Research Article

NC-Smarandache Ruled Surface and NW-Smarandache Ruled Surface according to Alternative Moving Frame in E^3

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In this study, original definitions of NC–Smarandache ruled surface and NW–Smarandache ruled surface according to the alternative moving frame are introduced in E^3. The main results of the study are presented in theorems that give necessary and sufficient conditions for those special surfaces to be developable and minimal. Finally, an example with illustrations is presented.

1. Introduction

In the classical differential geometry [1, 2], a ruled surface is a surface that can be swept out by moving a line in space. Hence, it is generated by a family of straight lines called rulings. It, therefore, has a parametrization of the form

$$\Psi: (s, v) \in I \times \mathbb{R} \mapsto \gamma(s) + v\overrightarrow{X}(s), \tag{1}$$

where I is an open interval of the real line $\mathbb{R}$, $\gamma = \gamma(s)$ is called the base curve of the ruled surface, and $\overrightarrow{X}(s)$ are the rulings directors of the ruled surface.

Ruled surfaces are useful and have practical applications in many areas such as mathematical physics, computer aided geometric design (CAGD), moving geometry, and kinematics.

The most important and interesting notions relative to the study of ruled surfaces are the developability and the minimalist notions.

Generally, developable ruled surfaces can be transformed into the plane without any deformation and distortion, and they form relatively small subsets that contain cylinders, cones, and tangent surfaces. They are characterized with vanishing Gaussian curvature [3–5]. On the contrary, minimal surfaces are surfaces that locally minimize their area. They refer to the fixed boundary curve of a surface

area that is minimal with respect to other surfaces with the same boundary. They are characterized with vanishing mean curvature [6–8].

Study of ruled surface according to some moving frames was and still being the point of interest for several researchers. Indeed, in [9], ruled surface was studied using the Darboux frame in E^3. Explicitly, we constructed the ruled surface Ψ generated by a curve c lying on an arbitrary regular surface φ and whose rulings are constant linear combinations of Darboux frame vectors of c on φ. We made a comparative study between both surfaces φ and Ψ along their common curve c, investigated properties of the built ruled surface Ψ, characterized it, and presented examples with illustrations.

However, in [10], we studied ruled surface using the alternative moving frame [11] in E^3. We built and studied the ruled surface whose rulings are constant linear combinations of alternative frame vectors of its base curve. We investigated properties of the constructed ruled surface, characterized it, and gave examples with illustrations in the case where the base curve is represented by some general helices [12] (resp. slant helices [13]).

In curve theory, Smarandache curves were introduced for the first time in Minkowski space-time by authors in [14]. Generally, a Smarandache curve defines a curve whose position vector is composed by a moving frame vectors on

another regular curve. In [15], one can find two Smarandache curves introduced according to the alternative moving frame. It concerns NC–Smarandache curves and NW–Smarandache curves.

In this study, we are inspired by Smarandache curves to introduce the notions of Smarandache ruled surface and NW–Smarandache ruled surface according to the alternative frame in E^3. We investigate theorems that give necessary and sufficient conditions for those two special ruled surfaces to be developable and minimal. Also, we derive some useful corollaries of the results. Furthermore, we present example with illustrations.

2. Preliminaries

Let E^3 be the Euclidean 3-space provided with the standard flat metric given by $\langle , \rangle = dx_1 + dx_2 + dx_3$, where (x_1, x_2, x_3) is a rectangular coordinate system of E^3. Recall that the norm of an arbitrary vector $\vec{u} \in E^3$ is given by $\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle}$.

Let

$$\Psi: (s, v) \in I \times \mathbb{R} \mapsto \gamma(s) + v\vec{X}(s) \tag{2}$$

be a ruled surface in E^3.

Let us denote by $\vec{m} = \vec{m}(s, v)$ the unit normal on the ruled surface Ψ at a regular point, and we have

$$\vec{m} = \frac{\Psi_s \wedge \Psi_v}{\|\Psi_s \wedge \Psi_v\|} = \frac{\left(\gamma' + v\vec{X}'\right) \times \vec{X}}{\left\|\left(\gamma' + v\vec{X}'\right) \times \vec{X}\right\|}, \tag{3}$$

where $\Psi_s = \partial\Psi(s, v)/\partial s$ and $\Psi_v = \partial\Psi(s, v)/\partial v$.

