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# On some Smarandache determinant sequences

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**Abstract** Murthy [1] introduced the concept of the Smarandache Cyclic Determinant Natural Sequence, the Smarandache Cyclic Arithmetic Determinant Sequence, the Smarandache Bisymmetric Determinant Natural Sequence, and the Smarandache Bisymmetric Arithmetic Determinant Sequence. In this paper, we derive the $n$-th terms of these four sequences.

**Keywords** The Smarandache cyclic determinant natural sequence, the Smarandache cyclic arithmetic determinant sequence, the Smarandache bisymmetric determinant natural sequence, the Smarandache bisymmetric arithmetic determinant sequence.

## §1. Introduction

Murthy [1] introduced the concept of the Smarandache cyclic determinant natural sequence, the Smarandache cyclic arithmetic determinant sequence, the Smarandache bisymmetric determinant natural sequence, and the Smarandache bisymmetric arithmetic determinant sequence as follows.

**Definition 1.1.** The Smarandache cyclic determinant natural sequence, $\{SCDNS(n)\}$ is

$$\left\{ |1|, \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}, \ldots \right\}.$$

Murthy conjectured that the $n-th$ term of the above sequence is

$$SCDNS(n) = (-1)^{\left[\frac{n}{2}\right]} \frac{\mathrm{n}+1}{2} n^{n-1},$$

where $[x]$ denotes the greatest integer less than or equal to $x$ .

**Definition 1.2.** The Smarandache cyclic arithmetic determinant sequence, $\{SCADS(n)\}$ is

$$\left\{ |1|, \begin{vmatrix} a & a+d \\ a+d & a \end{vmatrix}, \begin{vmatrix} a & a+d & a+2d \\ a+d & a+2d & a \\ a+2d & a & a+d \end{vmatrix}, \ldots \right\}.$$

Murthy conjectured, erroneously, that the $n-th$ term of the above sequence is

$$SCDNS(n) = (-1)^{\left[\frac{n}{2}\right]} \frac{\mathrm{a + (n-1)}d}{2} (nd)^{n-1}$$

where $[x]$ denotes the greatest integer less than or equal to $x$ .

**Definition 1.3.** The Smarandache bisymmetric determinant natural sequence, $\{SBDNS(n)\}$ is

$$\left\{ |1|\,, \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 3 \\ 3 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \end{vmatrix}, \cdots \right\}.$$

**Definition 1.4.** The Smarandache bisymmetric arithmetic determinant sequence, $\{SBADS(n)\}$ is

$$\left\{ |1|\,, \begin{vmatrix} a & a+d \\ a+d & a \end{vmatrix}, \begin{vmatrix} a & a+d & a+2d \\ a+d & a+2d & a+d \\ a+2d & a+d & a \end{vmatrix}, \cdots \right\}.$$

Murthy also conjectured about the $n$-th terms of the last two sequences, but those expressions are not correct.

In this paper, we derive explicit forms of the $n$-th terms of the four sequences. These are given in Section 3. Some preliminary results, that would be necessary in the derivation of the expressions of the $n$-th terms of the sequences, are given in Section 2.

## §2. Some preliminary results

In this section, we derive some results that would be needed later in proving the main results of this paper in Section 3. We start with the following result.

**Lemma 2.1.** Let $D \equiv |d_{ij}|$ be the determinant of order $n \geq 2$ with

$$d_{ij} = \begin{cases} a, \text{ if i = j} \geqslant 2; \\ 1, \text{ otherwise.} \end{cases}$$

where $a$ is a fixed number. Then,

$$D \equiv \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & a & 1 & \cdots & 1 & 1 \\ 1 & 1 & a & \cdots & 1 & 1 \\ \vdots & & & & & \\ 1 & 1 & 1 & \cdots & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & a \end{vmatrix} = (a-1)^{n-1}.$$

**Proof.** Performing the indicated column operations (where $C_i \to C_i - C_1$ indicates the column operation of subtracting the 1st column from the $ith$ column, $2 \leq i \leq n$), we get

$$D \equiv \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & a & 1 & \cdots & 1 & 1 \\ 1 & 1 & a & \cdots & 1 & 1 \\ \vdots & & & & & \\ 1 & 1 & 1 & \cdots & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & a \end{vmatrix} \begin{matrix} = \\ C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \\ \vdots \\ C_n \to C_n - C_1 \\ \phantom{x} \end{matrix} \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & a-1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & a-1 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 1 & 0 & 0 & \cdots & a-1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & a-1 \end{vmatrix}$$

$$= \begin{vmatrix} a-1 & 0 & \cdots & 0 & 0 \\ 0 & a-1 & \cdots & 0 & 0 \\ & & & & \\ 0 & 0 & \cdots & a-1 & 0 \\ 0 & 0 & \cdots & 0 & a-1 \end{vmatrix},$$

which is a determinant of order $n-1$ whose diagonal elements are all $a-1$ and off-diagonal elements are all zero. Hence,

$$D = (a-1)^{n-1}.$$

**Lemma 2.2.** Let $D^a = \left| d_{ij}^{(a)} \right|$ be the determinant of order $n \geq 2$ whose diagonal elements are all a (where a is a fixed number) and off-diagonal elements are all 1, that is,

$$d_{ij} = \begin{cases} a, \text{ if i = j} \geqslant 1; \\ 1, \text{ otherwise.} \end{cases}$$

