International J.Math. Combin. Vol.1(2016), 101-108

Quotient Cordial Labeling of Graphs

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Abstract: In this paper we introduce quotient cordial labeling of graphs. Let G be a (p,q) graph. Let $f : V(G) \to \{1,2,\cdots,p\}$ be a $1-1$ map. For each edge uv assign the label $\left[\frac{f(u)}{f(v)}\right]$ (or) $\left[\frac{f(v)}{f(u)}\right]$ according as $f(u) \geq f(v)$ or $f(v) > f(u)$. f is called a quotient cordial labeling of G if $|e_f(0) - e_f(1)| \leq 1$ where $e_f(0)$ and $e_f(1)$ respectively denote the number of edges labelled with even integers and number of edges labelled with odd integers. A graph with a quotient cordial labeling is called a quotient cordial graph. We investigate the quotient cordial labeling behavior of path, cycle, complete graph, star, bistar etc.

Key Words: Path, cycle, complete graph, star, bistar, quotient cordial labeling, Smarandachely quotient cordial labeling.

AMS(2010): 05C78.

§1. Introduction

Graphs considered here are finite and simple. Graph labeling is used in several areas of science and technology like coding theory, astronomy, circuit design etc. For more details refer Gallian [2]. The union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. Cahit [1], introduced the concept of cordial labeling of graphs. Recently Ponraj et al. [4], introduced difference cordial labeling of graphs. Motivated by these labelings we introduce quotient cordial labeling of graphs. Also in this paper we investigate the quotient cordial labeling behavior of path, cycle, complete graph, star, bistar etc. In [4], Ponraj et al. investigate the quotient cordial labeling behavior of subdivided star $S(K1,n)$, subdivided bistar $S(B_{n,n})$ and union of some star related graphs. $[x]$ denote the smallest integer less than or equal to x. Terms are not defined here follows from Harary [3].

[1]Received July 9, 2015, Accepted February 25, 2016.

§2. Quotient Cordial Labeling

Definition 2.1 *Let G be a (p, q) graph. Let $f : V(G) \to \{1, 2, \cdots, p\}$ be an injective map. For each edge uv assign the label $\left[\dfrac{f(u)}{f(v)}\right]$ (or) $\left[\dfrac{f(v)}{f(u)}\right]$ according as $f(u) \geq f(v)$ or $f(v) > f(u)$. Then f is called a quotient cordial labeling of G if $|e_f(0) - e_f(1)| \leq 1$ where $e_f(0)$ and $e_f(1)$ respectively denote the number of edges labelled with even integers and number of edges labelled with odd integers. A graph with a quotient cordial labeling is called a quotient cordial graph.*

Generally, a Smarandachely quotient cordial labeling of G respect to $S \subset V(G)$ is such a labelling of G that it is a quotient cordial labeling on $G \setminus S$. Clearly, a quotient cordial labeling is a Smarandachely quotient cordial labeling of G respect to $S = \emptyset$.

A simple example of quotient cordial graph is given in Figure 1.

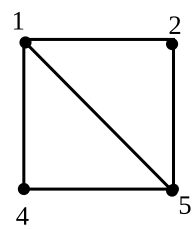

Figure 1.

§3. Main Results

First we investigate the quotient cordial labeling behavior of path.

Theorem 3.1 *Any path is quotient cordial.*

Proof Let P_n be the path $u_1u_2 \cdots u_n$. Assign the label 1 to u_1. Then assign $2, 4, 8, \cdots$ $(\leq n)$ to the consecutive vertices until we get $\left[\frac{n-1}{2}\right]$ edges with label 0, then choose the least number $\leq n$ that is not used as a label. That is consider the label 3. Assign the label to the next non labelled vertices consecutively by $3, 6, 12, \ldots (\leq n)$ until we get $\left[\frac{n-1}{2}\right]$ edges with label 0. If not, consider the next least number $\leq n$ that is not used as a label. That is choose 5. Then label the vertices $5, 10, 20, \cdots$ $(\leq n)$ consecutively. If the total number of edges with label 0 is $\left[\frac{n-1}{2}\right]$, then stop this process, otherwise repeat the same until we get the $\left[\frac{n-1}{2}\right]$ edges with label 0. Let S be the set of integer less than or equal to n that are not used as a label. Let t be the least integer such that u_t is not labelled. Then assign the label to the vertices $u_t, u_{t+1}, \cdots, u_n$ from the set S in descending order. Clearly the above vertex labeling is a quotient cordial labeling. $\square$

Illustration 3.2 A quotient cordial labeling of P_{15} is given in Figure 2.

