RESEARCH STUDIES ON VAGUE BINARY SOFT SET AND NEUTROSOPHIC VAGUE BINARY SET

Thesis submitted to Bharathiar University for the award of the Degree of

DOCTOR OF PHILOSOPHY IN MATHEMATICS

Submitted by

P.B. REMYA

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Under the Guidance of

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I, **P. B. Remya,** hereby declare that the thesis entitled **"Research Studies on Vague Binary Soft Set and Neutrosophic Vague Binary Set"** submitted to the Bharathiar University, Coimbatore, in partial fullfillment of the requirements for the award of the Degree of **Doctor of Philosophy in Mathematics,** is a record of original and independent research work done by me during **July 2018** to **April 2021** under the supervision and guidance of **Dr. A. Francina Shalini,** Assistant Professor, P. G & Research Department of Mathematics, Nirmala College for Women and it has not formed the basis for the award of any Degree/Diploma/Associateship/ Fellowship or other similar title to any candidate of any University.

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238

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"Just TRUST me"

– God

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- P. B. Remya

Contents

Contents

Chapter 1

Chapter 1

Introduction & Preliminaries

Classical Set Theory has a compact structure and hence it is considered as *Hard Set Theory*. Revolutionary and expeditious developments in research world made a talk among researchers for the need of loosening its tight structure at the earliest. Several researchers have been found on working in this area to find out new paths. Some of them messed up things and some other put narrow openings along with suffocating difficulties and some other people lead to total wrong destinations. These turbulences delivered vigorous outputs in the zone of invention. Long walk to this direction started in the early half of twenty's century itself but found a mile stone only in the end of twentyth century with the formulation of Soft Set and Neutrosophic Set. Both emerged out around the same period of time. Frame work of Soft Set is with its *unlimited parameter set* style which made loosened the tight structure of crisp set. By putting forward a new tool *indeterminacy* Neutrosophic Set also launched in research platform by showing *a green signal of hope* in research circle. Using existing data and tools, a theoretical enquiry towards the invisible areas of hidden knowledge, for paving new paths has been made in this research work.

Chapter Scheme :

1.1 Prologue Chronicle

Ever since from human history three major problems faced is that of data-loss, cloudy/blurred data and confused data from perplexed chances of life-situations. Our forefather's used stones, tally marks etc. for counting purpose. In a long course of time, it made them to think of more easier ways. Several works can be found along with time to handle this crisis namely, Classical Probability Theory (Gerolamo Cardano, 1501-1575) [68], Classical Set Theory (Georg Cantor, 1876-1894) [32], Fuzzy Set Theory (Lotfi Aliasker Zadeh, 1965)[50], Rough Set Theory (Pawlak, 1982) [61], Intuitionistic Fuzzy Set Theory (Krassimir. T.Attanassov, 1986) [48], Multiset Theory (Wayne.D.Blizard, 1991) [87], Vague Set Theory (Gau & Buehrer, 1993) [31], Theory of Interval Mathematics (Raymon. E.Moore, 1996) [72], Neutrosophic Set Theory (Florentin Smarandache, 1998)[28, 107], Soft Set Theory (D.Molodtsov, 1999) [53], Plithogenic Set Theory (Florentin Smarandache, 2016) [29, 102].

Probability theory may count as a pioneer work in ancient era. Later Crisp Set Theory stepped into the scence. It was a famous work of the Russian mathematician Georg Cantor [32]. Logic of fuzzy put forth more place for representative value instead of 0 and 1 of set logic. It generalized fixed boundries of crisp sets to reality and takes any value from [0,1]. Person behind this remarkable achievement, Lotfi Aliasker Zadeh[50], died on 6th september 2017. Elements with partial membership can't record inside a set. Upto a level, this issue got settled through fuzzy set theory.

Pawlak [61] pointed out soft sets and spoke on *why it is the need of the hour* in a concrete way during a workshop in 1993. This influenced Molodtsov [53] who was an attendee of it and he put a goal in mind to develop the idea presented by Pawalak. He succeeded in it and came out with feathers by gifting a *parameter set* to the axiom kit of set logic. It has loosened all the existing rigid structure of classical sets. Neutrosophic set also evolved around this time with an additional aid of cloudiness representor. Hybrid structures like fuzzy soft [13], soft fuzzy [13], vague soft [89] etc. developed later to make things more easier. All of them are found rich with parameterization tools.

Set theoretical ideas lying in the primary zone of this thesis are - *vague*, neutrosophic and soft. This dissertation presents three hybrid forms *vague binary soft*, *pythagorean vague binary soft*, and *neutrosophic vague binary* by insisting a binary universe to break the conventional way of single universe structure. An additional point used is the pythagorean concept of vague set. *Neutrosophic Vague* is a combined form of neutrosophic and vague. Binary universe concept has been applied towards this set to draw out its binary effect.

Being a backbone to this work, idea of vague deserves some more attention. Point to be noted is that Intuitionistic Fuzzy Set(in short, IFS)[48] is totally different from vague even it seems to be same. Difference is in its defining criteria of falsehood. In IFS falsehood is got by subtracting truth value from 1. But the argument of vague is that this deducted value will be a combination of false membership value & indeterminacy/ hesitancy membership value. This idea is important since these two values have their own place in research areas.

Neutrosophic Vague Binary Set has been framed via newly developed Vague Binary Set. Four different algebras namely, BCK/BCI [95], $BZMV^{dM}$ [14], K[24], G[73] -to *Neutrosophic Vague Binary Set* with its properties, notions and examples are also provided.

1.1 Historical Sketch

This section is divided into four subsections each of them will be a pointer to historical paths.

Various Pure Uncertain Set's

This section deals with various *pure uncertain sets* without any collaboration in its structure which found as base for chapter's 2 and 4. its operations, topology and related concepts are also provided.

In 1993, *Gau and Buehrer* [31]introduced false membership function for an element in a set and developed *vague set*. Truth membership value of an element gives the lower bound of an element's membership value. False membership value gives the lower bound of how much an element does not belong to the set.

In 1995, *Florentin Smarandache* [28, 30] coined the word indeterminacy additional to truth & false. In 1998, *neutrosophic set* has evolved using this new term. Based on the *dependent* and *independent* nature of truth, false and hesitant variables, sum of these three variables always lies in [0, 3]. More clearly, if all the three variables are independent, sum lies in [0, 3]. If two variables are dependent and one is independent, then sum lies in [0, 2]. Finally, if all the three variables are dependent then, sum lies between [0, 1], that means it becomes completely fuzzy. In 2012, *Salama. A. A and Alblowi. S. A* [77] introduced neutrosophic topological spaces and several basic ideas regarding neutrosophic set and its operations. In 2014, *Alblowi. S. A, Salama. A. A and Mohmed Eisa* [5] explained new concepts of neutrosophic sets. In 2018, *Vildan Cetkin and Halis Aygün* [85] presented an approach to neutrosophic ideals.

In 1999, *D. Molodtsov* [53] framed *soft set*, by adding a non - restricted parameter set. Lot of sophisticated real-life problems that are not - answered so far made use this new tool to solve the complicated situations. Pioneer work in *soft set operations* has done by *Maji. P. K, Biswas. R and Roy. A. R* [51] in 2003. In due course, so many modifications came on these definitions by different authors. Main change happened in the definition of intersection and subset notions. Redefined definitions and related works can be found in the papers of *Ping Zhu, Qiaoyan Wen* [67] etc.

Various Hybrid Set's

Hybrid sets act as foundation stones for chapter 2, 3 and 4. They are discussed in this section.

In 2010, *Wei Xu, Jian Ma, Shouyang Wang, Gang Hao* [89] presented *vague soft set* by applying vague concepts into soft set theory. It is a clear extension of soft set theory. In 2012, *Xiaokun Huang, Hongjie Li, Yunqiang Yin* [92] gave notes on vague soft sets and their properties.

In 2015, *Shawkat Alkhazaleh* [80] introduced *neutrosophic vague* set theory, by restricting truth and false values as dependent variables. It is an extension to neutrosophic set theory by inserting vague concepts into it. Clearly, sum total of variables in this theory lies between [0, 2]. In 2019, *Hazwani Hashim, Lazim Abdullah and Ashraf Al-Quran*[34] developed algebraic operations on new interval neutrosophic vague sets.