Then,

$$D^{(a)} \equiv \begin{vmatrix} a & 1 & 1 & \cdots & 1 & 1 \\ 1 & a & 1 & \cdots & 1 & 1 \\ 1 & 1 & a & \cdots & 1 & 1 \\ \vdots & & & & & \\ 1 & 1 & 1 & \cdots & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & a \end{vmatrix} = (a-1)^{n-1}(a+n-1).$$

**Proof.** We perform the indicated column operations (where $C_1 \to C_1 + C_2 + ... + C_n$ indicates the operation of adding all the columns and then replacing the 1st column by that sum, and $C_1 \to \dfrac{1}{C_1 + C_2 + ... + C_n}$ denotes the operation of taking out the common sum) to get

$$D^{(a)} \equiv \begin{vmatrix} a & 1 & 1 & \cdots & 1 & 1 \\ 1 & a & 1 & \cdots & 1 & 1 \\ 1 & 1 & a & \cdots & 1 & 1 \\ \vdots & & & & & \\ 1 & 1 & 1 & \cdots & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & a \end{vmatrix} \underset{\substack{C_1 \to C_1 + C_2 + ... + C_n \\ C_1 \to \frac{1}{C_1 + C_2 + ... + C_n}}}{=} (a+n-1) \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & a & 1 & \cdots & 1 & 1 \\ 1 & 1 & a & \cdots & 1 & 1 \\ \vdots & & & & & \\ 1 & 1 & 1 & \cdots & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & a \end{vmatrix}$$

$$= (a+n-1)(a-1)^{n-1},$$

where the last equality is by virtue of Lemma 2.1.

**Corollary 2.1.** The value of the following determinant of order $n \geq 2$ is

$$\begin{vmatrix} -n & 1 & 1 & \cdots & 1 & 1 \\ 1 & -n & 1 & \cdots & 1 & 1 \\ 1 & 1 & -n & \cdots & 1 & 1 \\ \vdots & & & & & \\ 1 & 1 & 1 & \cdots & -n & 1 \\ 1 & 1 & 1 & \cdots & 1 & -n \end{vmatrix} = (-1)^n \ (n+1)^{n-1}.$$

**Proof.** Follows immediately from Lemma 2.2 as a particular case when $a = -n$.

**Lemma 2.3.** Let $A_n = |a_{ij}|$ be the determinant of order $n \geq 2$, defined by

$$a_{ij} = \begin{cases} 1, \text{ if i} \leqslant \text{j}; \\ -1, \text{ otherwise.} \end{cases}$$

Then,

$$A_n \equiv \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & -1 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & & & & & & \\ -1 & -1 & -1 & \cdots & -1 & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & 1 \end{vmatrix} = 2^{n-1}.$$

**Proof.** The proof is by induction on $n$. Since

$$A_2 = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2,$$

the result is true for $n = 2$ . So, we assume the validity of the result for some integer $n \geq 2$. To prove the result for $n+1$, we consider the determinant of order $n+1$, and perform the indicated column operations (where $C_1 \to C_1 + C_n$ indicates the operation of adding the $n-th$ column to the $1^{st}$ column to get the new $1^{st}$ column), to get

$$A_{n+1} \equiv \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & -1 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & & & & & & \\ -1 & -1 & -1 & \cdots & -1 & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & 1 \end{vmatrix} \underset{C_1 \to C_1 + C_n}{=} \begin{vmatrix} 2 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & -1 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & & & & & & \\ 0 & -1 & -1 & \cdots & -1 & 1 & 1 \\ 0 & -1 & -1 & \cdots & -1 & -1 & 1 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & & & & & & \\ -1 & -1 & -1 & \cdots & -1 & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & 1 \end{vmatrix} = 2A_n = 2^n,$$

by virtue of the induction hypothesis. Thus, the result is true for $n+1$, which completes induction.

