Some Properties of the Harmonic Quadrilateral[1]

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Abstract. In this article, we review some properties of the harmonic quadrilateral related to triangle simedians and to Apollonius circles.

Keywords: harmonic quadrilateral, simedian, Apollonius circle.

MSC 2010: 51M04.

Definition 1. A convex quadrilateral $ABCD$ admitting a circumcicle and having the property $AB \cdot CD = BC \cdot AD$ is called a *harmonic quadrilateral.*

Definition 2. A *triangle simedian* is the isogonal cevian of a triangle median.

Proposition 1. *In the triangle ABC, the cevian AA_1, $A_1 \in (BC)$, is a simedian if and only if* $\dfrac{BA_1}{A_1C} = \left(\dfrac{AB}{AC}\right)^2$.

Proposition 2. *In an harmonic quadrilateral, the diagonals are simedians of the triangles determined by two consecutive sides of a quadrilateral with its diagonal.*

Proof. Let $ABCD$ be an harmonic quadrilateral and $\{K\} = AC \cap BD$ (Fig. 1). We prove that BK is simedian in the triangle ABC. From the similarity of the triangles ABK and DCK, we find that

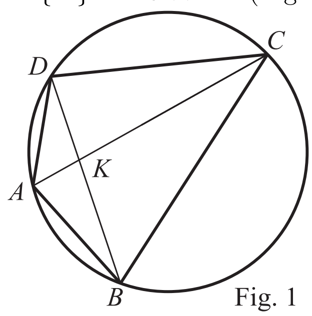

Fig. 1

(1) $$\frac{AB}{DC} = \frac{AK}{DK} = \frac{BK}{CK}.$$

From the similarity of the triangles BCK şi ADK, we conclude that

(2) $$\frac{BC}{AD} = \frac{CK}{DK} = \frac{BK}{AK}.$$

From the relations (1) and (2), by division, it follows that

(3) $$\frac{AB}{BC} \cdot \frac{AD}{DC} = \frac{AK}{CK}.$$

But $ABCD$ is an harmonic quadrilateral; consequently,

$$\frac{AB}{BC} = \frac{AD}{DC};$$

[1]See also **A. Reisner** - *Quadrange harmonique et nombres complexes*, this journal, 1/2014, 35-39.

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substituting this relation in (3), it follows that

$$\left(\frac{AB}{BC}\right)^2 = \frac{AK}{CK};$$

as shown by Proposition 1, BK is a simedian in the triangle ABC. Similarly, it can be shown that AK is a simedian in the triangle ABD, that CK is a simedian in the triangle BCD, and that DK is a simedian in the triangle ADC.

Remark 1. The converse of the Proposition 2 is proved similarly, i.e. we have:

Proposition 3. *If in a convex quadrilateral admitting a circumcicle a diagonal is a simedian in the triangle formed by the other diagonal with two consecutive sides of the quadrilateral, then the quadrilateral is an harmonic quadrilateral.*

Remark 2. From Propositions 2 and 3 above, it results a simple way to build an harmonic quadrilateral. In a circle, let a triangle ABC be considered; we construct the simedian of A, be it AK, and we denote by D the intersection of the simedian AK with the circle. The quadrilateral $ABCD$ is an harmonic quadrilateral.

Proposition 4. *In a triangle ABC, the points of the simedian of A are situated at proportional distances to the sides AB and AC.*

Proof. We have the simedian AA_1 in the triangle ABC (Fig. 2). We denote by D and E the projections of A_1 on AB, and AC respectively. We get:

$$\frac{BA_1}{CA_1} = \frac{Area\,(\Delta ABA_1)}{Area\,(\Delta ACA_1)} = \frac{AB \cdot A_1D}{AC \cdot A_1E}.$$

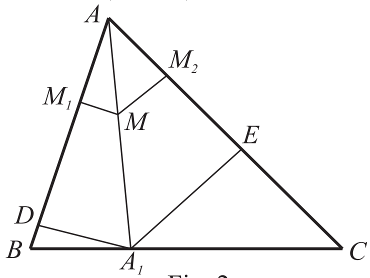

Fig. 2

Moreover, from Proposition 1, we know that $\frac{BA_1}{A_1C} = \left(\frac{AB}{AC}\right)^2$. Substituting in the previous relation, we obtain that

$$\frac{A_1D}{A_1E} = \frac{AB}{AC}.$$

On the other hand, $A_1D = AA_1 \cdot \sin\widehat{BAA_1}$ and $A_1E = AA_1 \cdot \sin\widehat{CAA_1}$, hence

$$\frac{A_1D}{A_1E} = \frac{\sin\widehat{BAA_1}}{\sin\widehat{CAA_1}} = \frac{AB}{AC}. \tag{4}$$

