Parastrophic invariance of Smarandache quasigroups *†

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Abstract

Every quasigroup $(L,\cdot)$ belongs to a set of 6 quasigroups, called parastrophes denoted by (L,π_i), $i\in\{1,2,3,4,5,6\}$. It is shown that (L,π_i) is a Smarandache quasigroup with associative subquasigroup (S,π_i) $\forall$ $i\in\{1,2,3,4,5,6\}$ if and only if for any of some four $j\in\{1,2,3,4,5,6\}$, (S,π_j) is an isotope of (S,π_i) or (S,π_k) for one $k\in\{1,2,3,4,5,6\}$ such that $i\neq j\neq k$. Hence, (L,π_i) is a Smarandache quasigroup with associative subquasigroup (S,π_i) $\forall$ $i\in\{1,2,3,4,5,6\}$ if and only if any of the six Khalil conditions is true for any of some four of (S,π_i).

1 Introduction

The study of the Smarandache concept in groupoids was initiated by W.B. Vasantha Kandasamy in [18]. In her book [16] and first paper [17] on Smarandache concept in loops, she defined a Smarandache loop as a loop with at least a subloop which forms a subgroup under the binary operation of the loop. Here, the study of Smarandache quasigroups is continued after their introduction in Muktibodh [9] and [10]. Let L be a non-empty set. Define a binary operation $(\cdot)$ on L : if $x\cdot y\in L$ $\forall$ $x,y\in L$, $(L,\cdot)$ is called a groupoid. If the system of equations ; $a\cdot x=b$ and $y\cdot a=b$ have unique solutions for x and y respectively, then $(L,\cdot)$ is called a quasigroup. Furthermore, if $\exists$ a ! element $e\in L$ called the identity element such that $\forall$ $x\in L$, $x\cdot e=e\cdot x=x$, $(L,\cdot)$ is called a loop. It can thus be seen clearly that quasigroups lie in between groupoids and loops. So, the Smarandache concept needed to be introduced into them and studied since it has been introduced and studied in groupoids and loops. Definitely, results of the Smarandache concept in groupoids will be true in quasigroup that are Smarandache and these together will be true in Smarandache loops.

*2000 Mathematics Subject Classification. Primary 20NO5 ; Secondary 08A05.
†**Keywords and Phrases :** parastrophes, Smarandache quasigroups, isotopic
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It has been noted that every quasigroup $(L,\cdot)$ belongs to a set of 6 quasigroups, called adjugates by Fisher and Yates [6], conjugates by Stein [15], [14] and Belousov [2] and parastrophes by Sade [12]. They have been studied by Artzy [1] and a detailed study on them can be found in [11], [4] and [5]. So for a quasigroup $(L,\cdot)$, its parastrophes are denoted by (L,π_i), $i\in\{1,2,3,4,5,6\}$ hence one can take $(L,\cdot)=(L,\pi_1)$. For more on quasigroup, loops and their properties, readers should check [11], [3],[4], [5], [7] and [16]. Let $(G,\oplus)$ and $(H,\otimes)$ be two distinct quasigroups. The triple (A,B,C) such that $A,B,C\ :\ (G,\oplus)\to(H,\otimes)$ are bijections is said to be an isotopism if and only if $xA\otimes yB=(x\oplus y)C\ \forall\ x,y\in G$. Thus, H is called an isotope of G and they are said to be isotopic.

In this paper, it will be shown that (L,π_i) is a Smarandache quasigroup with associative subquasigroup $(S,\pi_i)\ \forall\ i\in\{1,2,3,4,5,6\}$ if and only if for any of some four $j\in\{1,2,3,4,5,6\}$, (S,π_j) is an isotope of (S,π_i) or (S,π_k) for one $k\in\{1,2,3,4,5,6\}$ such that $i\neq j\neq k$. Hence, it can be concluded that (L,π_i) is a Smarandache quasigroup with associative subquasigroup $(S,\pi_i)\ \forall\ i\in\{1,2,3,4,5,6\}$ if and only if any of the six Khalil conditions is true for any of some four of (S,π_i).