Definition 1 (see [16]). A ruled surface is developable if the three vectors γ', $\vec{X}$, and $\vec{X}'$ are linearly dependant.

The first I and the second II fundamental forms of ruled surface Ψ at a regular point are defined, respectively, by

$$\text{I} \left(\Psi_s ds + \Psi_v dv\right) = E ds^2 + 2F ds dv + G dv^2,$$
$$\text{II} \left(\Psi_s ds + \Psi_v dv\right) = e ds^2 + 2f ds dv + g dv^2, \tag{4}$$

where

$$E = \|\Psi_s\|^2,$$
$$F = \langle \Psi_s, \Psi_v \rangle,$$
$$G = \|\Psi_v\|^2,$$
$$e = \langle \Psi_{ss}, \vec{m} \rangle, \tag{5}$$
$$f = \langle \Psi_{sv}, \vec{m} \rangle,$$
$$g = \langle \Psi_{vv}, \vec{m} \rangle = 0.$$

Definition 2 (see [16]). The Gaussian curvature K and the mean curvature H of the ruled surface Ψ at a regular point are given, respectively, by

$$K = -\frac{f^2}{EG - F^2},$$
$$H = \frac{Ge - 2Ff}{2\left(EG - F^2\right)}. \tag{6}$$

Proposition 1 (see [16]). *A ruled surface is developable if and only if its Gaussian curvature vanishes.*

Proposition 2 (see [16]). *A regular surface is minimal if and only if its mean curvature vanishes.*

Due to a unit speed curve $c: s \in I \subset \mathbb{R} \longrightarrow E^3$ with nonvanishing second derivative, there exists the Frenet–Serret frame denoted $\left\{\vec{T}, \vec{N}, \vec{B}\right\}$, where $\vec{T} = c'$ is the unit tangent vector, $\vec{N} = c''/\|c''\|$ is the principal normal vector, and $\vec{B} = \vec{T} \times \vec{N}$ is the binormal vector of the curve $c = c(s)$, respectively.

Then, the Frenet–Serret frame formulae is given by

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix}, \tag{7}$$

where $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are the curvature and the torsion of the curve $c = c(s)$, respectively.

The unit Darboux vector $\vec{W}$ of the curve $c = c(s)$ is given by

$$\vec{W} = \frac{1}{\sqrt{\kappa^2 + \tau^2}} (\tau\vec{T} + \kappa\vec{B}), \tag{8}$$

which is the angular velocity vector of $c = c(s)$ [11]. One can clearly see that the unit Darboux vector $\vec{W}$ is perpendicular to the principal normal $\vec{N}$. Then, defining a unit vector $\vec{C}$ by the cross product $\vec{C} = \vec{W} \times \vec{N}$ makes it possible to build another orthonormal moving frame along the curve $c = c(s)$. This frame is defined by $\left\{\vec{N}, \vec{C}, \vec{W}\right\}$ and is called the alternative frame of the curve $c = c(s)$.

The derivative formulae of the alternative frame is given by

$$\begin{bmatrix} \vec{N}' \\ \vec{C}' \\ \vec{W}' \end{bmatrix} = \begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & \beta \\ 0 & -\beta & 0 \end{bmatrix} \begin{bmatrix} \vec{N} \\ \vec{C} \\ \vec{W} \end{bmatrix}, \tag{9}$$

where $\alpha = \sqrt{\kappa^2 + \tau^2}$ and $\beta = (\kappa^2/(\kappa^2 + \tau^2))(\tau/\kappa)'$ [11].

Definition 3 (see [15]). NC-Smarandache curves according to the alternative frame of the curve $c = c(s)$ are given by

$$^{NC}c\left(s^*(s)\right) = \frac{1}{\sqrt{2}} \left(\vec{N}(s) + \vec{C}(s)\right). \tag{10}$$

Definition 4 (see [15]). NW-Smarandache curves according to the alternative frame of the curve $c = c(s)$ are given by

$$^{\text{NW}}c\left(s^{*}\left(s\right)\right) = \frac{1}{\sqrt{2}}\left(\vec{N}\left(s\right) + \vec{W}\left(s\right)\right). \tag{11}$$

Theorem 1 (see [1]). $c = c(s)$ *is a general helix if and only if the ratio*

$$\frac{\tau}{\kappa}, \tag{12}$$

is constant along the curve.