In 2016, *Ahu Acikgöz* [02] introduced *binary soft set* with its operations. Binary or double universe concept has made use in this extension work of soft sets. Here the parameter set is mapped to Cartesian Product of power set of universal sets. Elements of Cartesian Product are in ordered pairs. In ordered pairs, *order of elements* are very important. In turn, *binary soft set theory demands high concern while arranging its elements.* In 2017, *Shivanagappa. S. Benchalli, Prakashgouda. G. Pattil, Abeda. S. Dodamani and Pradeepkumar. J* [81] discussed on *binary soft topological Spaces* and also on its *Separation Axioms* [12]. In 2019, *Sabir Hussain* [75] discussed on some structures of Binary Soft Topological Spaces. In 2020, *P. G. Patil and Nagashree. N. Bhat* [58] gave *New Separation Axioms in Binary Soft Topological Spaces.* In 2020, *Sabir Hussain* [76] presented *Binary Soft Connected Spaces and an Application of Binary Soft Sets in Decision Making Problem*

In 2014, *Ronald. R. Yager* [74] introduced *Pythagorean Fuzzy Set* to extend Intuitionistic Fuzzy Set. In 1986, *Krassimir. T. Attanassov* [48] introduced classical *Intuitionistic Fuzzy Sets* and it's *second type* respectively. Second type allowed square sum of membership grades strictly ≤ 1. Pythagorean Fuzzy Set's handled both ≤ 1 and ≥ 1 cases flexibly. In 2016, *Harish Garg* [33] presented novel correlation coefficients between pythagorean fuzzy sets and its applications to decision making. In 2017, *Khaista Rahman, Saleem Abdullah, Muhammad Sajjad Ali Khan* [45] developed some basic operations on Pythagorean Fuzzy Sets. In 2019, *Peng. X* [62] introduced new operations for interval-valued Pythagorean Fuzzy Set. In 2019, *Murat Kirisci* [57] mentioned new type Pythagorean Fuzzy Soft Set with its application in decision making.

A word to *Distance* & *Similarity* Measure

A literature sketch can be viewed regarding various measures in Chapter 3.

Similarity measures are very useful to measure similarities between objects. Entropy and distance measures are also found to be useful to the same extent as similarity measures - while measuring uncertainties in day to day real- life. Different kinds of distance measures viz., Hamming, Normalized Hamming, Euclidean, Normalized Euclidean are found to be useful in an effective way in day to day life.

Ideas of similarity measure and distance measure between vague sets can be found in the papers [26, 27, 38, 83, 88, 90]. Similarity measures for soft sets are clearly mentioned in [10, 64]. Fuzzy soft set's entropy and similarity measures are given in papers [65, 70, 100]. Entropy, distance measures ans similarity measures for vague soft sets can be found in the papers [17, 18, 22, 66]. In 2016, *Kalyan Mondal, Surapati Pramanik, Florentin Smarandache* [40] mentioned Several Trigonometric Hamming Similarity Measures of Rough Neutrosophic Sets and their applications in decision making. Notions like entropy, distance measure and similarity measure for Pythagorean Fuzzy Sets is well mentioned in papers [52, 59, 60, 93]

Various Algebra's

This section deals with a preliminary historical outline to various algebras kept as base for chapter 5

In 1966, *Yasuyuki Imai* & *Kiyoshi Iseki* [95] introduced a logical algebra namely BCK/BCI-algebra. In 1999, *G. Catteaneo, R. Giuntini, R. Pilla*[14] introduced $BZMV^{dM}$*-algebra*. It is a logical algebra and is a combination of BZ and MV - algebra in de- Morgan atmosphere. In 2003, *Akram* & *Dar* [09] introduced K - algebra [24] and their work is published in 2005. In 2012, *Ravi Kumar Bandaru* & *N. Rafi* [73] introduced *G -algebra* as an extension work to QS - algebra. In 1999, Sun Shin Ahn and Hee Sik kim [84] introduced QS- algebra.

A deep discussion towards BCK/BCI- algebra in different directions can be found in the papers [1, 6, 8, 44, 46, 47, 49, 79, 99]. Similarly, papers connected to BZMV^{dM}-algebra are [7, 15, 16, 37, 82, 91, 96, 98]. Lot of discussions in different directions related to K-algebra can be seen in the papers [3, 4, 23, 41, 42, 43]. Several papers well-discussing different facets of G -algebra are [20, 25, 86]

Remark 1.2.1.
Part of homeomorphism can't discard in any topology. It is well-connected to continuity. Some base papers related to continuity used for chapter 2 and chapter 4 are [9, 11, 21, 36, 69, 78]. Some papers found as bulding bricks to coset notion developed in chapter 4 are [19, 56]. In the vast spectrum of algebras, some of them needs a special attention is properly mentioned in papers [35, 39, 54, 55, 71, 94, 97]

1.3 Preliminaries

Definition & Notions to Various Pure uncertain Sets

Definition 1.3.1. [31] ***(Vague Set)***
Let X be a space of points (objects), with a generic element of X denoted by x. A vague set V in X is characterized by a truth-membership function t_v and a false-membership function f_v. $t_v(x)$ is a lower bound on the grade of membership of x derived from the evidence for x, and $f_v(x)$ is a lower bound on the negation of x derived from the evidence against x. $t_v(x)$ and $f_v(x)$ both associate a real number in the interval [0,1] with each point in X, where $t_v(x) + f_v(x) \leq 1$. That is, $t_v : X \to [0,1]$, $f_v : X \to [0,1]$ This approach bounds the grade of membership of x to a subinterval $[t_v(x), 1 - f_v(x)]$ of $[0,1]$. In other words, the exact grade of membership $\mu_v(x)$ of x may be unknown but is bounded by $[t_v(x) \leq \mu_v(x) \leq 1 - f_v(x)]$, where $t_v(x) + f_v(x) \leq 1$.

The precision of our knowledge about x is immediately clear, with our uncertainty characterized by the difference $1 - f_v(x) - t_v(x)$. If this is small, our knowledge about x is relatively precise; if it is large, we know correspondingly little. If $1 - f_v(x)$ is equal to $t_v(x)$, our knowledge about x is exact, and the theory reverts back to that of fuzzy sets. If $1 - f_v(x)$ and $t_v(x)$ are both equal to 1 or 0, depending on whether x does or does not belong to V, our knowledge about x is very exact and the theory reverts back to that of ordinary sets (Le., sets with two-valued characteristic *functions).*

Definition 1.3.2. [31] ***(Operations on Vague Sets)***
Let a vague set is defined as in definition 1.3.1.

1. *A vague set is empty if and only if its truth membership and false membership functions are identically zero on X*
2. *The complement of a vague set A is denoted by A' and is defined by*
   (a) $t'_A(x) = f_A(x)$
   (b) $1 - f'_A(x) = 1 - t_A(x)$
3. *A vague set A is contained in the other vague set B, $A \subseteq B$, if and only if*
   (a) $t_A \leq t_B$
   (b) $1 - f_A \leq 1 - f_B$
4. *Two vague sets A and B are equal, written as $A = B$, if and only if $A \subseteq B$, and $B \subseteq A$; that is*
   (a) $t_A = t_B$
   (b) $1 - f_A = 1 - f_B$
5. *The union of two vague sets A and B with respective truth - membership and false - membership functions t_A, f_A, t_B and f_B is a vague set C, written as $C = A \cup B$, whose truth membership and false-membership functions are related to those of A and B by*
   (a) $t_C = max\,(t_A, t_B)$
   (b) $1 - f_C = max\,(1 - f_A, 1 - f_B) = 1 - \,min\,(f_A, f_B)$
6. *The intersection of two vague sets A and B with respective truth - membership and false - membership functions t_A, f_A, t_B and f_B is a vague set C, written as $C = A \cap B$, whose truth membership and false-membership functions are related to those of A and B by*
   (a) $t_C = min\,(t_A, t_B)$
   (b) $1 - f_C = min\,(1 - f_A, 1 - f_B) = 1 - \,max\,(f_A, f_B)$

Definition 1.3.3. [69] ***(Vague Continuous)***

*Let (U, τ) and $(Y.\vartheta)$ be any two topological spaces. A map $f : (U, \tau) \to (Y, \vartheta)$ is said to be **vague continuous** (V continuous in short) if $f^{-1}(V)$ is vague closed set in (U, τ) for every vague closed set V of $(Y.\vartheta)$*

Definition 1.3.4. [28, 30] ***(Neutrosophic Set)***

Let U be an universe of discourse then the neutrosophic set A is an object having the form $A = \{\langle x : T_A(x), I_A(x), F_A(x)\rangle, x \in U\}$ where the functions $T, I, F : U \to]^{-}0, 1^{+}[$ define respectively the degree of membership (or Truth), the degree of indeterminacy, and the degree of non-membership (or Falsehood) of the element $x \in U$ to the set A with the condition $^{-}0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^{+}$. From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^{-}0, 1^{+}[$ So instead of $]^{-}0, 1^{+}[$ we need to take the interval $[0, 1]$ for technical applications, because $]^{-}0, 1^{+}[$ will be difficult to apply in the real applications such as in scientific and engineering problems.