2 Preliminaries

Definition 2.1 *Let $(L,\cdot)$ be a quasigroup. If there exists at least a non-trivial subset $S\subset L$ such that $(S,\cdot)$ is an associative subquasigroup in L, then L is called a Smarandache quasigroup (SQ).*

Remark 2.1 *Definition 2.1 is equivalent to the definition of Smarandache quasigroup in [9] and [10].*

Definition 2.2 *Let (L,θ) be a quasigroup. The 5 parastrophes or conjugates or adjugates of (L,θ) are quasigroups whose binary operations θ^* , θ^{-1} , ${}^{-1}\theta$, $(\theta^{-1})^*$, $({}^{-1}\theta)^*$ defined on L are given by :*

(a) $(L,\theta^*)\ :\ y\theta^*x=z\Leftrightarrow x\theta y=z\ \forall\ x,y,z\in L.$

(b) $(L,\theta^{-1})\ :\ x\theta^{-1}z=y\Leftrightarrow x\theta y=z\ \forall\ x,y,z\in L.$

(c) $(L,{}^{-1}\theta)\ :\ z\ {}^{-1}\theta y=x\Leftrightarrow x\theta y=z\ \forall\ x,y,z\in L..$

(d) $(L,(\theta^{-1})^*)\ :\ z(\theta^{-1})^*x=y\Leftrightarrow x\theta y=z\ \forall\ x,y,z\in L.$

(e) $(L,({}^{-1}\theta)^*)\ :\ y({}^{-1}\theta)^*z=x\Leftrightarrow x\theta y=z\ \forall\ x,y,z\in L.$

Definition 2.3 *Let (L,θ) be a loop.*

(a) *R_x and L_x represent the left and right translation maps in $(L,\theta)\ \forall\ x\in L$.*

(b) *R_x^* and L_x^* represent the left and right translation maps in $(L,\theta^*)\ \forall\ x\in L$.*

(c) *$\mathcal{R}_x$ and $\mathcal{L}_x$ represent the left and right translation maps in $(L,\theta^{-1})\ \forall\ x\in L$.*

(d) $I\!R_x$ *and* $I\!L$ *represent the left and right translation maps in* $(L, {}^{-1}\theta)$ $\forall\ x \in L$.

(e) $\mathcal{R}_x^*$ *and* $\mathcal{L}_x^*$ *represent the left and right translation maps in* $(L, (\theta^{-1})^*)$ $\forall\ x \in L$.

(f) $I\!R_x^*$ *and* $I\!L^*$ *represent the left and right translation maps in* $(L, ({}^{-1}\theta)^*)$ $\forall\ x \in L$.

Remark 2.2 *If* (L, θ) *is a loop,* (L, θ^*) *is also a loop(and vice versa) while the other adjugates are quasigroups.*

Lemma 2.1 *If* (L, θ) *is a quasigroup, then*

1. $R_x^* = L_x$, $L_x^* = R_x$, $\mathcal{L}_x = L_x^{-1}$, $I\!R_x = R_x^{-1}$, $\mathcal{R}_x^* = L_x^{-1}$, $I\!L_x^* = R_x^{-1}$ $\forall\ x \in L$.

2. $\mathcal{L}_x = R_x^{*-1}$, $I\!R_x = L_x^{*-1}$, $\mathcal{R}_x^* = R_x^{*-1} = \mathcal{L}_x$, $I\!L_x^* = L_x^{*-1} = I\!R_x$ $\forall\ x \in L$.

Proof
The proof of these follows by using Definition 2.2 and Definition 2.3.

(1) $y\theta^* x = z \Leftrightarrow x\theta y = z \Rightarrow y\theta^* x = x\theta y \Rightarrow yR_x^* = yL_x \Rightarrow R_x^* = L_x$. Also, $y\theta^* x = x\theta y \Rightarrow xL_y^* = xR_y \Rightarrow L_y^* = R_y$.
$x\theta^{-1} z = y \Leftrightarrow x\theta y = z \Rightarrow x\theta(x\theta^{-1} z) = z \Rightarrow x\theta z\mathcal{L}_x = z \Rightarrow z\mathcal{L}_x L_x = z \Rightarrow \mathcal{L}_x L_x = I$. Also, $x\theta^{-1}(x\theta y) = y \Rightarrow x\theta^{-1} yL_x = y \Rightarrow yL_x\mathcal{L}_x = y \Rightarrow L_x\mathcal{L}_x = I$. Hence, $\mathcal{L}_x = L_x^{-1}$ $\forall\ x \in L$.
$z({}^{-1}\theta)y = x \Leftrightarrow x\theta y = z \Rightarrow (x\theta y)({}^{-1}\theta)y = x \Rightarrow xR_y({}^{-1}\theta)y = x \Rightarrow xR_y I\!R_y = x \Rightarrow R_y I\!R_y = I$. Also, $(z({}^{-1}\theta)y)\theta y = z \Rightarrow zI\!R_y\theta y = z \Rightarrow zI\!R_y R_y = z \Rightarrow I\!R_y R_y = I$. Thence, $I\!R_y = R_y^{-1}$ $\forall\ x \in L$.
$z(\theta^{-1})^* x = y \Leftrightarrow x\theta y = z$, so, $x\theta(z(\theta^{-1})^* x) = z \Rightarrow x\theta z\mathcal{R}_x^* = z \Rightarrow z\mathcal{R}_x^* L_x = z \Rightarrow \mathcal{R}_x^* L_x = I$. Also, $(x\theta y)(\theta^{-1})^* x = y \Rightarrow yL_x(\theta^{-1})^* x = y \Rightarrow yL_x\mathcal{R}_x^* = y \Rightarrow L_y\mathcal{R}_x^* = I$. Whence, $\mathcal{R}_x^* = L_x^{-1}$.
$y({}^{-1}\theta)^* z = x \Leftrightarrow x\theta y = z$, so, $y({}^{-1}\theta)^*(x\theta y) = x \Rightarrow y({}^{-1}\theta)^* xR_y = x \Rightarrow xR_y I\!L_y^* = x \Rightarrow R_y I\!L_y^* = I$. Also, $(y({}^{-1}\theta)^* z)\theta y = z \Rightarrow zI\!L_y^*\theta y = z \Rightarrow zI\!L_y^* R_y = z \Rightarrow I\!L_y^* R_y = I$. Thus, $I\!L_y^* = R_y^{-1}$.