Theorem 2 (see [17]). $c = c(s)$ *is a slant helix if and only if*

$$\sigma(s) = \frac{\kappa^2(s)}{\kappa^2(s) + \tau^2(s)}\left(\frac{\tau}{\kappa}(s)\right)', \tag{13}$$

is a constant function.

3. NC-Smarandache Ruled Surface and NW-Smarandache Ruled Surface according to the Alternative Frame in E^3

We start our section by giving new and original definitions of two Smarandache ruled surfaces according to the alternative frame of a curve in E^3.

Definition 5. Let $c = c(s)$ be the C^2-class differentiable unit speed curve in E^3. Let us denote by $\left\{\vec{N}, \vec{C}, \vec{W}\right\}$ the alternative frame of $c = c(s)$. The following ruled surfaces generated by NC-Smarandache curve and NW-Smarandache curve, which are defined, respectively, by

$$\begin{cases} ^{\text{NC}}\Psi\left(s, v\right) = \frac{1}{\sqrt{2}}\left(\vec{N}\left(s\right) + \vec{C}\left(s\right)\right) + v\vec{W}\left(s\right), ^{\text{NW}}\Psi(s, v) = \frac{1}{\sqrt{2}}\left(\vec{N}\left(s\right) + \vec{W}\left(s\right)\right) + v\vec{C}\left(s\right), \end{cases} \tag{14}$$

are called NC-Smarandache ruled surface and NW-Smarandache ruled surface, according to alternative moving frame of the curve $c = c(s)$, respectively.

In the rest of this section, we will investigate theorems that give necessary and sufficient conditions for both surfaces (14) to be developable and minimal.

Theorem 3. *NC-Smarandache ruled surface of (14) is developable if and only if the invariants of the alternative frame of the curve $c = c(s)$ assume the equality $\alpha\beta = 0$.*

Theorem 4. *NC-Smarandache ruled surface of (14) is minimal if and only if the invariants of the alternative frame of the curve $c = c(s)$ assume the equality $\alpha(-2\alpha^2 + \beta^2) + v\sqrt{2}\alpha(\alpha\beta + \alpha'\beta + \alpha^2\beta - \beta') - v^2 2\alpha^2\beta^2 = 0$.*

Proof. Differentiating the first line of (14) with respect to s and v, respectively, and using (9), we obtain

$$\begin{cases} ^{\text{NC}}\Psi_s = -\frac{\alpha}{\sqrt{2}}\vec{N} + \left(\frac{\alpha}{\sqrt{2}} - v\beta\right)\vec{C} + \frac{\beta}{\sqrt{2}}\vec{W}, ^{\text{NC}}\Psi_v = \vec{W}. \end{cases} \tag{15}$$

The cross product of these two vectors gives the normal vector on NC-Smarandache ruled surface of (14):

$$^{\text{NC}}\Psi_s \times {}^{\text{NC}}\Psi_v = \left(\frac{\alpha}{\sqrt{2}} - v\beta\right)\vec{N} + \frac{\alpha}{\sqrt{2}}\vec{C}$$
$$= \frac{1}{\sqrt{2}}\left[(\alpha - v\sqrt{2}\beta)\vec{N} + \alpha\vec{C}\right]. \tag{16}$$

Thus, under regularity condition, the unit normal takes the form

$$\frac{^{\text{NC}}\Psi_s \times {}^{\text{NC}}\Psi_v}{\left\|^{\text{NC}}\Psi_s \times {}^{\text{NC}}\Psi_v\right\|}$$
$$= \frac{1}{\sqrt{2\alpha^2 - v2\sqrt{2}\alpha\beta + v2\beta^2}}\left[(\alpha - v\sqrt{2}\beta)\vec{N} + \alpha\vec{C}\right]. \tag{17}$$

Making the norms for (15), we get the components of the first fundamental form of NC-Smarandache ruled surface of (14), at regular points, as follows:

$$\begin{cases} ^{\text{NC}}E = \frac{2\alpha^2 + \beta^2}{2} - v\sqrt{2}\alpha\beta + v^2\beta^2, {}^{\text{NC}}F = \frac{\beta}{\sqrt{2}}, {}^{\text{NC}}G = 1. \end{cases} \tag{18}$$