If M is a point on the simedian and MM_1 and MM_2 are its projections on AB, and AC respectively, we have: $MM_1 = AM \cdot \sin\widehat{BAA_1}$, $MM_2 = AM \cdot \sin\widehat{CAA_1}$, hence:

$$\frac{MM_1}{MM_2} = \frac{\sin\widehat{BAA_1}}{\sin\widehat{CAA_1}}.$$

Taking into account (4), we obtain that

$$\frac{MM_1}{MM_2} = \frac{AB}{AC}.$$

Remark 3. The converse of the property in the statement above is valid, meaning that, *if M is a point inside a triangle, its distances to two sides are proportional to the lengths of these sides, then M the point belongs to the simedian of the triangle having the vertex joint to the two sides.*

Proposition 5. *In an harmonic quadrilateral, the point of intersection of the diagonals is located towards the sides of the quadrilateral to proportional distances to these sides.*

The **Proof** of this statement relies on Propositions 2 and 4.

Proposition 6 (*R. Tucker*). *The point of intersection of the diagonals of an harmonic quadrilateral minimizes the sum of squares of distances from a point inside the quadrilateral to the quadrilateral sides.*

Proof. Let $ABCD$ be an harmonic quadrilateral and M any point within. We denote by x, y, z, u the distances of M to the AB, BC, CD, DA sides of lenghts $a, b, c,$ and d (Fig. 3). Let S be the area of the quadrilateral $ABCD$. We have: $ax + by + cz + du = 2S$. Following Cauchy-Buniakowski-Schwarz Inequality, we get:

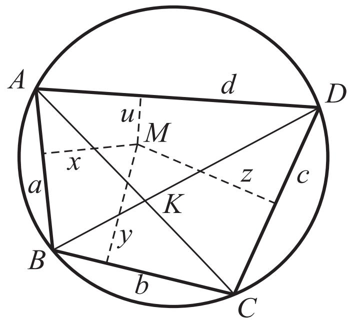

Fig. 3

$$\left(a^2 + b^2 + c^2 + d^2\right)\left(x^2 + y^2 + z^2 + u^2\right) \geq (ax + by + cz + du)^2,$$

and it is obvious that

$$x^2 + y^2 + z^2 + u^2 \geq \frac{4S^2}{a^2 + b^2 + c^2 + d^2}.$$

We note that the minimum sum of squared distances is $\frac{4S^2}{a^2 + b^2 + c^2 + d^2} = const.$ In Cauchy-Buniakowski-Schwarz Inequality, the equality occurs if

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{u}{d}.$$

Since $\{K\} = AC \cap BD$ is the only point with this property, it ensues that $M = K$, so K has the property of the minimum in the statement.

Definition 3. We call *external simedian* of ABC triangle a cevian $AA_1^{'}$ corresponding to the vertex A, where $A^{'}$ is the harmonic conjugate of the point A_1 – simedian′s foot from A relative to points B and C.

Remark 4. In Fig. 4, the cevian AA_1 is an internal simedian, and $AA_1^{'}$ is an external simedian.

We have $\dfrac{A_1B}{A_1C} = \dfrac{A_1^{'}B}{A_1^{'}C}$. In view of Proposition 1, we get that

$$\frac{A_1'B}{A_1'C} = \left(\frac{AB}{AC}\right)^2.$$

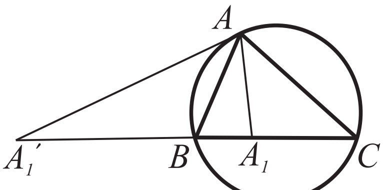

Fig. 4

Proposition 7. *The tangents taken to the extremes of a diagonal of a circle circumscribed to the harmonic quadrilateral intersect on the other diagonal.*

Proof. Let P be the intersection of a tangent taken in D to the circle circumscribed to the harmonic quadrilateral $ABCD$ with AC (Fig. 5). Since triangles PDC and PAD are alike, we conclude that:

$$\text{(5)} \qquad \frac{PD}{PA} = \frac{PC}{PD} = \frac{DC}{AD}.$$

From relations (5), we find that:

$$\text{(6)} \qquad \frac{PA}{PC} = \left(\frac{AD}{DC}\right)^2.$$

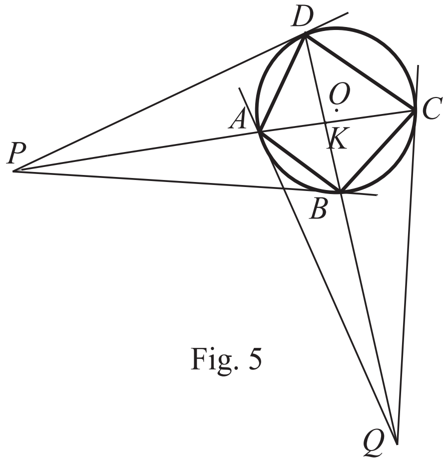

Fig. 5

This relationship indicates that P is the harmonic conjugate of K with respect to A and C, so DP is an external simedian from D of the triangle ADC.

