An Introduction to Neutrosophy, Neutrosophic Logic, Neutrosophic Set, and Neutrosophic Probability and Statistics

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0.1 Introduction to Non-Standard Analysis.

In 1960s Abraham Robinson has developed the non-standard analysis, a formalization of analysis and a branch of mathematical logic, that rigorously defines the infinitesimals. Informally, an infinitesimal is an infinitely small number. Formally, x is said to be infinitesimal if and only if for all positive integers n one has $|x| < 1/n$. Let $\varepsilon>0$ be a such infinitesimal number. The hyper-real number set is an extension of the real number set, which includes classes of infinite numbers and classes of infinitesimal numbers. Let's consider the non-standard finite numbers $1^+ = 1+\varepsilon$, where "1" is its standard part and "ε" its non-standard part, and $^-0 = 0-\varepsilon$, where "0" is its standard part and "ε" its non-standard part.

Then, we call $]^-0, 1^+[$ a non-standard unit interval. Obviously, 0 and 1, and analogously non-standard numbers infinitely small but less than 0 or infinitely small but greater than 1, belong to the non-standard unit interval. Actually, by "^-a" one signifies a monad, i.e. a set of hyper-real numbers in non-standard analysis:

$\mu(^-a)= \{a-x: x\in\mathbb{R}^*, x \text{ is infinitesimal}\}$,

and similarly "b^+" is a monad:

$\mu(b^+)= \{b+x: x\in\mathbb{R}^*, x \text{ is infinitesimal}\}$.

Generally, the left and right borders of a non-standard interval $]^-0, 1^+[$ are vague, imprecise, themselves being non-standard (sub)sets $\mu(^-a)$ and $\mu(b^+)$ as defined above.

Combining the two before mentioned definitions one gets, what we would call, a binad of "$^-c^+$":

$\mu(^-c^+)= \{c-x: x\in\mathbb{R}^*, x \text{ is infinitesimal}\} \cup \{c+x: x\in\mathbb{R}^*, x \text{ is infinitesimal}\}$, which is a collection of open punctured neighborhoods (balls) of c.

Of course, $^-a < a$ and $b^+ > b$. No order between $^-c^+$ and c.

Addition of non-standard finite numbers with themselves or with real numbers:

$^-a + b = {^-(a + b)}$
$a + b^+ = (a + b)^+$
$^-a + b^+ = {^-(a + b)^+}$
$^-a + {^-b} = {^-(a + b)}$ (the left monads absorb themselves)
$a^+ + b^+ = (a + b)^+$ (analogously, the right monads absorb themselves)

Similarly for subtraction, multiplication, division, roots, and powers of non-standard finite numbers with themselves or with real numbers.

By extension let $\inf\ ]^-0, 1^+[= {^-a}$ and $\sup\ ]^-0, 1^+[= b^+$.

0.2 Definition of Neutrosophic Components.

Let T, I, F be standard or non-standard real subsets of $]^-0, 1^+[$,

with $\sup T = t\_sup$, $\inf T = t\_inf$,
$\sup I = i\_sup$, $\inf I = i\_inf$,

and $\sup F = f\_sup$, $\inf F = f\_inf$,
$n\_sup = t\_sup+i\_sup+f\_sup$,
$n\_inf = t\_inf+i\_inf+f\_inf$.

The sets T, I, F are not necessarily intervals, but may be any real sub-unitary subsets: discrete or continuous; single-element, finite, or (countably or uncountably) infinite; union or intersection of various subsets; etc.
They may also overlap. The real subsets could represent the relative errors in determining t, i, f (in the case when the subsets T, I, F are reduced to points).
Statically T, I, F are subsets.

But dynamically, looking therefore from another perspective, the components T, I, F are at each instance dependant on many parameters, and therefore they can be considered set-valued vector functions or even operators. The parameters can be: time, space, etc. (some of them are hidden/unknown parameters): T(t, s, …), I(t, s, …), F(t, s, …), where t=time, s=space, etc., that's why the neutrosophic logic can be used in quantum physics.
The Dynamic Neutrosophic Calculus can be used in psychology.
Neutrosophics try to reflect the dynamics of things and ideas.
See an example:
The proposition "Tomorrow it will be raining" does not mean a fixed-valued components structure; this proposition may be say 40% true, 50% indeterminate, and 45% false at time t_1; but at time t_2 may change at 50% true, 49% indeterminate, and 30% false (according with new evidences, sources, etc.); and tomorrow at say time t_{145} the same proposition may be 100%, 0% indeterminate, and 0% false (if tomorrow it will indeed rain). This is the dynamics: the truth value changes from a time to another time.
In other examples: the truth value of a proposition may change from a place to another place, for example: the proposition "It is raining" is 0% true, 0% indeterminate, and 100% false in Albuquerque (New Mexico), but moving to Las Cruces (New Mexico) the truth value changes and it may be (1, 0, 0).
Also, the truth value depends/changes with respect to the observer (subjectivity is another parameter of the functions/operators T, I, F). For example: "John is smart" can be (.35, .67, .60) according to his boss, but (.80, .25, .10) according to himself, or (.50, .20, .30) according to his secretary, etc.

In the this book T, I, F, called *neutrosophic components*, will represent the truth value, indeterminacy value, and falsehood value respectively referring to neutrosophy, neutrosophic logic, neutrosophic set, neutrosophic probability, neutrosophic statistics.