Definition 1.3.5. [77] ***(Complement of a Neutrosophic Set)***

Let $A = \langle \mu_A, \sigma_A, \gamma_A\rangle$ a Neutrosophic Set (NS, in short)on X, then the complement of the set A $\quad (C(A) \quad for \quad short)$, may be defined as three kinds of complements

(C_1) $\quad C(A) = \{\langle x, 1 - \mu_A(x), 1 - \sigma_A(x), 1 - \gamma_A(x)\rangle : x \in X\}$

(C_2) $\quad C(A) = \{\langle x, \gamma_A(x), \sigma_A(x), \mu_A(x)\rangle : x \in X\}$

(C_3) $\quad C(A) = \{\langle x, \gamma_A(x), 1 - \sigma_A(x), \mu_A(x)\rangle : x \in X\}$

Definition 1.3.6. [77] ***(Subset of Neutrosophic Set)***

Let X be a non-empty set, and Neutrosophic Subsets (NSS, in short) A and B in the form $A = \langle \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle$ and $B = \langle \mu_B(x), \sigma_B(x), \gamma_B(x)\rangle$, then we may consider two possible definitions for subsets $(A \subseteq B)$ may be defined as

(1) $A \subseteq B \Leftrightarrow \quad \mu_A(x) \leq \mu_B(x), \quad \gamma_A(x) \geq \gamma_B(x), \quad \sigma_A(x) \leq \sigma_B(x) \quad \forall x \in X$

(2) $A \subseteq B \Leftrightarrow \quad \mu_A(x) \leq \mu_B(x), \quad \gamma_A(x) \geq \gamma_B(x), \quad \sigma_A(x) \geq \sigma_B(x) \quad \forall x \in X$

Definition 1.3.7. [77] ***(Intersection** & **Union of Neutrosophic Set)***

Let X be a non-empty set, and $A = \langle \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle$, $B = \langle \mu_B(x), \sigma_B(x), \gamma_B(x)\rangle$ are NSS. Then

(1) $(A \cap B)$ *may be defined as*

(I_1) $\quad (A \cap B) = \langle x, \quad \mu_A(x).\mu_B(x), \quad \sigma_A(x).\sigma_B(x), \quad \gamma_A(x).\gamma_B(x)\rangle$

(I_2) $\quad (A \cap B) = \langle x, \quad \mu_A(x) \wedge \mu_B(x), \quad \sigma_A(x) \wedge \sigma_B(x), \quad \gamma_A(x) \vee \gamma_B(x)\rangle$

(I_3) $\quad (A \cap B) = \langle x, \quad \mu_A(x) \wedge \mu_B(x), \quad \sigma_A(x) \vee \sigma_B(x), \quad \gamma_A(x) \vee \gamma_B(x)\rangle$

(2) $(A \cup B)$ *may be defined as*

(U_1) $(A \cup B) = \langle x, \quad \mu_A(x) \vee \mu_B(x), \quad \sigma_A(x) \vee \sigma_B(x), \quad \gamma_A(x) \wedge \gamma_B(x) \rangle$

(U_2) $(A \cup B) = \langle x, \quad \mu_A(x) \vee \mu_B(x), \quad \sigma_A(x) \wedge \sigma_B(x), \quad \gamma_A(x) \wedge \gamma_B(x) \rangle$

Definition 1.3.8. [77] ***(*** 0_N & 1_N***)***

0_N *in* X *may be defined as :*

(0_1) $0_N = \{\langle 0,0,1 \rangle : x \in X\}$

(0_2) $0_N = \{\langle 0,1,1 \rangle : x \in X\}$

(0_3) $0_N = \{\langle 0,1,0 \rangle : x \in X\}$

(0_4) $0_N = \{\langle 0,0,0 \rangle : x \in X\}$

1_N *in* X *may be defined as :*

(1_1) $1_N = \{\langle 1,0,0 \rangle : x \in X\}$

(1_2) $1_N = \{\langle 1,0,1 \rangle : x \in X\}$

(1_3) $1_N = \{\langle 1,1,0 \rangle : x \in X\}$

(1_4) $1_N = \{\langle 1,1,1 \rangle : x \in X\}$

Definition 1.3.9. [77] ***(Neutrosophic Topology)***

A neutrosophic topology (NT, for short) an a non empty set X *is a family* τ *of neutrosophic subsets in* X *satisfying the following axioms*

$(NT_1)\, 0_N, 1_N \in \tau$

$(NT_2)\, G_1 \cap G_2 \in \tau$ *for any* $G_1, G_2 \in \tau$

$(NT_3) \cup G_i \in \tau \forall \{G_i : i \in J\} \subseteq \tau$

In this case the pair (X, τ) *is called a neutrosophic topological space* $(NTS \quad for \quad short)$ *and any neutrosophic set in* τ *is known as neuterosophic open set* $(NOS \quad for \quad short)$ *in* X*. The elements of* τ *are called open neutrosophic sets, A neutrosophic set* F *is closed if and only if it* $C(F)$ *is neutrosophic open.*

Definition 1.3.10. [53] ***(Soft Set)***

Let U *be an initial universe and let* E *be a set of parameters. A pair* (F, E) *is called a soft set (over U) if and only if F is a mapping of E into the set of all subsets of the set U. A soft set can be viewed as :* $(F, A) = \{F(e)/e \in A\}$*. Every set F(e) from this family may be considered as the set of e - approximate element of the soft set (F, A)*

Definition 1.3.11. [67] ***(Redefined Definition - Intersection of Soft Sets)***

The intersection of two soft sets (F, A) *and* (G, B) *over a common universe* U *is the soft set* (H, C) *where* $C = (A \cup B)$*, and* $\forall e \in C$*,*

$$H(e) = \begin{cases} F(e), & if \quad e \in A - B \\ G(e), & if \quad e \in B - A \\ F(e) \cap G(e), & if \quad e \in A \cap B \end{cases}$$

It is denoted as $(F, A) \tilde{\cup} (G, B) = (H, C)$

Definition 1.3.12. [51] ***(Union of Soft Sets)***

The union of two soft sets (F, A) *and* (G, B) *over a common universe* U *is the soft set* (H, C) *where* $C = (A \cup B)$, *and* $\forall e \in C$,

$$H(e) = \begin{cases} F(e), & if \quad e \in A - B \\ G(e), & if \quad e \in B - A \\ F(e) \cup G(e), & if \quad e \in A \cap B \end{cases}$$

It is denoted as $(F, A) \tilde{\cap} (G, B) = (H, C)$

Definition 1.3.13. [51, 67] ***(Complement of a Soft Set)***

The complement of a soft set (F, A) *denoted by* $(F, A)^c$ *and is defined by* $(F, A)^c = (F^c, A)$; $F^c : A \rightarrow P(U)$ *is a mapping given by* $F^c(e) = U - F(e)$, *for each* $e \in A.F^c$ *is called the soft complement function of* F.

Definition 1.3.14. [67] ***(Redefined Definition - Soft Subset)***

For two soft sets (F, A) *and* (G, B) *over a universe* U, *we say that* (F, A) *is a soft subset of* (G, B) *if*

1. $A \subseteq B$
2. $\forall e \in A, F(e) \subseteq G(e)$.

Remark 1.3.15 (53).

