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Refined Literal Indeterminacy and the Multiplication Law of Subindeterminacies

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Abstract. In this paper, we make a short history about: the neutrosophic set, neutrosophic numerical components and neutrosophic literal components, neutrosophic numbers, neutrosophic intervals, neutrosophic dual number, neutrosophic special dual number, neutrosophic special quasi dual number, neutrosophic linguistic number, neutrosophic linguistic interval-style number, neutrosophic hypercomplex numbers of dimension n, and elementary neutrosophic algebraic structures. Afterwards, their generalizations to refined neutrosophic set, respectively refined neutrosophic numerical and literal components, then refined neutrosophic numbers and refined neutrosophic algebraic structures, and set-style neutrosophic numbers. The aim of this paper is to construct examples of splitting the literal indeterminacy ($\boldsymbol{I}$) into literal sub-indeterminacies ($\boldsymbol{I_1}, \boldsymbol{I_2}, \dots, \boldsymbol{I_r}$), and to define a multiplication law of these literal sub-indeterminacies in order to be able to build refined $\boldsymbol{I}$ −neutrosophic algebraic structures. Also, examples of splitting the numerical indeterminacy ($\boldsymbol{i}$) into numerical sub-indeterminacies, and examples of splitting neutrosophic numerical components into neutrosophic numerical sub-components are given.

Keywords: neutrosophic set, elementary neutrosophic algebraic structures, neutrosophic numerical components, neutrosophic literal components, neutrosophic numbers, refined neutrosophic set, refined elementary neutrosophic algebraic structures, refined neutrosophic numerical components, refined neutrosophic literal components, refined neutrosophic numbers, literal indeterminacy, literal sub-indeterminacies, $\boldsymbol{I}$-neutrosophic algebraic structures.

1 Introduction

Neutrosophic Set was introduced in 1995 by Florentin Smarandache, who coined the words „neutrosophy" and its derivative „neutrosophic". The first published work on neutrosophics was in 1998 {see [1]}.

There exist two types of neutrosophic components: numerical and literal.

2 Neutrosophic Numerical Components

Of course, the *neutrosophic numerical components* (t, i, f) are crisp numbers, intervals, or in general subsets of the unitary standard or nonstandard unit interval.

Let $\mathcal{U}$ be a universe of discourse, and M a set included in $\mathcal{U}$. A generic element x from $\mathcal{U}$ belongs to the set M in the following way: $x(t, i, f) \in M$, meaning that x's degree of membership/truth with respect to the set M is t, x's degree of indeterminacy with respect to the set M is i, and x's degree of non-membership/falsehood with respect to the set M is f, where t, i, f are independent standard subsets of the interval $[0, 1]$, or non-standard subsets of the non-standard interval $]^-0, 1^+[$ in the case when one needs to make distinctions between *absolute and relative* truth, indeterminacy, or falsehood.

Many papers and books have been published for the cases when t, i, f were single values (crisp numbers), or t, i, f were intervals.

3 Neutrosophic Literal Components

In 2003, W. B. Vasantha Kandasamy and Florentin Smarandache [4] introduced the *literal indeterminacy* "I", such that $I^2 = I$ (whence $I^n = I$ for $n \geq 1$, n integer). They extended this to *neutrosophic numbers* of the form: $a + bI$, where a, b are real or complex numbers, and

$$(a_1 + b_1 I) + (a_2 + b_2 I) = (a_1 + a_2) + (b_1 + b_2)I \quad (1)$$

$$(a_1 + b_1 I)(a_2 + b_2 I) = (a_1 a_2) + (a_1 b_2 + a_2 b_1 + b_1 b_2)I \quad (2)$$

and developed many I-neutrosophic algebraic structures based on sets formed of neutrosophic numbers.

Working with imprecisions, Vasantha Kandasamy & Smarandache have proposed (approximated) I^2 *by* I; yet different approaches may be investigated by the interested researchers where $I^2 \neq I$ (in accordance with their believe and with the practice), and thus a new field would arise in the neutrosophic theory.