Differentiating (15) with respect to s and v, respectively, and using (9), we obtain

$$\begin{cases} ^{\text{NC}}\Psi_{ss} = -\frac{1}{\sqrt{2}}\left[\alpha' + \alpha^2 - v\sqrt{2}\alpha\beta\right]\vec{N} + \frac{1}{\sqrt{2}}\left[-\alpha^2 - \beta^2 + \alpha' - v\sqrt{2}\beta'\right]\vec{C} + \frac{1}{\sqrt{2}}\left[\beta' + \alpha\beta - v\sqrt{2}\beta^2\right]\vec{W}, {}^{\text{NC}}\Psi_{vs} = -\beta\vec{C}, {}^{\text{NC}}\Psi_{vv} = 0. \end{cases} \tag{19}$$

Hence, from (17) and (19), the components of the second fundamental form of NC-Smarandache ruled surface of (14) are obtained at regular points as follows:

$$\left\{ {}^{\mathrm{NC}}e = \frac{\alpha\left(-2\alpha^2 - \beta^2\right) + v\sqrt{2}\,\alpha\left(\alpha\beta + \alpha'\beta + \alpha^2\beta - \beta'\right) - v^2 2\alpha^2\beta^2}{\sqrt{2}\,\sqrt{2\alpha^2 - v2\sqrt{2}\,\alpha\beta + v2\beta^2}}, {}^{\mathrm{NC}}f = \frac{-\alpha\beta}{\sqrt{2\alpha^2 - v2\sqrt{2}\,\alpha\beta + v2\beta^2}}, {}^{\mathrm{NC}}g = 0. \right. \tag{20}$$

Consequently, from (18) and (20), we obtain the Gaussian curvature and the mean curvature of NC-Smarandache ruled surface of (14) at regular points:

$$\left\{ {}^{\mathrm{NC}}K = -2\left(\frac{\alpha\beta}{2\alpha^2 - v2\sqrt{2}\,\alpha\beta + v2\beta^2}\right)^2, {}^{\mathrm{NC}}H = \frac{\alpha\left(-2\alpha^2 + \beta^2\right) + v\sqrt{2}\,\alpha\left(\alpha\beta + \alpha'\beta + \alpha^2\beta - \beta'\right) - v^2 2\alpha^2\beta^2}{\sqrt{2}\left(\left(\alpha - v\sqrt{2}\,\beta\right)^2 + \alpha^2\right)^{3/2}}, \right. \tag{21}$$

which replies to both the above theorems.   □

Corollary 1. *If $c = c(s)$ is a planar curve (resp. general helix), then NC- Smarandache ruled surface is developable but nonminimal.*

Theorem 6. *NW-Smarandache ruled surface of (14) is developable.*

Theorem 7. *NW-Smarandache ruled surface of (14) is minimal if and only if $c = c(s)$ is a slant helix.*

Proof. Differentiating the second line of (14) with respect to s and v, respectively, and using (9), we obtain

$$\left\{ {}^{\mathrm{NW}}\Psi_s = -v\alpha\overrightarrow{N} + \frac{\alpha - \beta}{\sqrt{2}}\overrightarrow{C} + v\beta\overrightarrow{W}, {}^{\mathrm{NW}}\Psi_v = \overrightarrow{C}, \right. \tag{22}$$

which implies that the normal vector on NW-Smarandache ruled surface of (14) is

$$\left\{ {}^{\mathrm{NW}}\Psi_{ss} = -\frac{1}{\sqrt{2}}\left[\alpha(\alpha - \beta) + v\sqrt{2}\,\alpha'\right]\overrightarrow{N} + \frac{1}{\sqrt{2}}\left[\alpha' - \beta' - v\sqrt{2}\left(\alpha^2 + \beta^2\right)\right]\overrightarrow{C} + \frac{1}{\sqrt{2}}\left[\beta(\alpha - \beta) + v\sqrt{2}\,\beta'\right]\overrightarrow{W}, {}^{\mathrm{NW}}\Psi_{vs} = -\alpha\overrightarrow{N} + \beta\overrightarrow{W}, {}^{\mathrm{NW}}\Psi_{vv} = 0. \right. \tag{26}$$