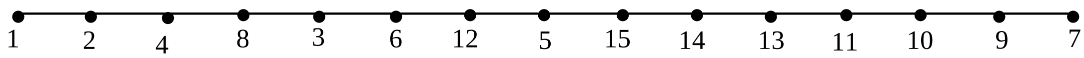

Figure 2

Here, $S = \{5, 7, 9, 10, 11, 13, 14, 15\}$

Corollary 3.3 *If n is odd then the cycle C_n is quotient cordial.*

Proof The quotient cordial labeling of path P_n, n odd, given in Theorem 3.1 is obviously a quotient cordial labeling of the cycle C_n. □

Next is the complete graph.

Theorem 3.4 *The complete graph K_n is quotient cordial iff $n \leq 4$.*

Proof Obviously $K_n, n \leq 4$ is quotient cordial. Assume $n > 4$. Suppose f is a quotient cordial labeling of K_n.

Case 1. n is odd.

Consider the sets,

$$\begin{aligned}
S_1 &= \left\{ \left[\frac{n}{n-1}\right], \left[\frac{n}{n-2}\right], \ldots, \left[\frac{n}{\frac{n+1}{2}}\right] \right\} \cup \left\{ \left[\frac{n}{1}\right] \right\} \\
S_2 &= \left\{ \left[\frac{n-2}{n-3}\right], \left[\frac{n-2}{n-4}\right], \ldots, \left[\frac{n-2}{\frac{n-1}{2}}\right] \right\} \cup \left\{ \left[\frac{n-2}{1}\right] \right\} \\
&\vdots \\
S_{\frac{n-1}{2}} &= \left\{ \left[\frac{3}{2}\right] \right\} \cup \left\{ \left[\frac{3}{1}\right] \right\}
\end{aligned}$$

Clearly, S_1 contains $\frac{n+1}{2}$ integers. S_2 contains $\frac{n-1}{2}$ integers. S_3 contains $\frac{n-3}{2}$ integers. $\ldots$, $S_{\frac{n-2}{2}}$ contains 2 integers. Each S_i obviously contributes edges with label 1.T herefore

$$\begin{aligned}
e_f(1) &\geq |S_1| + |S_2| + \ldots + \left|S_{\frac{n-1}{2}}\right| \\
&= \frac{n+1}{2} + \frac{n-1}{2} + \frac{n-3}{2} + \ldots + 2 \\
&= 2 + 3 + \ldots + \frac{n+1}{2} \\
&= \left[1 + 2 + 3 + \ldots + \frac{n+1}{2}\right] - 1 \\
&= \frac{\left(\frac{n+1}{2}\right)\left(\frac{n+1}{2} + 1\right)}{2} - 1 = \frac{(n+1)(n+3)}{8} - 1 \qquad (1)
\end{aligned}$$

Next consider the sets,

$$\begin{aligned}
S'_1 &= \left\{\left[\frac{n-1}{n-2}\right], \left[\frac{n-1}{n-3}\right], \ldots, \left[\frac{n-1}{\frac{n+1}{2}}\right]\right\} \\
S'_2 &= \left\{\left[\frac{n-3}{n-4}\right], \left[\frac{n-3}{n-5}\right], \ldots, \left[\frac{n-3}{\frac{n-1}{2}}\right]\right\} \\
&\vdots \\
S'_{\frac{n-3}{2}} &= \left\{\left[\frac{4}{3}\right]\right\}
\end{aligned}$$

Clearly each of the sets S'_i also contributes edges with label 1. Therefore

$$\begin{aligned}
e_f(1) &\geq |S'_1| + |S'_2| + \ldots + \left|S'_{\frac{n-3}{2}}\right| \\
&= \frac{n-3}{2} + \frac{n-5}{2} + \frac{n-7}{2} + \ldots + 1 \\
&= 1 + 2 + 4 + \ldots + \frac{n-3}{2} \\
&= \frac{\left(\frac{n-3}{2}\right)\left(\frac{n-3}{2}+1\right)}{2} = \frac{(n-3)(n-1)}{8} \qquad (2)
\end{aligned}$$

From (1) and (2), we get

$$\begin{aligned}
e_f(1) &\geq \frac{(n+1)(n+3)}{8} - 1 + \frac{(n-3)(n-1)}{8} \\
&\geq \frac{n^2+4n+3+n^2-4n+3-8}{8} \\
&\geq \frac{2n^2-2}{8} \geq \frac{n^2-1}{4} > \left[\frac{n(n-1)}{4}\right] + 1,
\end{aligned}$$

a contradiction to that f is a quotient cordial labeling.