1. *Family of all soft sets over a common universe* U *with a fixed parameter set* A *is denoted by* $SS(U)_A$
2. *A soft set* (F, A) *over* U *is said to be*
   (a) *a null soft set denoted by* (Φ, A)*if for all* $e \in A, F(e) = \Phi$.
   (b) *an absolute soft set denoted by* (U, A)*if for all* $e \in A, F(e) = U$.

Definition 1.3.16. [53] ***(Soft Point)***

The soft set $(F, A) \in SS(U)_A$ is called a soft point in (U, A) denoted by $e_{(F,A)}$ or e_F if for each element $e \in A, F(e) \neq \Phi$ and $F(e^{'}) = \Phi, \forall e^{'} \in A - \{e\}$

Definitions & Notions to Various Hybrid Set's

Definition 1.3.17. [92] ***(Vague Soft Set)***

Let U be a universe, E be a set of parameters, V(U) be the power set of vague set on U and $A \subseteq B$. A pair (F, A) is called a Vague Soft Set over U is a parameterized family of vague sets of the universe U. For $e \in A, \mu_F(e) : U \to [0,1]$ is regarded as the set of e - approximate elements of the vague soft sets.

Definition 1.3.18. [80] ***(Neutrosophic Vague Set)***

A Neutrosophic Vague Set A_{NV} (NVS in short) on the universe of discourse U can be written as $A_{NV} = \left\{\left\langle u; \hat{T}_{A_{NV}}(x), \hat{I}_{A_{NV}}(x), \hat{F}_{A_{NV}}(x); x \in U \right\rangle\right\}$ whose truth membership, indeterminacy membership and falsity membership functions are defined as $\hat{T}_{A_{NV}}(x) = [T^-, T^+], \hat{I}_{A_{NV}}(x) = [I^-, I^+], \hat{F}_{A_{NV}}(x) = [F^-, F^+]$ where

1. $T^+ = (1 - F^-), F^+ = (1 - T^-)$ *and*
2. *(a)* $^-0 \leq T^- + I^- + F^- \leq 2^+$

   (b) $^-0 \leq T^+ + I^+ + F^+ \leq 2^+$

Definition 1.3.19. [02] ***(Binary Soft Set)***

Let U_1, U_2 be two initial universal sets and E be a set of parameters. Let $P(U_1), P(U_2)$ denote the power set of U_1, U_2 respectively.

Also let A, $B \subseteq E$. A pair (F, A) is called a Binary Soft Set over U_1, U_2 where F is defined as below : $F : A \to P(U_1) \times P(U_2), F(e) = (X, Y)$ for each $e \in A$ such that $X \subseteq U_1, Y \subseteq U_2$

Definition 1.3.20. [57] ***(Pythagorean Fuzzy Soft Set)***

Given an initial universe set U and a universe set of parameters E.

A pair (F, A) is referred to as a Pythagorean Fuzzy Soft Set(PFSS, in short)on U if $A \subseteq E$ and $F : A \to PF(U)$, where PF(U) is the family of all Pythagorean fuzzy subsets of U.

Formulae & Notions to Various Measures

Definition 1.3.21. [64] ***(Various Distance Measures between two Soft Sets)*** *Let $U = \{x_1, x_2, --, x_n\}$ be a universal set and $E = \{e_1, e_2, --, e_m\}$ be a fixed set of parameters. For two soft sets (F_1, E) and (F_2, E) over U based on E we define $D^s(F_1, F_2)$ between two soft sets as*

1. *Mean Hamming distance:* $D^s(F_1, F_2)$ *between two soft sets as*

$$D^s(F_1, F_2) = \frac{1}{m}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}|F_1(e_i)(x_j) - F_2(e_i)(x_j)|\right\}$$

2. *Normalized Hamming distance:* $L^s(F_1, F_2)$ *between two soft sets as*

$$L^s(F_1, F_2) = \frac{1}{m.n}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}|F_1(e_i)(x_j) - F_2(e_i)(x_j)|\right\}$$

3. *Euclidean distance:* $E^s(F_1, F_2)$ *between two soft sets as*

$$E^s(F_1, F_2) = \sqrt{\frac{1}{m}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}(F_1(e_i)(x_j) - F_2(e_i)(x_j))^2\right\}}$$

4. *Normalized Euclidean distance:* $Q^s(F_1, F_2)$ *between two soft sets as*

$$Q^s(F_1, F_2) = \sqrt{\frac{1}{m.n}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}(F_1(e_i)(x_j) - F_2(e_i)(x_j))^2\right\}}$$

Definition 1.3.22. [88]
(Szmidt and Kaeprzyk's Distances between Vague Sets)

$$d_H^{Sz}(A, B) = \frac{1}{2}\Sigma_{i=1}^{n}\left[|\Delta_t(i)| + |\Delta_f(i)| + |\Delta_\Pi(i)|\right]$$
$$d_{nH}^{Sz}(A, B) = \frac{1}{2n}\Sigma_{i=1}^{n}\left[|\Delta_t(i)| + |\Delta_f(i)| + |\Delta_\Pi(i)|\right]$$
$$d_E^{Sz}(A, B) = \sqrt{\frac{1}{2}\Sigma_{i=1}^{n}\left[|\Delta_t(i)|^2 + |\Delta_f(i)|^2 + |\Delta_\Pi(i)|^2\right]}$$
$$d_{nE}^{Sz}(A, B) = \sqrt{\frac{1}{2n}\Sigma_{i=1}^{n}\left[|\Delta_t(i)|^2 + |\Delta_f(i)|^2 + |\Delta_\Pi(i)|^2\right]}$$

where

$$\Delta_t(i) = t_A(x_i) - t_B(x_i),$$
$$\Delta_f(i) = f_A(x_i) - f_B(x_i),$$
$$\Delta_\Pi(i) = \Pi_A(x_i) - \Pi_B(x_i)$$

Definition 1.3.23. [83]

(Similarity Measure between Vague Sets)

A and B are two vague sets over the universe of discourse $U = \{x_1, x_2, ---, x_n\}$. $V_A(x_i) = [t_A(x_i), 1 - f_A(x_i)]$ *is the membership value of* x_i *in vague set A and* $V_B(x_i) = [t_B(x_i), 1 - f_B(x_i)]$ *is the membership value of* x_i *in vague set B. Let*

$$A = \sum_{i=1}^{n} \left[\frac{t_A(x_i), 1 - f_A(x_i)}{x_i}\right] \quad ; \quad B = \sum_{i=1}^{n} \left[\frac{t_B(x_i), 1 - f_B(x_i)}{x_i}\right]$$

Similarity between vague sets A and B can be obtained by the function,

$$T(A, B) = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{1 - |S(V_A(x_i)) - S(V_B(x_i))|}{2}\right)$$

T(A, B) $\in [0, 1]$.

Larger value of T(A, B) indicates more similarity between the sets.

Definitions to Various Algebra's

This section deals with definitions to various algebras acted as base for chapter 5

Definition 1.3.24. [95]

Let U be a non-empty set with a binary operation $\star$ *and a constant 0. Then* $(U, \star, 0)$ *is called a* ***BCI -algebra*** *if it satisfies the following conditions: for all x, y, z* $\in U$

(1) $((x \star y) \star (x \star z)) \star (z \star y) = 0$

(2) $(x \star (x \star y)) \star y = 0$

(3) $(x \star x) = 0$

(4) $(x \star x) = 0$ *and* $(x \star x) = 0$ *imply* $x = y$

If a BCI - algebra X satisfies $(0 \star x) = 0$ *for all* $x \in U$*, then U is a* ***BCK - algebra***

Definition 1.3.25. [14]

A ***de Morgan BZMV****-algebra* (*denoted as* $BZMV^{dM}$), *is a BZMV algebra* $\mathcal{A} = \langle A, \oplus, \neg, \sim, 0\rangle$ *which satisfies the following axioms: for all x, y, z* $\in A$

$(BZMV1)\ (x \oplus y) \oplus z = (y \oplus z) \oplus x$

$(BZMV2)\ (x \oplus 0) = x$

$(BZMV3)\ \neg(\neg x) = x$

$(BZMV4)\, \neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$

$(BZMV5) \sim x \oplus \neg\neg x = \neg 0$

$(BZMV6)\, x \oplus \sim\sim x = \sim\sim x$

$(BZMV7) \sim \neg[(\neg(x \oplus \neg y) \oplus \neg y)] = \neg(\sim\sim x \oplus \neg \sim\sim y) \oplus \neg \sim\sim y$