**Corollary 2.2.** The value of the following determinant of order $n \geq 2$ is

$$B_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & -1 \\ 1 & 1 & 1 & \cdots & 1 & -1 & -1 \\ \vdots & & & & & & \\ 1 & 1 & -1 & \cdots & -1 & -1 & -1 \\ 1 & -1 & -1 & \cdots & -1 & -1 & -1 \end{vmatrix} = (-1)^{\left[\frac{n}{2}\right]} 2^{n-1}.$$

**Proof.** To prove the result, note that the determinant $B_n$ can be obtained from the determinant $A_n$ of Lemma 2.3 by successive interchange of columns. To get the determinant $B_n$ from the determinant $A_n$, we consider the two cases depending on whether $n$ is even or odd.

**Case 1 :** When $n$ is even, say, $n = 2m$ for some integer $m \geq 1$.

In this case, starting with the determinant $B_n = B_{2m}$, we perform the indicated column operations.

$$B_n = B_{2m} = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 & -1 \\ \vdots & & & & & \\ 1 & 1 & \cdots & -1 & -1 & -1 \\ 1 & -1 & \cdots & -1 & -1 & -1 \end{vmatrix} \underset{\substack{C_1 \to C_{2m} \\ C_2 \to C_{2m-1} \\ \vdots \\ C_m \to C_{m+1}}}{=} (-1)^m \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & & & & & \\ -1 & -1 & \cdots & -1 & 1 & 1 \\ -1 & -1 & \cdots & -1 & -1 & 1 \end{vmatrix}$$

$$= (-1)^m 2^{n-1}.$$

**Case 2 :** When n is odd, say, $n = 2m+1$ for some integer $m \geq 1$ . In this case,

$$B_n = B_{2m+1} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & -1 \\ 1 & 1 & 1 & \cdots & 1 & -1 & -1 \\ \vdots & & & & & & \\ 1 & 1 & -1 & \cdots & -1 & -1 & -1 \\ 1 & -1 & -1 & \cdots & -1 & -1 & -1 \end{vmatrix}$$

$$\begin{array}{c} = \\ C_1 \to C_{2m+1} \\ C_2 \to C_{2m} \\ \vdots \\ C_m \to C_{m+2} \end{array} (-1)^m \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & -1 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & & & & & & \\ -1 & -1 & -1 & \cdots & -1 & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & 1 \end{vmatrix} = (-1)^m 2^{n-1}.$$

Since, in either case, $m = \left[\frac{n}{2}\right]$ , the result is established.

## §3. Main results

In this section, we derive the explicit expressions of the $n$-th terms of the four determinant sequences, namely, the Smarandache cyclic determinant natural sequence, the Smarandache cyclic arithmetic determinant sequence, the Smarandache bisymmetric determinant natural sequence, and the Smarandache bisymmetric arithmetic determinant sequence. These are given in Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 3.4 respectively.

**Theorem 3.1.** The $n$-th term of the Smarandache cyclic determinant natural sequence, $SCDNS(n)$ is

$$SCDNS(n) = \begin{vmatrix} 1 & 2 & 3 & 4 & \cdots & n-2 & n-1 & n \\ 2 & 3 & 4 & 5 & \cdots & n-1 & n & 1 \\ 3 & 4 & 5 & 6 & \cdots & n & 1 & 2 \\ 4 & 5 & 6 & 7 & \cdots & 1 & 2 & 3 \\ \vdots & & & & & & & \\ n-1 & n & 1 & 2 & \cdots & n-4 & n-3 & n-2 \\ n & 1 & 2 & 3 & \cdots & n-3 & n-2 & n-1 \end{vmatrix} = (-1)^{\left[\frac{n}{2}\right]} \frac{n+1}{2} n^{n-1}.$$

**Proof.** We consider separately the possible two cases.

**Case 1 :** When $n$ is even, say, $n = 2m$ for some integer $m \geq 1$ (so that $\left[\frac{n}{2}\right] = m$ ).
We now perform the indicated operations on $SCDNS(n)$ (where $C_i \leftrightarrow C_j$ denotes the operation

of interchanging the $i$ -thcolumn and the $j$-th column, and $R_i \to R_i - R_j$ means that the $j$-th row is subtracted from the $i$-th row to get the new $i$-th row). Note that, there are in total, $m$ interchanges of columns, each changing the value of $SCDNS(n)$ by $-1$. Then,

$$SCDNS(n) = \begin{vmatrix} 1 & 2 & 3 & 4 & \cdots & 2m-2 & 2m-1 & 2m \\ 2 & 3 & 4 & 5 & \cdots & 2m-1 & 2m & 1 \\ 3 & 4 & 5 & 6 & \cdots & 2m & 1 & 2 \\ 4 & 5 & 6 & 7 & \cdots & 1 & 2 & 3 \\ \vdots & & & & & & & \\ 2m-1 & 2m & 1 & 2 & \cdots & 2m-4 & 2m-3 & 2m-2 \\ 2m & 1 & 2 & 3 & \cdots & 2m-3 & 2m-2 & 2m-1 \end{vmatrix}$$

