' + (\tau_g^1)'\Big) - 3\kappa_g^2(\kappa_g^2)',$$
$$\mu_3 = (\kappa_g^2)'' - (\kappa_g^2)^3 - \kappa_g^2\tau_g^1(\kappa_n + \tau_g^1),$$
$$\mu_4 = \kappa_n'' + (\tau_g^1)'' - \tau_g^1(\kappa_g^2)^2 - (\kappa_n + \tau_g^1)\Big((\kappa_n)^2 + (\tau_g^1)^2\Big).$$

Using (5), (8) and (9) gets

$$\beta' \otimes \beta'' \otimes \beta''' = \frac{1}{2\sqrt{2}}\Big(\nu_1 \mathsf{T} + \nu_2 \mathsf{E} + \nu_3 \mathsf{D} + \nu_4 \mathsf{N}\Big),$$

where

$$\nu_1 = -(\kappa_n + \tau_g^1)\Big(\mu_2(\kappa_g^2)' + \mu_3\Big((\kappa_g^2)^2 + \tau_g^1(\kappa_n + \tau_g^1)\Big) - \mu_4\kappa_g^2\tau_g^1\Big) + \mu_4(\kappa_g^2)^3 + \mu_2\kappa_g^2\Big(\kappa_n' + (\tau_g^1)'\Big),$$
$$\nu_2 = (\kappa_n + \tau_g^1)\Big(\mu_1(\kappa_g^2)' + \mu_3\kappa_n(\kappa_n + \tau_g^1) - \mu_4\kappa_n\kappa_g^2\Big) - \mu_1\kappa_g^2\Big(\kappa_n' + (\tau_g^1)'\Big),$$
$$\nu_3 = (\kappa_n + \tau_g^1)\Big(\mu_1(\kappa_g^2) + (\mu_1\tau_g^1 - \mu_2\kappa_n)(\kappa_n + \tau_g^1)\Big),$$
$$\nu_4 = \kappa_g^2(\kappa_n + \tau_g^1)(\mu_2\kappa_n - \mu_1\tau_g^1) - \mu_1(\kappa_g^2)^3.$$

Since

$$\| \beta' \otimes \beta'' \otimes \beta''' \| = \frac{1}{2\sqrt{2}}\sqrt{\nu_1^2 + \nu_2^2 + \nu_3^2 + \nu_4^2},$$

from (1) we find the second binormal vector of TE-Smarandache curve β as

$$\mathsf{b}_2^* = \frac{1}{\sqrt{\nu_1^2 + \nu_2^2 + \nu_3^2 + \nu_4^2}}(\nu_1 \mathsf{T} + \nu_2 \mathsf{E} + \nu_3 \mathsf{D} + \nu_4 \mathsf{N}). \tag{10}$$

Also using (10), (5) and (8), the first binormal vector of TE-Smarandache curve β is obtained as

$$\mathsf{b}_1^* = \frac{1}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2}}(\lambda_1 \mathsf{T} + \lambda_2 \mathsf{E} + \lambda_3 \mathsf{D} + \lambda_4 \mathsf{N}), \tag{11}$$

where

$$\lambda_1 = \Big((\kappa_g^2)^2 + \tau_g^1(\kappa_n + \tau_g^1)\Big)\Big(\nu_4\kappa_g^2 - \nu_3(\kappa_n + \tau_g^1)\Big) + \nu_2\Big(\kappa_g^2\kappa_n' - \kappa_n(\kappa_g^2)' + \kappa_g^2(\tau_g^1)' - \tau_g^1(\kappa_g^2)'\Big),$$
$$\lambda_2 = -\kappa_n(\kappa_n + \tau_g^1)\Big(\nu_4\kappa_g^2 - \nu_3(\kappa_n + \tau_g^1)\Big) - \nu_1\Big(\kappa_g^2\kappa_n' - \kappa_n(\kappa_g^2)' + \kappa_g^2(\tau_g^1)' - \tau_g^1(\kappa_g^2)'\Big),$$
$$\lambda_3 = \nu_1(\kappa_g^2)^2(\kappa_n + \tau_g^1) + (\kappa_n + \tau_g^1)^2(\nu_1\tau_g^1 - \nu_2\kappa_n),$$
$$\lambda_4 = -\nu_1(\kappa_g^2)^3 - \kappa_g^2(\kappa_n + \tau_g^1)(\nu_1\tau_g^1 - \nu_2\kappa_n).$$