The neutrosophic number $N = a + bI$ can be interpreted as: "a" represents the determinate part of number N, while "bI" the indeterminate part of number N, where indeterminacy I may belong to a known (or unknown) set (not necessarily interval).

For example, $\sqrt{7} = 2.6457...$ that is irrational has infinitely many decimals. We cannot work with this exact number in our real life, we need to approximate it. Hence, we may write it as $2 + I$ with $I \in (0.6, 0.7)$, or as $2.6 + 3I$ with $I \in (0.01, 0.02)$, or $2.64 + 2I$ with $I \in (0.002, 0.004)$, etc. depending on the problem to be solved and on the needed accuracy.

Jun Ye [9] applied the neutrosophic numbers to decision making in *2014*.

The neutrosophic number $a+bI$ can be extended to a *Set-Style Neutrosophic Number A+BI*, where A and B are sets, while I is indeterminacy. As an interesting particular case one has when A and B are intervals, which is called *Interval-Style Neutrosophic Number.*

For example, $\{2, 3, 5\} + \{0, 4, 8, 12\}I$, with $I \in (0.5, 0.9)$, is a set-style neutrosophic number.

While $[30, 40] + [-10, -20]I$, with $I \in [7, 14]$, is an interval-style neutrosophic number.

4 Generalized Complex Numbers

For a *generalized complex number*, which has the form $N = (a+bI_1) + (c+dI_2)i$, where $i = \sqrt{-1}$, one has I_1 = the indeterminacy of the real part of N, while I_2 = indeterminacy of the complex part of N. In particular cases we may have $I_1 = I_2$.

5 Neutrosophic Dual Numbers

A *dual number* [13] is a number

$$D = a + bg, \quad (3)$$

where a and b are real numbers, while g is an element such that $g^2 = 0$.

Then, a *neutrosophic dual number*

$$ND = (a_0+a_1I_1) + (b_1+b_2I_2)g \quad (4)$$

where a_0, a_1, b_1, b_2 are real numbers, I_1 and I_2 are subindeterminacies, and g is an element such that $g^2 = 0$.

A *dual number of dimension n* has the form

$$D_n = a_0 + b_1g_1+b_2g_2+...+b_{n-1}g_{n-1} \quad (5)$$

where $a_0, b_1, b_2, ..., b_{n-1}$ are real numbers, while all g_j are elements such that $g_j^2 = 0$ and $g_j\, g_k = g_k\, g_j = 0$ for all $j \neq k$.

One can generalize this to a *dual complex number of dimension n*, considering the same definition as (5), but taking $a_0, b_1, b_2, ..., b_{n-1}$ as complex numbers.

Now, a *neutrosophic dual number of dimension n* has the form:

$$ND_n = (a_{00}+a_{01}I_0) + (b_{11}+b_{12}I_1)g_1 + (b_{21}+b_{22}I_2)g_2 + \dots + (b_{n-1,1}+b_{n-1,2}I_{n-1})g_{n-1} \quad (6)$$

where a_{00}, a_{01}, and all b_{jk} are real or complex numbers, while $I_0, I_1, ..., I_{n-1}$ are subindeterminacies.

Similarly for *special dual numbers*, introduced by W. B. Vasantha & F. Smarandache [14], i.e. numbers of the form:

$$SD = a + bg, \quad (7)$$

where a and b are real numbers, while g is an element such that $g^2 = g$ [for dimension n one has $g_jg_k = g_kg_j = 0$ for $j \neq k$]; to observe that $g \neq I$ = indeterminacy, and in general the product of subindeterminacies

$$I_jI_k \neq 0 \text{ for } j \neq k], \quad (8)$$

and *special quasi dual number*, introduced by Vasantha-Smarandache [15], which has the definition:

$$SQD = a + bg, \quad (9)$$

where a and b are real numbers, while g is an element such that $g^2 = -g$ [for dimension n one also has $g_jg_k = g_kg_j = 0$ for $j \neq k$], (10)

and their corresponding forms for dimension n.

