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RESEARCH ARTICLE

A Different View on Dynamics of Space Curves Geometry

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Abstract

In this study, we define the X-torque curves, $X-$equilibrium curves, X-moment conservative curves, $X-$gyroscopic curves as new curves derived from a regular space curve by using the Frenet vectors of a space curve and its position vector, where $X \in \{T(s), N(s), B(s)\}$ and we examine these curves and we give their properties.

Keywords Dual curves · Torque curves · Equilibrium · Gyroscobic curves

MSC Classification 53Z05 · 53A04 · 53A17

1 Introduction

From the physical point of view, the concepts of linear velocity and angular velocity and hence the concepts of angular moment and torque are very important in the dynamics of the motion of a body in a certain orbit. In terms of physics, unless an external force is applied to an object, the object is stationary and will not have a velocity and an orbit, therefore its moment vector and torque vector will not occur. When a force is applied to the object, the object will have a linear velocity and hence linear momentum, angular velocity and angular moment relative to a reference point taken in space. In addition, an orbit and angular moment of the object relative to the reference point and a torque vector relative to the position of the object will occur. The linear moment of a body of mass m with its linear velocity v is $P = mv$, and the moment vector (or angular moment vector) of the object relative to the point taken, with its position r relative to the reference point taken, is defined by $L = r \times P$. Torque vector is defined as the change of the angular moment of the object with respect to time during the motion of the object as $\Gamma = \frac{dL}{dt}$ such that

$$\Gamma = \frac{dL}{dt} = \frac{dr}{dt} \times P + r \times \frac{dP}{dt} \tag{1}$$

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where $\frac{dr}{dt} = v$, $\frac{dP}{dt} = F$ and since P and v are linearly dependent then we have $\Gamma = r \times F$ [1]. Γ is the torque vector caused by the force F acting on the object with respect to the reference point and its magnitude is the size of the torque vector. If the torque is zero, angular momentum is conserved during the movement, if the torque vector and angular moment vector are linearly dependent, the body rotates only in the plane perpendicular to the torque vector with respect to the reference point, if the torque vector and angular moment vector are perpendicular, the body has a gyroscopic effect [2]. There are three types of equilibrium in physics which are called stable, unstable and neutral systems. If the direction of the displacement vector of an object and the directions of the torque vector are opposite, the body is in stable equilibrium in its orbit. If the direction of the displacement vector of an object and the directions of the torque vector are the same, it means that the object is in unstable equilibrium in its orbit. If the balance of the object is independent of the displacement vector from its original position, it is said that the object is in a neutral equilibrium state in its orbit [3]. If we look at the subject in terms of differential geometry, we assume that every regular $\alpha(s)$ curve is an orbit of a moving body with unit mass and with unit velocity which is the same of the drawing velocity of the curve. The body has a helix motion at every point of the curve. This motion is called Frenet motion [4]. This motion is the movement of linear independent orthonormal $T(s)$, $N(s)$ and $B(s)$ vectors (which are called tangent, principal normal and binormal) at each point of the curve. During this movement, it has a curvature $\kappa(s)$ and torsion $\tau(s)$.

In this study, we will define and examine new curves by using the physical vector quantities such as torque, equilibrium moment conservative and gyroscope mentioned above in the theory of curves in differential geometry.

Let $\alpha(s)$ be a regular curve with arclenght parameter and with the Frenet vectors $T(s)$, $N(s)$ $B(s)$ and with the curvature $\kappa(s)$ and torsion $\tau(s)$. The variation of the Frenet frame according to time is defined as

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \kappa(s)\varphi(s) \\ 0 & -\kappa(s)\varphi(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix} \tag{2}$$

where $\varphi(s) = \frac{\tau(s)}{\kappa(s)}$ and the functions. $f(s)$, $g(s)$ and $h(s)$ are at least C^0–functions and so the position vector of the curve can be given as

$$\alpha(s) = f(s)T(s) + g(s)N(s) + h(s)B(s). \tag{3}$$

Here, the functions $f(s)$, $g(s)$ and $h(s)$ satisfy the relations

$$f'(s) - g(s)\kappa(s) = 1 \tag{4}$$

$$g'(s) + f(s)\kappa(s) - h(s)\kappa(s)\varphi(s) = 0 \tag{5}$$

$$h'(s) + g(s)\kappa(s)\varphi(s) = 0 \tag{6}$$

[5]. The planes $\{N(s), B(s)\}, \{T(s), B(s)\}$ and $\{T(s), N(s)\}$ at each point of the curve are called normal, rectifying and osculatory planes, respectively. An $\alpha(s)$ curve is called a normal, rectifying, and osculatory curve if the position vector lies in the normal, rectifying, and osculatory plane, respectively [6, 7]. Helices are known as famous Lancret's curves and occupy a very special place in the theory of curves. For a curve to be a generalized helix, the necessary and sufficient condition is $\varphi(s) =$ constant. Venant first proved this characterization [8–10]. In particular, if curvature and torsion are constant, it is a curved circular helix. If a curve is a linear combination of constant multiples of Frenet vectors of another curve, the curve is called a Smarandache curve [11]. Tuncer, defined and examined the moment vectors ($T-$dual, $N-$dual and $B-$dual curve) of the curve with respect to the origin of the vector by using $T(s)$, $N(s)$, $B(s)$ vector and by using the position vector of the curve [5]. Şenyurt et al. examined the vectorial moment of the unit Darboux vector [12]. Şenyurt and Çalışkan expressed and examined the vectorial moments in terms of alternative frame and they applied these to ruled surfaces [13]. If $T-$dual, $N-$dual and $B-$dual curves of a curve are denoted by $L_T(s)$, $L_N(s)$ and $L_B(s)$ according to the origin, then these curves are defined as

$$L_T(s) = \alpha(s) \times T(s) \tag{7}$$

$$L_N(s) = \alpha(s) \times N(s) \tag{8}$$

$$L_B(s) = \alpha(s) \times B(s) \tag{9}$$

respectively [5].

On the other hand, while a body with the unit mass and with the unit speed moving along the curve $\alpha(s)$, it has displacements on the $T-$direction, $N-$direction and $B-$direction at every moment of the motion due to the curvature and torsion of the $\alpha(s)$ which are geometrically deflected from the tangent direction (or rotation around the center of curvature) and deflected from the osculatory plane (or rotation around the torsion center).

Throughout this study, external forces such as gravity and friction were neglected.

2 T-torque curve, T-equilibrium and T-gyroscobic curve of a space curve

In displacement motion on $T-$direction along $\alpha(s)$, from equations (3) ve (7), $T-$dual curve according to origin is

$$L_T(s) = h(s)N(s) - g(s)B(s) \tag{10}$$

and as s changes, angular moment vector changes, $T-$torque vector is

$$\Gamma_T\left(s_*\right) = \frac{dL_T(s)}{ds} = \kappa(s)\alpha(s) \times N(s).$$

according to the origin. By using (3), we can give the following definition.

Definition 1 Let $\alpha(s)$ be at least regular C^3-curve with arc-length parameter, with Frenet vectors $T(s)$, $N(s)$, $B(s)$ and with the curvatures $\kappa(s)$ and $\tau(s)$. The curve $\Gamma_T\left(s_*\right) = \frac{dL_T(s)}{ds}$ is called $T-$torque curve of $\alpha(s)$ where s_* is the arc-length parameter, where $L_T(s)$ is the position vector of $T-$dual curve. We can express $\Gamma_T\left(s_*\right)$ as the following form

$$\Gamma_T\left(s_*\right) = -\kappa(s)h(s)T(s) + f(s)\kappa(s)B(s). \tag{11}$$

If $\frac{dL_T(s)}{ds} = 0$ then we called $\alpha(s)$ is $T-$moment conservative curve.