If we use (11), (10) and (5), we calculate the principal normal vector of TE-Smarandache curve β as

$$\mathsf{n}^* = \frac{1}{\sqrt{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2}}(\sigma_1\mathsf{T}+\sigma_2\mathsf{E}+\sigma_3\mathsf{D}+\sigma_4\mathsf{N}),$$

where

$$\sigma_1 = (\kappa_n+\tau_g^1)(\lambda_2\nu_3-\lambda_3\nu_2)-\kappa_g^2(\lambda_2\nu_4-\lambda_4\nu_2),$$

$$\sigma_2 = (\kappa_n+\tau_g^1)(\lambda_3\nu_1-\lambda_1\nu_3)+\kappa_g^2(\lambda_1\nu_4-\lambda_4\nu_1),$$

$$\sigma_3 = (\kappa_n+\tau_g^1)(\lambda_1\nu_2-\lambda_2\nu_1),$$

$$\sigma_4 = -\kappa_g^2(\lambda_1\nu_2-\lambda_2\nu_1).$$

From (2), the first curvature of TE-Smarandache curve β is

$$k_1^* = \omega\Delta,$$

where

$$\omega = \frac{\sqrt{2}}{\left((\kappa_g^2)^2+(\kappa_n+\tau_g^1)^2\right)\sqrt{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2}}$$

and

$$\Delta = -\sigma_1\kappa_n(\kappa_n+\tau_g^1)-\sigma_2\left((\kappa_g^2)^2+\tau_g^1(\kappa_n+\tau_g^1)\right)$$
$$+\sigma_3(\kappa_g^2)'+\sigma_4\left(\kappa_n'+(\tau_g^1)'\right).$$

Also using (2), the second curvature of TE-Smarandache curve β is obtained as

$$k_2^* = \frac{\vartheta}{\Delta}\sum_{i=1}^{4}\lambda_i\mu_i$$

where

$$\vartheta = \frac{\sqrt{2(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2)}}{\sqrt{(\kappa_g^2)^2+(\kappa_n+\tau_g^1)^2}\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2}}.$$

Moreover, we get

$$\beta^{(4)} = \frac{1}{\sqrt{2}}\Big((\mu_1'-\mu_4\kappa_n)\mathsf{T}+(\mu_2'-\mu_3\kappa_g^2-\mu_4\tau_g^1)\mathsf{E}$$
$$+(\mu_3'+\mu_2\kappa_g^2)\mathsf{D}+(\mu_4'+\mu_1\kappa_n+\mu_2\tau_g^1)\mathsf{N}\Big).$$

So from (2), the third curvature of TE-Smarandache curve β is found as

$$k_3^* = \Gamma\Big(\nu_1(\mu_1'-\mu_4\kappa_n)+\nu_2(\mu_2'-\mu_3\kappa_g^2-\mu_4\tau_g^1)$$
$$+\nu_3(\mu_3'+\mu_2\kappa_g^2)+\nu_4(\mu_4'+\mu_1\kappa_n+\mu_2\tau_g^1)\Big),$$

where

$$\Gamma = \frac{\sqrt{2(\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2)}}{\sqrt{(\kappa_g^2)^2+(\kappa_n+\tau_g^1)^2}\sqrt{\nu_1^2+\nu_2^2+\nu_3^2+\nu_4^2}\sum_{i=1}^{4}\lambda_i\mu_i}.$$