Similarly, if we denote by $P^{'}$ the intersection of the tangent taken in B to the circle circumscribed with AC, we get:

$$\text{(7)} \qquad \frac{P^{'}A}{P^{'}C} = \left(\frac{BA}{BC}\right)^2.$$

From (6) and (7), as well as from the properties of the harmonic quadrilateral, we know that $\dfrac{AB}{BC} = \dfrac{AD}{DC}$, which means that $\dfrac{PA}{PC} = \dfrac{P^{'}A}{P^{'}C}$, hence $P = P^{'}$.

Similarly, it is shown that the tangents taken to A and C intersect at point Q located on the diagonal BD.

Remark 5. 1) The points P and Q are the diagonal poles of BD and AC in relation to the circle circumscribed to the quadrilateral.

2) From the previous proposition, it follows that in a triangle the internal simedian of an angle is consecutive to the external simedians of the other two angles.

Proposition 8. *Let $ABCD$ be an harmonic quadrilateral inscribed in the circle of center O and let P and Q be the intersections of the tangents taken in B and D, respectively in A and C to the circle circumscribed to the quadrilateral. If $\{K\} = AC \cap BD$, then the orthocenter of triangle PKQ is O.*

Proof. From the properties of tangents taken from a point to a circle, we conclude that $PO \perp BD$ and $QO \perp AC$. These relations show that in the triangle PKQ, PO and QO are heights, so O is the orthocenter of this triangle.

Definition 4. The *Apollonius circle* related to the vertex A of the triangle ABC is the circle built on the segment $[DE]$ in diameter, where D and E are the feet of the internal, respectively, external bisectors taken from A to the triangle ABC.

If the triangle ABC is isosceles with $AB = AC$, the Apollonius circle corresponding to vertex A is not defined.

Proposition 9. *The Apollonius circle relative to the vertex A of the triangle ABC has as center the feet of the external simedian taken from A.*

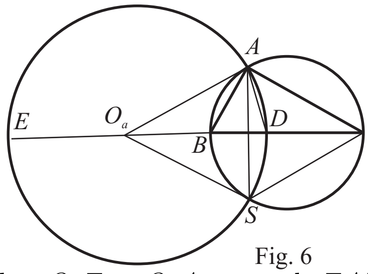

Fig. 6

Proof. Let O_a be the intersection of the external simedian of the triangle ABC with BC (Fig. 6). Assuming that $m(\widehat{B}) > m(\widehat{C})$, we find that $m\left(\widehat{EAB}\right) = \frac{1}{2}\left[m\left(\widehat{B}\right) + m\left(\widehat{C}\right)\right]$. O_aA being a tangent, we find that $m\left(\widehat{O_aAB}\right) = m\left(\widehat{C}\right)$. From, $m\left(\widehat{EAO_a}\right) = \frac{1}{2}\left[m\left(\widehat{B}\right) - m\left(\widehat{C}\right)\right]$ and $m\left(\widehat{AEO_a}\right) = \frac{1}{2}\left[m\left(\widehat{B}\right) - m\left(\widehat{C}\right)\right]$ it results that $O_aE = O_aA$; onward, EAD being a right angled triangle, we obtain $O_aA = O_aD$, hence O_a is the center of Apollonius circle corresponding to the vertex A.

Proposition 10. *Apollonius circle relative to the vertex A of the triangle ABC cuts the circle circumscribed to the triangle following the internal simedian taken from A.*

Proof. Let S be the second point of intersection of Apollonius circles relative to vertex A and the circle circumscribe the triangle ABC. Because O_aA is tangent in A to the circle circumscribed, it results, for reasons of symmetry, that O_aS will be tangent in S to the circumscribed circle.

For triangle ACS, O_aA and O_aS are external simedians; it results that CO_a a is internal simedian in the triangle ACS, furthermore, it results that the quadrilateral $ABSC$ is an harmonic quadrilateral. Consequently, AS is the internal simedian of the triangle ABC and the property is proven.

Remark 6. From this, in view of Fig. 5, it results that the circle of center Q passing through A and C is an Apollonius circle relative to the vertex A for the triangle ABD. This circle (of center Q and radius QC) is also an Apollonius circle relative to the vertex C of the triangle BCD.