The position vector $\Gamma_T\left(s_*\right)$ always lies in rectifying plane of the curve $\alpha(s)$ and it is perpendicular to the principal normal vector. Let the parameter s_* be arc lenght parameter of $T-$torque curve then there is the relation

$$\frac{ds}{ds_*} = \frac{1}{\sqrt{\left(\{\kappa(s)h(s)\}'\right)^2 + \left(\kappa(s)^2(h(s) + \varphi(s)f(s))\right)^2 + \left(\{\kappa(s)f(s)\}'\right)^2}}$$

between the arclenght parameters of both the $T-$torque and the curve $\alpha(s)$. As the parameter s changes in an interval $I \subset IR$, the condition for the vector $\Gamma_T\left(s_*\right)$ to construct a regular curve is that $\Gamma'_T\left(s_*\right) \neq 0$, where

$$\Gamma'_T\left(s_*\right)\frac{ds_*}{ds} = \left\{ \begin{array}{c} -\{\kappa(s)h(s)\}'T(s) - \kappa(s)^2(h(s) + \varphi(s)f(s))N(s) \\ +\{\kappa(s)f(s)\}'B(s) \end{array} \right\}. \tag{12}$$

Definition 2 Let $\alpha(s)$ be a regular space curve, the curve $\alpha(s)$ is called unstable $T-$equilibrium curve, stable $T-$equilibrium curve if the vectors $\Gamma_T\left(s_*\right)$ and $T(s)$ are of the same direction, the vectors $\Gamma_T\left(s_*\right)$ and $T(s)$ are of the opposite direction at each points of the curve $\alpha(s)$, respectively. The curve $\alpha(s)$ is called neutral $T-$equilibrium curve if the vector $\Gamma_T\left(s_*\right)$ doesn't have any components on $T(s)$.

Definition 3 Let $\alpha(s)$ be a regular space curve, the curve $\alpha(s)$ is called $T-$gyroscobic curve, if the vectors $\Gamma_T\left(s_*\right)$ and $L_T(s)$ are perpendicular at each point of the curve $\alpha(s)$.

From the Eqs. (4), (5) and (6), it is easy to see that angular moment is conservative in the motion of displacement on direction $T(s)$ along the curve if and only if the curve $\alpha(s)$ is to be a linear orbit. If the $T-$torque vector is a constant vector then $\Gamma'_T\left(s_*\right) = 0$. From (12), for $\kappa(s)h(s) = c_0$ and $\kappa(s)f(s) = c_1$, we have $\varphi(s) = -\frac{c_0}{c_1}$ and from (4) and (6), we get $c_0 = h(s) = 0$ then, this requires the curve $\alpha(s)$ to be a planar curve. For this reason, we can say that there are no any curves, whose $T-$torque vectors are constant. For the same reason, $\Gamma_T\left(s_*\right)$ is not a Smarandache curve of $\alpha(s)$.

When the $T-$torque and $T-$moment vectors are perpendicular, then from (4) and (11), for $\kappa(s) \neq 0$, we get$g(s)f(s) = 0$, hence we can say all the rectifying and normal curves are $T-$gyroscobic curves.

If the $T-$torque and $T-$dual(angular moment) vectors are linearly dependent then from (4) and (11), for $\kappa(s) \neq 0$, we get $h(s) = 0$, $-c_0 g(s) = f(s)\kappa(s)$ and from (5), we have $g(s) = c_1 e^{c_0 s}$ and $f(s) = -\frac{c_0 c_1 e^{c_0 s}}{\kappa(s)}$ also from (4), following differential equation satisfies

$$c_0 c_1 e^{c_0 s} \kappa'(s) - c_1 e^{c_0 s} \kappa(s)^3 - \kappa(s)^2 - \left(c_0\right)^2 c_1 e^{c_0 s} \kappa(s) = 0.$$

In this case, $\alpha(s)$ is a planar curve and for these curves, we can say the motion of displacement on direction $T(s)$ is the rotation around the origin for the planar curves with nonzero curvature. The position vector of $\alpha(s)$ is

$$\alpha(s) = -\frac{c_0 c_1 e^{c_0 s}}{\kappa(s)} T(s) + c_1 e^{c_0 s} N(s),$$

the position vector of $L_T(s)$ is

$$L_T(s) = -c_1 e^{c_0 s} B(s),$$

and the position vector of $\Gamma_T\left(s_*\right)$ is

$$\Gamma_T\left(s_*\right) = c_0 L_T(s).$$

Here, the vector $B(s)$ is the constant normal vector of plane on which curve $\alpha(s)$ is drawn. If $\Gamma_T\left(s_*\right) = \epsilon T(s)$ for $\epsilon = \pm 1$, then we find $f(s) = 0$ and $\epsilon = -\kappa(s)h(s)$, from (4), we obtain $g(s) = \frac{1}{\kappa(s)}$, $h(s) = \frac{-\epsilon}{\kappa(s)} = -\epsilon g(s)$ from (5) and (6) we obtain

$$g(s) = -\epsilon \int \varphi(s) ds + c_0$$

also, we get

$$\kappa'(s) - \epsilon \tau(s) \kappa(s) = 0$$

the solution is

$$\kappa(s) = c e^{\epsilon \int \tau(s) ds}.$$

Thus, $g(s) = c e^{-\epsilon \int \tau(s) ds}$ and $h(s) = -\epsilon c e^{-\epsilon \int \tau(s) ds}$. In this case, the position vectors of $\alpha(s)$, $L_T(s)$ and $\Gamma_T\left(s_*\right)$ are

$$\alpha(s) = f(s)T(s) + c e^{-\epsilon \int \tau(s) ds} N(s) - \epsilon c e^{-\epsilon \int \tau(s) ds} B(s),$$
$$L_T(s) = -\epsilon c e^{-\epsilon \int \tau(s) ds} N(s) - c e^{-\epsilon \int \tau(s) ds} B(s)$$

and

$$\Gamma_T\left(s_*\right) = \epsilon \kappa(s) c e^{-\epsilon \int \tau(s) ds} T(s)$$

respectively. If $h(s) = 0$ in (11), then $T-$torque vector doesn't have any components on $T(s)$, in this case $\alpha(s)$ is a neutral $T-$equilibrium curve. We can give the following theorem.

Theorem 1 *For a regular space curve* $\alpha(s)$, *the followings are true.*

i. The curve $\alpha(s)$ is a line if and only if angular moment of displacement motion on direction $T(s)$ along the curve is conservative.
ii. There are no any space curves with constant $T-$torque vectors, but all the plane curves are of constant $T-$torque vector.
iii. All of the rectifying curves and normal curves are $T-$gyroscobic curves.
iv. If the position vector of $\alpha(s)$ is $\alpha(s) = -\frac{c_0 c_1 e^{c_0 s}}{\kappa(s)} T(s) + c_1 e^{c_0 s} N(s)$, then the motion of displacement on direction $T(s)$ of $\alpha(s)$ is the rotation around the origin.
v. The curve $\alpha(s)$ is stable or unstable $T-$equilibrium curve (for $\epsilon = -1$ or $\epsilon = 1$ respectively) if and only if its position vector is

$$\alpha(s) = f(s)T(s) + ce^{-\epsilon \int \tau(s)ds} N(s) - \epsilon ce^{-\epsilon \int \tau(s)ds} B(s)$$

for $f(s) \in C^0$.

vi. Osculating curves are neutral $T-$equilibrium curves.