3.2 TD-Smarandache curve in $\mathbb{E}^4$

Definition 3. Let $\mathcal{M}$ be an oriented hypersurface in $\mathbb{E}^4$ and α be a Frenet curve with arc-length parameter s on $\mathcal{M}$. Denoting the ED-frame field of $\alpha(s)$ by $\{\mathsf{T}(s),\mathsf{E}(s),\mathsf{D}(s),\mathsf{N}(s)\}$, TD-Smarandache curve β can be defined as

$$\beta(s) = \frac{1}{\sqrt{2}}\big(\mathsf{T}(s)+\mathsf{D}(s)\big). \tag{12}$$

Let us compute the Frenet apparatus of TD-Smarandache curve depending on the ED-frame invariants. Let s^* be the arc-length parameter of β. Differentiating (12) with respect to s and using (3) yields

$$\beta' = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(-\kappa_g^2\mathsf{E}+\kappa_n\mathsf{N}\right). \tag{13}$$

Then substituting

$$\|\beta'\| = \frac{ds^*}{ds} = \sqrt{\frac{(\kappa_g^2)^2+\kappa_n^2}{2}}$$

into (13), we obtain the unit tangent vector T^* of TD-Smarandache curve β as

$$\mathsf{T}^* = \frac{1}{\sqrt{(\kappa_g^2)^2+\kappa_n^2}}\left(-\kappa_g^2\mathsf{E}+\kappa_n\mathsf{N}\right). \tag{14}$$

Using (13) gets

$$\beta'' = \frac{-1}{\sqrt{2}}\Big(\kappa_n^2\mathsf{T}+\left((\kappa_g^2)'+\kappa_n\tau_g^1\right)\mathsf{E}$$
$$+(\kappa_g^2)^2\mathsf{D}+\left(\kappa_g^2\tau_g^1-\kappa_n'\right)\mathsf{N}\Big) \tag{15}$$

and

$$\beta''' = \frac{-1}{\sqrt{2}}\left(\rho_1\mathsf{T}+\rho_2\mathsf{E}+\rho_3\mathsf{D}+\rho_4\mathsf{N}\right), \tag{16}$$

where

$$\rho_1 = \kappa_n(3\kappa_n'-\kappa_g^2\tau_g^1),$$

$$\rho_2 = (\kappa_g^2)''+2\kappa_n'\tau_g^1+\kappa_n(\tau_g^1)'-(\kappa_g^2)^3-\kappa_g^2(\tau_g^1)^2,$$

$$\rho_3 = \kappa_g^2\left(3(\kappa_g^2)'+\kappa_n\tau_g^1\right),$$

$$\rho_4 = -\kappa_n''+2(\kappa_g^2)'\tau_g^1+\kappa_g^2(\tau_g^1)'+\kappa_n^3+\kappa_n(\tau_g^1)^2.$$

If we use (13), (15) and (16), we have

$$\beta'\otimes\beta''\otimes\beta''' = \frac{1}{2\sqrt{2}}\left(\epsilon_1\mathsf{T}+\epsilon_2\mathsf{E}+\epsilon_3\mathsf{D}+\epsilon_4\mathsf{N}\right),$$

where

$$\epsilon_1 = \rho_3\tau_g^1\left((\kappa_g^2)^2+\kappa_n^2\right)-\rho_3\left(\kappa_g^2\kappa_n'-(\kappa_g^2)'\kappa_n\right)$$
$$-(\kappa_g^2)^2(\rho_2\kappa_n+\rho_4\kappa_g^2),$$

$$\epsilon_2 = \kappa_n\left(\rho_1(\kappa_g^2)^2-\rho_3\kappa_n^2\right),$$

$$\epsilon_3 = -\rho_1\tau_g^1\left((\kappa_g^2)^2+\kappa_n^2\right)+\rho_1\left(\kappa_g^2\kappa_n'-(\kappa_g^2)'\kappa_n\right)$$
$$+\kappa_n^2(\rho_2\kappa_n+\rho_4\kappa_g^2),$$

$$\epsilon_4 = \kappa_g^2\left(\rho_1(\kappa_g^2)^2-\rho_3\kappa_n^2\right)$$

and

$$\|\beta'\otimes\beta''\otimes\beta'''\| = \frac{1}{2\sqrt{2}}\sqrt{\epsilon_1^2+\epsilon_2^2+\epsilon_3^2+\epsilon_4^2}.$$