(2) These ones follow from (1).

Lemma 2.2 *Every quasigroup which is a Smarandache quasigroup has at least a subgroup.*

Proof
If a quasigroup $(L, \cdot)$ is a SQ, then there exists a subquasigroup $S \subset L$ such that $(S, \cdot)$ is associative. According [8], every quasigroup satisfying the associativity law has an identity hence it is a group. So, S is a subgroup of L.

Theorem 2.1 *(Khalil Conditions [13]) A quasigroup is an isotope of a group if and only if any one of some six identities are true in the quasigroup.*

3 Main Results

Theorem 3.1 *(L,θ) is a Smarandache quasigroup with associative subquasigroup (S,θ) if and only if any of the following equivalent statements is true.*

1. *(S,θ) is isotopic to $(S,(\theta^{-1})^*)$.*
2. *(S,θ^*) is isotopic to (S,θ^{-1}).*
3. *(S,θ) is isotopic to $(S,(^{-1}\theta)^*)$.*
4. *(S,θ^*) is isotopic to $(S,{}^{-1}\theta)$.*

Proof
L is a SQ with associative subquasigroup S if and only if $s_1\theta(s_2\theta s_3)=(s_1\theta s_2)\theta s_3 \Leftrightarrow R_{s_2}R_{s_3}=R_{s_2\theta s_3} \Leftrightarrow L_{s_1\theta s_2}=L_{s_2}L_{s_1}\ \forall\ s_1,s_2,s_3\in S$.

The proof of the equivalence of (1) and (2) is as follows. $L_{s_1\theta s_2}=L_{s_2}L_{s_1} \Leftrightarrow \mathcal{L}^{-1}_{s_1\theta s_2}=\mathcal{L}^{-1}_{s_2}\mathcal{L}^{-1}_{s_1} \Leftrightarrow \mathcal{L}_{s_1\theta s_2}=\mathcal{L}_{s_1}\mathcal{L}_{s_2} \Leftrightarrow (s_1\theta s_2)\theta^{-1}s_3=s_2\theta^{-1}(s_1\theta^{-1}s_3) \Leftrightarrow (s_1\theta s_2)\mathcal{R}_{s_3}=s_2\theta^{-1}s_1\mathcal{R}_{s_3}=s_1\mathcal{R}_{s_3}(\theta^{-1})^*s_2 \Leftrightarrow (s_1\theta s_2)\mathcal{R}_{s_3}=s_1\mathcal{R}_{s_3}(\theta^{-1})^*s_2 \Leftrightarrow (s_2\theta^* s_1)\mathcal{R}_{s_3}=s_2\theta^{-1}s_1\mathcal{R}_{s_3} \Leftrightarrow (\mathcal{R}_{s_3},I,\mathcal{R}_{s_3})\ :\ (S,\theta)\to(S,(\theta^{-1})^*) \Leftrightarrow (I,\mathcal{R}_{s_3},\mathcal{R}_{s_3})\ :\ (S,\theta^*)\to(S,\theta^{-1}) \Leftrightarrow (S,\theta)$ is isotopic to $(S,(\theta^{-1})^*) \Leftrightarrow (S,\theta^*)$ is isotopic to (S,θ^{-1}).