In the case that $\alpha(s)$ is a generalised helix, then $\varphi(s) = \varphi = constant$ and from (5)

$$\left(1 - \frac{\kappa'(s)}{\kappa(s)}\right) g'(s) + g(s)\kappa(s)^2 \left(1 + \varphi^2\right) + \kappa(s) = 0$$

and we get the solution

$$g(s) = \left\{ \int \frac{r(s)ds}{\kappa(s)q(s)} + c_0 \right\} q(s)$$

also from (4) and (6), we get

$$f(s) = s + \int \left\{ \int \frac{r(s)ds}{\kappa(s)q(s)} + c_0 \right\} q(s)\kappa(s)ds + c_1$$
$$h(s) = -\varphi \int \left\{ \int \frac{r(s)ds}{\kappa(s)q(s)} + c_0 \right\} q(s)ds + c_2$$

where $q(s) = e^{(1+\varphi^2)\int r(s)ds}$ and $r(s) = \frac{\kappa(s)^3}{-\kappa(s)+\kappa'(s)}$. In the special case of $\alpha(s)$ is a circular helix then

$$g(s) = -\frac{1 - c_0\kappa\left(1 + \varphi^2\right)e^{-\kappa^2\left(1+\varphi^2\right)s}}{\kappa\left(1 + \varphi^2\right)}$$
$$f(s) = \frac{\kappa\varphi^2 s + c_0 e^{-\kappa^2\left(1+\varphi^2\right)s}}{\kappa\left(1 + \varphi^2\right)} + c_1$$
$$h(s) = \frac{\kappa\varphi s + c_0 e^{-\kappa^2\left(1+\varphi^2\right)s}}{\kappa\left(1 + \varphi^2\right)} + c_2.$$

Thus, we give the following remark.

Remark 1 For a regular space curve $\alpha(s)$,

i. If $\alpha(s)$ is a generalised helix then its $T-$dual curve is

$$L_T(s) = \left\{ \begin{array}{c} \left(-\varphi \int \left\{ \int \frac{r(s)ds}{\kappa(s)q(s)} + c_0 \right\} q(s)ds + c_2 \right) N(s) \\ -\left\{ \int \frac{r(s)ds}{\kappa(s)q(s)} + c_0 \right\} q(s)B(s) \end{array} \right\}$$

and its $T-$torque curve is

$$\Gamma_T\left(s_*\right) = -\kappa(s) \left\{ \begin{array}{c} \left(-\varphi \int \left\{ \int \frac{r(s)ds}{\kappa(s)q(s)} + c_0 \right\} q(s)ds + c_2 \right) T(s) \\ +\left(s + \int \left\{ \int \frac{r(s)ds}{\kappa(s)q(s)} + c_0 \right\} q(s)\kappa(s)ds + c_1 \right) B(s) \end{array} \right\}$$

where $q(s) = e^{\left(1+\varphi^2\right)\int r(s)ds}$ and $r(s) = \frac{\kappa(s)^3}{-\kappa(s)+\kappa'(s)}$.

ii. If $\alpha(s)$ is a circular helix then its $T-$dual curve is

$$L_T(s) = \left\{ \left(\frac{\kappa\varphi p(s)s + c_0}{\kappa\left(1 + \varphi^2\right)p(s)} + c_2 \right) N(s) + \frac{p(s) - c_0\kappa\left(1 + \varphi^2\right)}{\kappa\left(1 + \varphi^2\right)p(s)} B(s) \right\}$$

and its $T-$torque curve is

$$\Gamma_T\left(s_*\right) = -\kappa \left\{ \left(\frac{\kappa\varphi p(s)s + c_0}{\kappa\left(1 + \varphi^2\right)p(s)} + c_2 \right) T(s) + \left(\frac{\kappa\varphi^2 p(s)s + c_0}{\kappa\left(1 + \varphi^2\right)p(s)} + c_1 \right) B(s) \right\}$$

where $p(s) = e^{\kappa^2\left(1+\varphi^2\right)s}$.

3 N-torque curve, N-equilibrium and N-gyroscobic curve of a space curve

In displacement motion on N-direction along $\alpha(s)$, from (3) and (8), position vector of $N-$dual curve is

$$L_N(s) = -h(s)T(s) + f(s)B(s) \tag{13}$$

and as s changes, angular moment vector changes, $N-$torque vector is

$$\Gamma_N(s_*) = \frac{dL_N(s)}{ds} = -\kappa(s)\alpha(s) \times T(s) + \kappa(s)\varphi(s)\alpha(s) \times B(s) + B(s)$$

where s_* is the arc-lenght parameter. The vector $\Gamma_N(s_*)$ defines a curve, from (3), we can give the following definition.

Definition 4 Let $\alpha(s)$ be at least regular C^3-curve with arc-length parameter, wit Frenet vectors $T(s)$, $N(s)$, $B(s)$ and with the curvatures $\kappa(s)$ and $\tau(s)$. The curve $\Gamma_N(s_*) = \frac{dL_N(s)}{ds}$ is called $N-$torque curve of $\alpha(s)$ where s_* is the arc-length parameter, where $L_N(s)$ is the position vector of $N-$dual curve. We can express $\Gamma_N(s_*)$ as the following form

$$\Gamma_N(s_*) = \left\{ \begin{array}{c} \kappa(s)\varphi(s)g(s)T(s) - \kappa(s)Q(s)N(s) \\ +(1 + \kappa(s)g(s))B(s) \end{array} \right\} \tag{14}$$

where $Q(s) = h(s) + f(s)\varphi(s)$. If $\frac{dL_N(s)}{ds} = 0$, then we call $\alpha(s)$ is $N-$moment conservative curve.

There is the relation

$$\frac{ds}{ds_*} = \frac{1}{\sqrt{\left(\{\kappa(s)h(s)\}'\right)^2 + \kappa(s)^4 Q(s)^2 + \left(\{\kappa(s)f(s)\}'\right)^2}}$$

between the arc-lenght parameters of both the $N-$torque and the curve $\alpha(s)$. The regularity condition requires $\Gamma'_N(s_*) \neq 0$, where

$$\Gamma'_N(s_*)\frac{ds_*}{ds} = \left\{ \begin{array}{c} \left\{ \left[g(s)\kappa(s)\varphi(s)\right]' + \kappa(s)^2 Q(s) \right\} T(s) \\ -\left\{ \kappa(s)\varphi(s) + [\kappa(s)Q(s)]' \right\} N(s) \\ +\left\{ (1 + g(s)\kappa(s))' - \kappa(s)^2\varphi(s)Q(s) \right\} B(s) \end{array} \right\}. \tag{15}$$

Definition 5 Let $\alpha(s)$ be a regular space curve, the curve $\alpha(s)$ is called unstable $N-$equilibrium curve, stable $N-$equilibrium curve if the vectors $\Gamma_N(s_*)$ and $N(s)$ are of the same direction, the vectors $\Gamma_N(s_*)$ and $N(s)$ are of the opposite direction at each point of the curve $\alpha(s)$, respectively. The curve $\alpha(s)$ is called neutral $N-$equilibrium curve if the vector $\Gamma_B(s_*)$ doesn't have any components on $B(s)$.