So the second binormal vector b_2^* of TD-Smarandache curve β is found as

$$\mathsf{b}_2^* = \frac{1}{\sqrt{\epsilon_1^2+\epsilon_2^2+\epsilon_3^2+\epsilon_4^2}}(\epsilon_1\mathsf{T}+\epsilon_2\mathsf{E}+\epsilon_3\mathsf{D}+\epsilon_4\mathsf{N}). \tag{17}$$

Moreover, the first binormal vector b_1^* of TD-Smarandache curve β is obtained as

$$\mathsf{b}_1^* = \frac{1}{\sqrt{\zeta_1^2+\zeta_2^2+\zeta_3^2+\zeta_4^2}}(\zeta_1\mathsf{T}+\zeta_2\mathsf{E}+\zeta_3\mathsf{D}+\zeta_4\mathsf{N}), \tag{18}$$

where

$$\zeta_1 = -\epsilon_3\tau_g^1\Big((\kappa_g^2)^2+\kappa_n^2\Big)+\epsilon_3\Big(\kappa_g^2\kappa_n{}'-(\kappa_g^2)'\kappa_n\Big)$$
$$+(\kappa_g^2)^2(\epsilon_2\kappa_n+\epsilon_4\kappa_g^2),$$
$$\zeta_2 = \kappa_n\Big(\epsilon_3\kappa_n^2-\epsilon_1(\kappa_g^2)^2\Big),$$
$$\zeta_3 = \epsilon_1\tau_g^1\Big((\kappa_g^2)^2+\kappa_n^2\Big)-\epsilon_1\Big(\kappa_g^2\kappa_n{}'-(\kappa_g^2)'\kappa_n\Big)$$
$$-\kappa_n^2(\epsilon_2\kappa_n+\epsilon_4\kappa_g^2),$$
$$\zeta_4 = \kappa_g^2\Big(\epsilon_3\kappa_n^2-\epsilon_1(\kappa_g^2)^2\Big).$$

Using (18), (17) and (13) yields the principal normal vector n^* of TD-Smarandache curve β as

$$\mathsf{n}^* = \frac{1}{\sqrt{\delta_1^2+\delta_2^2+\delta_3^2+\delta_4^2}}(\delta_1\mathsf{T}+\delta_2\mathsf{E}+\delta_3\mathsf{D}+\delta_4\mathsf{N}),$$

where

$$\delta_1 = \kappa_n(\epsilon_3\zeta_2-\epsilon_2\zeta_3)+\kappa_g^2(\epsilon_3\zeta_4-\epsilon_4\zeta_3),$$
$$\delta_2 = \kappa_n(\epsilon_1\zeta_3-\epsilon_3\zeta_1),$$
$$\delta_3 = \kappa_n(\epsilon_2\zeta_1-\epsilon_1\zeta_2)+\kappa_g^2(\epsilon_4\zeta_1-\epsilon_1\zeta_4),$$
$$\delta_4 = \kappa_g^2(\epsilon_1\zeta_3-\epsilon_3\zeta_1).$$

If we use (2), we find the first curvature k_1^* of TD-Smarandache curve β as

$$k_1^* = \varpi\,\Omega,$$

where

$$\varpi = \frac{-\sqrt{2}}{\Big((\kappa_g^2)^2+\kappa_n^2\Big)\sqrt{\delta_1^2+\delta_2^2+\delta_3^2+\delta_4^2}}$$

and

$$\Omega = \delta_1\kappa_n^2+\delta_2\Big((\kappa_g^2)'+\kappa_n\tau_g^1\Big)+\delta_3(\kappa_g^2)^2+\delta_4\Big(\kappa_g^2\tau_g^1-\kappa_n{}'\Big).$$