$$\underset{\substack{C_1 \leftrightarrow C_{2m} \\ C_2 \leftrightarrow C_{2m-1} \\ \vdots \\ C_m \leftrightarrow C_{m+1}}}{=} (-1)^m \begin{vmatrix} 2m & 2m-1 & 2m-2 & \cdots & 3 & 2 & 1 \\ 1 & 2m & 2m-1 & \cdots & 4 & 3 & 2 \\ 2 & 1 & 2m & \cdots & 5 & 4 & 3 \\ 3 & 2 & 1 & \cdots & 6 & 5 & 4 \\ \vdots & & & & & & \\ 2m-3 & 2m-4 & 2m-5 & & 2m & 2m-1 & 2m-2 \\ 2m-2 & 2m-3 & 2m-4 & \cdots & 1 & 2m & 2m-1 \\ 2m-1 & 2m-2 & 2m-3 & \cdots & 2 & 1 & 2m \end{vmatrix}$$

$$\underset{\substack{C_{2m} \to C_1 + C_2 + ... + C_{2m} \\ C_{2m} \to \frac{1}{C_1 + C_2 + ... + C_{2m}}}}{=} (-1)^m \frac{2m(2m+1)}{2} \begin{vmatrix} 2m & 2m-1 & \cdots & 3 & 2 & 1 \\ 1 & 2m & \cdots & 4 & 3 & 1 \\ \vdots & & & & & \\ 2m-3 & 2m-4 & & 2m & 2m-1 & 1 \\ 2m-2 & 2m-3 & \cdots & 1 & 2m & 1 \\ 2m-1 & 2m-2 & \cdots & 2 & 1 & 1 \end{vmatrix}$$

$$\underset{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_2 \\ \vdots \\ R_{2m} \to R_{2m} - R_{2m-1}}}{=} (-1)^m \frac{2m(2m+1)}{2} \begin{vmatrix} 2m & 2m-1 & 2m-2 & \cdots & 2 & 1 \\ 1-2m & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1-2m & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1-2m & \cdots & 1 & 0 \\ \vdots & & & & & \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1-2m & 0 \end{vmatrix}$$

$$= (-1)^m \frac{2m(2m+1)}{2} (-1)^{2m+1} \begin{vmatrix} 1-2m & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1-2m & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1-2m & \cdots & 1 & 1 \\ \vdots & & & & & \\ 1 & 1 & 1 & & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1-2m & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1-2m \end{vmatrix}$$

$$= (-1)^{m+1} \frac{2m(2m+1)}{2} \left\{ (-1)^{2m-1} (2m)^{2(m-1)} \right\} = (-1)^m \frac{2m+1}{2} (2m)^{2m-1}.$$

**Case 2 :** When $n$ is odd, say, $n = 2m+1$ for some integer $m \geq 1$ (so that $\left[\frac{n}{2}\right] = m$). Here,

$$SCDNS(n) = \begin{vmatrix} 1 & 2 & 3 & 4 & \cdots & 2m-1 & 2m & 2m+1 \\ 2 & 3 & 4 & 5 & \cdots & 2m & 2m+1 & 1 \\ 3 & 4 & 5 & 6 & \cdots & 2m+1 & 1 & 2 \\ 4 & 5 & 6 & 7 & \cdots & 1 & 2 & 3 \\ \vdots & & & & & & & \\ 2m & 2m+1 & 1 & 2 & \cdots & 2m-3 & 2m-2 & 2m-1 \\ 2m+1 & 1 & 2 & 3 & \cdots & 2m-2 & 2m-1 & 2m \end{vmatrix}$$

$$\begin{array}{c} = \\ C_1 \leftrightarrow C_{2m+1} \\ C_2 \leftrightarrow C_{2m} \\ C_3 \leftrightarrow C_{2m-1} \\ \vdots \\ C_m \leftrightarrow C_{m+2} \end{array} (-1)^m \begin{vmatrix} 2m+1 & 2m & 2m-1 & \cdots & 3 & 2 & 1 \\ 1 & 2m+1 & 2m & \cdots & 4 & 3 & 2 \\ 2 & 1 & 2m+1 & \cdots & 5 & 4 & 3 \\ 3 & 2 & 1 & \cdots & 6 & 5 & 4 \\ \vdots & & & & & & \\ 2m-2 & 2m-3 & 2m-4 & & 2m+1 & 2m & 2m-1 \\ 2m-1 & 2m-2 & 2m-3 & \cdots & 1 & 2m+1 & 2m \\ 2m & 2m-1 & 2m-2 & \cdots & 2 & 1 & 2m+1 \end{vmatrix}$$