Definition 6 Let $\alpha(s)$ be a regular space curve, the curve $\alpha(s)$ is called $N-$gyroscobic curve, if the vectors $\Gamma_N\left(s_*\right)$ and $L_N(s)$ are perpendicular at each point of the curve $\alpha(s)$.

If there is no change of $N-$dual(angular moment) vector, then $\Gamma_N\left(s_*\right) = 0$, in this case, from (14), $\varphi(s) = 0$, $g(s) = -\frac{1}{\kappa(s)}$ and $h(s) = 0$, and from (4), $f(s) = c_0$ so the curve $\alpha(s)$ is a planar curve, from (5), we obtain the differential equation

$$\kappa'(s) + c_0\kappa(s)^3 = 0.$$

The solution is $\kappa(s) = \frac{1}{\sqrt{2c_0 s + c1}}$, therefore, the curve

$$\alpha(s) = c_0 T(s) - \sqrt{2c_0 s + c_1} N(s)$$

is $N-$moment conservative curve.

Assume that $\alpha(s)$ is the curve with nonzero constant $N-$torque vector then $\Gamma'_N\left(s_*\right) = 0$, from (15), we have the following equations,

$$\left[g(s)\kappa(s)\varphi(s)\right]' + \kappa(s)^2 Q(s) = 0, \tag{16}$$

$$\kappa(s)\varphi(s) + [\kappa(s)Q(s)]' = 0, \tag{17}$$

$$(1 + g(s)\kappa(s))' - \kappa(s)^2\varphi(s)Q(s) = 0. \tag{18}$$

Let us take $\frac{(g(s)\kappa(s))'}{\varphi(s)} = \kappa(s)^2 Q(s)$ in (18) and replace this in (16), we obtain

$$\left[g(s)\kappa(s)\right]' + \frac{\varphi'(s)\varphi(s)}{\left(\varphi(s)^2 + 1\right)}\left[g(s)\kappa(s)\right] = 0,$$

we obtain

$$g(s)\kappa(s) = \frac{c_0}{\sqrt{\varphi(s)^2 + 1}} \tag{19}$$

and from (4) and (6), also we get

$$f(s) = s + c_0 \int \frac{1}{\sqrt{\varphi(s)^2 + 1}} ds + c_1$$

and

$$h(s) = -c_0 \int \frac{\varphi(s)}{\sqrt{\varphi(s)^2 + 1}} ds + c_2.$$

Therefore, if we rewrite $\frac{(g(s)\kappa(s))'}{\kappa(s)\varphi(s)} = \kappa(s)Q(s)$ in (17), then we obtain

$$\left[\frac{(g(s)\kappa(s))'}{\kappa(s)\varphi(s)}\right]' = \kappa(s)\varphi(s)$$

by using $g(s)\kappa(s) = \frac{c_0}{\sqrt{\varphi(s)^2+1}}$, we get

$$\left[\frac{(g(s)\kappa(s))'}{\kappa(s)\varphi(s)}\right]' = \int \kappa(s)\varphi(s)ds$$

and also

$$g(s)\kappa(s) = \frac{1}{2}\left(\int \kappa(s)\varphi(s)ds\right)^2 + c_3. \tag{20}$$

Thus, from (19) and (20), we obtain

$$\kappa(s) = \frac{c_0\varphi'(s)}{\left(\varphi(s)^2+1\right)^{5/4}\sqrt{2c_0 - 2c_3\sqrt{\varphi(s)^2+1}}}.$$

But, in this case, for these values of $f(s)$, $g(s)$ and $h(s)$, we find $\varphi(s) = const.$ from (5) and also $\kappa(s) = 0$. Thus, there are no any curves except lines, whose $N-$torque vectors are constant. In the case that the components on $T(s)$, $N(s)$ and $B(s)$ of $N-$torque curve are constant then from (14), we have the following equations

$$\kappa(s)\varphi(s)g(s) = c_0, \tag{21}$$

$$\kappa(s)(h(s) + \varphi(s)f(s)) = c_1, \tag{22}$$

$$(1 + \kappa(s)g(s)) = c_2. \tag{23}$$

From (4) and (23), we obtain $f(s) = c_2 s + c_3$ and $g(s)\kappa(s) = c_2 - 1$. By using (20), (6) and (22), we get $\varphi(s) = \frac{c_0}{c_2-1}$, $h(s) = -c_0 s + c_4$ and

$$\kappa(s) = \frac{c_1\left(c_2-1\right)}{c_0 s + c_0 c_3 + c_4\left(c_2-1\right)}.$$

If we rewrite these values in (5) after arranging, then we obtain

$$\left\{\begin{array}{c} \left\{\left(c_1\right)^2\left(\left(c_0\right)^2 + \left(c_2\right)^2 - c_2\right) + \left(c_0\right)^2\right\}s \\ +\left(\left(c_1\right)^2\left(c_2-1\right) + \left(c_0\right)^2\right)c_3 + \left(c_0\left(c_2-1\right) - c_0\left(c_1\right)^2\right)c_4 \end{array}\right\} = 0$$

where

$$c_2 = \frac{1}{2} + \frac{\pm\sqrt{(c_1)^2 - 4(c_0)^2\left((c_1)^2 - 1\right)}}{2c_1}$$

and

$$c_4 = \frac{c_3\left((c_1)^2 - (c_0)^2 - (c_1)^2 c_2\right)}{c_0\left(c_2 - (c_1)^2 - 1\right)}.$$

In this case, $\alpha(s)$ is a generalised helix and then, we give the following theorem.

Theorem 2 *Let $\alpha(s)$ be a regular space curve in E^3, then the followings are true.*

i. The curve $\alpha(s)$ is $N-$moment conservative if and only if its position vector is

$$\alpha(s) = c_0 T(s) - \sqrt{2c_0 s + c_1} N(s),$$

ii. There is nospace curve, whose $N-$torque vector is a constant vector, except lines in E^3,

iii. If the $N-$torque curve of $\alpha(s)$ is a Smarandach curve then $\alpha(s)$ is a generalised helix and its position vector is

$$\alpha(s) = \left(c_2 s + c_3\right)T(s) + \frac{c_0 s + c_0 c_3 + c_4\left(c_2 - 1\right)}{c_1}N(s) + \left(-c_0 s + c_4\right)B(s),$$

whose curvature and torsion are

$$\kappa(s) = \frac{c_1\left(c_2 - 1\right)}{c_0 s + c_0 c_3 + c_4\left(c_2 - 1\right)}, \tau(s) = \frac{c_0 c_1}{c_0 s + c_0 c_3 + c_4\left(c_2 - 1\right)}$$

Also, position vectors of the $N-$dual curve and the $N-$torque curves are

$$L_N(s) = \left(c_0 s - c_4\right)T(s) + \left(c_2 s + c_3\right)B(s)$$

and

$$\Gamma_N\left(s_*\right) = c_0 T(s) - c_1 N(s) + c_2 B(s)$$

respectively, where c_0 and c_1 are non-zero constants, $c_2 \neq 1$ and $c_2 = \frac{1}{2} + \frac{\pm\sqrt{(c_1)^2 - 4(c_0)^2\left((c_1)^2 - 1\right)}}{2c_1}, c_4 = \frac{c_3\left((c_1)^2 - (c_0)^2 - (c_1)^2 c_2\right)}{c_0\left(c_2 - (c_1)^2 - 1\right)}$.