They all can be extended to *neutrosophic special dual number* and respectively *neutrosophic special quasi dual number* (of dimension *2*, and similarly for dimension *n*) in a same way.

6 Neutrosophic Linguistic Numbers

A *neutrosophic linguistic number* has the shape:

$$N = L_{j+aI}, \quad (11)$$

where "L" means label or instance of a linguistic variable

$$V = \{L_0, L_1, L_2, ..., L_p\}, \text{ with } p \geq 1, \quad (12)$$

j is a positive integer between 0 and p-1, a is a real number, and I is indeterminacy that belongs to some real set, such that

$$0 \leq min\{j+aI\} \leq max\{j+aI\} \leq p. \quad (13)$$

Neutrosophic linguistic interval-style number has the form:

$$N = [L_{j+aI}, L_{k+bI}] \quad (14)$$

with similar restrictions (5) for L_{k+bI}.

7 Neutrosophic Intervals

We now for the first time extend the neutrosophic number to (open, closed, or half-open half-closed) neutrosophic interval. A *neutrosophic interval A* is an (open, closed, or half-open half-closed) interval that has some indeterminacy in one of its extremes, i.e. it has the form $A = [a, b] \cup \{cI\}$, or $A = \{cI\} \cup [a, b]$, where $[a, b]$ is the determinate part of the neutrosophic interval A, and I is the indeterminate part of it (while a, b, c are real numbers, and $\cup$ means union). (Herein I is an interval.)

We may even have neutrosophic intervals with double indeterminacy (or refined indeterminacy): one to the left (I_1), and one to the right (I_2):

$$A = \{c_1I_1\} \cup [a, b] \cup \{c_2I_2\}. \quad (15)$$

A classical real interval that has a neutrosophic number as one of its extremes becomes a neutrosophic interval. For example: $[0, \sqrt{7}]$ can be represented as $[0, 2] \cup I$ with $I = (2.0, 2.7)$, or $[0, 2] \cup \{10I\}$ with $I = (0.20, 0.27)$, or $[0, 2.6] \cup \{10I\}$ with $I = (0.26, 0.27)$, or $[0, 2.64] \cup \{10I\}$ with $I =$

(0.264, 0.265), etc. in the same way depending on the problem to be solved and on the needed accuracy.

We gave examples of closed neutrosophic intervals, but the open and half-open half-closed neutrosophic intervals are similar.

8 Notations

In order to make distinctions between the numerical and literal neutrosophic components, we start denoting the *numerical indeterminacy* by lower case letter "i" (whence consequently similar notations for *numerical truth* "t", and for *numerical falsehood* "f"), and *literal indeterminacy* by upper case letter "I" (whence consequently similar notations for *literal truth* "T", and for *literal falsehood* "F").

9 Refined Neutrosophic Components

In *2013*, F. Smarandache [3] introduced the refined neutrosophic components in the following way: the neutrosophic numerical components t, i, f can be refined (split) into respectively the following refined neutrosophic numerical sub-components:

$$\langle t_1, t_2, \dots t_p;\ i_1, i_2, \dots i_r;\ f_1, f_2, \dots f_s; \rangle, \tag{16}$$

where p, r, s are integers ≥ 1 and $\max\{p, r, s\} \geq 2$, meaning that at least one of p, r, s is ≥ 2; and t_j represents types of numeral truths, i_k represents types of numeral indeterminacies, and f_l represents types of numeral falsehoods, for $j = 1, 2, \dots, p$; $k = 1, 2, \dots, r$; $l = 1, 2, \dots, s$.

t_j, i_k, f_l are called numerical subcomponents, or respectively *numerical sub-truths*, *numerical sub-indeterminacies*, and *numerical sub-falsehoods*.