$$\underset{\substack{C_{2m+1} \to C_1 + C_2 + \ldots + C_{2m+1} \\ C_{2m+1} \to \frac{1}{C_1 + C_2 + \ldots + C_{2m+1}}}}{=} (-1)^m \frac{(2m+1)(2m+2)}{2} \begin{vmatrix} 2m+1 & 2m & 2m-1 & \cdots & 3 & 2 & 1 \\ 1 & 2m+1 & 2m & \cdots & 4 & 3 & 1 \\ 2 & 1 & 2m+1 & \cdots & 5 & 4 & 1 \\ 3 & 2 & 1 & \cdots & 6 & 5 & 1 \\ \vdots & & & & & & \\ 2m-2 & 2m-3 & 2m-4 & & 2m+1 & 2m & 1 \\ 2m-1 & 2m-2 & 2m-3 & \cdots & 1 & 2m+1 & 1 \\ 2m & 2m-1 & 2m-2 & \cdots & 2 & 1 & 1 \end{vmatrix}$$

$$\underset{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_2 \\ \vdots \\ R_{2m+1} \to R_{2m+1} - R_{2m}}}{=} (-1)^m \frac{2m(2m+1)}{2} \begin{vmatrix} 2m+1 & 2m & 2m-1 & \cdots & 3 & 2 & 1 \\ -2m & 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & -2m & 1 & \cdots & 1 & 1 & 0 \\ 1 & 1 & -2m & \cdots & 1 & 1 & 0 \\ \vdots & & & & & & \\ 1 & 1 & 1 & & 1 & 1 & 0 \\ 1 & 1 & 1 & \cdots & -2m & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & -2m & 0 \end{vmatrix}$$

$$= (-1)^m \frac{(2m+1)(2m+2)}{2} (-1)^{2m+2} \begin{vmatrix} -2m & 1 & 1 & \cdots & 1 & 1 \\ 1 & -2m & 1 & \cdots & 1 & 1 \\ 1 & 1 & -2m & \cdots & 1 & 1 \\ \vdots & & & & & \\ 1 & 1 & 1 & & 1 & 1 \\ 1 & 1 & 1 & \cdots & -2m & 1 \\ 1 & 1 & 1 & \cdots & 1 & -2m \end{vmatrix}$$

$$= (-1)^m \frac{(2m+1)(2m+2)}{2} \left\{(-1)^{2m}(2m+1)^{2m-1}\right\} = (-1)^m \frac{2m+2}{2}(2m+1)^{2m}.$$

Thus, the result is true both when n is even and when n is odd, completing the proof.

**Theorem 3.2.** The $n$-th term of the Smarandache cyclic arithmetic determinant sequence,

$SCADS(n)$ is

$$SCADS(n) = \begin{vmatrix} a & a+d & a+2d & \cdots & a+(n-2)d & a+(n-1)d \\ a+d & a+2d & a+3d & \cdots & a+(n-1)d & a \\ a+2d & a+3d & a+4d & \cdots & a & a+d \\ a+3d & a+4d & a+5d & \cdots & a+d & a+2d \\ \vdots & & & & & \\ a+(n-2)d & a+(n-1)d & a & \cdots & a+(n-4)d & a+(n-3)d \\ a+(n-1)d & a & a+d & \cdots & a+(n-3)d & a+(n-2)d \end{vmatrix}$$

$$= (-1)^{\left[\frac{n}{2}\right]} \left(a + \frac{n-1}{2}d\right) (nd)^{n-1}.$$

**Proof.** Here also, we consider separately the possible two cases.

**Case 1 :** If $n = 2m$ for some integer $m \geq 1$ (so that $\left[\frac{n}{2}\right] = m$). In this case, performing the indicated column and row operations, we get successively

$$SCADS(n) = \begin{vmatrix} a & a+d & \cdots & a+(2m-2)d & a+(2m-1)d \\ a+d & a+2d & \cdots & a+(2m-1)d & a \\ a+2d & a+3d & \cdots & a & a+d \\ a+3d & a+4d & \cdots & a+d & a+2d \\ \vdots & & & & \\ a+(2m-2)d & a+(2m-1)d & \cdots & a+(2m-4)d & a+(2m-3)d \\ a+(2m-1)d & a & \cdots & a+(2m-3)d & a+(2m-2)d \end{vmatrix}$$

$$\underset{\substack{C_1 \leftrightarrow C_{2m} \\ C_2 \leftrightarrow C_{2m-1} \\ \vdots \\ C_m \leftrightarrow C_{m+1}}}{=} (-1)^m \begin{vmatrix} a+(2m-1)d & a+(2m-2)d & \cdots & a+d & a \\ a & a+(2m-1)d & \cdots & a+2d & a+d \\ a+d & a & \cdots & a+3d & a+2d \\ a+2d & a+d & \cdots & a+4d & a+3d \\ \vdots & & & & \\ a+(2m-4)d & a+(2m-5)d & & a+(2m-2)d & a+(2m-3)d \\ a+(2m-3)d & a+(2m-4)d & \cdots & a+(2m-1)d & a+(2m-2)d \\ a+(2m-2)d & a+(2m-3)d & \cdots & a & a+(2m-1)d \end{vmatrix}$$