Similarly, the Apollonius circles corresponding to vertexes B and D and to the triangles ABC, and ADC respectively, coincide.

Proposition 11. *In an harmonic quadrilateral, the Apollonius circles - associated with the vertexes of a diagonal and to the triangles determined by those vertexes to the other diagonal - coincide. Radical axis of the Apollonius circles is the right determined* *by the center of the circle circumscribed to the harmonic quadrilateral and by the intersection of its diagonals.*

Proof. Referring to Fig. 5, we observe that the power of O towards the Apollonius circles relative to vertexes B and C of triangles ABC and BCU is $OB^2 = OC^2$. So O belongs to the radical axis of the circles.

We also have $KA \cdot KC = KB \cdot KD$, relatives indicating that the point K has equal powers towards the highlighted Apollonius circles.

Bibliografie

1. **R.A. Johnson** – *Advanced Euclidean Geometry*, Dover Publications, Inc. Mineola, New York, USA, 2007.
2. **F. Smarandache, I. Pătraşcu** – *The Geometry of Homological Triangles*, The Education Publisher, Inc. Columbus, Ohio, USA, 2012.
3. **F. Smarandache, I. Pătraşcu** – *Variance on Topics of plane Geometry*, The Education Publisher, Inc. Columbus, Ohio, USA, 2013.

Al doilea autor va împlini în curând 60 de ani. Redacţia revistei foloseşte acest prilej pentru a-i ura ,,MULŢI ANI !", trăiţi în sănătate, cu bucurii şi noi şi remarcabile succese. Dintr-un material amplu primit de redacţie, selectăm un număr de aspecte care să dea contur personalităţii sale atipice, caracterizată prin energie explozivă, creativitate, diversitatatea domeniilor abordate, prolificitate.

Florentin Smarandache s-a născut la 10 decembrie 1954 în Bălceşti, jud. Vâlcea. Studiile, de la clasele primare la cele superioare, le-a făcut în România. A absolvit în 1979 secţia informatică a Facultăţii de Ştiinţe a Universităţii din Craiova, ca şef de promoţie.

În 1988 emigrează din motive politice. Din anul 2008 ocupă o poziţie de profesor universitar la University of New Mexico (Gallup). În 1997 obţine titlul de doctor în matematici, în domeniul teoriei numerelor, la Universitatea de Stat din Chişinău.

Are contribuţii în matematică (teoria numerelor, geometrie, logică), fizică, filozofie, literatură (poeme, nuvele, povestiri, un roman, piese de teatru, eseuri, traduceri, interviuri) şi artă (experimente în desene, picturi, colaje, fotografii, artă pe calculator).

În matematică a introdus gradul de negare al unei axiome ori teoreme (vezi geometriile smarandache, care pot fi parţial euclidiene şi parţial neeuclidiene), multi-structurile (vezi n-structurile smarandache, unde o structură mai slabă conţine insule de structuri mai puternice) şi multi-spaţiile (combinaţii de spaţii heterogene). Lucrările sale de teoria numerelor s-au bucurat de o anumită popularitate, fiind preluate şi dezvoltate de matematicieni români şi străini din multe ţări. A propus extinderea probabilităţilor clasice şi imprecise la ,,probabilitate neutrosofică", ca un vector tridimensional ale cărui componente sunt submulţimi ale intervalului ne-standard]-0,1+[. Împreună cu Jean Dezert extinde Teoria Dempster-Shafer la o nouă teorie de fuzionare a informaţiei plauzibile şi paradoxiste (numită Teoria Dezert-Smarandache).

În fizică a introdus paradoxuri cuantice şi noţiunea de ,,nematerie", formată din combinaţii de materie şi antimaterie, ca o a treia formă posibilă de materie. A emis ipoteza că ,,nu există o barieră a vitezei în univers şi se pot construi orice viteze", pe baza căreia a propus o Teorie Absolută a relativităţii, care să nu producă dilatare a timpului sau contractare a spaţiului.

În filozofie a introdus conceptul de ,,neutrosofie", ca o generalizare a dialecticii lui Hegel, care stă la baza cercetărilor sale în matematică şi economie, precum şi ,,logica neutrosofică", ,,mulţime neutrosofică", ,,probabilitate neutrosofică", ,,statistica neutrosofică".

În literatură şi artă a fondat în 1980 curentul de avangardă numit paradoxism, care constă în folosirea excesivă în creaţii a contradicţiilor, antitezelor, antinomiilor, oximoronilor, paradoxurilor. A fost nominalizat pentru premiul Nobel în literatură pe anul 2011.