If position vectors of $N-$torque and $N-$dual curves are linearly dependent then at first, from (13) and (14), later by using (4), (5) and (6), we obtain $f(s) = \varepsilon c_0 e^{\varepsilon s}$, $\varphi(s) = h(s) = 0$, $g(s) = \pm\varepsilon\sqrt{c_1 + 2c_0\varepsilon e^{\varepsilon s} - \left(c_0\right)^2 e^{2\varepsilon s}}$ and $\kappa(s) = \frac{c_0 \varepsilon e^{\varepsilon s} - 1}{\pm\varepsilon\sqrt{c_1 + 2c_0\varepsilon e^{\varepsilon s} - (c_0)^2 e^{2\varepsilon s}}}$.

Thus, $L_N(s) = c_0 e^{\varepsilon s} B(s)$ and $\Gamma_N\left(s_*\right) = c_0 \varepsilon e^{\varepsilon s} B(s)$, in this case $\alpha(s)$ is a plane curve on the plane with unit constant normal vector $B(s)$.

Now, assume that position vectors of $N-$torque and $N-$dual curves are perpendicular then from (13) and (14), we have

$$f(s)(1+\kappa(s)g(s)) - \kappa(s)\varphi(s)g(s)h(s) = 0$$

by using (4) and (6), we get

$$f(s)f'(s) + h(s)h'(s) = 0$$

Here, we can take $f(s) = c_0 \cos\psi(s)$ and $h(s) = c_0 \sin\psi(s)$, where $\psi(s)$ is the angle between the position vector of $\alpha(s)$ and $T(s)$. Therefore, from (4) and (6)

$$\varphi(s) = \frac{c_0\psi'(s)\cos\psi(s)}{1 + c_0\psi'(s)\sin\psi(s)}$$

and

$$g(s)\kappa(s) = -\left(1 + c_0\psi'(s)\sin\psi(s)\right)$$
$$\kappa(s) = -\frac{\left(1 + c_0\psi'(s)\sin\psi(s)\right)}{g(s)}$$

for $\psi'(s) \neq 0$, from (5) we obtain

$$g'(s)g(s) - c_0\cos\psi(s) = 0$$

so the solution is

$$g(s) = \pm\sqrt{c_1 + 2c_0\left(\int \cos\psi(s)ds\right)}.$$

Let the vectors $\left\{\Gamma_N\left(s_*\right), N(s)\right\}$ be linearly dependent such as $\Gamma_N\left(s_*\right) = \epsilon N(s)$ for $\epsilon = \pm 1$. We have $\varphi(s)g(s) = 1 + \kappa(s)g(s) = 0$ and then we obtain $\varphi(s) = 0$, $g(s) = \frac{-1}{\kappa(s)}$, from (4) and (5), we obtain $f(s) = c_0$ and

$$g'(s)g(s) - c_0 = 0$$

The solution is $g(s) = \pm\sqrt{c_1 + 2c_0 s}$. Also from (6), we get $h(s) = c_2$. Thus, $\Gamma_N\left(s_*\right) = -\kappa(s)c_2 N(s)$, where $\kappa(s) = \frac{-\epsilon}{c_2}$, in this case, c_0 has to be zero.

Let the $N-$torque vector be independent from $N(s)$ then $Q(s) = h(s) + f(s)\varphi(s) = 0$ for $\kappa(s) \neq 0$. From (4) and (6), we obtain $f(s) = -\frac{\varphi(s)}{\varphi'(s)}$, $h(s) = \frac{\varphi(s)^2}{\varphi'(s)}$ and from (5) we get

$$g'(s)g(s) - \frac{\varphi(s)^2\varphi''(s)\left(1+\varphi(s)^2\right)}{\varphi'(s)^3} = 0$$

so the solution is

$$g(s) = \pm\sqrt{c_1 + 2\int \frac{\varphi(s)^2\varphi''(s)\left(1+\varphi(s)^2\right)}{\varphi'(s)^3}ds}.$$

Hence, we have the following theorem.

Theorem 3 *Let $\alpha(s)$ be a regular space curve in E^3, then the followings are true.*

i. The displacement motion on N-direction along $\alpha(s)$ is rotation around of origin if the position vector of $\alpha(s)$ is

$$\alpha(s) = \varepsilon c_0 e^{\varepsilon s}T(s) \pm \varepsilon\sqrt{c_1 + 2c_0\varepsilon e^{\varepsilon s} - \left(c_0\right)^2 e^{2\varepsilon s}}N(s),$$

ii. If the curve $\alpha(s)$ is $N-$gyroscobic curve, then for any C^0- function $g(s)$, the position vector $\alpha(s)$ is

$$\alpha(s) = c_0\cos\psi(s)T(s) \pm \sqrt{c_1 + 2c_0\left(\int \cos\psi(s)ds\right)}N(s) + c_0\sin\psi(s)B(s),$$

where $\psi(s)$ is the angle between the position vector of $\alpha(s)$ and $T(s)$.

iii. The curve $\alpha(s)$ is stable or unstable $N-$equilibrium curve if and only if its position vector is

$$\alpha(s) = \pm\sqrt{c_1}N(s) + c_2B(s),$$

iv. If the curve $\alpha(s)$ is neutral $N-$ equilibrium curve then its position vector is

$$\alpha(s) = -\frac{\varphi(s)}{\varphi'(s)}T(s) \pm \sqrt{c_1 + 2\int \frac{\varphi(s)^2\varphi''(s)\left(1+\varphi(s)^2\right)}{\varphi'(s)^3}ds}N(s) + \frac{\varphi(s)^2}{\varphi'(s)}B(s).$$

4 B-torque curve, B-equilibrium and B-gyroscobic curve of a space curve

In displacement motion on $B-$direction along $\alpha(s)$, from (3) and (9), $B-$dual curve is defined by

$$L_B(s) = g(s)T(s) - f(s)N(s) \tag{24}$$

and the $B-$torque vector which is denoted by $\Gamma_B\left(s_*\right)$ is defined as

$$\Gamma_B\left(s_*\right) = \frac{dL_B(s)}{ds} = -N(s) - \kappa(s)\varphi(s)\alpha(s)\Lambda N(s)$$

according to origin. From (3), we can give the following definition.

Definition 7 Let $\alpha(s)$ be at least regular C^3-curve with arc-length parameter, Frenet vectors $T(s)$, $N(s)$, $B(s)$ and curvature and torsion $\kappa(s)$, $\tau(s)$. The curve $\Gamma_B\left(s_*\right) = \frac{dL_B(s)}{ds}$ is called $B-$torque curve of $\alpha(s)$ where s_* is the arc-length parameter, where $L_B(s)$ is the position vector of $B-$dual curve. We can express $\Gamma_B\left(s_*\right)$ as the following form

$$\Gamma_B\left(s_*\right) = \kappa(s)\varphi(s)h(s)T(s) - N(s) - \kappa(s)\varphi(s)f(s)B(s). \tag{25}$$

If $\frac{dL_B(s)}{ds} = 0$, then we called $\alpha(s)$ is $B-$moment conservative curve.