This representation is closer to the human mind reasoning. It characterizes/catches the imprecision of knowledge or linguistic inexactitude received by various observers (that's why T, I, F are subsets - not necessarily single-elements), uncertainty due to incomplete knowledge or acquisition errors or stochasticity (that's why the subset I exists), and vagueness due to lack of clear contours or limits (that's why T, I, F are subsets and I exists; in particular for the appurtenance to the neutrosophic sets).
One has to specify the superior (x_sup) and inferior (x_inf) limits of the subsets because in many problems arises the necessity to compute them.

0.3. Operations with Sets.

Let S_1 and S_2 be two (unidimensional) real standard or non-standard subsets, then one defines:

Addition of Sets:

$S_1 \oplus S_2 = \{x \mid x=s_1+s_2, \text{ where } s_1 \in S_1 \text{ and } s_2 \in S_2\}$,
with $\inf S_1 \oplus S_2 = \inf S_1 + \inf S_2$, $\sup S_1 \oplus S_2 = \sup S_1 + \sup S_2$;
and, as some particular cases, we have
$\{a\} \oplus S_2 = \{x \mid x=a+s_2, \text{ where } s_2 \in S_2\}$
with $\inf \{a\} \oplus S_2 = a + \inf S_2$, $\sup \{a\} \oplus S_2 = a + \sup S_2$.

Subtraction of Sets:

$S_1 \ominus S_2 = \{x \mid x=s_1-s_2, \text{ where } s_1 \in S_1 \text{ and } s_2 \in S_2\}$.
For real positive subsets (most of the cases will fall in this range) one gets
$\inf S_1 \ominus S_2 = \inf S_1 - \sup S_2$, $\sup S_1 \ominus S_2 = \sup S_1 - \inf S_2$;
and, as some particular cases, we have
$\{a\} \ominus S_2 = \{x \mid x=a-s_2, \text{ where } s_2 \in S_2\}$,
with $\inf \{a\} \ominus S_2 = a - \sup S_2$, $\sup \{a\} \ominus S_2 = a - \inf S_2$;
also $\{1^+\} \ominus S_2 = \{x \mid x=1^+-s_2, \text{ where } s_2 \in S_2\}$,
with $\inf \{1^+\} \ominus S_2 = 1^+ - \sup S_2$, $\sup \{1^+\} \ominus S_2 = 100 - \inf S_2$.

Multiplication of Sets:

$S_1 \odot S_2 = \{x \mid x=s_1 \cdot s_2, \text{ where } s_1 \in S_1 \text{ and } s_2 \in S_2\}$.
For real positive subsets (most of the cases will fall in this range) one gets
$\inf S_1 \odot S_2 = \inf S_1 \cdot \inf S_2$, $\sup S_1 \odot S_2 = \sup S_1 \cdot \sup S_2$;
and, as some particular cases, we have
$\{a\} \odot S_2 = \{x \mid x=a \cdot s_2, \text{ where } s_2 \in S_2\}$,
with $\inf \{a\} \odot S_2 = a * \inf S_2$, $\sup \{a\} \odot S_2 = a \cdot \sup S_2$;
also $\{1^+\} \odot S_2 = \{x \mid x=1 \cdot s_2, \text{ where } s_2 \in S_2\}$,
with $\inf \{1^+\} \odot S_2 = 1^+ \cdot \inf S_2$, $\sup \{1^+\} \odot S_2 = 1^+ \cdot \sup S_2$.

Division of a Set by a Number:

Let $k \in \mathbb{R}^*$, then $S_1 \oslash k = \{x \mid x=s_1/k, \text{ where } s_1 \in S_1\}$.

1. NEUTROSOPHY:

1.1 Definition:

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.
It is the base of *neutrosophic logic*, a multiple value logic that generalizes the fuzzy logic and deals with paradoxes, contradictions, antitheses, antinomies.

1.2 Etymology:

Neutro-sophy [French *neutre* < Latin *neuter*, neutral, and Greek *sophia*, skill/wisdom] means knowledge of neutral thought.

1.3 Characteristics of this mode of thinking:

- proposes new philosophical theses, principles, laws, methods, formulas, movements;
- reveals that world is full of indeterminacy;
- interprets the uninterpretable;
- regards, from many different angles, old concepts, systems:
showing that an idea, which is true in a given referential system, may be false in another one, and vice versa;
- attempts to make peace in the war of ideas,
and to make war in the peaceful ideas;
- measures the stability of unstable systems,
and instability of stable systems.

Let's note by <A> an idea, or proposition, theory, event, concept, entity, by <Non-A> what is not <A>, and by <Anti-A> the opposite of <A>. Also, <Neut-A> means what is neither <A> nor <Anti-A>, i.e. neutrality in between the two extremes. And <A'> a version of <A>.

<Non-A> is different from <Anti-A>.

1.4 Main Principle:

Between an idea <A> and its opposite <Anti-A>, there is a continuum-power spectrum of neutralities <Neut-A>.

1.5 Fundamental Thesis of Neutrosophy:

Any idea <A> is T% true, I% indeterminate, and F% false, where $T, I, F \subset]^-0, 1^+[$.

1.6 Main Laws of Neutrosophy:

Let <α> be an attribute, and $(T, I, F) \subset]^-0, 1^+[^3$. Then:
- There is a proposition <P> and a referential system {R}, such that <P> is T% <α>, I% indeterminate or <Neut-α>, and F% <Anti-α>.
- For any proposition <P>, there is a referential system {R}, such that <P> is T% <α>, I% indeterminate or <Neut-α>, and F% <Anti-α>.
- <α> is at some degree <Anti-α>, while <Anti-α> is at some degree <α>.