Similarly, the neutrosophic literal components T, I, F can be refined (split) into respectively the following neutrosophic literal subcomponents:

$$\langle T_1, T_2, \dots T_p;\ I_1, I_2, \dots I_r;\ F_1, F_2, \dots F_s; \rangle, \tag{17}$$

where p, r, s are integers ≥ 1 too, and $\max\{p, r, s\} \geq 2$, meaning that at least one of p, r, s is ≥ 2; and similarly T_j represent types of literal truths, I_k represent types of literal indeterminacies, and F_l represent types of literal falsehoods, for $j = 1, 2, \dots, p$; $k = 1, 2, \dots, r$; $l = 1, 2, \dots, s$.

T_j, I_k, F_l are called literal subcomponents, or respectively *literal sub-truths*, *literal sub-indeterminacies*, and *literal sub-falsehoods*.

Let consider a *simple example of refined numerical components*.

Suppose that a country C is composed of two districts D_1 and D_2, and a candidate John Doe competes for the position of president of this country C. Per whole country, NL(Joe Doe) $= (0.6, 0.1, 0.3)$, meaning that 60% of people voted for him, 10% of people were indeterminate or neutral – i.e. didn't vote, or gave a black vote, or a blank vote –, and 30% of people voted against him, where NL means the neutrosophic logic values.

But a political analyst does some research to find out what happened to each district separately. So, he does a refinement and he gets:

$$\begin{pmatrix} 0.40 & 0.20 \\ t_1 & t_2 \end{pmatrix}; \begin{pmatrix} 0.08 & 0.02 \\ i_1 & i_2 \end{pmatrix}; \begin{pmatrix} 0.05 & 0.25 \\ f_1 & f_2 \end{pmatrix} \tag{18}$$

which means that 40% of people that voted for Joe Doe were from district D_1, and 20% of people that voted for Joe Doe were from district D_2; similarly, 8% from D_1 and 2% from D_2 were indeterminate (neutral), and 5% from D_1 and 25% from D_2 were against Joe Doe.

It is possible, in the same example, to refine (split) it in a different way, considering another criterion, namely: what percentage of people did not vote (i_1), what percentage of people gave a blank vote – cutting all candidates on the ballot – (i_2), and what percentage of people gave a blank vote – not selecting any candidate on the ballot (i_3). Thus, the numerical indeterminacy (i) is refined into i_1, i_2, and i_3:

$$\begin{pmatrix} 0.60 \\ t \end{pmatrix}; \begin{pmatrix} 0.05 & 0.04 & 0.01 \\ i_1 & i_2 & i_3 \end{pmatrix}; \begin{pmatrix} 0.30 \\ f \end{pmatrix} \tag{19}$$

10 Refined Neutrosophic Numbers

In 2015, F. Smarandache [6] introduced the *refined literal indeterminacy* (I), which was split (refined) as $I_1, I_2, \dots, I_r$, with $r \geq 2$, where I_k, for $k = 1, 2, \dots, r$ represent types of literal sub-indeterminacies. A refined neutrosophic number has the general form:

$$N_r = a + b_1 I_1 + b_2 I_2 + \cdots + b_r I_r, \tag{20}$$

where $a, b_1, b_2, \dots, b_r$ are real numbers, and in this case N_r is called a *refined neutrosophic real number*; and if at least one of $a, b_1, b_2, \dots, b_r$ is a complex number (i.e. of the form $\alpha + \beta\sqrt{-1}$, with $\beta \neq 0$, and α, β real numbers), then N_r is called a *refined neutrosophic complex number*.