As the parameter s changes, the vector $\Gamma_B\left(s_*\right)$ draws a curve, we call this curve as the $B-$torque curve of $\alpha(s)$, where s_* is the arc-lenght parameter of $B-$torque curve. The condition of regularity of the curve $\Gamma_B\left(s_*\right)$ requires $\Gamma'_B\left(s_*\right) \neq 0$ which is

$$\Gamma'_B\left(s_*\right)\frac{ds_*}{ds} = \left\{ \begin{array}{c} \left\{[\kappa(s)\varphi(s)h(s)]' - \kappa(s)\right\}T(s) + \kappa(s)^2\varphi(s)Q(s)N(s) \\ -\left\{\kappa(s)\varphi(s) + \left[\kappa(s)\varphi(s)f(s)\right]'\right\}B(s) \end{array} \right\} \tag{26}$$

where

$$\frac{ds}{ds_*} = \frac{1}{\sqrt{\left\{ \begin{array}{c} \left\{[\kappa(s)\varphi(s)h(s)]' - \kappa(s)\right\}^2 + \kappa(s)^4\varphi(s)^2Q(s)^2 \\ +\left\{\kappa(s)\varphi(s) + \left[\kappa(s)\varphi(s)f(s)\right]'\right\}^2 \end{array} \right\}}}.$$

and $Q(s) = h(s) + f(s)\varphi(s)$.

Definition 8 Let $\alpha(s)$ be a regular space curve, the curve $\alpha(s)$ is called unstable $B-$equilibrium curve, stable $B-$equilibrium curve if the vectors $\Gamma_B\left(s_*\right)$ and $B(s)$ are of the same direction, the vectors $\Gamma_B\left(s_*\right)$ and $B(s)$ are of the opposite direction at each point of the curve $\alpha(s)$, respectively. The curve $\alpha(s)$ is called neutral $B-$equilibrium curve if the vector $\Gamma_B\left(s_*\right)$ does not have any components on $B(s)$.

Definition 9 Let $\alpha(s)$ be a regular space curve, the curve $\alpha(s)$ is called $B-$gyroscobic curve, if the vectors $\Gamma_B\left(s_*\right)$ and $L_B(s)$ are perpendicular at each point of the curve $\alpha(s)$.

It is clear from (25) that space curves are not $B-$moment conservative curves. If $\Gamma'_B\left(s_*\right) = 0$, then from (26), we have the followings,

$$[\kappa(s)\varphi(s)h(s)]' - \kappa(s) = 0, \tag{27}$$

$$\kappa(s)^2\varphi(s)Q(s) = 0, \tag{28}$$

$$\kappa(s)\varphi(s) + \left[\kappa(s)\varphi(s)f(s)\right]'. \tag{29}$$

When the Eqs. (4), (5) and (6) are considered, there are no any solutions for (27), (28) and (29) except $\kappa(s) = 0$, so there are no any space curves, whose $B-$torque vectors are constant in E^3. Assume that the coefficients on $T(s)$ and $B(s)$ of $\Gamma_B(s_*)$ are constants such that $\kappa(s)\varphi(s)h(s) = c_0$ and $\kappa(s)\varphi(s)f(s) = c_1$ then $h(s) = \frac{c_0}{c_1}f(s)$ and from (6), we have $\kappa(s)g(s) = \frac{-c_0}{c_1\varphi(s)+c_0}$ and by using (4) and (6), we obtain $f(s) = c_1 \int \frac{\varphi(s)}{c_1\varphi(s)+c_0}ds + c_3$ and $h(s) = c_0 \int \frac{\varphi(s)}{c_1\varphi(s)+c_0}ds + \frac{c_0c_3}{c_1}$. From (5), we have

$$g(s)g'(s) - \frac{c_0\left(c_1 - c_0\varphi(s)\right)}{c_1\varphi(s) + c_0}\left\{\int \frac{\varphi(s)}{c_1\varphi(s) + c_0}ds + \frac{c_0c_3}{c_1}\right\} = 0$$

and we get the solution

$$g(s) = \pm\sqrt{2\int \frac{c_0\left(c_1 - c_0\varphi(s)\right)}{c_1\varphi(s) + c_0}\left\{\int \frac{\varphi(s)}{c_1\varphi(s) + c_0}ds + \frac{c_0c_3}{c_1}\right\}ds + c_4}$$

In this case, the curve $\Gamma_B(s_*)$ is the Smarandach curve and the position vector is

$$\Gamma_B(s_*) = \left\{\begin{array}{c} \kappa(s)\varphi(s)\left(c_1 \int \frac{\varphi(s)}{c_1\varphi(s)+c_0}ds + c_3\right)T(s) - N(s) \\ -\kappa(s)\varphi(s)\left(c_0 \int \frac{\varphi(s)}{c_1\varphi(s)+c_0}ds + \frac{c_0c_3}{c_1}\right)B(s) \end{array}\right\}.$$

In the case of the position vectors of $\Gamma_B(s_*)$ and $L_B(s)$ are perpendicular then from (24) and (25), we get

$$f(s) = h'(s)h(s) \tag{30}$$

from (4), (5) and (6), we have the following equations

$$\begin{aligned} h'(s)^2 + h(s)h''(s) + g(s)\kappa(s) + 1 &= 0 \\ g'(s) - h(s)\kappa(s)\left(h'(s) + \varphi(s)\right) &= 0 \\ h'(s) + g(s)\kappa(s)\varphi(s) &= 0. \end{aligned} \tag{31}$$

For $\varphi(s) \neq 0$, the common solution of (31) is

$$\begin{aligned} \varphi(s) &= \frac{-h'(s)}{h'(s)^2 + h(s)h''(s) - 1} \\ \kappa(s) &= \frac{h'(s)^2 + h(s)h''(s) - 1}{g(s)} \\ g(s) &= \pm\sqrt{c_0 - h^2(s)h'(s)^2}. \end{aligned} \tag{32}$$

In this case, the position vectors of $B-$dual curve and $B-$torque curve are

$$L_B(s) = \pm\sqrt{c_0 - h^2(s)h'(s)^2}T(s) - h'(s)h(s)N(s)$$

and

$$\Gamma_B(s_*) = \kappa(s)\varphi(s)h(s)T(s) - N(s) - \kappa(s)\varphi(s)h'(s)h(s)B(s)$$

respectively.

On the other hand, it is easy to see that since $\{\Gamma_B(s_*), B(s)\}$ cannot be inearly dependent such that $\Gamma_B(s_*) = \pm B(s)$, then there are no stable or unstable B–equilibrium curves. From (25), if $f(s) = 0$ then binormal vector is perpendicular to $\Gamma_B(s_*)$, so we can say all the normal curves are neutral B–equilibrium curves. Thus, we give the following theorem.