Case 2. n is even.

Similar to Case 1, we get a contradiction. □

Theorem 3.5 *Every graph is a subgraph of a connected quotient cordial graph.*

Proof Let G be a (p,q) graph with $V(G) = \{u_i : 1 \leq i \leq p\}$. Consider the complete graph K_p with vertex set $V(G)$. Let $f(u_i) = i$, $1 \leq i \leq p$. By Theorem 3.4, we get $e_f(1) > e_f(0)$. Let $e_f(1) = m + e_f(0)$, $m \in \mathbb{N}$. Consider the two copies of the star $K_{1,m}$. The super graph G^* of G is obtained from K_p as follows: Take one star $K_{1,m}$ and identify the central vertex of the star with u_1. Take another star $K_{1,m}$ and identify the central vertex of the same with u_2. Let $S_1 = \{x :\ x$ is an even number and $p < x < p+2m\}$ and $S_2 = \{x :\ x$ is an odd number and $p <$

$x < p + 2m\}$. Assign the label to the pendent vertices adjacent to u_1 from the set S_1in any order and then assign the label to the pendent vertices adjacent to u_2 from the set S_2. Clearly this vertex labeling is a quotient cordial labeling of G^*. □

Illustration 3.6 K_5 is not quotient cordial but it is a subgraph of quotient cordial graph G^* given in Figure 3.

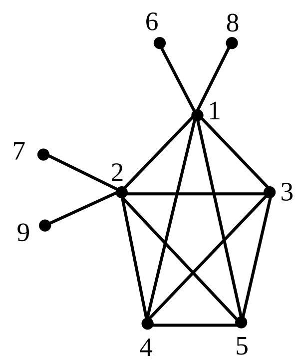

Figure 3

Theorem 3.7 *Any star $K_{1,n}$ is quotient cordial.*

Proof Let $V(K_{1,n}) = \{u, u_i : 1 \leq i \leq n\}$ and $E(K_{1,n}) = \{uu_i : 1 \leq i \leq n\}$. Assign the label 1 to the central vertex u and then assign the labels $2, 3, \cdots, n+1$ to the pendent vertices $u_1, u_2, \cdots, u_n$. f is a quotient cordial labeling follows from the following Table 1. □

Nature of n	$e_f\,(0)$	$e_f\,(1)$
even	$\frac{n}{2}$	$\frac{n}{2}$
odd	$\frac{n+1}{2}$	$\frac{n-1}{2}$

Table 1

Now we investigate the complete bipartite graph $K_{2,n}$.

Theorem 3.8 *$K_{2,n}$ is quotient cordial.*

Proof Let $V(K_{2,n}) = \{u, v, u_i : 1 \leq i \leq n\}$ and $E(K_{2,n}) = \{uu_i, vv_i : 1 \leq i \leq n\}$. Assign the label $1, 2$ respectively to the vertices u, v. Then assign the label $3, 4, 5, \cdots, m+2$ to the remaining vertices. Clearly f is a quotient cordial labeling since $e_f(0) = m+1$, $e_f(1) = m$. □

Theorem 3.9 *$K_{1,n} \cup K_{1,n} \cup K_{1,n}$ is quotient cordial.*

Proof Let $V(K_{1,n} \cup K_{1,n} \cup K_{1,n}) = \{u, u_i, v, v_i, w, w_i : 1 \leq i \leq n\}$ and $E(K_{1,n} \cup K_{1,n} \cup K_{1,n}) = \{uu_i, vv_i, ww_i : 1 \leq i \leq n\}$. Define a map $f : V(K_{1,n} \cup K_{1,n} \cup K_{1,n}) \to \{1, 2, 3, \ldots, 3n\}$ by $f(u) = 1$, $f(v) = 2$, $f(w) = 3$,

$$\begin{aligned} f(u_i) &= 3i+1, \quad 1 \le i \le n \\ f(v_i) &= 3i+3, \quad 1 \le i \le n \\ f(w_i) &= 3i+2, \quad 1 \le i \le n. \end{aligned}$$