$$=$$

$$C_{2m} \to C_1 + C_2 + ... + C_{2m}$$
$$C_{2m} \to \frac{1}{C_1 + C_2 + ... + C_{2m}}$$

$$(-1)^m S_{2m} \begin{vmatrix} a+(2m-1)d & a+(2m-2)d & \cdots & a+d & 1 \\ a & a+(2m-1)d & \cdots & a+2d & 1 \\ a+d & a & \cdots & a+3d & 1 \\ a+2d & a+d & \cdots & a+4d & 1 \\ \vdots & & & & \\ a+(2m-4)d & a+(2m-5)d & & a+(2m-2)d & 1 \\ a+(2m-3)d & a+(2m-4)d & \cdots & a+(2m-1)d & 1 \\ a+(2m-2)d & a+(2m-3)d & \cdots & a & 1 \end{vmatrix}$$

$$\left(S_{2m} = a + (a+d) + (a+2d) + \ldots + \{a+(2m-1)d\} = 2ma + \frac{2m(2m-1)}{2}d\right)$$

$$\begin{array}{c} = \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_2 \\ \vdots \\ R_{2m} \to R_{2m} - R_{2m-1} \end{array} \quad (-1)^m\ S_{2\mathrm{m}} \begin{vmatrix} a+(2m-1)d & a+(2m-2)d & \cdots & a+d & 1 \\ (1-2m)d & d & \cdots & d & 0 \\ d & (1-2m)d & \cdots & d & 0 \\ d & d & \cdots & d & 0 \\ \vdots & & & & \\ d & d & \cdots & d & 0 \\ d & d & \cdots & (1-2m)d & 0 \end{vmatrix}$$

$$= (-1)^m \left\{2ma + \frac{2m(2m-1)}{2}d\right\}\ (-1)^{2m+1}\ d^{2m-1} \begin{vmatrix} 1-2m & 1 & \cdots & 1 \\ 1 & 1-2m & \cdots & 1 \\ \vdots & & & \\ 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1-2m \end{vmatrix}$$

$$= \quad (-1)^{m+1} \left\{2ma + \frac{2m(2m-1)}{2}d\right\} \left\{d^{2m-1}(-1)^{2m-1}(2m)^{2(m-1)}\right\}$$

$$= \quad (-1)^m \left\{a + (\frac{2m-1}{2})d\right\}\ d^{2m-1}\ (2m)^{2m-1}.$$

**Case 2 :** If $n = 2m+1$ for some integer $m \geq 1$ (so that $\left[\frac{n}{2}\right] = m$).

In this case, performing the indicated column and row operations, we get successively

$$\mathrm{SCADS(n)} = \begin{vmatrix} a & a+d & a+2d & \cdots & a+(2m-1)d & a+2md \\ a+d & a+2d & a+3d & \cdots & a+2md & a \\ a+2d & a+3d & a+4d & \cdots & a & a+d \\ a+3d & a+4d & a+5d & \cdots & a+d & a+2d \\ \vdots & & & & & \\ a+(2m-1)d & a+2md & a & \cdots & a+(2m-3)d & a+(2m-2)d \\ a+2md & a & a+d & \cdots & a+(2m-2)d & a+(2m-1)d \end{vmatrix}$$

$$\begin{gathered} = \\ C_1 \leftrightarrow C_{2m+1} \\ C_2 \leftrightarrow C_{2m} \\ \vdots \\ C_m \leftrightarrow C_{m+2} \end{gathered}$$

$$(-1)^m \begin{vmatrix} a+2md & a+(2m-1)d & \cdots & a+d & a \\ a & a+2md & \cdots & a+2d & a+d \\ a+d & a & \cdots & a+3d & a+2d \\ a+2d & a+d & \cdots & a+4d & a+3d \\ \vdots & & & & \\ a+(2m-3)d & a+(2m-4)d & & a+(2m-1)d & a+(2m-2)d \\ a+(2m-2)d & a+(2m-3)d & \cdots & a+2md & a+(2m-1)d \\ a+(2m-1)d & a+(2m-2)d & \cdots & a & a+2md \end{vmatrix}$$

$$\begin{gathered} = \\ C_{2m+1} \to C_1 + C_2 ... + C_{2m+1} \\ C_{2m+1} \to \frac{1}{C_1 + C_2 ... + C_{2m+1}} \end{gathered}$$