An example of refined neutrosophic number, with three types of indeterminacies resulted from the cubic root (I_1), from Euler's constant e (I_2), and from number π (I_3):

$$N_3 = -6 + \sqrt[3]{59} - 2e + 11\pi \tag{21}$$

Roughly

$N_3 = -6 + (3 + I_1) - 2(2 + I_2) + 11(3 + I_3)$

$= (-6 + 3 - 4 + 33) + I_1 - 2I_2 + 11I_3 = 26 + I_1 - 2I_2 + 11I_3$

where $I_1 \in (0.8, 0.9)$, $I_2 \in (0.7, 0.8)$, and $I_3 \in (0.1, 0.2)$, since $\sqrt[3]{59} = 3.8929\dots$, $e = 2.7182\dots$, $\pi = 3.1415\dots$.

Of course, other *3*-valued refined neutrosophic number representations of N_3 could be done depending on accuracy.

Then F. Smarandache [6] defined the *refined I-neutrosophic algebraic structures* in 2015 as algebraic structures based on sets of refined neutrosophic numbers.

Soon after this definition, Dr. Adesina Agboola wrote a paper on refined *I*-neutrosophic algebraic structures [7].

They were called "I-neutrosophic" because the refinement is done with respect to the literal indeterminacy (I), in order to distinguish them from the refined (t, i, f)-neutrosophic algebraic structures, where "(t, i, f)-neutrosophic" is referred to as refinement of the neutrosophic numerical components t, i, f.

Said Broumi and F. Smarandache published a paper [8] on refined neutrosophic numerical components in 2014.

11 Neutrosophic Hypercomplex Numbers of Dimension n

The *Hypercomplex Number of Dimension n* (or *n-Complex Number*) was defined by S. Olariu [10] as a number of the form:

$u = x_o + h_1x_1 + h_2x_2 + \ldots + h_{n-1}x_{n-1}$ (22)

where $n \geq 2$, and the variables $x_0, x_1, x_2, \ldots, x_{n-1}$ are real numbers, while $h_1, h_2, \ldots, h_{n-1}$ are the complex units, $h_o = 1$, and they are multiplied as follows:

$h_jh_k = h_{j+k}$ *if* $0 \leq j+k \leq n-1$, and $h_jh_k = h_{j+k-n}$ *if* $n \leq j+k \leq 2n-2$. (23)

We think that the above (11) complex unit multiplication formulas can be written in a simpler way as:

$h_jh_k = h_{j+k \,(mod\, n)}$ (24)

where *mod n* means *modulo n*.

For example, if $n = 5$, then $h_3h_4 = h_{3+4(mod\, 5)} = h_{7(mod 5)} = h_2$.

Even more, formula (12) allows us to multiply many complex units at once, as follows:

$h_{j1}h_{j2}\ldots h_{jp} = h_{j1+j2+\ldots+jp\,(mod\, n)}$, *for* $p \geq 1$. (25)

We now define for the first time the *Neutrosophic Hypercomplex Number of Dimension n* (or *Neutrosophic n-Complex Number*), which is a number of the form:

$u+vI$, (26)

where u and v are n-complex numbers and I = indeterminacy.

We also introduce now the *Refined Neutrosophic Hypercomplex Number of Dimension n* (or *Refined Neutrosophic n-Complex Number*) as a number of the form:

$u+v_1I_1+v_2I_2+\ldots+v_rI_r$ (27)

where $u, v_1, v_2, \ldots, v_r$ are n-complex numbers, and $I_1, I_2, \ldots, I_r$ are sub-indeterminacies, for $r \geq 2$.

Combining these, we may define a *Hybrid Neutrosophic Hypercomplex Number* (or *Hybrid Neutrosophic n-Complex Number*), which is a number of the form $u+vI$, where either u or v is a n-complex number while the other one is different (may be an m-complex number, with $m \neq n$, or a real number, or another type of number).

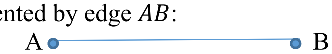
And a *Hybrid Refined Neutrosophic Hypercomplex Number* (or *Hybrid Refined Neutrosophic n-Complex Number*), which is a number of the form $u+v_1I_1+v_2I_2+\ldots+v_rI_r$, where at least one of $u, v_1, v_2, \ldots, v_r$ is a n-complex number, while the others are different (may be m-complex numbers, with $m \neq n$, and/or a real numbers, and/or other types of numbers).