Theorem 4 *Let $\alpha(s)$ be a space curve then, the followings are true.*

i. There are no any B–moment conservative space curves in E^3,

ii. There are no any space curves whose B–torque vectors are constant vectors, except lines in E^3,

iii. if $\Gamma_B(s_*)$ is a Smarandach curve, then position vector of $\alpha(s)$ is

$$\alpha(s) = \left\{ \begin{array}{l} \left(c_1 \int \frac{\varphi(s)}{c_1\varphi(s)+c_0} ds + c_3\right)T(s) + \left(c_0 \int \frac{\varphi(s)}{c_1\varphi(s)+c_0} ds + \frac{c_0c_3}{c_1}\right)B(s) \\ \pm\left(\sqrt{2\int \frac{c_0(c_1 - c_0\varphi(s))}{c_1\varphi(s)+c_0}\left\{\int \frac{\varphi(s)}{c_1\varphi(s)+c_0} ds + \frac{c_0c_3}{c_1}\right\} ds + c_4}\right)N(s) \end{array} \right\}.$$

iv. $\alpha(s)$ is a B–gyroscobic curve with $\varphi(s) = \frac{-h'(s)}{h'(s)^2 + h(s)h''(s) - 1}$ and $\kappa(s) = \frac{h'(s)^2 + h(s)h''(s) - 1}{\pm\sqrt{c_0 - h^2(s)h'(s)^2}}$ if and only if the position vector of $\alpha(s)$ is

$$\alpha(s) = h'(s)h(s)T(s) \pm \sqrt{c_0 - h^2(s)h'(s)^2}N(s) + h(s)B(s)$$

for $h(s) \in C^0$.

v. There is no any space curve which are stable or unstable B–equilibrium in E^3,

vi. All the normal curves are neutral B–equilibrium curves.

vii. The space curve $\alpha(s)$ is neutral B–equilibrium curve if and only if $\alpha(s)$ is a normal curve.

In this part of our study, we give some generic properties about vectorial moments of space curves and torque curves. Assume that total angular moments vectors of a space curve $\alpha(s)$ satisfy

$$L_T(s) + L_N(s) + L_B(s) = 0,$$

then from the Eqs. (3), (4), (5) and (6), we have

$$(g(s) - h(s))T(s) + (h(s) - f(s))N(s) + (f(s) - g(s))B(s) = 0.$$

Since $\{T(s), N(s), B(s)\}$ is linearly independent,

$$f(s) = g(s) = h(s)$$

For $f(s) = g(s) = h(s) = \mu(s)$ Eqs. (4), (5) and (6) turn to

$$\begin{aligned} \mu'(s) - \mu(s)\kappa(s) &= 1 \\ \mu'(s) + \mu(s)\kappa(s) - \mu(s)\kappa(s)\varphi(s) &= 0 \\ \mu'(s) + \mu(s)\kappa(s)\varphi(s) &= 0 \end{aligned}$$

and the common solution for $\mu(s)$, $\kappa(s)$ and $\varphi(s)$, obtained as $\mu(s) = \frac{1}{3}s + c_0$, $\kappa(s) = -\frac{2}{s+3c}$, $\varphi(s) = \frac{1}{2}$. Thus $f(s) = g(s) = h(s) = \frac{1}{3}s + c_0$ and $\tau(s) = 2\kappa(s)$. On the contrary, we assume that the curve $\alpha(s)$ is a curve with position vector

$$\alpha(s) = \left(\frac{1}{3}s + c_0\right)\{T(s) + N(s) + B(s)\}.$$

From the equation (4), we obtain $\varphi(s) = \frac{1}{2}$, $\kappa(s) = -\frac{2}{s+3c}$ and also we get $\tau(s) = 2\kappa(s)$, thus $\alpha(s)$ is a generalised helix with total vectorial moment zero.

If the total torque vectors is zero, $\Gamma_T(s_*) + \Gamma_N(s_*) + \Gamma_B(s_*) = 0$, then from (11), (14) and (25), we can write

$$\left\{ \begin{array}{c} \{\kappa(s)\varphi(s)g(s) + (\varphi(s) - 1)\kappa(s)h(s)\}T(s) \\ -\{\kappa(s)Q(s) + 1\}N(s) \\ +\{1 + \kappa(s)g(s) - (\varphi(s) - 1)f(s)\kappa(s)\}B(s) \end{array} \right\} = 0 \tag{33}$$

where $Q(s) = h(s) + f(s)\varphi(s)$ and we have

$$-h'(s) + (\varphi(s) - 1)\kappa(s)h(s) = 0 \tag{34}$$

$$\kappa(s)\{h(s) + \varphi(s)f(s)\} + 1 = 0 \tag{35}$$

$$f'(s) - (\varphi(s) - 1)f(s)\kappa(s) = 0. \tag{36}$$

by using $f'(s) = 1 + \kappa(s)g(s)$ and $h'(s) = -\kappa(s)\varphi(s)g(s)$ from (4) and (6). From the Eqs. (34) and (36), we get

$$f(s) = c_0 e^{\int (\varphi(s)-1)\kappa(s)ds}$$

and

$$h(s) = c_1 e^{\int (\varphi(s)-1)\kappa(s)ds}.$$

By rewriting these values in (35), we get

$$e^{\int (\varphi(s)-1)\kappa(s)ds} = -\frac{1}{\kappa(s)\left(\varphi(s)c_0 + c_1\right)} \tag{37}$$

From (5), we have

$$g(s) = \int \kappa(s)\big(\varphi(s)c_1 - c_0\big)e^{\int (\varphi(s)-1)\kappa(s)ds}ds + c_2$$

from (37), we get

$$g(s) = -s + c_2$$

for $\kappa(s)\big(\varphi(s)c_0 + c_1\big) \neq 0$. Thus, the position vector is

$$\alpha(s) = -\frac{c_0}{\kappa(s)\big(\varphi(s)c_0 + c_1\big)}T(s) + (-s + c_2)N(s) - \frac{c_1}{\kappa(s)\big(\varphi(s)c_0 + c_1\big)}B(s).$$

We give the following theorem.

Theorem 5 *Let $\alpha(s)$ be a regular space curve in E^3, then the followings are true.*

i. A regular space curve $\alpha(s)$ is the curve with zero total vectorial moment curve if and only if $\alpha(s)$ is the generalized helix with the position vector

$$\alpha(s) = \left(\frac{1}{3}s + c_0\right)\{T(s) + N(s) + B(s)\}.$$

ii. If a regular space curve $\alpha(s)$ is the curve with zero total torque curve, then the curvature and the torsion satisfy $e^{\int (\varphi(s)-1)\kappa(s)ds} = -\frac{1}{\kappa(s)\left(\varphi(s)c_0+c_1\right)}$. In this case, the position vector of $\alpha(s)$ is

$$\alpha(s) = -\frac{c_0}{\kappa(s)\big(\varphi(s)c_0 + c_1\big)}T(s) + (-s + c_2)N(s) - \frac{c_1}{\kappa(s)\big(\varphi(s)c_0 + c_1\big)}B(s).$$

iii. There is a relation between $N-$dual curve and $T-$torque curve such that

$$\Gamma_T\big(s_*\big) = \kappa(s)L_N(s).$$

iv. There is a relation between $T-$dual curve, $B-$dual curve, $N-$torque curve and binormal vector such that