The unary operation $\neg : A \to A$ is a Kleene (or Zadeh) orthocomplementation (negation).The unary operation $\sim : A \to A$ is a Brouwer orthocomplementation (negation)

Definition 1.3.26. [24]

*Let $(G, ., e)$ be a group in which each non-identity element is not of order 2. Then **K - algebra** is a structure $\mathcal{K} = (G, ., \odot, e)$ on a group G in which induced binary operation $\odot : G \times G \to G$ is defined by $\odot(x, y) = (x \odot y) = x.y^{-1}$ and satisfies the following $\odot$ - axioms: for all x, y, $z \in G$*

(K1) $(x \odot y) \odot (x \odot z) = (x \odot ((e \odot z) \odot (e \odot y))) \odot x$

(K2) $x \odot (x \odot y) = (x \odot y^{-1}) \odot x = (x \odot (e \odot y)) \odot x$

(K3) $(x \odot x) = e$

(K4) $(x \odot e) = x$

(K5) $(e \odot x) = x^{-1}$

Definition 1.3.27. [73]

*A **G - algebra** is a non-empty set A with a constant 0 and a binary operation $\star$ satisfying axioms: for all x, $y \in A$*

$(B_3)\,(x \star x) = 0 \quad (B_{12})\,(x \star (x \star y)) = y.$

A G - algebra is denoted by $(A, \star, 0)$

Definition 1.3.28. [08]

*A vague set A of X is called a**vague BCK/BCI- subalgebra** of X if it satisfies the following condition:*

$V_A(x * y) \succeq rmin\,\{V_A(x), V_A(y)\}$, *for all x, $y \in X$. That is,*

1. $t_A(x * y) \geq min\,\{t_A(x), t_A(y)\}$

2. $1 - f_A(x * y) \geq min\,\{1 - f_A(x), 1 - f_A(y)\}$

Remark 1.3.29. [08]

Let D[0, 1] denote the family of all closed sub-intervals of [0, 1]. Refined Minimum *(briefly, r min) and an order* $0 \leq 1$ *on elements* $D_1 = [a_1, b_1]$ *and* $D_2 = [a_2, b_2]$ *of* $D[0,1]$ *as :* $r\ min\ (D_1, D_2) = [min\{a_1, a_2\}, min\{b_1, b_2\}]$. *Similarly, define* $\geq, =$ *and r max. Then the concept of r min and r max could be extended to define r inf and r sup of infinite number of elements of* $D[0,1]$. *It is a known fact that* $L = \{D[0,1], rinf, rsup, \leq\}$ *is a lattice with universal bounds* $[0,0]$ *and* $[1,1]$.

Definition 1.3.30. [08]

*A vague set A of a BCI - algebra X is called a **vague ideal** of X if the following conditions are true :*

1. $V_A(0) \succeq V_A(x); \forall x \in X$
2. $V_A(x * y) \succeq rmin\{V_A(x), V_A(y)\}; \forall x, y \in X$

Definition 1.3.31. [99] ***(Various ideals under Vague Concept)***

1. ***Vague p -ideal:*** *A vague set A of X is called a vague p -ideal of a BCI - algebra of X if it satisfies*
   (a) $V_A(0) \succeq V_A(x); \forall x \in X$
   (b) $V_A(x) \succeq rmin\{V_A((x * z) * (y * z))\}; \forall x, y, z \in X$
2. ***Vague q -ideal:*** *A vague set A of X is called a vague p -ideal of a BCI - algebra of X if it satisfies*
   (a) $V_A(0) \succeq V_A(x); \forall x \in X$
   (b) $V_A(x * z) \succeq rmin\{V_A(x * (y * z)), V_A(y)\}; \forall x, y, z \in X$
3. ***Vague a -ideal:*** *A vague set A of X is called a vague a -ideal of a BCI - algebra of X if it satisfies*
   (a) $V_A(0) \succeq V_A(x); \forall x \in X$
   (b) $V_A(y * x) \succeq rmin\{V_A((x * z) * (0 * y)), V_A(z)\}; \forall x, y, z \in X$
4. ***Vague H -ideal:*** *A vague set A of X is called a vague p -ideal of a BCI - algebra of X if it satisfies*
   (a) $V_A(0) \succeq V_A(x); \quad \forall x \in X$
   (b) $V_A(x * z) \succeq rmin\{V_A(x * (y * z)), V_A(y)\}; \quad \forall x, y, z \in X$

Remark 1.3.32. *[101]*

In 2018 Smarandache generalized the Soft Set to the Hyper Soft set by transforming the classical uni-argument function F into multi-argument function.

Definition 1.3.33. *[29, 102]* **(Plithogenic Set)**

A plithogenic set P is a set whose elements are characterized by one or more attributes, and each attribute may have many values. Each attribute value v has a corresponding degree of appurtenance d(x, v) of the element x to the set P, with respect to some given criteria. In order to get a better accuracy for the plithogenic aggregation operators, a contradiction (dissimilarity) degree is defined between each attribute value and the dominant (most important) attribute value. The plithogenic aggregation operators (intersection, union, complement, inclusion, equality) are based on contradiction degrees between attribute values and the first two are linear combinations of the fuzzy operator t-norm and t-conorm Plithogenic Set was introduced by Smarandache in 2017 and it is a generalization of the crisp set, fuzzy set, neutrosophic set, since these types of sets are characterized by a single attribute value (appurtenance): which has one value membership for the crisp set and fuzzy set, two values (membership and nonmembership) for intuitionistic fuzzy set, or three values (membership, nonmembership and indeterminacy) for neutrosophic set.

Definition 1.3.34. *[103, 104, 105, 106]:* **(Neutro-Algebra and Anti-algebra)**

In 2019 and 2020 Smarandache generalized the classical algebraic structures to Neutro Algebraic Structures (or Neutro Algebras) whose operations and axioms are partially true, partially indeterminate, and partially false as extensions of Partial Algebra and to Anti Algebraic Structures (or Anti Algebras) whose operations and axioms are totally false. And general he extended any classical structure, in no matter what field of knowledge, to a neutro structure and Anti Structure

1.4 Author's Contributions

Following are the contributions.

1. Three different sets are developed
   (a) Vague Binary Soft Set
   (b) Pythagorean Vague Binary Soft Set
   (c) Neutrosophic Vague Binary Set
2. Operations & Properties of the above mentioned Sets
3. Different Algebraic Structures of Neutrosophic Vague Binary Sets

Notations

$VBSS\,(U_1, U_2)_E$ - Set of all vague binary soft sets over the binary universe (U_1, U_2) under the fixed parameter set E

VBSS - Vague Binary Soft Set

VBSSS-Vague Binary Soft SubSet

VBST - Vague Binary Soft Topology

VBSTS - Vague Binary Soft Topological Space

VBSOS - Vage Binary Soft open Set

VBSCS - Vague Binary Soft Closed Set

PVBSS - Pythagorean Vague Binary Soft Set

NVBS - Neutrosophic Vague Binary Set

NVBST - Neutrosophic Vague Binary Soft Topology

NVBSTS - Neutrosophic Vague Binary Soft Topological Space

VBSS (U_1, U_2) - Set of all VBSS's over a binary universe (U_1, U_2)

PVBSS (U_1, U_2) - Set of all PVBSS's over a binary universe (U_1, U_2)

$\left(\ddot{F}, A\right)$ - Vague Binary Soft Set with a fixed parameter set A under the mapping F

$(U_1, U_2, \ddot{\tau}_\Delta, E)_A$ - Vague Binary Soft Topological Space with a Binary Universe (U_1, U_2) with a fixed parameter set A $\subseteq$ E

Chapter 2

Chapter 2

Vague Binary Soft Set

One of the integral parts in the configuration of Molodtsov's [53] soft set theory is *single* universe. In literature meaning, word *Binary* indicates *Two*. Ahu Acikgöz [02] extended *single universe* concept of soft set theory to *double/binary* universe notion and kept the name as *Binary Soft Set theory*. *Wei Xu*, *Jian Ma* [89] have done vague extension of soft sets and named it as *vague soft set theory*. New set developed in this chapter is a combination of the above two theories.