2. NEUTROSOPHIC LOGIC:

2.1 Introduction.

As an alternative to the existing logics we propose the Neutrosophic Logic to represent a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction. It is a non-classical logic.

Eksioglu (1999) explains some of them:

"Imprecision of the human systems is due to the imperfection of knowledge that humain receives (observation) from the external world. Imperfection leads to a doubt about the value of a variable, a decision to be taken or a conclusion to be drawn for the actual system. The sources of uncertainty can be stochasticity (the case of intrinsic imperfection where a typical and single value does not exist), incomplete knowledge (ignorance of the totality, limited view on a system because of its complexity) or the acquisition errors (intrinsically imperfect observations, the quantitative errors in measures)."

"Probability (called sometimes the objective probability) process uncertainty of random type (stochastic) introduced by the chance. Uncertainty of the chance is clarified by the

time or by events' occurrence. The probability is thus connected to the frequency of the events' occurrence."
"The vagueness which constitutes another form of uncertainty is the character of those with contours or limits lacking precision, clearness. […] For certain objects, the fact to be in or out of a category is difficult to mention. Rather, it is possible to express a partial or gradual membership."
Indeterminacy means degrees of uncertainty, vagueness, imprecision, undefined, unknown, inconsistency, redundancy.

A question would be to try, if possible, to get an axiomatic system for the neutrosophic logic. Intuition is the base for any formalization, because the postulates and axioms derive from intuition.

2.2 Definition:

A logic in which each proposition is estimated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F, where T, I, F are defined above, is called *Neutrosophic Logic*.

We use a subset of truth (or indeterminacy, or falsity), instead of a number only, because in many cases we are not able to exactly determine the percentages of truth and of falsity but to approximate them: for example a proposition is between 30-40% true and between 60-70% false, even worst: between 30-40% or 45-50% true (according to various analysers), and 60% or between 66-70% false.
The subsets are not necessary intervals, but any sets (discrete, continuous, open or closed or half-open/half-closed interval, intersections or unions of the previous sets, etc.) in accordance with the given proposition.
A subset may have one element only in special cases of this logic.

Constants: (T, I, F) truth-values, where T, I, F are standard or non-standard subsets of the non-standard interval $]^-0, 1^+[$, where $n_{inf} = \inf T + \inf I + \inf F \geq {}^-0$, and $n_{sup} = \sup T + \sup I + \sup F \leq 3^+$.
Atomic formulas: a, b, c, … .
Arbitrary formulas: A, B, C, … .

The neutrosophic logic is a formal frame trying to measure the truth, indeterminacy, and falsehood.
My hypothesis is that **no theory is exempted from paradoxes**, because of the language imprecision, metaphoric expression, various levels or meta-levels of understanding/interpretation which might overlap.

2.3 Definition of Neutrosophic Logical Connectives:

The connectives (rules of inference, or operators), in any non-bivalent logic, can be defined in various ways, giving rise to lots of distinct logics. For example, in three-valued logic, where three possible values are possible: true, false, or undecided, there are 3072 such logics! (Weisstein, 1998) A single change in one of any connective's truth table is enough to form a (completely) different logic.

The rules are hypothetical or factual. How to choose them? The philosopher Van Fraassen (1980) [see Shafer, 1986] commented that such rules may always be controvertible "for it always involves the choice of one out of many possible but nonactual worlds". There are general rules of combination, and ad hoc rules.

For an applied logic to artificial intelligence, a better approach, the best way would be to define the connectives recursively (Dubois, Prade), changing/adjusting the definitions after each step in order to improve the next result. This might be comparable to approximating the limit of a convergent sequence, calculating more and more terms, or by calculating the limit of a function successively substituting the argument with values closer and closer to the critical point. The recurrence allows evolution and self-improvement.

Or to use *greedy algorithms*, which are combinatorial algorithms that attempt at each iteration as much improvement as possible unlike myopic algorithms that look at each iteration only at very local information as with steepest descent method.

As in non-monotonic logic, we make assumptions, but we often err and must jump back, revise our assumptions, and start again. We may add rules which don't preserve monotonicity.

In bio-mathematics Heitkoetter and Beasley (1993-1999) present the *evolutionary algorithms* which are used "to describe computer-based problem solving systems which employ computational models of some of the known mechanisms of evolution as key elements in their design and implementation". They simulate, via processes of selection, mutation, and reproduction, the evolution of individual structures. The major evolutionary algorithms studied are: genetic algorithm (a model of machine learning based on genetic operators), evolutionary programming (a stochastic optimization strategy based on linkage between parents and their offspring; conceived by L. J. Fogel in 1960s), evolution strategy, classifier system, genetic programming.

Pei Wang devised a Non-Axiomatic Reasoning System as an intelligent reasoning system, where intelligence means working and adopting with insufficient knowledge and resources.

The inference mechanism (endowed with rules of transformation or rules of production) in neutrosophy should be non-monotonic and should comprise ensembles of recursive rules, with preferential rules and secondary ones (priority order), in order to design a good expert system. One may add new rules and eliminate old ones proved unsatisfactory. There should be strict rules, and rules with exceptions. Recursivity is seen as a computer program that learns from itself. The statistical regression method may be employed as well to determine a best algorithm of inference.

