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Radical Axis of Lemoine Circles

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Abstract

In this short paper, a theorem stating that the radical axis of of the Lemoine circles of a triangle is perpendicular to a line raised on the symmedian is proven.

In this article, the emphasis is on the radical axis of the Lemoine Circles. We open with some definitions.

Definition: In a triangle, **a symmedian** is a line constructed by first drawing a line from a vertex to the midpoint of the opposite side and then reflecting that line across the line that bisects the angle. You will then have three segments emanating from a vertex with the angle bisector in the middle and the symmedian on the side opposite the segment connected to the midpoint of the opposite side. There are of course three symmedians in a triangle.

Definition: The **symmedian center** of a triangle is the point where the three symmedian segments mutually intersect.

Definition: A **Lemoine parallel** is a line through the symmedian point of a triangle that is parallel to a side of the triangle.

Definition: A **Lemoine circle** is the circle determined by the points where the Lemoine parallels intersect the sides of the triangle.

We open by reminding the reader of the statements of two theorems.

Theorem 1: The parallels taken through the symmedian center K of a triangle to the sides of the triangle determine on them six concyclic points (The First Lemoine Circle).

Theorem 2: The antiparallels taken through the symmedian center of a triangle to the sides of a triangle determine on them six concyclic points (The Second Lemoine Circle).

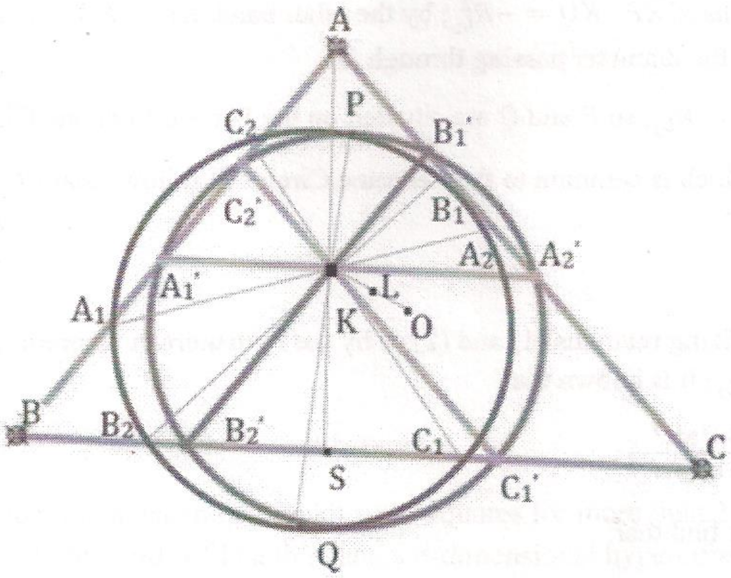
Remark 1 : If ABC is a scalene triangle and K is its symmedian center, then L, the center of the First Lemoine Circle, is the middle of the segment [OK], where O is the center of the circumscribed circle, and the center of the Second Lemoine Circle is K. It follows that the radical axis of Lemoine circles is perpendicular on the line of the centers LK, therefore on the line OK.

Proposition 1 : The radical axis of Lemoine Circles is perpendicular on the line OK raised in the symmedian center K.

Proof : The proof references figure 1. Let A_1A_2 be the antiparallel to BC taken through K. KA_1 is the radius R_{L_2} of the Second Lemoine Circle; we have:

$$R_{L_2} = \frac{abc}{a^2+b^2+c^2}.$$

Figure 1



Let $A_1'A_2'$ be the Lemoine parallel taken to BC; we evaluate the power of K towards the First Lemoine Circle. We have:

$$\overrightarrow{KA_1'} \cdot \overrightarrow{KA_2'} = LK^2 - R_{L_1}^2. \qquad (1)$$

Let S be the simmedian leg from A; it follows that:

$$\frac{KA_1'}{BS} = \frac{AK}{AS} - \frac{KA_2'}{SC}.$$

We obtain:

$$KA_1' = BS \cdot \frac{AK}{AS} \text{ and } KA_2' = SC \cdot \frac{AK}{AS},$$

but $\frac{BS}{SC} = \frac{c^2}{b^2}$ and $\frac{AK}{AS} = \frac{b^2+c^2}{a^2+b^2+c^2}$.

Therefore:

$$\overrightarrow{KA_1'} \cdot \overrightarrow{KA_2'} = -BS \cdot SC \cdot \left(\frac{AK}{AS}\right)^2 = \frac{-a^2b^2c^2}{(b^2+c^2)^2} \cdot \frac{(b^2+c^2)^2}{(a^2+b^2+c^2)^2} = -R_{L_2}^2. \qquad (2)$$

We draw the perpendicular in K on the line LK and denote by P and Q its inter
First Lemoine Circle; we have $\overrightarrow{KP} \cdot \overrightarrow{KQ} = -R_{L_2}^2$; by the other hand, $KP = KQ$
which is perpendicular to the diameter passing through K).

It follows that $KP = KQ = R_{L_2}$, so P and Q are situated on the Second Lemoine

Because PQ is a chord which is common to the Lemoine Circles, it follows that
radical axis.

Comment 1: After equalizing relations (1) and (2) or by the Pythagorean theorem
PKL, we can calculate R_{L_1}. It is known that:

$$OK^2 = R^2 - \frac{3a^2b^2c^2}{(a^2+b^2+c^2)^2};$$

and since $LK = \frac{1}{2}OK$, we find that:

$$R_{L_1}^2 = \frac{1}{4} \cdot \left[R^2 + \frac{a^2b^2c^2}{(a^2+b^2+c^2)^2}\right].$$

Remark 2 : The proven *Proposition* regarding the radical axis of the Lemoine Circles is a particular case of the following *Proposition*, which we leave it to the reader to prove.

Proposition 2: If $\mathcal{C}(O_1, R_1)$ și $\mathcal{C}(O_2, R_2)$ are two circles such as the power of ce
$\mathcal{C}(O_2, R_2)$ is $-R_1^2$, then the radical axis of the circles is the perpendicular in O_1
centers O_1O_2.

References

1. F. Smarandache, Ion Patrascu: *The Geometry of Homological Triangles*, Education Publisher, Ohio, USA, 2012.
2. Ion Patrascu, F. Smarandache: *Variance on Topics of Plane Geometry*, Educa
Publisher, Ohio, USA, 2013.