Chapter Scheme:

Section 2.1 : Vague Binary Soft Sets & its operations
Section 2.2 : Algebraic Properties of Vague Binary Soft Sets
Section 2.3 : Vague Binary Soft Topology
Section 2.4 : Vague Binary Soft Continuity

2.1 Vague Binary Soft Set and Its Operations

Vague Binary Soft Set (In Short, VBSS) is the newly framed set presented in this section by combining *vague soft* and *binary soft* concepts. In this section, its operations have also been constructed

Definition 2.1.1. ***(Vague Binary Soft Set)***

Let (U_1, U_2) *be a binary universe with* $U_1 = \{u_1, u_2, --, u_r, --, u_i\}$,
$U_2 = \{v_1, v_2, --, v_s, --, v_j\}$. $E = \{e_1, e_2, --, e_t, --, e_k\}$ *be a fixed parameter set where* $A = \{e_1, e_2, --, e_p, --, e_m\}$ *with* $A \subseteq E$. *Let* $V(U_1)$, $V(U_2)$ *denote power set*

of vague sets on U_1, U_2 respectively.

A pair $\left(\ddot{F}, A\right)$ is said to be a Vague Binary Soft Set (in short, VBSS) over (U_1, U_2) where $\ddot{F}$ is a mapping given by, $\ddot{F} : A \longrightarrow V(U_1) \times V(U_2)$.
Here, $\left(\ddot{F}, A\right) = \left\{e_p \in A / \left(e_p, \ddot{F}(e_p)\right)\right\}$
where

$$\ddot{F}(e_p) = \left\{ \left(\left\langle \frac{V_{\ddot{F}(e_p)}(u_r)}{u_r}; \forall e_p \in A, \forall u_r \in U_1 \right\rangle, \ \left\langle \frac{V_{\ddot{F}(e_p)}(v_s)}{v_s}; \forall e_p \in A, \forall v_s \in U_2 \right\rangle \right) \right\}$$

$$= \left\{ \left(\begin{array}{l} \left\langle \frac{\left[t_{\ddot{F}(e_p)}(u_r), 1 - f_{\ddot{F}(e_p)}(u_r)\right]}{u_r}; \forall e_p \in A, \forall u_r \in U_1 \right\rangle, \\ \left\langle \frac{\left[t_{\ddot{F}(e_p)}(v_s), 1 - f_{\ddot{F}(e_p)}(v_s)\right]}{v_s}; \forall e_p \in A, \forall v_s \in U_2 \right\rangle \end{array} \right) \right\}$$

Truth Membership function $t_{\ddot{F}(e_p)}(u_r)$ is a lower bound on the grade of membership of u_r derived from the evidence for $u_r \in U_1$ for the parameter e_p under the mapping $\ddot{F}$. False Membership function $f_{\ddot{F}(e_p)}(u_r)$ is a lower bound on the grade of membership of u_r derived from the evidence for $u_r \in U_1$ for the parameter e_p under the mapping $\ddot{F}$ & Similarly for $t_{\ddot{F}(e_p)}(v_s)$ and $f_{\ddot{F}(e_p)}(v_s)$ where $v_s \in U_2$ &
Grade of membership of $u_r \in U_1$ in the vague binary soft set $\left(\ddot{F}, A\right)$ is bounded to a sub-interval $\left[t_{\ddot{F}(e_p)}(u_r), 1 - f_{\ddot{F}(e_p)}(u_r)\right]$ of [0, 1]
$t_{\ddot{F}(e_p)}(u_r), f_{\ddot{F}(e_p)}(u_r) : U_1 \to [0, 1]$; $\quad 0 \leq t_{\ddot{F}(e_p)}(u_r) + f_{\ddot{F}(e_p)}(u_r) \leq 1$;
Grade of membership of $v_s \in U_2$ in the vague binary soft set $\left(\ddot{F}, A\right)$ is bounded to a sub-interval $\left[t_{\ddot{F}(e_p)}(v_s), 1 - f_{\ddot{F}(e_p)}(v_s)\right]$ of [0, 1]
$t_{\ddot{F}(e_p)}(v_s), f_{\ddot{F}(e_p)}(v_s) : U_2 \to [0, 1]$; $\quad 0 \leq t_{\ddot{F}(e_p)}(v_s) + f_{\ddot{F}(e_p)}(v_s) \leq 1$
Vague Binary Soft Value, $V_{\ddot{F}(e_p)}(u_r) = \left[t_{\ddot{F}(e_p)}(u_r), 1 - f_{\ddot{F}(e_p)}(u_r)\right]$ indicates that the exact grade of membership $\mu_{\ddot{F}(e_p)}(u_r)$ of $u_r \in U_1$ may be unknown, but it is bounded by : $\quad t_{\ddot{F}(e_p)}(u_r) \leq \mu_{\ddot{F}(e_p)}(u_r) \leq 1 - f_{\ddot{F}(e_p)}(u_r)$
Vague Binary Soft Value, $V_{\ddot{F}(e_p)}(v_s) = \left[t_{\ddot{F}(e_p)}(v_s), 1 - f_{\ddot{F}(e_p)}(v_s)\right]$ indicates that the exact grade of membership $\mu_{\ddot{F}(e_p)}(v_s)$ of $v_s \in U_2$ may be unknown, but it is bounded by : $\quad t_{\ddot{F}(e_p)}(v_s) \leq \mu_{\ddot{F}(e_p)}(v_s) \leq 1 - f_{\ddot{F}(e_p)}(v_s)$

Example 2.1.2.
Let $U_1 = \{b_1, b_2, b_3\}$ and $U_2 = \{l_1, l_2, l_3\}$ be food varieties for break - fast and lunch

respectively. Add a parameter set for characteristics of different hotels say

$$E = \left\{ \begin{array}{l} e_1 = luxury, \quad e_2 = famous, \quad e_3 = warm - serviced, \\ e_4 = early \quad check \quad in \quad and \quad late \quad check - out, \\ e_5 = honest \quad information \quad e_6 = hospitality \end{array} \right\}$$

A customer's preferences are given by the parameter set $A = \{e_3, e_6\}$.
Vague Binary Soft Set (VBSS) according to customer's interest is given as,
$\left(\ddot{F}, A\right) =$

$$\left\{ \begin{array}{l} \left(e_3, \left(\left\langle \frac{[0.3,0.5]}{b_1}, \frac{[0.5,0.7]}{b_2}, \frac{[0.7,0.9]}{b_3} \right\rangle, \left\langle \frac{[0.4,0.5]}{l_1}, \frac{[0.6,0.8]}{l_2}, \frac{[0.8,0.9]}{l_3} \right\rangle\right)\right) \\ \left(e_6, \left(\left\langle \frac{[0.4,0.6]}{b_1}, \frac{[0.6,0.8]}{b_2}, \frac{[0.8,0.9]}{b_3} \right\rangle, \left\langle \frac{[0.4,0.8]}{l_1}, \frac{[0.3,0.5]}{l_2}, \frac{[0.7,0.9]}{l_3} \right\rangle\right)\right) \end{array} \right\}$$

Notation

1. Soft Set is denoted with a mapping. To indicate the mapping any capital letter from english alphabet is allowed (F is used in common). To denote vague soft set, 'hat' symbol on the top of the letter is used (for example, $\hat{F}$). For binary soft set, $\ddot{}$ symbol is used on the top of the letter(for example, $\ddot{\tilde{F}}$.

2. To indicate the mapping in VBSS a new symbol is provided. All english capital alphabets are allowed with a 'double dot' on top of the letter used for underlying mapping ($\ddot{F}$ is used in common). Double-dot, . . , gives a typical representaion of two universes U_1 and U_2

$$\text{Literature Meaning for symbolic representation} \left\{ \begin{array}{cc} . & . \\ \downarrow & \downarrow \\ U_1 & U_2 \end{array} \right\}$$

Definition 2.1.3. ***(Null & Absolute - Vague Binary Soft Set)***
Let (U_1, U_2) *be a binary universe with* $U_1 = \{u_1, u_2, --, u_r, --, u_i\}$,
$U_2 = \{v_1, v_2, --, v_s, --, v_j\}$. $E = \{e_1, e_2, --, e_t, --, e_k\}$ *be a fixed parameter set where* $A = \{e_1, e_2, --, e_p, --, e_m\}$ *with* $A \subseteq E$.