Non-monotonic reasoning means to make assumptions about things we don't know. Heuristic methods may be involved in order to find successive approximations.

In terms of the previous results, a default neutrosophic logic may be used instead of the normal inference rules. The distribution of possible neutrosophic results serves as an orientating frame for the new results. The flexible, continuously refined, rules obtain iterative and gradual approaches of the result.

A comparison approach is employed to check the result (conclusion) p by studying the opposite of this: what would happen if a non-p conclusion occurred? The inconsistence of information shows up in the result, if not eliminated from the beginning. The data bases should be stratified. There exist methods to construct preferable coherent sub-bases within incoherent

bases. In Multi-Criteria Decision one exploits the complementarity of different criteria and the complementarity of various sources.
For example, the Possibility Theory (Zadeh 1978, Dubois, Prade) gives a better approach than the Fuzzy Set Theory (Yager) due to self-improving connectives. The Possibility Theory is proximal to the Fuzzy Set Theory, the difference between these two theories is the way the fusion operators are defined.

One uses the definitions of neutrosophic probability and neutrosophic set operations.
Similarly, there are many ways to construct such connectives according to each particular problem to solve; here we present the easiest ones:

One notes the neutrosophic logical values of the propositions A_1 and A_2 by
$NL(A_1) = (T_1, I_1, F_1)$ and $NL(A_2) = (T_2, I_2, F_2)$.
For all neutrosophic logical values below: if, after calculations, one obtains numbers < 0 or > 1, one replaces them by ${}^-0$ or 1^+ respectively.

2.3.1 Negation:
$NL(\neg A_1) = (\{1^+\}\ominus T_1, \{1^+\}\ominus I_1, \{1^+\}\ominus F_1)$.

2.3.2 Conjunction:
$NL(A_1 \wedge A_2) = (T_1\odot T_2, I_1\odot I_2, F_1\odot F_2)$.
(And, in a similar way, generalized for n propositions.)

2.3.3 Weak or inclusive disjunction:
$NL(A_1 \vee A_2) = (T_1\oplus T_2\ominus T_1\odot T_2, I_1\oplus I_2\ominus I_1\odot I_2, F_1\oplus F_2\ominus F_1\odot F_2)$.
(And, in a similar way, generalized for n propositions.)

2.3.4 Strong or exclusive disjunction:
$NL(A_1 \veebar A_2) =$
$(T_1\odot (\{1\}\ominus T_2) \oplus T_2\odot (\{1\}\ominus T_1) \ominus T_1\odot T_2\odot (\{1\}\ominus T_1) \odot (\{1\}\ominus T_2),$
$I_1 \odot (\{1\}\ominus I_2) \oplus I_2 \odot (\{1\}\ominus I_1) \ominus I_1 \odot I_2 \odot (\{1\}\ominus I_1) \odot (\{1\}\ominus I_2),$
$F_1\odot (\{1\}\ominus F_2) \oplus F_2\odot (\{1\}\ominus F_1) \ominus F_1\odot F_2 \odot (\{1\}\ominus F_1) \odot (\{1\}\ominus F_2))$.
(And, in a similar way, generalized for n propositions.)

2.3.5 Material conditional (implication):
$NL(A_1 \mapsto A_2) = (\{1^+\}\ominus T_1\oplus T_1\odot T_2, \{1^+\}\ominus I_1\oplus I_1\odot I_2, \{1^+\}\ominus F_1\oplus F_1\odot F_2)$.

2.3.6 Material biconditional (equivalence):
$NL(A_1 \leftrightarrow A_2) = ((\{1^+\}\ominus T_1\oplus T_1\odot T_2) \odot (\{1^+\}\ominus T_2\oplus T_1\odot T_2),$
$(\{1^+\}\ominus I_1\oplus I_1\odot I_2) \odot (\{1^+\}\ominus I_2\oplus I_1 \odot I_2),$
$(\{1^+\}\ominus F_1\oplus F_1\odot F_2) \odot (\{1^+\}\ominus F_2\oplus F_1\odot F_2))$.

2.3.7 Sheffer's connector:
$NL(A_1 \mid A_2) = NL(\neg A_1 \vee \neg A_2) = (\{1^+\}\ominus T_1\odot T_2, \{1^+\}\ominus I_1\odot I_2, \{1^+\}\ominus F_1\odot F_2)$.

2.3.8 Peirce's connector:

$NL(A_1 \downarrow A_2) = NL(\neg A_1 \wedge \neg A_2) =$

$$= ((\{1^+\}\ominus T_1) \odot (\{1^+\}\ominus T_2), (\{1^+\}\ominus I_1) \odot (\{1^+\}\ominus I_2), (\{1^+\}\ominus F_1) \odot (\{1^+\}\ominus F_2)).$$

2.4 Generalizations:

When the sets are reduced to an element only respectively, then
$t\_sup = t\_inf = t$, $i\_sup = i\_inf = i$, $f\_sup = f\_inf = f$,
and $n\_sup = n\_inf = n = t+i+f$)

Hence, the neutrosophic logic generalizes:

- the *intuitionistic logic*, which supports incomplete theories (for $0 < n < 1$ and $i=0$, $0 \le t, i, f \le 1$);
- the *fuzzy logic* (for $n = 1$ and $i = 0$, and $0 \le t, i, f \le 1$);

from "CRC Concise Concise Encyclopedia of Mathematics", by Eric W. Weisstein, 1998, the fuzzy logic is "an extension of two-valued logic such that statements need not to be True or False, but may have a degree of truth between 0 and 1";

- the *Boolean logic* (for $n = 1$ and $i = 0$, with t, f either 0 or 1);
- the *multi-valued logic* (for $0 \le t, i, f \le 1$);

definition of <many-valued logic> from "The Cambridge Dictionary of Phylosophy", general editor Robert Audi, 1995, p. 461: "propositions may take many values beyond simple truth and falsity, values functionally determined by the values of their components"; Lukasiewicz considered three values (1, 1/2, 0). Post considered m values, etc. But they varied in between 0 and 1 only. In the neutrosophic logic a proposition may take values even greater than 1 (in percentage greater than 100%) or less than 0.