From (2), the second curvature k_2^* of TD-Smarandache curve β is

$$k_2^* = \frac{\eta}{\Omega}\sum_{i=1}^{4}\zeta_i\rho_i$$

where

$$\eta = \frac{\sqrt{2}\sqrt{\delta_1^2+\delta_2^2+\delta_3^2+\delta_4^2}}{\sqrt{(\kappa_g^2)^2+\kappa_n^2}\sqrt{\zeta_1^2+\zeta_2^2+\zeta_3^2+\zeta_4^2}}.$$

Besides, we have

$$\beta^{(4)} = \frac{-1}{\sqrt{2}}\Big((\rho_1{}'-\rho_4\kappa_n)\mathsf{T}+(\rho_2{}'-\rho_3\kappa_g^2-\rho_4\tau_g^1)\mathsf{E}$$
$$+(\rho_3{}'+\rho_2\kappa_g^2)\mathsf{D}+(\rho_4{}'+\rho_1\kappa_n+\rho_2\tau_g^1)\mathsf{N}\Big)$$

and using (2), we obtain the third curvature k_3^* of TD-Smarandache curve β as

$$k_3^* = \Lambda\Big(\epsilon_1(\rho_1{}'-\rho_4\kappa_n)+\epsilon_2(\rho_2{}'-\rho_3\kappa_g^2-\epsilon_4\tau_g^1)$$
$$+\epsilon_3(\rho_3{}'+\rho_2\kappa_g^2)+\epsilon_4(\rho_4{}'+\rho_1\kappa_n+\rho_2\tau_g^1)\Big),$$

where

$$\Lambda = \frac{\sqrt{2(\zeta_1^2+\zeta_2^2+\zeta_3^2+\zeta_4^2)}}{\sqrt{(\kappa_g^2)^2+\kappa_n^2}\sqrt{\epsilon_1^2+\epsilon_2^2+\epsilon_3^2+\epsilon_4^2}\sum_{i=1}^{4}\zeta_i\rho_i}.$$

3.3 TN-Smarandache curve in $\mathbb{E}^4$

Definition 4. Let α be a Frenet curve with arc-length parameter s on an oriented hypersurface $\mathcal{M}$ in $\mathbb{E}^4$ and $\{\mathsf{T}(s),\mathsf{E}(s),\mathsf{D}(s),\mathsf{N}(s)\}$ denotes the ED-frame field of $\alpha(s)$. TN-Smarandache curve β is defined by

$$\beta(s) = \frac{1}{\sqrt{2}}\Big(\mathsf{T}(s)+\mathsf{N}(s)\Big). \tag{19}$$

Let us now obtain the Frenet apparatus of TN-Smarandache curve β depending on the ED-frame invariants. Let s^* be the arc-length parameter of β. If we differentiate (19) with respect to s and use (3), we get

$$\beta' = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{-1}{\sqrt{2}}\Big(\kappa_n\mathsf{T}+\tau_g^1\mathsf{E}-\kappa_n\mathsf{N}\Big). \tag{20}$$

Then from (20), we have

$$\|\beta'\| = \frac{ds^*}{ds} = \sqrt{\frac{2\kappa_n^2+(\tau_g^1)^2}{2}} \tag{21}$$

and substituting (21) into (20) gives the unit tangent vector T^* of TN-Smarandache curve β, i.e.:

$$\mathsf{T}^* = \frac{-1}{\sqrt{2\kappa_n^2+(\tau_g^1)^2}}\Big(\kappa_n\mathsf{T}+\tau_g^1\mathsf{E}-\kappa_n\mathsf{N}\Big). \tag{22}$$