A Vague Binary Soft Set $\left(\ddot{F}, A\right)$ *over a binary universe* (U_1, U_2) *with a fixed parameter set E having* $A \subseteq E$ *is said to be a*

1. *Null Vague Binary Soft Set (in short, Null − VBSS) denoted by* $\left(\ddot{\Phi}, A\right)$ *if* $[t_{\ddot{F}(e_p)}(u_r), \quad 1 - f_{\ddot{F}(e_p)}(u_r)] = [0,0]; \quad [t_{\ddot{F}(e_p)}(v_s)), 1 - f_{\ddot{F}(e_p)}(v_s)] = [0,0]$

2. *Absolute Vague Binary Soft Set (in short, Absolute-VBSS) denoted by* $\left(\ddot{U}, A\right)$ *if* $[t_{\ddot{F}(e_p)}(u_r), 1 - f_{\ddot{F}(e_p)}(u_r)] = [1,1]; \quad [t_{\ddot{F}(e_p)}(v_s)), 1 - f_{\ddot{F}(e_p)}(v_s)] = [1,1]$

$$\forall e_p \in A, \quad \forall \quad u_r \in U_1, \forall \quad v_s \in U_2$$

Example 2.1.4.

Let $U_1 = \{p_1, p_2\}$ *and* $U_2 = \{c_1\}$ *be a binary universe and* $A = \{e_1, e_2\}$.
A Null-VBSS and Absolute - VBSS according to this context is given by,
$\left(\ddot{\Phi}, A\right) =$

$$\left\{ \left(e_1, \left(\left\langle \frac{[0,0]}{p_1}, \frac{[0,0]}{p_2} \right\rangle, \left\langle \frac{[0,0]}{c_1} \right\rangle\right)\right) \left(e_2, \left(\left\langle \frac{[0,0]}{p_1}, \frac{[0,0]}{p_2} \right\rangle, \left\langle \frac{[0,0]}{c_1} \right\rangle\right)\right)\right\}$$

$\left(\ddot{U}, A\right) =$

$$\left\{ \left(e_1, \left(\left\langle \frac{[1,1]}{p_1}, \frac{[1,1]}{p_2} \right\rangle, \left\langle \frac{[1,1]}{c_1} \right\rangle\right)\right) \left(e_2, \left(\left\langle \frac{[1,1]}{p_1}, \frac{[1,1]}{p_2} \right\rangle, \left\langle \frac{[1,1]}{c_1} \right\rangle\right)\right)\right\}$$

Vague Binary Soft Set Operations

Remark 2.1.5.

1. *Cantor set operation symbol is not applicable to VBSS. Therefore, notations for VBSS operations have to change.*

2. *Soft Set Operation is denoted with* $\sim$ *symbol over Cantor set operation symbol. For example,* $\widetilde{\cup}$ *for soft set union, here* $\cup$*, is the Cantor set union. Similarly soft set intersection is denoted as* $\widetilde{\cap}$ *instead of* $\cap$

3. *Vague Soft Set operation is indicated with* $\wedge$ *symbol over cantor set operation symbol. For example, vague soft union is denoted as,* $\widehat{\cup}$.

4. *Binary Soft Set operation is indicated with* $=$ *symbol over cantor set operation symbol. For example, binary soft union is denoted as,* $\tilde{\tilde{\cup}}$.

5. *For VBSS, double dot - symbol over, cantor set operation symbol is made use with! For example, vague binary soft union is indicated as* $\ddot{\cup}$ *instead of* $\cup$*. Same work to other vague binary soft operations !*

Definition 2.1.6. ***(Vague Binary Soft Complement)***

Let (U_1, U_2) be a binary universe. E be a fixed set of parameters with $A \subseteq E$. Vague Binary Soft Complement of a Vague Binary Soft Set $\left(\ddot{F}, A\right)$ with respect to Absolute Vague Binary Soft Set $\left(\ddot{U}, A\right)$ is a mapping given by
$\ddot{F}^c : A \to V(U_1) \times V(U_2)$ *where,* $\left(\ddot{F}^c, A\right) = \left\{e_p \in A / \left(e_p, \ddot{F}^c(e_p)\right)\right\}$
$\ddot{F}^c(e_p) =$

$$\left\{\left(\left\langle \frac{V_{\ddot{F}^c(e_p)}(u_r)}{u_r}; \forall e_p \in A, \forall u_r \in U_1 \right\rangle, \left\langle \frac{V_{\ddot{F}^c(e_p)}(v_s)}{v_s}; \forall e_p \in A, \forall v_s \in U_2 \right\rangle\right)\right\}$$

$$= \left\{\left(\begin{array}{l} \left\langle \frac{\left[t_{\ddot{F}^c(e_p)}(u_r), 1 - f_{\ddot{F}^c(e_p)}(u_r)\right]}{u_r}; \forall e_p \in A, \forall u_r \in U_1 \right\rangle, \\ \left\langle \frac{\left[t_{\ddot{F}^c(e_p)}(v_s), 1 - f_{\ddot{F}^c(e_p)}(v_s)\right]}{v_s}; \forall e_p \in A, \forall v_s \in U_2 \right\rangle \end{array}\right)\right\}$$

where

$$\begin{cases} t_{\ddot{F}^c(e_p)}(u_r) = f_{\ddot{F}(e_p)}(u_r) & ; 1 - f_{\ddot{F}^c(e_p)}(u_r) = 1 - t_{\ddot{F}(e_p)}(u_r) \quad ; \forall e_p \in A; \forall u_r \in U_1 \\ t_{\ddot{F}^c(e_p)}(v_s) = f_{\ddot{F}(e_p)}(v_s) & ; 1 - f_{\ddot{F}^c(e_p)}(v_s) = 1 - t_{\ddot{F}(e_p)}(v_s) \quad ; \forall e_p \in A; \forall v_s \in U_2 \end{cases}$$

It is denoted by $\left(\ddot{F}, A\right)^c = \left(\ddot{F}^c, A\right)$

Example 2.1.7.

$$\text{Let} \quad \left\{\begin{array}{l} U_1 = \{b_1, b_2, b_3, b_4, b_5\} \quad \text{be the set of books} \\ U_2 = \{p_1, p_2, p_3, p_4\} \quad \text{be the set of pencils} \end{array}\right\} \text{be the binary universe}$$

$$E = \left\{\begin{array}{l} e_1 = cheap, e_2 = expensive, e_3 = attractive, \\ e_4 = small, e_5 = long, e_6 = colorful \end{array}\right\} \text{be the set of parameters.}$$

A stationary shop proprietor decided to purchase some books and pencils from a wholesale dealer. Let a VBSS $\left(\ddot{F}, A\right)$ describes his taste, where $A = \{e_2, e_6\}$ gives features of the items as per his taste.

$$\ddot{F}(e_2) = \left(\begin{array}{l} \left\langle \frac{[0.2, 0.3]}{b_1}, \frac{[0.4, 0.6]}{b_2}, \frac{[0.2, 0.8]}{b_3}, \frac{[0.5, 0.8]}{b_4}, \frac{[0.7, 0.8]}{b_5} \right\rangle, \\ \left\langle \frac{[0.5, 0.6]}{p_1}, \frac{[0.2, 0.6]}{p_2}, \frac{[0.8, 0.9]}{p_3}, \frac{[0.4, 0.7]}{p_4} \right\rangle \end{array}\right)$$

$$\ddot{F}(e_6) = \left(\begin{array}{l} \left\langle \frac{[0.3,0.4]}{b_1}, \frac{[0.5,0.7]}{b_2}, \frac{[0.5,0.6]}{b_3}, \frac{[0.6,0.7]}{b_4}, \frac{[0.8,0.9]}{b_5} \right\rangle, \\ \left\langle \frac{[0.7,0.8]}{p_1}, \frac{[0.4,0.7]}{p_2}, \frac{[0.2,0.8]}{p_3}, \frac{[0.5,0.5]}{p_4} \right\rangle \end{array} \right)$$