- the *paraconsistent logic* (for $n > 1$ and $i = 0$, with both $t, f < 1$);
- the *dialetheism*, which says that some contradictions are true (for $t = f = 1$ and $i = 0$; some paradoxes can be denoted this way too);
- the *faillibilism*, which says that uncertainty belongs to every proposition (for $i > 0$);
- the *paradoxist logic*, based on paradoxes ($i > 1$);
- the *pseudoparadoxist logic*, based on pseudoparadoxes ($0 < i < 1$, $t + f > 1$);
- the *tautologic logic*, based on tautologies ($i < 0$, $t > 1$).

Compared with all other logics, the neutrosophic logic and intuitionistic fuzzy logic introduce a percentage of "indeterminacy" - due to unexpected parameters hidden in some propositions, or unknowness, or God's will, but only neutrosophic logic let each component t, i, f be even boiling *over 1* (overflooded) or freezing *under 0* (underdried): to be able to make distinction between relative truth and absolute truth, and between relative falsity and absolute falsity.

For example: in some tautologies $t > 1$, called "overtrue". Similarly, a proposition may be "overindeterminate" (for $i > 1$, in some paradoxes), "overfalse" (for $f > 1$, in some unconditionally false propositions); or "undertrue" (for $t < 0$, in some unconditionally false propositions), "underindeterminate" (for $i < 0$, in some unconditionally true or false propositions), "underfalse" (for $f < 0$, in some unconditionally true propositions).

This is because we should make a distinction between unconditionally true ($t > 1$, and $f < 0$ or $i < 0$) and conditionally true propositions ($t \le 1$, and $f \le 1$ or $i \le 1$).

3 NEUTROSOPHIC SET:

3.1 Definition:

Let T, I, F be real standard or non-standard subsets of $]^{-}0, 1^{+}[$,

with sup T = t_sup, inf T = t_inf,
sup I = i_sup, inf I = i_inf,
sup F = f_sup, inf F = f_inf,

and n_sup = t_sup+i_sup+f_sup,
n_inf = t_inf+i_inf+f_inf.

Let U be a universe of discourse, and M a set included in U. An element x from U is noted with respect to the set M as x(T, I, F) and belongs to M in the following way:

it is t% true in the set, i% indeterminate (unknown if it is) in the set, and f% false, where t varies in T, i varies in I, f varies in F.

Statically T, I, F are subsets, but dynamically the components T, I, F are set-valued vector functions/operators depending on many parameters, such as: time, space, etc. (some of them are hidden parameters, i.e. unknown parameters).

3.2 General Examples:

Let A and B be two neutrosophic sets.

One can say, by language abuse, that any element neutrosophically belongs to any set, due to the percentages of truth/indeterminacy/falsity involved, which varies between 0 and 1 or even less than 0 or greater than 1.

Thus: x(50,20,30) belongs to A (which means, with a probability of 50% x is in A, with a probability of 30% x is not in A, and the rest is undecidable); or y(0,0,100) belongs to A (which normally means y is not for sure in A); or z(0,100,0) belongs to A (which means one does know absolutely nothing about z's affiliation with A).

More general, x((20-30), (40-45)∪[50-51], {20,24,28}) belongs to the set A, which means:

- with a probability in between 20-30% x is in A (one cannot find an exact approximate because of various sources used);
- with a probability of 20% or 24% or 28% x is not in A;
- the indeterminacy related to the appurtenance of x to A is in between 40-45% or between 50-51% (limits included);

The subsets representing the appurtenance, indeterminacy, and falsity may overlap, and n_sup = 30+51+28 > 100 in this case.

3.3 Physics Examples:

a) For example the Schrodinger's Cat Theory says that the quantum state of a photon can basically be in more than one place in the same time, which translated to the neutrosophic set means that an element (quantum state) belongs and does not belong to a set (one place) in the same time; or an element (quantum state) belongs to two different sets (two different places) in the same time. It is a question of "alternative worlds" theory very well represented by the neutrosophic set theory.

In Schroedinger's Equation on the behavior of electromagnetic waves and "matter waves" in quantum theory, the wave function Psi which describes the superposition of possible states may be simulated by a neutrosophic function, i.e. a function whose values are not unique for each argument from the domain of definition (the vertical line test fails, intersecting the graph in more points).

Don’t we better describe, using the attribute “neutrosophic” than “fuzzy” or any others, a quantum particle that neither exists nor non-exists?

b) How to describe a particle ζ in the infinite micro-universe that belongs to two distinct places P_1 and P_2 in the same time? $\zeta \in P_1$ and $\zeta \notin P_1$ as a true contradiction, or $\zeta \in P_1$ and $\zeta \in \neg P_1$.