From (20), we get

$$\beta'' = \frac{-1}{\sqrt{2}}\Big((\kappa_n{}'+\kappa_n^2)\mathsf{T}+\Big((\tau_g^1)'+\kappa_n\tau_g^1\Big)\mathsf{E}$$
$$+\tau_g^1\kappa_g^2\mathsf{D}+\Big(\kappa_n^2-\kappa_n{}'+(\tau_g^1)^2\Big)\mathsf{N}\Big) \tag{23}$$

and

$$\beta''' = \frac{1}{\sqrt{2}}\Big(\xi_1\mathsf{T}+\xi_2\mathsf{E}+\xi_3\mathsf{D}+\xi_4\mathsf{N}\Big), \tag{24}$$

where

$$\xi_1 = \kappa_n\Big(\kappa_n^2+(\tau_g^1)^2\Big)-3\kappa_n\kappa_n{}'-\kappa_n{}'',$$
$$\xi_2 = \tau_g^1\Big(\kappa_n^2+(\tau_g^1)^2+(\kappa_g^2)^2-2\kappa_n{}'\Big)-\kappa_n(\tau_g^1)'-(\tau_g^1)'',$$
$$\xi_3 = -2\kappa_g^2(\tau_g^1)'-\tau_g^1\Big(\kappa_n\kappa_g^2+(\kappa_g^2)'\Big),$$
$$\xi_4 = -\kappa_n\Big(\kappa_n^2+(\tau_g^1)^2\Big)-3\Big(\kappa_n\kappa_n{}'+\tau_g^1(\tau_g^1)'\Big)+\kappa_n{}''.$$

Equations (20), (23) and (24) yields

$$\beta'\otimes\beta''\otimes\beta''' = \frac{1}{2\sqrt{2}}\Big(\Psi_1\mathsf{T}+\Psi_2\mathsf{E}+\Psi_3\mathsf{D}+\Psi_4\mathsf{N}\Big),$$

where

$$\Psi_1 = -\xi_3\tau_g^1\left(2\kappa_n^2+(\tau_g^1)^2\right)+\xi_3\left(\kappa_n{}'\tau_g^1-\kappa_n(\tau_g^1)'\right)$$
$$+\kappa_g^2\tau_g^1(\xi_2\kappa_n+\xi_4\tau_g^1),$$
$$\Psi_2 = \xi_3\kappa_n\left(2\kappa_n^2+(\tau_g^1)^2\right)-\kappa_n\kappa_g^2\tau_g^1(\xi_1+\xi_4),$$
$$\Psi_3 = (\xi_1\tau_g^1-\xi_2\kappa_n)\left(2\kappa_n^2+(\tau_g^1)^2\right)$$
$$-(\xi_1+\xi_4)(\kappa_n{}'\tau_g^1-\kappa_n(\tau_g^1)'),$$
$$\Psi_4 = \kappa_g^2\tau_g^1(\xi_2\kappa_n-\xi_1\tau_g^1)+\xi_3\left(\kappa_n{}'\tau_g^1-\kappa_n(\tau_g^1)'\right).$$

So, we have

$$\| \beta'\otimes\beta''\otimes\beta'''\| = \frac{1}{2\sqrt{2}}\sqrt{\Psi_1^2+\Psi_2^2+\Psi_3^2+\Psi_4^2},$$

which enables us to find the second binormal vector b_2^* of TN-Smarandache curve β as

$$\mathsf{b}_2^* = \frac{1}{\sqrt{\Psi_1^2+\Psi_2^2+\Psi_3^2+\Psi_4^2}}(\Psi_1\mathsf{T}+\Psi_2\mathsf{E}+\Psi_3\mathsf{D}+\Psi_4\mathsf{N}). \quad (25)$$

From (25), (20) and (23), we obtain the first binormal vector b_1^* of TN-Smarandache curve β as

$$\mathsf{b}_1^* = \frac{1}{\sqrt{\gamma_1^2+\gamma_2^2+\gamma_3^2+\gamma_4^2}}(\gamma_1\mathsf{T}+\gamma_2\mathsf{E}+\gamma_3\mathsf{D}+\gamma_4\mathsf{N}), \quad (26)$$