Therefore, from (24) and (26), the components of the second fundamental form of NW-Smarandache ruled surface of (14) are obtained at regular points as follows:

$$\left\{ {}^{\mathrm{NW}}e = \frac{v\left(\alpha'\beta - \alpha\beta'\right)}{\sqrt{\alpha^2 + \beta^2}}, {}^{\mathrm{NW}}f = \frac{\alpha\beta - \alpha\beta}{\sqrt{\alpha^2 + \beta^2}} = 0, {}^{\mathrm{NW}}g = 0. \right. \tag{27}$$

$$ {}^{\mathrm{NW}}\Psi_s \times {}^{\mathrm{NW}}\Psi_v = -v(\beta\overrightarrow{N} + \alpha\overrightarrow{W}). \tag{23}$$

Thus, under regularity condition, the unit normal is given by

$$\frac{{}^{\mathrm{NW}}\Psi_s \times {}^{\mathrm{NW}}\Psi_v}{\left\|{}^{\mathrm{NW}}\Psi_s \times {}^{\mathrm{NW}}\Psi_v\right\|} = -\frac{1}{\sqrt{\alpha^2 + \beta^2}}(\beta\overrightarrow{N} + \alpha\overrightarrow{W}). \tag{24}$$

Making the norms for (22), we get the components of the first fundamental form of NW-Smarandache ruled surface of (14) at regular points:

$$\left\{ {}^{\mathrm{NW}}E = \frac{(\alpha - \beta)^2}{2} + v^2\left(\alpha^2 + \beta^2\right), {}^{\mathrm{NW}}F = \frac{\alpha - \beta}{\sqrt{2}}, {}^{\mathrm{NW}}G = 1. \right. \tag{25}$$

Differentiating (22) with respect to s and v, respectively, and using (9), we obtain

Consequently, from (25) and (27), we obtain the Gaussian curvature and the mean curvature of NW-Smarandache ruled surface of (14) at regular points:

$$\left\{ {}^{\mathrm{NW}}K = 0, {}^{\mathrm{NW}}H = \frac{\alpha\beta' - \alpha'\beta}{2v\left(\alpha^2 + \beta^2\right)^{3/2}}, \right. \tag{28}$$

which means clearly that NW-Smarandache ruled surface is developable. However, it is minimal if and only if

$\alpha\beta' - \alpha'\beta = 0$, so $(\beta/\alpha) = \sigma = $ constant, which is equivalent to have $c = c(s)$ as slant helix. □

Corollary 2. *If $c = c(s)$ is a planar curve (resp. general helix and resp. slant helix), then NW-Smarandache ruled surface is developable and minimal.*

Here follows, we give example investigating NC-Smarandache ruled surface and NW-Smarandache ruled surface according to alternative frame of a special cuvre. In addition, we present illustrations of both surfaces.

Example 1. Let us consider the regular curve $c(s) = (c_1(s), c_2(s), c_3(s))$ defined by

$$
\begin{cases}
c_1(s) = \dfrac{3}{\sqrt{2}} \sin(\sqrt{2}\,s)\cos(s) - 2\sin(s)\cos(\sqrt{2}\,s), \\[2mm]
c_2(s) = \dfrac{3}{\sqrt{2}} \cos(\sqrt{2}\,s)\cos(s) + 2\sin(s)\sin(\sqrt{2}\,s), \\[2mm]
c_3(s) = -\dfrac{1}{\sqrt{2}}\cos(s).
\end{cases}
\tag{29}
$$

The alternative frame vectors of curve (29) are given, respectively, by

$$
\overrightarrow{N}(s) = \begin{pmatrix} -\dfrac{1}{\sqrt{2}}\sin(\sqrt{2}\,s) \\[3mm] -\dfrac{1}{\sqrt{2}}\cos(\sqrt{2}\,s) \\[3mm] \dfrac{1}{\sqrt{2}} \end{pmatrix},
$$

$$
\overrightarrow{C}(s) = \begin{pmatrix} -\cos(\sqrt{2}\,s) \\[2mm] \sin(\sqrt{2}\,s) \\[2mm] 0 \end{pmatrix},
\tag{30}
$$

$$
\overrightarrow{W}(s) = \begin{pmatrix} -\dfrac{1}{\sqrt{2}}\sin(\sqrt{2}\,s) \\[3mm] -\dfrac{1}{\sqrt{2}}\cos(\sqrt{2}\,s) \\[3mm] -\dfrac{1}{\sqrt{2}} \end{pmatrix}.
$$