3.4 Philosophical Examples:

Or, how to calculate the truth-value of Zen (in Japanese) / Chan (in Chinese) doctrine philosophical proposition: the present is eternal and comprises in itself the past and the future?
In Eastern Philosophy the contradictory utterances form the core of the Taoism and Zen/Chan (which emerged from Buddhism and Taoism) doctrines.
How to judge the truth-value of a metaphor, or of an ambiguous statement, or of a social phenomenon which is positive from a standpoint and negative from another standpoint?

There are many ways to construct them, in terms of the practical problem we need to simulate or approach. Below there are mentioned the easiest ones:

3.5 Application:

A cloud is a neutrosophic set, because its borders are ambiguous, and each element (water drop) belongs with a neutrosophic probability to the set (e.g. there are a kind of separated water drops, around a compact mass of water drops, that we don't know how to consider them: in or out of the cloud).
Also, we are not sure where the cloud ends nor where it begins, neither if some elements are or are not in the set. That's why the percent of indeterminacy is required and the neutrosophic probability (using subsets - not numbers - as components) should be used for better modeling: it is a more organic, smooth, and especially accurate estimation. Indeterminacy is the zone of ignorance of a proposition’s value, between truth and falsehood.

3.6 Neutrosophic Set Operations:

One notes, with respect to the sets A and B over the universe U,
$x = x(T_1, I_1, F_1) \in A$ and $x = x(T_2, I_2, F_2) \in B$, by mentioning x’s *neutrosophic membership appurtenance*.
And, similarly, $y = y(T', I', F') \in B$.
For all neutrosophic set operations: if, after calculations, one obtains numbers < 0 or > 1, one replaces them by $^-0$ or 1^+ respectively.

3.6.1 Complement of A:

If $x(T_1, I_1, F_1) \in A$,
then $x(\{1^+\}\ominus T_1, \{1^+\}\ominus I_1, \{1^+\}\ominus F_1) \in C(A)$.

3.6.2 Intersection:

If $x(T_1, I_1, F_1) \in A$, $x(T_2, I_2, F_2) \in B$,
then $x(T_1 \odot T_2, I_1 \odot I_2, F_1 \odot F_2) \in A \cap B$.

3.6.3 Union:

If $x(T_1, I_1, F_1) \in A$, $x(T_2, I_2, F_2) \in B$, then $x(T_1 \oplus T_2 \ominus T_1 \odot T_2, I_1 \oplus I_2 \ominus I_1 \odot I_2, F_1 \oplus F_2 \ominus F_1 \odot F_2) \in A \cup B$.

3.6.4 Difference:
If $x(T_1, I_1, F_1) \in A$, $x(T_2, I_2, F_2) \in B$,
then $x(T_1 \ominus T_1 \odot T_2, I_1 \ominus I_1 \odot I_2, F_1 \ominus F_1 \odot F_2) \in A \setminus B$,
because $A \setminus B = A \cap C(B)$.

3.6.5 Cartesian Product:
If $x(T_1, I_1, F_1) \in A$, $y(T', I', F') \in B$,
then $(x(T_1, I_1, F_1), y(T', I', F')) \in A \times B$.

3.6.6 M is a subset of N:
If $x(T_1, I_1, F_1) \in M \Rightarrow x(T_2, I_2, F_2) \in N$,
where $\inf T_1 \leq \inf T_2$, $\sup T_1 \leq \sup T_2$, and $\inf F_1 \geq \inf F_2$, $\sup F_1 \geq \sup F_2$.

3.7 Generalizations and Comments:

From the intuitionistic logic, paraconsistent logic, dialetheism, faillibilism, paradoxes, pseudoparadoxes, and tautologies we transfer the "adjectives" to the sets, i.e. to intuitionistic set (set incompletely known), paraconsistent set, dialetheist set, faillibilist set (each element has a percenatge of indeterminacy), paradoxist set (an element may belong and may not belong in the same time to the set), pseudoparadoxist set, and tautologic set respectively.

Hence, the neutrosophic set generalizes:
- the *intuitionistic set*, which supports incomplete set theories (for $0 < n < 1$ and $i = 0$, $0 \leq t, i, f \leq 1$) and incomplete known elements belonging to a set;
- the *fuzzy set* (for $n = 1$ and $i = 0$, and $0 \leq t, i, f \leq 1$);
- the *classical set* (for $n = 1$ and $i = 0$, with t, f either 0 or 1);
- the *paraconsistent set* (for $n > 1$ and $i = 0$, with both $t, f < 1$);
- the *faillibilist set* ($i > 0$);
- the *dialetheist set*, which says that the intersection of some disjoint sets is not empty (for $t = f = 1$ and $i = 0$; some paradoxist sets can be denoted this way too);
- the *paradoxist set* ($i > 1$);
- the *pseudoparadoxist set* ($0 < i < 1$, $t + f > 1$);
- the *tautological set* ($i < 0$).

Compared with all other types of sets, in the neutrosophic set each element has three components which are subsets (not numbers as in fuzzy set) and considers a subset, similarly to intuitionistic fuzzy set, of "indeterminacy" - due to unexpected parameters hidden in some sets, and let the superior limits of the components to even boil *over 1* (overflooded) and the inferior limits of the components to even freeze *under 0* (underdried).

For example: an element in some tautological sets may have $t > 1$, called "overincluded". Similarly, an element in a set may be "overindeterminate" (for $i > 1$, in some paradoxist sets), "overexcluded" (for $f > 1$, in some unconditionally false appurtenances); or "undertrue" (for $t < 0$, in some unconditionally false appurtenances), "underindeterminate" (for $i < 0$, in some unconditionally true or false appurtenances), "underfalse" (for $f < 0$, in some unconditionally true appurtenances).