$$(-1)^m \, S_{\mathrm{2m\,+\,1}} \begin{vmatrix} a+2md & a+(2m-1)d & \cdots & a+2d & a+d & 1 \\ a & a+2md & \cdots & a+3d & a+2d & 1 \\ a+d & a & \cdots & a+4d & a+3d & 1 \\ a+2d & a+d & \cdots & a+5d & a+4d & 1 \\ \vdots & & & & & \\ a+(2m-2)d & a+(2m-3)d & & a & a+2md & 1 \\ a+(2m-1)d & a+(2m-2)d & \cdots & a+d & a & 1 \end{vmatrix}$$

$$\left( S_{2m+1} = a + (a+d) + (a+2d) \ldots + (a+2md) = (2m+1)a + \frac{2m(2m+1)}{2} d \right)$$

$$\begin{array}{c} = \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_2 \\ \vdots \\ R_{2m+1} \to R_{2m+1} - R_{2m} \end{array} \quad (-1)^m\, S_{2\mathrm{m}\,+\,1} \begin{vmatrix} a+2md & a+(2m-1)d & \cdots & a+d & 1 \\ -2md & d & \cdots & d & 0 \\ d & -2md & \cdots & d & 0 \\ \vdots & & & & \\ d & d & & d & 0 \\ d & d & \cdots & -2md & 0 \end{vmatrix}$$

$$= (-1)^m \left\{(2m+1)a + \frac{2m(2m+1)}{2} d\right\} (-1)^{2m+2} d^{2m} \begin{vmatrix} -2m & 1 & \cdots & 1 \\ 1 & -2m & \cdots & 1 \\ \vdots & & & \\ 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & -2m \end{vmatrix}$$

$$\begin{aligned} &= (-1)^m \left\{(2m+1)a + \frac{2m(2m+1)}{2} d\right\} \left\{d^{2m}(-1)^{2m}(2m+1)^{2m-1}\right\} \\ &= (-1)^m \left\{a + \frac{2m}{2} d\right\} d^{2m}(2m+1)^{2m}. \end{aligned}$$

Thus, in both the cases, the result holds true. This completes the proof.

**Theorem 3.3.** The $n$-th term of the Smarandache bisymmetric determinant natural sequence, $\{SBDNS(n)\}$, $n \geq 5$ , is

$$SBDNS(n) = \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & n-1 \\ 3 & 4 & 5 & \cdots & n-1 & n-2 \\ \vdots & & & & & \\ n-2 & n-1 & n & & 4 & 3 \\ n-1 & n & n-1 & \cdots & 3 & 2 \\ n & n-1 & n-2 & \cdots & 2 & 1 \end{vmatrix} = (-1)^{\left[\frac{n}{2}\right]}(n+1)2^{n-2}.$$

**Proof.** We perform the indicated row and column operations to reduce the determinant

$SBDNS(n)$ to the form $B_{n-1}$ (of Corollary 2.2) as follows :

$$SBDNS(n)=\begin{vmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ 2 & 3 & 4 & \cdots & n-1 & n & n-1 \\ 3 & 4 & 5 & \cdots & n & n-1 & n-2 \\ \vdots & & & & & & \\ n-2 & n-1 & n & & 5 & 4 & 3 \\ n-1 & n & n-1 & \cdots & 4 & 3 & 2 \\ n & n-1 & n-2 & \cdots & 3 & 2 & 1 \end{vmatrix}$$

$$\begin{matrix} = \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_2 \\ \vdots \\ R_n \to R_n - R_{n-1} \end{matrix}\begin{vmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ 1 & 1 & 1 & \cdots & 1 & 1 & -1 \\ 1 & 1 & 1 & \cdots & 1 & -1 & -1 \\ \vdots & & & & & & \\ 1 & 1 & 1 & & -1 & -1 & -1 \\ 1 & 1 & -1 & \cdots & -1 & - & -1 \\ 1 & -1 & -1 & \cdots & -1 & -1 & -1 \end{vmatrix}$$

$$\begin{matrix} = \\ C_n \to C_n + C_1 \end{matrix}\begin{vmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n+1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & -1 & 0 \\ \vdots & & & & & & \\ 1 & 1 & 1 & & -1 & -1 & 0 \\ 1 & 1 & -1 & \cdots & -1 & -1 & 0 \\ 1 & -1 & -1 & \cdots & -1 & -1 & 0 \end{vmatrix}$$

$$=(-1)^{n+1}(n+1)\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & -1 \\ \vdots & & & & & \\ 1 & 1 & 1 & & -1 & -1 \\ 1 & 1 & -1 & \cdots & -1 & -1 \\ 1 & -1 & -1 & \cdots & -1 & -1 \end{vmatrix} = (-1)^{n+1}(n+1)B_{n-1}$$

$$=(-1)^{n+1}(n+1)\left\{(-1)^{\left[\frac{n-1}{2}\right]}2^{n-2}\right\}.$$

Now, if $n=2m+1$, then $(-1)^{n+1+\left[\frac{n-1}{2}\right]}=(-1)^{(2m+2)+m}=(-1)^m=(-1)^{\left[\frac{n}{2}\right]}$, and if $n=2m$, then $(-1)^{n+1+\left[\frac{n-1}{2}\right]}=(-1)^{2m+1+(m-1)}=(-1)^m=(-1)^{\left[\frac{n}{2}\right]}$.