where

$$\gamma_1 = -\Psi_3\tau_g^1\left(2\kappa_n^2+(\tau_g^1)^2\right)+\Psi_3\left(\tau_g^1\kappa_n{}'-\kappa_n(\tau_g^1)'\right)$$
$$+\kappa_g^2\tau_g^1\left(\Psi_2\kappa_n+\Psi_4\tau_g^1\right),$$
$$\gamma_2 = \Psi_3\kappa_n\left(2\kappa_n^2+(\tau_g^1)^2\right)-\kappa_n\tau_g^1\kappa_g^2(\Psi_1+\Psi_4),$$
$$\gamma_3 = (\Psi_1\tau_g^1-\Psi_2\kappa_n)\left(2\kappa_n^2+(\tau_g^1)^2\right)$$
$$+(\Psi_1+\Psi_4)\left(\kappa_n(\tau_g^1)'-\tau_g^1\kappa_n{}'\right),$$
$$\gamma_4 = -\kappa_g^2\tau_g^1(\Psi_1\tau_g^1-\Psi_2\kappa_n)-\Psi_3\left(\kappa_n(\tau_g^1)'-\tau_g^1\kappa_n{}'\right).$$

If we use (26), (25) and (20), we find the principal normal vector n^* of TN-Smarandache curve β as

$$\mathsf{n}^* = \frac{1}{\sqrt{\varphi_1^2+\varphi_2^2+\varphi_3^2+\varphi_4^2}}(\varphi_1\mathsf{T}+\varphi_2\mathsf{E}+\varphi_3\mathsf{D}+\varphi_4\mathsf{N}),$$

where

$$\varphi_1 = \kappa_n(\gamma_2\Psi_3-\gamma_3\Psi_2)+\tau_g^1(\gamma_4\Psi_3-\gamma_3\Psi_4),$$
$$\varphi_2 = \kappa_n(\gamma_3\Psi_1-\gamma_1\Psi_3+\gamma_3\Psi_4-\gamma_4\Psi_3),$$
$$\varphi_3 = \kappa_n(\gamma_1\Psi_2-\gamma_2\Psi_1+\gamma_4\Psi_2-\gamma_2\Psi_4)$$
$$+\tau_g^1(\gamma_1\Psi_4-\gamma_4\Psi_1),$$
$$\varphi_4 = \kappa_n(\gamma_2\Psi_3-\gamma_3\Psi_2)+\tau_g^1(\gamma_3\Psi_1-\gamma_1\Psi_3).$$

Using (2), the first curvature k_1^* of TN-Smarandache curve β is obtained as

$$k_1^* = \psi\,\Phi,$$

where

$$\psi = \frac{-\sqrt{2}}{\left(2\kappa_n^2+(\tau_g^1)^2\right)\sqrt{\varphi_1^2+\varphi_2^2+\varphi_3^2+\varphi_4^2}}$$

and

$$\Phi = \varphi_1(\kappa_n{}'+\kappa_n^2)+\varphi_2\left((\tau_g^1)'+\kappa_n\tau_g^1\right)$$
$$+\varphi_3\tau_g^1\kappa_g^2+\varphi_4\left(\kappa_n^2-\kappa_n{}'+(\tau_g^1)^2\right).$$

If we use (2), we calculate the second curvature k_2^* of TN-Smarandache curve β as

$$k_2^* = \frac{\varepsilon}{\Phi}\sum_{i=1}^{4}\gamma_i\xi_i$$

where

$$\varepsilon = \frac{-\sqrt{2(\varphi_1^2+\varphi_2^2+\varphi_3^2+\varphi_4^2)}}{\sqrt{2\kappa_n^2+(\tau_g^1)^2}\sqrt{\gamma_1^2+\gamma_2^2+\gamma_3^2+\gamma_4^2}}.$$