This is because we should make a distinction between unconditionally true ($t > 1$, and $f < 0$ or $i < 0$) and conditionally true appurtenances ($t \leq 1$, and $f \leq 1$ or $i \leq 1$).

In a *rough set* RS, an element on its boundary-line cannot be classified neither as a member of RS nor of its complement with certainty. In the neutrosophic set a such element may be characterized by x(T, I, F), with corresponding set-values for $T, I, F \subseteq]^-0, 1^+[$.

4. NEUTROSOPHIC PROBABILITY:

4.1 Definition:

Let T, I, F be real standard or non-standard subsets included in $]^-0, 1^+[$,
with $\sup T = t\_sup$, $\inf T = t\_inf$,
$\sup I = i\_sup$, $\inf I = i\_inf$,
$\sup F = f\_sup$, $\inf F = f\_inf$,
and $n\_sup = t\_sup+i\_sup+f\_sup$,
$n\_inf = t\_inf+i\_inf+f\_inf$.

The *neutrosophic probability* is a generalization of the classical probability and imprecise probability in which the chance that an event A occurs is t% true - where t varies in the subset T, i% indeterminate - where i varies in the subset I, and f% false - where f varies in the subset F. *Statically T, I, F are subsets, but dynamically the components T, I, F are set-valued vector functions/operators depending on many parameters*, such as: time, space, etc. (some of them are hidden parameters, i.e. unknown parameters).

In classical probability $n\_sup \leq 1$, while in neutrosophic probability $n\_sup \leq 3^+$.

In imprecise probability: the probability of an event is a subset $T \subset [0, 1]$, not a number $p \in [0, 1]$, what's left is supposed to be the opposite, subset F (also from the unit interval [0, 1]); there is no indeterminate subset I in imprecise probability.

One notes NP(A) = (T, I, F), a triple of sets.

4.2 Neutrosophic Probability Space:

The universal set, endowed with a neutrosophic probability defined for each of its subset, forms a neutrosophic probability space.

Let A and B be two neutrosophic events, and $NP(A) = (T_1, I_1, F_1)$, $NP(B) = (T_2, I_2, F_2)$ their neutrosophic probabilities. Then we define:

$(T_1, I_1, F_1) \boxplus (T_2, I_2, F_2) = (T_1 \oplus T_2, I_1 \oplus I_2, F_1 \oplus F_2)$,

$(T_1, I_1, F_1) \boxminus (T_2, I_2, F_2) = (T_1 \ominus T_2, I_1 \ominus I_2, F_1 \ominus F_2)$,

$(T_1, I_1, F_1) \boxdot (T_2, I_2, F_2) = (T_1 \odot T_2, I_1 \odot I_2, F_1 \odot F_2)$.

$NP(A \cap B) = NP(A) \boxdot NP(B)$;

$NP(\neg A) = \{1^+\} \boxminus NP(A)$, [this second axiom may be replaced, in specific applications, with $NP(\neg A) = (F_1, I_1, T_1)$];

$NP(A \cup B) = NP(A) \boxplus NP(B) \boxminus NP(A) \boxdot NP(B)$.

Neutrosophic probability is also a non-additive probability, like the classical probability, but even for independent events, i.e. $P(A \cup B) \neq P(A)+P(B)$. We have equality only when A or B are impossible events.

A probability-function P is called additive if $P(A \cup B) = P(A)+P(B)$, sub-additive if $P(A \cup B) \leq P(A)+P(B)$, and super-additive if $P(A \cup B) \geq P(A)+P(B)$.

In the Dempster-Shafer Theory $P(A) + P(\neg A)$ may be $\neq 1$, in neutrosophic probability almost all the time $P(A) + P(\neg A) \neq 1$.

1. NP(impossible event) = $(T_{imp}, I_{imp}, F_{imp})$,
where $\sup T_{imp} \leq 0$, $\inf F_{imp} \geq 1$; no restriction on I_{imp}.
NP(sure event) = $(T_{sur}, I_{sur}, F_{sur})$,
where $\inf T_{sur} \geq 1$, $\sup F_{sur} \leq 0$.
NP(totally indeterminate event) = $(T_{ind}, I_{ind}, F_{ind})$;
where $\inf I_{ind} \geq 1$; no restrictions on T_{ind} or F_{ind}.
2. $NP(A) \in \{(T, I, F)$, where T, I, F are real standard or non-standard subsets included in $]^-0, 1^+[$ that may overlap$\}$.
3. $NP(A \cup B) = NP(A) \boxplus NP(B) \boxminus NP(A \cap B)$.
4. $NP(A) = \{1\} \boxminus NP(\neg A)$.

4.3 Applications:

#1. From a pool of refugees, waiting in a political refugee camp in Turkey to get the American visa, a% have the chance to be accepted - where a varies in the set A, r% to be rejected - where r varies in the set R, and p% to be in pending (not yet decided) - where p varies in P.
Say, for example, that the chance of someone Popescu in the pool to emigrate to USA is (between) 40-60% (considering different criteria of emigration one gets different percentages, we have to take care of all of them), the chance of being rejected is 20-25% or 30-35%, and the chance of being in pending is 10% or 20% or 30%. Then the neutrosophic probability that Popescu emigrates to the Unites States is

NP(Popescu) = ((40-60), (20-25)U(30-35), {10,20,30}), closer to the life.