Hence, NC-Smarandache ruled surface according to the alternative frame (30) of curve (29) is given by the parametric representation:
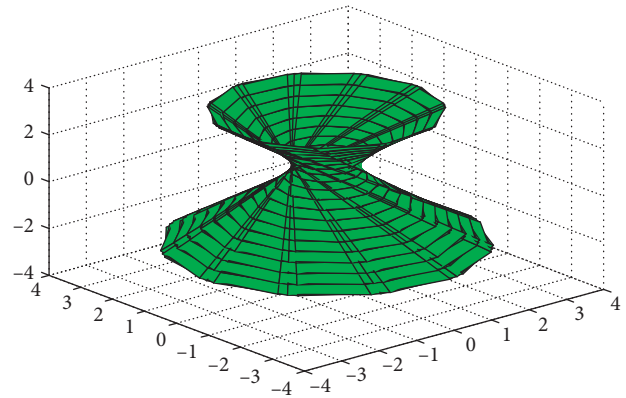

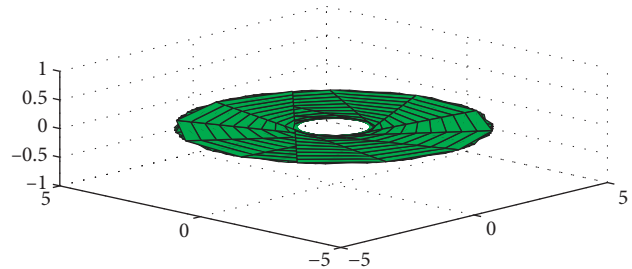FIGURE 1: NC-smarandache ruled surface (31).

FIGURE 2: NW-smarandache ruled surface (32).

$$
^{\mathrm{NC}}\Psi(s,v) = \frac{1}{\sqrt{2}} \begin{pmatrix} -\dfrac{1}{\sqrt{2}}\sin(\sqrt{2}\,s) - \cos(\sqrt{2}\,s) \\[3mm] -\dfrac{1}{\sqrt{2}}\cos(\sqrt{2}\,s) + \sin(\sqrt{2}\,s) \\[3mm] \dfrac{1}{\sqrt{2}} \end{pmatrix}
$$

$$
+ v \begin{pmatrix} -\dfrac{1}{\sqrt{2}}\sin(\sqrt{2}\,s) \\[3mm] -\dfrac{1}{\sqrt{2}}\cos(\sqrt{2}\,s) \\[3mm] -\dfrac{1}{\sqrt{2}} \end{pmatrix}.
\tag{31}
$$

In addidtion, its illustration is represented by the following Figure 1 which is drawn for $(s, v) \in\,]-2\pi, 2\pi[\,\times\,]-4, 4[$.

On the contrary, NW-Smarandache ruled surface according to the alternative frame (30) of curve (29) is given by

$$^{NW}\Psi(s,v) = \begin{pmatrix} -\sin(\sqrt{2}\,s) \\ -\cos(\sqrt{2}\,s) \\ 0 \end{pmatrix} + v\begin{pmatrix} -\cos(\sqrt{2}\,s) \\ \sin(\sqrt{2}\,s) \\ 0 \end{pmatrix}, \quad (32)$$

and its illustration is represented in Figure 2 which is drawn for $(s,v) \in\,]-2\pi, 2\pi[\, \times\,]-4, 4[$.

4. Conclusion

NC-Smarandache ruled surface and NW-Smarandache ruled surface, according to the alternative frame of a curve, are introduced in E^3. The results of the paper are presented in the form of theorems that give necessary and sufficient conditions for those special surfaces to be developable and minimal. The investigated characterizations give relationships according to the invariants of the alternative frame, the notion of general helix, and the notion of slant helix. Finally, an example is presented with illustrations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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