12 Neutrosophic Graphs

We now introduce for the first time the general definition of a *neutrosophic graph* [12], which is a (directed or undirected) graph that has some indeterminacy with respect to its edges, or with respect to its vertexes (nodes), or with respect to both (edges and vertexes simultaneously). We have four main categories of neutrosophic graphs:

1) The (t, i, f)-*Edge Neutrosophic Graph.*

In such a graph, the connection between two vertexes A and B, represented by edge AB:



has the neutroosphic value of (t, i, f).

2) *I-Edge Neutrosophic Graph.*

This one was introduced in 2003 in the book "Fuzzy Cognitive Maps and Neutrosophic Cognitive Maps", by Dr. Vasantha Kandasamy and F. Smarandache, that used a different approach for the edge:

which can be just I = literal indeterminacy of the edge, with $I^2 = I$ (as in I-Neutrosophic algebraic structures). Therefore, simply we say that the connection between vertex A and vertex B is indeterminate.

3) *Orientation-Edge Neutrosophic Graph.*

At least one edge, let's say AB, has an unknown orientation (i.e. we do not know if it is from A to B, or from B to A).

4) *I-Vertex Neutrosophic Graph.*

Or at least one literal indeterminate vertex, meaning we do not know what this vertex represents.

5) (t, i, f)-*Vertex Neutrosophic Graph.*

We can also have at least one neutrosophic vertex, for example vertex A only partially belongs to the graph (t), indeterminate appurtenance to the graph (i), does not partially belong to the graph (f), we can say $A(t, i, f)$.

And combinations of any two, three, four, or five of the above five possibilities of neutrosophic graphs.

If (t, i, f) or the literal I are refined, we can get corresponding *refined neurosophic graphs*.

13 Example of Refined Indeterminacy and Multiplication Law of Subindeterminacies

Discussing the development of Refined I-Neutrosophic Structures with Dr. W.B. Vasantha Kandasamy, Dr. A.A.A. Agboola, Mumtaz Ali, and Said Broumi, a question has arisen: if I is refined into $I_1, I_2, \ldots, I_r$, with $r \geq 2$, how to define (or compute) $I_j * I_k$, for $j \neq k$?

We need to design a Sub-Indeterminacy $*$ Law Table.

Of course, this depends on the way one defines the algebraic binary multiplication law $*$ on the set:

$$\{N_r = a + b_1I_1 + b_2I_2 + \cdots + b_rI_r | a, b_1, b_2, \ldots, b_r \in M\}, \quad (28)$$

where M can be $\mathbb{R}$ (the set of real numbers), or $\mathbb{C}$ (the set of complex numbers).

We present the below example.

But, first, let's present several (possible) interconnections between logic, set, and algebra.

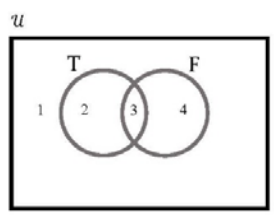
	Logic	*Set*	*Algebra*
operators	Disjunction (or) $\vee$	Union $\cup$	Addition $+$
	Conjunction (and) $\wedge$	Intersection $\cap$	Multiplication $\cdot$
	Negation $\neg$	Complement $\complement$	Subtraction $-$
	Implication $\rightarrow$	Inclusion $\subseteq$	Subtraction, Addition $-, +$
	Equivalence $\leftrightarrow$	Identity $\equiv$	Equality $=$

Table 1: *Interconnections between logic, set, and algebra.*

In general, if a Venn Diagram has n sets, with $n \geq 1$, the number of disjoint parts formed is 2^n. Then, if one combines the 2^n parts either by none, or by one, or by 2, ..., or by 2^n, one gets:

$$C_{2^n}^0 + C'_{2^n} + C_{2^n}^2 + \cdots + C_{2^n}^{2^n} = (1+1)^{2^n} = 2^{2^n}. \quad (29)$$