This is a better approach than the classical probability, where $40 \leq P(\text{Popescu}) \leq 60$, because from the pending chance - which will be converted to acceptance or rejection - Popescu might get extra percentage in his will to emigration,
and also the superior limit of the subsets sum

$60+35+30 > 100$

and in other cases one may have the inferior sum < 0,
while in the classical fuzzy set theory the superior sum should be 100 and the inferior sum ≥ 0.
In a similar way, we could say about the element Popescu that
Popescu((40-60), (20-25)U(30-35), {10,20,30}) belongs to the set of accepted refugees.

#2. The probability that candidate C will win an election is say 25-30% true (percent of people voting for him), 35% false (percent of people voting against him), and 40% or 41% indeterminate (percent of people not coming to the ballot box, or giving a blank vote - not selecting anyone, or giving a negative vote - cutting all candidates on the list).
Dialectic and dualism don't work in this case anymore.

#3. Another example, the probability that tomorrow it will rain is say 50-54% true according to meteorologists who have investigated the past years' weather, 30 or 34-35% false according to today's very sunny and droughty summer, and 10 or 20% undecided (indeterminate).

#4. The probability that Yankees will win tomorrow versus Cowboys is 60% true (according to their confrontation's history giving Yankees' satisfaction), 30-32% false (supposing Cowboys are actually up to the mark, while Yankees are declining), and 10 or 11 or 12% indeterminate (left to the hazard: sickness of players, referee's mistakes, atmospheric conditions during the game). These parameters act on players' psychology.

4.4 Remarks:

Neutrosophic probability are useful to those events which involve some degree of indeterminacy (unknown) and more criteria of evaluation - as above. This kind of probability is necessary because it provides a better approach than classical probability to uncertain events.

This probability uses a subset-approximation for the truth-value (like *imprecise probability*), but also subset-approximations for indeterminacy- and falsity-values.

Also, it makes a distinction between "*relative sure event*", event which is sure only in some particular world(s): NP(*rse*) = 1, and "*absolute sure event*", event which is sure in all possible worlds: NP(*ase*) = 1^+; similarly for "*relative impossible event*" / "*absolute impossible event*", and for "*relative indeterminate event*" / "*absolute indeterminate event*".

In the case when the truth- and falsity-components are complementary, i.e. no indeterminacy and their sum is 100, one falls to the classical probability. As, for example, tossing dice or coins, or drawing cards from a well-shuffled deck, or drawing balls from an urn.

4.5 Generalizations:

An interesting particular case is for $n = 1$, with $0 \le t, i, f \le 1$, which is closer to the classical probability.

For $n = 1$ and $i = 0$, with $0 \le t, f \le 1$, one obtains the classical probability.

If I disappear and F is ignored, while the non-standard unit interval $]^-0, 1^+[$ is transformed into the classical unit interval [0, 1], one gets the imprecise probability.

From the intuitionistic logic, paraconsistent logic, dialetheism, faillibilism, paradoxism, pseudoparadoxism, and tautologism we transfer the "adjectives" to probabilities, i.e. we define the ***intuitionistic probability*** (when the probability space is incomplete), ***paraconsistent probability***, ***faillibilist probability***, ***dialetheist probability***, ***paradoxist probability***, ***pseudoparadoxist probability***, and ***tautologic probability*** respectively.

Hence, the neutrosophic probability generalizes:

- the *intuitionistic probability*, which supports incomplete (not completely known/determined) probability spaces (for $0 < n < 1$ and $i = 0$, $0 \le t, f \le 1$) or incomplete events whose probability we need to calculate;
- the *classical probability* (for $n = 1$ and $i = 0$, and $0 \le t, f \le 1$);
- the *paraconsistent probability* (for $n > 1$ and $i = 0$, with both $t, f < 1$);
- the *dialetheist probability*, which says that intersection of some disjoint probability spaces is not empty (for $t = f = 1$ and $i = 0$; some paradoxist probabilities can be denoted this way);
- the *faillibilist probability* (for $i > 0$);
- the *pseudoparadoxism* (for n_sup > 1 or n_inf < 0);
- the *tautologism* (for t_sup > 1).

Compared with all other types of classical probabilities, the neutrosophic probability introduces a percentage of "indeterminacy" - due to unexpected parameters hidden in some probability spaces, and let each component t, i, f be even boiling *over 1* (overflooded) or freezing *under 0* (underdried).

For example: an element in some tautological probability space may have $t > 1$, called "overprobable". Similarly, an element in some paradoxist probability space may be "overindeterminate" (for $i > 1$), or "overunprobable" (for $f > 1$, in some unconditionally false appurtenances); or "underprobable" (for $t < 0$, in some unconditionally false appurtenances), "underindeterminate" (for $i < 0$, in some unconditionally true or false appurtenances), "underunprobable" (for $f < 0$, in some unconditionally true appurtenances).

This is because we should make a distinction between unconditionally true ($t > 1$, and $f < 0$ or $I < 0$) and conditionally true appurtenances ($t \leq 1$, and $f \leq 1$ or $I \leq 1$).

5 NEUTROSOPHIC STATISTICS:

Analysis of events, described by the neutrosophic probability, means *neutrosophic statistics*. This is also a generalization of classical statistics.

In accordance with the development of neutrosophic probability the neutrosophic statistics could be better studied. Here, above, it is only a definition in order to give scientists an impulse for research.

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