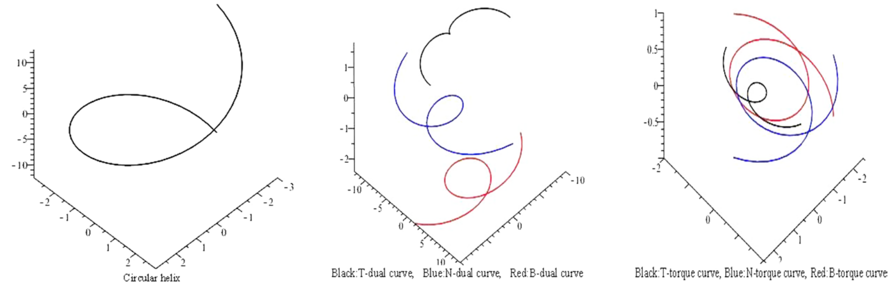
$$\Gamma_N\big(s_*\big) = -\kappa(s)L_T(s) + \kappa(s)\varphi(s)L_B(s) + B(s).$$

v. There is a relation between $T-$torque curve, $B-$torque curve and principal normal vector such that

$$\Gamma_B\big(s_*\big) = -\varphi(s)\Gamma_T\big(s_*\big) - N(s).$$

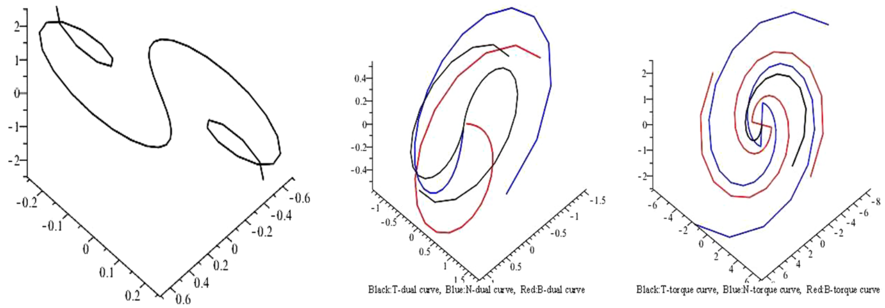
Example 1 Circular helix, T-dual, N-dual, B-dual, T-torque, N-torque, and B-torque curves (Fig. 1).

Example 2 Euler spiral curve



(a) Circular helix. **(b)** Dual curves of circular helix. **(c)** Torque curves of circular helix.

Fig. 1 Dual curves and torque curves of a circular helix



(a) Euler spiral. **(b)** Dual curves of Euler spiral. **(c)** Torque curves of Euler spiral.

Fig. 2 Dual curves and torque curves of Euler spiral

$$\alpha(s) = \left(\frac{3}{5}\int \sin\left(s^2+1\right)ds, \frac{3}{5}\int \cos\left(s^2+1\right)ds, \frac{4}{5}s\right)$$

[14] (Fig. 2).

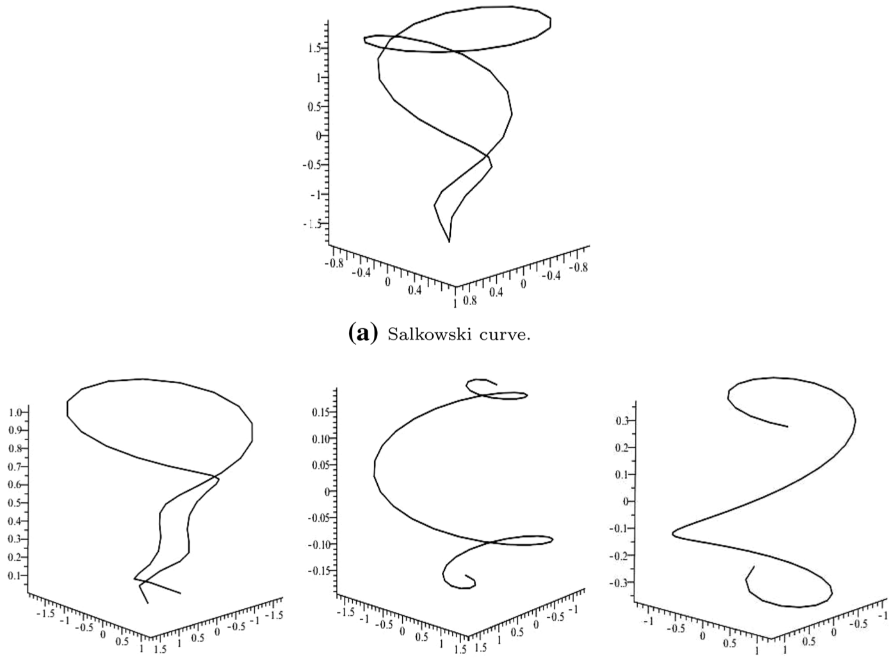
Example 3 Salkowski curve $\alpha_m(s) = \frac{1}{\sqrt{1+m^2}}\left(\alpha_m^1(s), \alpha_m^2(s), \alpha_m^3(s)\right)$, $n = \frac{m}{\sqrt{1+m^2}}$, $m \neq \frac{\pm 1}{\sqrt{3}}, m \neq 0, m \in IR$ and

$$\begin{aligned}
\alpha_m^1(s) =& \frac{-(1-n)\sin((1+2n)s)}{4(1+2n)} - \frac{(1+n)\sin((1-2n)s)}{4(1-2n)} - \frac{\sin(s)}{2},\\
\alpha_m^2(s) =& \frac{(1-n)\cos((1+2n)s)}{4(1+2n)} + \frac{(1+n)\cos((1-2n)s)}{4(1-2n)} + \frac{\cos(s)}{2},\\
\alpha_m^3(s) =& \frac{\cos(2ns)}{4m},
\end{aligned}$$

for $m = \frac{1}{8}$ [14, 15] (Figs. 3, 4).

5 Conclusions

The physical investigation of the movement of the Frenet frame along the curve in the theory of curves has led to the emergence of new curves, both in terms of pairs of curves and by incorporating physical concepts into the calculations. Especially with the inclusion of physical concepts in the calculations of Frenet motion, X-moment curves, X-torque curves, X-gyroscopic curves, X-equilibrium curves and X-moment conservative curves entered the active field of study as new curves, where $X \in \{T(s), N(s), B(s)\}$. All the curves presented in this article can be described and analyzed by the position vector of a curve. All rectifying curves and normal curves are T-gyroscopic curves, osculating curves are neutral T-equilibrium curves, if the curve is N-torque curve then it is a Smarandache curve, and in the case the curve is generalized helix, normal curves are also neutral B-equilibrium curves, which are



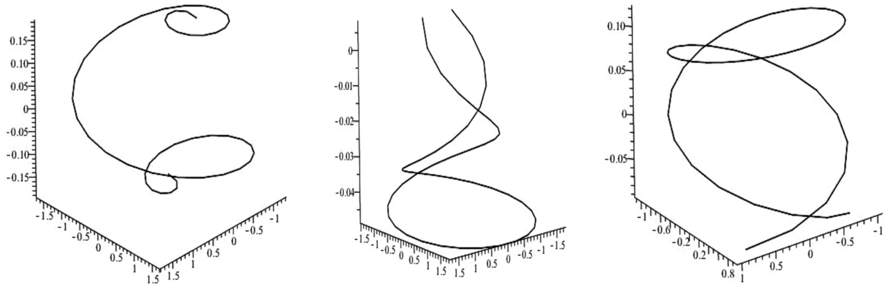
(a) Salkowski curve.

(b) T-dual curve of Salkowski curve.

(c) N-dual curve of Salkowski curve.

(d) B-dual curve of Salkowski curve.

Fig. 3 Dual curves of Salkowski curve



(a) T-torque curve of Salkowski curve.

(b) N-torque curve of Salkowski curve.

(c) B-torque curve of Salkowski curve.

Fig. 4 Torque curves of Salkowski curve

some of the important results obtained. Also, in many special cases, the position vectors of the new curves mentioned have been obtained.

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Availability of Data and Materials The authors confirm that the data supporting the findings of this study are available within the article or its supplementary materials.

Declarations

Conflict of Interest No potential confict of interest was reported by the author.

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