Hence, finally, we get $\qquad SBDNS(n) = (-1)^{\left[\frac{n}{2}\right]}(n+1)2^{n-2}.$

**Remark 3.1.** The values of $SBDNS(3)$ and $SBDNS(4)$ can be obtained by proceeding as in Theorem 3.3. Thus,

$$SBDNS(3) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 1 \end{vmatrix} \underset{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_2}}{=} \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{vmatrix} \underset{C_3 \to C_3 + C_1}{=} \begin{vmatrix} 1 & 2 & 4 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} = -8,$$

$$SBDNS(4) = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 3 \\ 3 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \end{vmatrix} \underset{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_3}}{=} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{vmatrix}$$

$$\underset{C_4 \to C_4 + C_1}{=} \begin{vmatrix} 1 & 2 & 3 & 5 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & -1 & 0 \end{vmatrix} = (-5) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{vmatrix} = (-5)\{(-1)^{\left[\frac{3}{2}\right]}2^{3-1}\} = 20.$$

**Theorem 3.4.** The $n$-th term of the Smarandache bisymmetric arithmetic determinant sequence, $\{SBADS(n)\}$, $n \geq 5$, is

$$\text{SBADS(n)} = \begin{vmatrix} a & a+d & \cdots & a+(n-2)d & a(n-1)d \\ a+d & a+2d & \cdots & a+(n-1)d & a+(n-2)d \\ \vdots & & & & \\ a+(n-2)d & a+(n-1)d & \cdots & a+2d & a+d \\ a+(n-1)d & a+(n-2)d & \cdots & a+d & a \end{vmatrix}$$

$= (-1)^{\left[\frac{n}{2}\right]}\left(a + \frac{n-1}{2}d\right)(2d)^{n-1}.$

**Proof.** We get the desired result, starting from $SBADS(n)$, expressing this in terms of the determinant $B_{n-1}$ (of Corollary 2.2) by performing the indicated row and column operations.

$$\text{SBADS(n)} = \begin{vmatrix} a & a+d & \cdots & a+(n-2)d & a+(n-1)d \\ a+d & a+2d & \cdots & a+(n-1)d & a+(n-2)d \\ \vdots & & & & \\ a(n-2)d & a+(n-1)d & \cdots & a+2d & a+d \\ a+(n-1)d & a+(n-2)d & \cdots & a+d & a \end{vmatrix}$$

$$\underset{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_2 \\ \vdots \\ R_n \to R_n - R_{n-1}}}{=} \begin{vmatrix} a & a+d & a+2d & \cdots & a+(n-3)d & a+(n-2)d & a+(n-1)d \\ \text{d} & d & d & \cdots & d & d & -d \\ \text{d} & d & d & \cdots & d & -d & -d \\ \vdots \\ d & d & -d & \cdots & -d & -d & -d \\ d & -d & -d & \cdots & -d & -d & -d \end{vmatrix}$$

$$\underset{C_n \to C_n + C_1}{=} d^{n-1} \begin{vmatrix} a & a+d & a+2d & \cdots & a+(n-3)d & a+(n-2)d & 2a+(n-1)d \\ 1 & 1 & 1 & \cdots 1 & 1 & 0 & \\ 1 & 1 & 1 & \cdots & 1 & -1 & 0 \\ \vdots \\ 1 & 1 & 1 & & -1 & -1 & 0 \\ 1 & 1 & -1 & \cdots & -1 & -1 & 0 \\ 1 & -1 & -1 & \cdots & -1 & -1 & 0 \end{vmatrix}$$

$$= (-1)^{n+1}\, d^{n-1}\ \{2a+(n-1)d\} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & -1 \\ \vdots \\ 1 & 1 & 1 & & -1 & -1 \\ 1 & 1 & -1 & \cdots & -1 & -1 \\ 1 & -1 & -1 & \cdots & -1 & -1 \end{vmatrix}$$

$$= (-1)^{n+1} d^{n-1}\{2a+(n-1)d\}\left((-1)^{\left[\frac{n-1}{2}\right]} 2^{n-2}\right) = (-1)^{\left[\frac{n}{2}\right]} d^{n-1}\{2a+(n-1)d\}\, 2^{n-2}.$$

# References

[1] Amarnath Murthy, Smarandache Determinant Sequences, Smarandache Notions Journal, **12**(2001), 275-278.