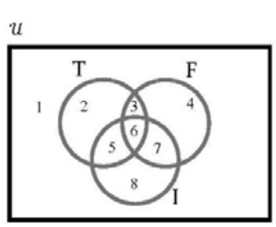
Hence, for $n = 2$, the Venn Diagram, with literal truth (T), and literal falsehood (F), will make $2^2 = 4$ disjoint parts, where the whole rectangle represents the whole universe of discourse $(\mathcal{U})$.



Venn Diagram for n =2.

Then, combining the four disjoint parts by none, by one, by two, by three, and by four, one gets

$$C_4^0 + C_4^1 + C_4^2 + C_4^3 + C_4^4 = (1+1)^4 = 2^4 = 16 = 2^{2^2}. \quad (30)$$

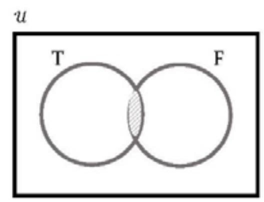


Venn Diagram for n = 3.

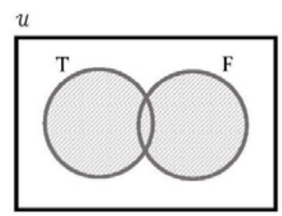
For $n = 3$, one has $2^3 = 8$ disjoint parts, and combining them by none, by one, by two, and so on, by eight, one gets $2^8 = 256$, or $2^{2^3} = 256$.

For the case when $n = 2 = \{T, F\}$ one can make up to *16* sub-indeterminacies, such as:

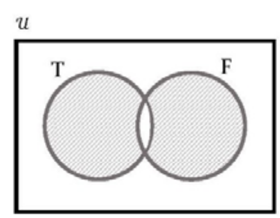
$I_1 = C =$ **contradiction** = True and False $= T \wedge F$



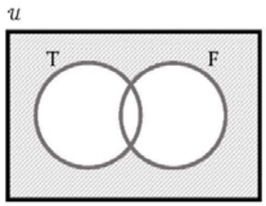
$I_2 = Y =$ **uncertainty** = True or False $= T \vee F$



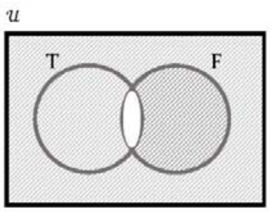
$I_3 = S =$ **unsureness** = either True or False $= T \underline{\vee} F$



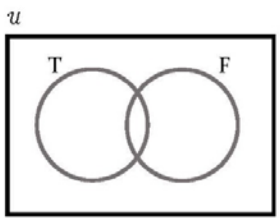
$I_4 = H =$ **nihilness** = neither True nor False $= \neg T \wedge \neg F$



$I_5 = V =$ **vagueness** $=$ not True or not False
$= \neg T \vee \neg F$



$I_6 = E =$ **emptiness** $=$ neither True nor not True
$= \neg T \wedge \neg(\neg T) = \neg T \wedge T$



Let's consider the literal indeterminacy (I) refined into only six literal sub-indeterminacies as above.

The binary multiplication law

$$*: \{I_1, I_2, I_3, I_4, I_5, I_6\}^2 \to \{I_1, I_2, I_3, I_4, I_5, I_6\} \quad (31)$$

defined as:

$I_j * I_k$ = intersections of their Venn diagram representations;
or $I_j * I_k$ = application of $\wedge$ operator, i.e. $I_j \wedge I_k$.