Clearly Table 3 shows that f is a quotient cordial labeling. $\square$

Nature of n	$e_f(0)$	$e_f(1)$
n is even	$\frac{3n}{2}$	$\frac{3n}{2}$
$n \equiv 1 \pmod 4$	$\frac{3n-1}{2}$	$\frac{3n+1}{2}$
$n \equiv 3 \pmod 4$	$\frac{3n+1}{2}$	$\frac{3n-1}{2}$

Table 3

Next is the bistar $B_{n,n}$.

Theorem 3.10 *The bistar $B_{n,n}$ is quotient cordial.*

Proof Let $V(B_{n,n}) = \{u, u_i, v, v_i : 1 \le i \le n\}$ and $E(B_{n,n}) = \{uv, uu_i, vv_i : 1 \le i \le n\}$. Assign the label 1 to u and assign the label 2 to v. Then assign the labels $3, 4, 5, \ldots, n+2$ to the vertices $u_1, u_2, \cdots, u_n$. Next assign the label $n+3, n+4, \ldots, 2n+2$ to the pendent vertices $v_1, v_2, \cdots, v_n$. The edge condition is given in Table 2.

Nature of n	$e_f(0)$	$e_f(1)$
$n \equiv 0, 1, 2 \pmod n$	$n+1$	n
$n \equiv 3 \pmod n$	n	$n+1$

Table 3

Hence f is a quotient cordial labeling. $\square$

The final investigation is about the graph obtained from a triangle and three stars.

Theorem 3.11 *Let C_3 be the cycle $u_1u_2u_3u_1$. Let G be a graph obtained from C_3 with $V(G) = V(C_3) \cup \{v_i, w_i, z_i : 1 \le i \le n\}$ and $E(G) = E(C_3) \cup \{u_1v_i, u_2w_i, u_3z_i : 1 \le i \le n\}$. Then G is quotient cordial.*

Proof Define $f : V(G) \to \{1, 2, 3, \cdots, 3n+3\}$ by $f(u_1) = 1$, $f(u_2) = 2$, $f(u_3) = 3$.

Case 1. $n \equiv 0, 2, 3 \pmod 4$.

Define

$$f(v_i) = 3i+1, \quad 1 \le i \le n$$
$$f(w_i) = 3i+3, \quad 1 \le i \le n$$
$$f(z_i) = 3i+2, \quad 1 \le i \le n.$$

Case 2. $n \equiv 1 \pmod 4$.

Define

$$f(v_i) = 3i+2, \quad 1 \le i \le n$$
$$f(w_i) = 3i+1, \quad 1 \le i \le n$$
$$f(z_i) = 3i+3, \quad 1 \le i \le n.$$

The Table 4 shows that f is a quotient cordial labeling.

values of n	$e_f(0)$	$e_f(1)$
$n \equiv 1, 3 \pmod 4$	$\frac{3n+3}{2}$	$\frac{3n+3}{2}$
$n \equiv 0, 2 \pmod 4$	$\frac{3n+2}{2}$	$\frac{3n+4}{2}$

Table 3

□

Illustration 3.12 A quotient cordial labeling of G obtained from C_3 and $K_{1,7}$ is given in Figure 4.

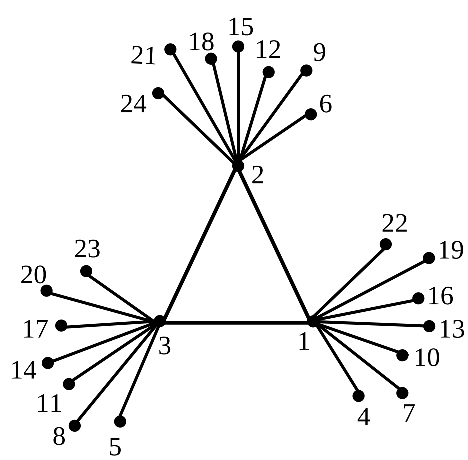

Figure 4

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