The proof of the equivalence of (3) and (4) is as follows. $R_{s_2}R_{s_3}=R_{s_2\theta s_3} \Leftrightarrow \mathbb{R}^{-1}_{s_2}\mathbb{R}^{-1}_{s_3}=\mathbb{R}^{-1}_{s_2\theta s_3} \Leftrightarrow \mathbb{R}_{s_3}\mathbb{R}_{s_2}=\mathbb{R}_{s_2\theta s_3} \Leftrightarrow (s_1{}^{-1}\theta s_3)^{-1}\theta s_2=s_1{}^{-1}\theta(s_2\theta s_3) \Leftrightarrow (s_2\theta s_3)\mathbb{L}_{s_1}=s_3\mathbb{L}_{s_1}{}^{-1}\theta s_2=s_2(^{-1}\theta)^*s_3\mathbb{L}_{s_1} \Leftrightarrow (s_2\theta s_3)\mathbb{L}_{s_1}=s_2(^{-1}\theta)^*s_3\mathbb{L}_{s_1} \Leftrightarrow (s_3\theta^* s_2)\mathbb{L}_{s_1}=s_3\mathbb{L}_{s_1}{}^{-1}\theta s_2 \Leftrightarrow (I,\mathbb{L}_{s_1},\mathbb{L}_{s_1})\ :\ (S,\theta)\to(S,(^{-1}\theta)^*) \Leftrightarrow (\mathbb{L}_{s_1},I,\mathbb{L}_{s_1})\ :\ (S,\theta^*)\to(S,{}^{-1}\theta) \Leftrightarrow (S,\theta)$ is isotopic to $(S,(^{-1}\theta)^*) \Leftrightarrow (S,\theta^*)$ is isotopic to $(S,{}^{-1}\theta)$.

Remark 3.1 *In the proof of Theorem 3.1, it can be observed that the isotopisms are triples of the forms (A,I,A) and (I,B,B). All weak associative identities such as the Bol, Moufang and extra identities have been found to be isotopic invariant in loops for any triple of the form (A,B,C) while the central identities have been found to be isotopic invariant only under triples of the forms (A,B,A) and (A,B,B). Since associativity obeys all the Bol-Moufang identities, the observation in the theorem agrees with the latter stated facts.*

Corollary 3.1 *(L,θ) is a Smarandache quasigroup with associative subquasigroup (S,θ) if and only if any of the six Khalil conditions is true for some four parastrophes of (S,θ).*

Proof
Let (L,θ) be the quasigroup in consideration. By Lemma 2.1, (S,θ) is a group. Notice that $R_{s_2}R_{s_3}=R_{s_2\theta s_3} \Leftrightarrow L^*_{s_2\theta s_3}=L^*_{s_3}L^*_{s_2}$. Hence, (S,θ^*) is also a group. In Theorem 3.1, two of the parastrophes are isotopes of (S,θ) while the other two are isotopes of (S,θ^*). Since the Khalil conditions are neccessary and sufficient conditions for a quasigroup to be an isotope of a group, then they must be necessarily and sufficiently true in the four quasigroup parastrophes of (S,θ).

Lemma 3.1 *(L, θ^*) is a Smarandache quasigroup with associative subquasigroup (S, θ^*) if and only if any of the following equivalent statements is true.*

1. *(S, θ^*) is isotopic to $(S, {}^{-1}\theta)$.*
2. *(S, θ) is isotopic to $(S, ({}^{-1}\theta)^*)$.*
3. *(S, θ^*) is isotopic to (S, θ^{-1}).*
4. *(S, θ) is isotopic to $(S, (\theta^{-1})^*)$.*

Proof
Replace (L, θ) with (L, θ^*) in Theorem 3.1.

Corollary 3.2 *(L, θ^*) is a Smarandache quasigroup with associative subquasigroup (S, θ^*) if and only if any of the six Khalil conditions is true for some four parastrophes of (S, θ).*

Proof
Replace (L, θ) with (L, θ^*) in Corollary 3.1.

Lemma 3.2 *(L, θ^{-1}) is a Smarandache quasigroup with associative subquasigroup (S, θ^{-1}) if and only if any of the following equivalent statements is true.*

1. *(S, θ^{-1}) is isotopic to (S, θ^*) .*
2. *$(S, (\theta^{-1})^*)$ is isotopic to (S, θ).*
3. *(S, θ^{-1}) is isotopic to $(S, {}^{-1}\theta)$.*
4. *$(S, (\theta^{-1})^*)$ is isotopic to $(S, ({}^{-1}\theta)^*)$.*

Proof
Replace (L, θ) with (L, θ^{-1}) in Theorem 3.1.