We make the following:

*	I_1	I_2	I_3	I_4	I_5	I_6
I_1	I_1	I_1	I_6	I_6	I_6	I_6
I_2	I_1	I_2	I_3	I_6	I_3	I_6
I_3	I_6	I_3	I_3	I_6	I_3	I_6
I_4	I_6	I_6	I_6	I_4	I_4	I_6
I_5	I_6	I_3	I_3	I_4	I_5	I_6
I_6	I_6	I_6	I_6	I_6	I_6	I_6

Table 2: Sub-Indeterminacies Multiplication Law

14 Remark on the Variety of Sub-Indeterminacies Diagrams

One can construct in various ways the diagrams that represent the sub-indeterminacies and similarly one can define in many ways the $*$ algebraic multiplication law, $I_j * I_k$, depending on the problem or application to solve.

What we constructed above is just an example, not a general procedure.

Let's present below several calculations, so the reader gets familiar:

$I_1 * I_2 =$ (shaded area of I_1) $\cap$ (shaded area of I_2) $=$ shaded area of I_1,
or $I_1 * I_2 = (T \wedge F) \wedge (T \vee F) = T \wedge F = I_1$.
$I_3 * I_4 =$ (shaded area of I_3) $\cap$ (shaded area of I_4) $=$ empty set $= I_6$,
or $I_3 * I_4 = (T \underline{\vee} F) \wedge (\neg T \wedge \neg F) = [T \wedge (\neg T \wedge \neg F)] \underline{\vee} [F \wedge (\neg T \wedge \neg F)] = (T \wedge \neg T \wedge \neg F) \underline{\vee} (F \wedge \neg T \wedge \neg F) =$ (impossible) $\underline{\vee}$ (impossible)
because of $T \wedge \neg T$ in the first pair of parentheses and because of $F \wedge \neg F$ in the second pair of parentheses
$=$ (impossible) $= I_6$.
$I_5 * I_5 =$ (shaded area of I_5) $\cap$ (shaded area of I_5) $=$ (shaded area of I_5) = I_5,
or $I_5 * I_5 = (\neg T \vee \neg F) \wedge (\neg T \vee \neg F) = \neg T \vee \neg F = I_5$.

Now we are able to build refined I-neutrosophic algebraic structures on the set

$$S_6 = \{a_0 + a_1 I_1 + a_2 I_2 + \cdots + a_6 I_6, \text{for } a_0, a_1, a_2, \ldots\ a_6 \in \mathbb{R}\}, \quad (32)$$

by defining the addition of refined I-neutrosophic numbers:

$$(a_0 + a_1 I_1 + a_2 I_2 + \cdots + a_6 I_6) + (b_0 + b_1 I_1 + b_2 I_2 + \cdots + b_6 I_6) = (a_0 + b_0) + (a_1 + b_1) I_1 + (a_2 + b_2) I_2 + \cdots + (a_6 + b_6) I_6 \in S_6. \quad (33)$$

And the multiplication of refined neutrosophic numbers:

$$(a_0 + a_1 I_1 + a_2 I_2 + \cdots + a_6 I_6) \cdot (b_0 + b_1 I_1 + b_2 I_2 + \cdots + b_6 I_6) = a_0 b_0 + (a_0 b_1 + a_1 b_0) I_1 + (a_0 b_2 + a_2 b_0) I_2 + \cdots + (a_0 b_6 + a_6 b_0) I_6 +$$
$$+ \sum_{j,k=1}^{6} a_j b_k \left(I_j * I_k\right) = a_0 b_0 + \sum_{k=1}^{6} (a_0 b_k + a_k b_0) I_k + \sum_{j,k=1}^{6} a_j b_k \left(I_j * I_k\right) \in S_6, \quad (34)$$

where the coefficients (scalars) $a_m \cdot b_n$, for $m = 0, 1, 2, \ldots, 6$ and $n = 0, 1, 2, \ldots, 6$, are multiplied as any real numbers, while $I_j * I_k$ are calculated according to the previous Sub-Indeterminacies Multiplication Law (Table 2).

Clearly, both operators (addition and multiplication of refined neutrosophic numbers) are well-defined on the set S_6.

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