Also, we get

$$\beta^{(4)} = \frac{1}{\sqrt{2}}\Big((\xi_1{}'-\xi_4\kappa_n)\mathsf{T}+(\xi_2{}'-\xi_3\kappa_g^2-\xi_4\tau_g^1)\mathsf{E}$$
$$+(\xi_3{}'+\xi_2\kappa_g^2)\mathsf{D}+(\xi_4{}'+\xi_1\kappa_n+\xi_2\tau_g^1)\mathsf{N}\Big)$$

and from (2), we find the third curvature of TN-Smarandache curve β as

$$k_3^* = \Theta\Big(\Psi_1(\xi_1{}'-\xi_4\kappa_n)+\Psi_2(\xi_2{}'-\xi_3\kappa_g^2-\xi_4\tau_g^1)$$
$$+\Psi_3(\xi_{3'}+\xi_2\kappa_g^2)+\Psi_4(\xi_4{}'+\xi_1\kappa_n+\xi_2\tau_g^1)\Big),$$

where

$$\Theta = \frac{\sqrt{2(\gamma_1^2+\gamma_2^2+\gamma_3^2+\gamma_4^2)}}{\sqrt{2\kappa_n^2+(\tau_g^1)^2}\sqrt{\Psi_1^2+\Psi_2^2+\Psi_3^2+\Psi_4^2}\sum_{i=1}^{4}\gamma_i\xi_i}.$$

4. Conclusion

In this study, according to the ED-frame in $\mathbb{E}^4$, TE-Smarandache curve, TD-Smarandache curve and TN-Smarandache curve are defined and considering the ED-frame of second kind, the Frenet apparatus of these curves depending on the invariants of the ED-frame are obtained. Similarly, the other special Smarandache curves such as ED-Smarandache curve, EN-Smarandache curve, DN-Smarandache curve and etc. can be defined. Also considering the ED-frame of first kind, the Frenet apparatus of TE, TD, TN-Smarandache curves and the other special Smarandache curves depending on the invariants of the ED-frame can be calculated.

References

[1] M. Turgut, S. Yılmaz, “Smarandache curves in Minkowski space-time”, *International Journal of Mathematical Combinatorics*, vol. 3, pp. 51-55, 2008.

[2] A. T. Ali, “Special Smarandache curves in the Euclidean space”, *International Journal of Mathematical Combinatorics*, vol. 2, pp. 30-36, 2010.

[3] H. S. Abdel-Aziz, M. K. Saad, “Computation of Smarandache curves according to Darboux frame in Minkowski 3-space”, *Journal of the Egyptian Mathematical Society*, vol. 25, no. 4, pp. 382-390, 2017.

[4] Ö. Bektaş, S. Yüce, "Special Smarandache curves according to Darboux frame in E^3", *Romanian Journal of Mathematics and Computer Science*, vol. 3, no. 1, pp. 48-59, 2013.

[5] M. Elzawy, "Smarandache curves in Euclidean 4-space E^4", *Journal of the Egyptian Mathematical Society*, vol. 25, no. 3, pp. 268-271, 2017.

[6] H. S. Abdel-Aziz, M. K. Saad, "Smarandache curves of some special curves in the Galilean 3-space", *Honam Mathematical Journal*, vol. 37, no. 2, pp. 253-264, 2015.

[7] M. Elzawy, S. Mosa, "Smarandache curves in the Galilean 4-space G_4", *Journal of the Egyptian Mathematical Society*, vol. 25, no. 1, pp. 53-56, 2017.

[8] M. K. Saad, "Spacelike and timelike admissible Smarandache curves in pseudo-Galilean space", *Journal of the Egyptian Mathematical Society*, vol. 24, no. 3, pp. 416-423, 2016.

[9] M. Düldül, B. Uyar Düldül, N. Kuruoğlu, E. Özdamar, "Extension of the Darboux frame into Euclidean 4-space and its invariants", *Turkish Journal of Mathematics*, vol. 41, no. 6, pp. 1628-1639, 2017.

[10] M. Z. Williams, F. M. Stein, "A triple product of vectors in four-space", *Mathematics Magazine*, vol. 37, no. 4, pp. 230-235, 1964.

[11] O. Alessio, "Differential geometry of intersection curves in R^4 of three implicit surfaces", *Computer Aided Geometric Design*, vol. 26, no. 4, pp. 455-471, 2009.