Corollary 3.3 *(L, θ^{-1}) is a Smarandache quasigroup with associative subquasigroup (S, θ^{-1}) if and only if any of the six Khalil conditions is true for some four parastrophes of (S, θ).*

Proof
Replace (L, θ) with (L, θ^{-1}) in Corollary 3.1.

Lemma 3.3 *$(L, {}^{-1}\theta)$ is a Smarandache quasigroup with associative subquasigroup $(S, {}^{-1}\theta)$ if and only if any of the following equivalent statements is true.*

1. *$(S, {}^{-1}\theta)$ is isotopic to (S, θ^{-1}).*
2. *$(S, ({}^{-1}\theta)^*)$ is isotopic to $(S, (\theta^{-1})^*)$.*
3. *$(S, {}^{-1}\theta)$ is isotopic to (S, θ^*).*

4. $(S, ({}^{-1}\theta)^*)$ *is isotopic to* (S, θ).

Proof
Replace (L, θ) with $(L, {}^{-1}\theta)$ in Theorem 3.1.

Corollary 3.4 *$(L, {}^{-1}\theta)$ is a Smarandache quasigroup with associative subquasigroup $(S, {}^{-1}\theta)$ if and only if any of the six Khalil conditions is true for some four parastrophes of (S, θ).*

Proof
Replace (L, θ) with $(L, {}^{-1}\theta)$ in Corollary 3.1.

Lemma 3.4 *$(L, (\theta^{-1})^*)$ is a Smarandache quasigroup with associative subquasigroup $(S, (\theta^{-1})^*)$ if and only if any of the following equivalent statements is true.*

1. $(S, (\theta^{-1})^*)$ *is isotopic to* $(S, ({}^{-1}\theta)^*)$.
2. (S, θ^{-1}) *is isotopic to* $(S, {}^{-1}\theta)$.
3. $(S, (\theta^{-1})^*)$ *is isotopic to* (S, θ).
4. $(S, \theta^{-1}))$ *is isotopic to* (S, θ^*).

Proof
Replace (L, θ) with $(L, (\theta^{-1})^*)$ in Theorem 3.1.

Corollary 3.5 *$(L, (\theta^{-1})^*)$ is a Smarandache quasigroup with associative subquasigroup $(S, (\theta^{-1})^*)$ if and only if any of the six Khalil conditions is true for some four parastrophes of (S, θ).*

Proof
Replace (L, θ) with $(L, (\theta^{-1})^*)$ in Corollary 3.1.

Lemma 3.5 *$(L, ({}^{-1}\theta)^*)$ is a Smarandache quasigroup with associative subquasigroup $(S, ({}^{-1}\theta)^*)$ if and only if any of the following equivalent statements is true.*

1. $(S, ({}^{-1}\theta)^*)$ *is isotopic to* (S, θ).
2. $(S, {}^{-1}\theta)$ *is isotopic to* (S, θ^*).
3. $(S, ({}^{-1}\theta)^*)$ *is isotopic to* $(S, (\theta^{-1})^*)$.
4. $(S, {}^{-1}\theta)$ *is isotopic to* (S, θ^{-1}).

Proof
Replace (L, θ) with $(L, ({}^{-1}\theta)^*)$ in Theorem 3.1.

Corollary 3.6 *$(L, (^{-1}\theta)^*)$ is a Smarandache quasigroup with associative subquasigroup $(S, (^{-1}\theta)^*)$ if and only if any of the six Khalil conditions is true for some four parastrophes of (S, θ).*

Proof
Replace (L, θ) with $(L, (^{-1}\theta)^*)$ in Corollary 3.1.

Theorem 3.2 *(L, π_i) is a Smarandache quasigroup with associative subquasigroup (S, π_i) $\forall\ i \in \{1, 2, 3, 4, 5, 6\}$ if and only if for any of some four $j \in \{1, 2, 3, 4, 5, 6\}$, (S, π_j) is an isotope of (S, π_i) or (S, π_k) for one $k \in \{1, 2, 3, 4, 5, 6\}$ such that $i \neq j \neq k$.*

Proof
This is simply the summary of Theorem 3.1, Lemma 3.1, Lemma 3.2, Lemma 3.3, Lemma 3.4 and Lemma 3.5.

Corollary 3.7 *(L, π_i) is a Smarandache quasigroup with associative subquasigroup (S, π_i) $\forall\ i \in \{1, 2, 3, 4, 5, 6\}$ if and only if any of the six Khalil conditions is true for any of some four of (S, π_i).*

Proof
This can be deduced from Theorem 3.2 and the Khalil conditions or by combining Corollary 3.1, Corollary 3.2, Corollary 3.3, Corollary 3.4, Corollary 3.5 and Corollary 3.6.

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