Vague Binary Soft Complement of above VBSS is given by
$\left(\ddot{F}, A\right)^c = \left\{ \left(e_2, \ddot{F}^c(e_2)\right), \left(e_6, \ddot{F}^c(e_6)\right) \right\}$, *where*

$$\ddot{F}^c(e_2) = \left\{ \left(\begin{array}{l} \left\langle \frac{[0.7,0.8]}{b_1}, \frac{[0.4,0.6]}{b_2}, \frac{[0.2,0.8]}{b_3}, \frac{[0.2,0.5]}{b_4}, \frac{[0.2,0.3]}{b_5} \right\rangle, \\ \left\langle \frac{[0.4,0.5]}{p_1}, \frac{[0.4,0.8]}{p_2}, \frac{[0.1,0.2]}{p_3}, \frac{[0.3,0.6]}{p_4} \right\rangle \end{array} \right) \right\}$$

$$\ddot{F}^c(e_6) = \left\{ \left(\begin{array}{l} \left\langle \frac{[0.6,0.7]}{b_1}, \frac{[0.3,0.5]}{b_2}, \frac{[0.4,0.5]}{b_3}, \frac{[0.3,0.4]}{b_4}, \frac{[0.1,0.2]}{b_5} \right\rangle, \\ \left\langle \frac{[0.2,0.3]}{p_1}, \frac{[0.3,0.6]}{p_2}, \frac{[0.2,0.8]}{p_3}, \frac{[0.5,0.5]}{p_4} \right\rangle \end{array} \right) \right\}$$

Definition 2.1.8. ***(Intersection & Union of Vague Binary Soft Set's)***
Let (U_1, U_2) be a binary universe. E be a fixed set of parameters.
Take $U_1 = \{u_1, u_2, --, u_r, --, u_i\}, U_2 = \{v_1, v_2, --, v_s, --, v_j\}$.
$E = \{e_1, e_2, --, e_t, --, e_k\}$ *where*
$A = \{e_1, e_2, --, e_p, --, e_m\}$ $B = \{e_1, e_2, --, e_q, --, e_n\}$ *with $A, B \subseteq E$.*
Let $C = (A \cup B) = \{e_1, e_2, --, e_p, --, e_m\} \cup \{e_1, e_2, --, e_q, --, e_n\}$
$= \{e_1, e_2, --, e_g, --, e_h\}$ [$\cup$ *is Cantor Set Union Operation*]

1. ***Vague Binary Soft Intersection***
   Consider two VBSS's $\left(\ddot{F}, A\right)$ and $\left(\ddot{G}, B\right)$ over (U_1, U_2).
   Let output of $(\ddot{F}, A)\ddot{\cap}(\ddot{G}, B)$ is $(\ddot{K}, C)$ where $\ddot{K} : C \to V(U_1) \times V(U_2)$.
   where $C = (A \cup B)$; $\cup$ represents union operation in Cantor Set Theory
   Then $\forall e_g \in C$,

$$\ddot{K}(e_g) = \begin{cases} \ddot{F}(e_p); & \forall e_p \in B^c;\ c \text{ is Cantor Set Complement with respect to } C \\ \ddot{G}(e_q); & \forall e_q \in A^c;\ c \text{ is Cantor Set Complement with respect to } C \\ \ddot{F}(e_p) \cap \ddot{G}(e_q); & \forall e_p, e_q \in (A \cap B);\ \cap \text{ is Cantor Set Intersection} \end{cases}$$

$$t_{\ddot{K}(e_g)}(u_r) = \begin{cases} t_{\ddot{F}(e_p)}(u_r) & ; \forall e_p \in B^c \quad ; \forall u_r \in U_1 \\ t_{\ddot{G}(e_p)}(u_r) & ; \forall e_q \in A^c \quad ; \forall u_r \in U_1 \\ min\left\{ t_{\ddot{F}(e_p)}(u_r), t_{\ddot{G}(e_q)}(u_r) \right\} & ; \forall e_p, e_q \in (A \cap B)\, ;\ \forall u_r \in U_1 \end{cases}$$

$$1-f_{\ddot{K}(e_r)}(u_r)=\begin{cases} 1-f_{\ddot{F}(e_p)}(u_r) & ;\forall e_p\in B^c \quad ;\forall u_r\in U_1 \\ 1-f_{\ddot{G}(e_q)}(u_r) & ;\forall e_q\in A^c \quad ;\forall u_r\in U_1 \\ min\left\{1-f_{\ddot{F}(e_p)}(u_r),1-f_{\ddot{G}(e_q)}(u_r)\right\} & ;\forall e_p,e_q\in(A\cap B);\forall u_r\in U_1 \end{cases}$$

$$t_{\ddot{K}(e_g)}(v_s)=\begin{cases} t_{\ddot{F}(e_p)}(v_s) & ;\forall e_p\in B^c \quad ;\forall v_s\in U_2 \\ t_{\ddot{G}(e_q)}(v_s) & ;\forall e_q\in A^c \quad ;\forall v_s\in U_2 \\ min\left\{t_{\ddot{F}(e_p)}(v_s),t_{\ddot{G}(e_q)}(v_s)\right\} & ;\forall e_p,e_q\in(A\cap B)\ ;\forall v_s\in U_2 \end{cases}$$

$$1-f_{\ddot{K}(e_g)}(v_j)=\begin{cases} 1-f_{\ddot{F}(e_p)}(v_s) & ;\forall e_p\in B^c \quad ;\forall v_s\in U_2 \\ 1-f_{\ddot{G}(e_q)}(v_s) & ;\forall e_q\in A^c \quad ;\forall v_s\in U_2 \\ min\left\{1-f_{\ddot{F}(e_p)}(v_s),1-f_{\ddot{G}(e_q)}(v_s)\right\} & ;\forall e_p,e_q\in(A\cap B);\forall v_s\in U_2 \end{cases}$$

2. ***Vague Binary Soft Union***
Consider two VBSS's $\left(\ddot{F},A\right)$ *and* $\left(\ddot{G},B\right)$ *over* (U_1,U_2).
Let output of $(\ddot{F},A)\,\ddot{\cup}\,(\ddot{G},B)$ *is* $(\ddot{H},C)$, *where* $\ddot{H}:C\to V(U_1)\times V(U_2)$
where $C=(A\cup B)$; $\cup$ *represents union operation in cantor set theory.*
Then $\forall e_g\in C$,

$$\ddot{H}(e_g)=\begin{cases} \ddot{F}(e_p) & ;\forall e_p\in B^c; c\ is\ Cantor\ Set\ Complement\ with\ respect\ to\ C \\ \ddot{G}(e_q) & ;\forall e_q\in A^c; c\ is\ Cantor\ Set\ Complement\ with\ respect\ to\ C \\ \ddot{F}(e_p)\cup\ddot{G}(e_q) & ;\forall e_p,e_q\in(A\cap B);\cup\ \&\ \cap\ are\ as\ in\ Cantor\ Set\ Theory \end{cases}$$

$$t_{\ddot{H}(e_g)}(u_r)=\begin{cases} t_{\ddot{F}(e_p)}(u_r) & ;\forall e_p\in B^c \quad ;\forall u_r\in U_1 \\ t_{\ddot{G}(e_q)}(u_r) & ;\forall e_q\in A^c \quad ;\forall u_r\in U_1 \\ max\left\{t_{\ddot{F}(e_p)}(r),t_{\ddot{G}(e_q)}(u_r)\right\} & ;\forall e_p,e_q\in(A\cap B);\ \forall u_r\in U_1 \end{cases}$$

$$1-f_{\ddot{H}(e_g)}(u_r)=\begin{cases} 1-f_{\ddot{F}(e_p)}(u_r) & ;\forall e_p\in B^c \quad ;\forall u_r\in U_1 \\ 1-f_{\ddot{G}(e_q)}(u_r) & ;\forall e_q\in A^c \quad ;\forall u_r\in U_1 \\ max\left\{1-f_{\ddot{F}(e_p)}(u_r),1-f_{\ddot{G}(e_q)}(u_r)\right\} & ;\forall e_p,e_q\in(A\cap B);\forall u_r\in U_1 \end{cases}$$

$$t_{\ddot{H}(e_g)}(v_s)=\begin{cases} t_{\ddot{F}(e_p)}(v_s) & ;\forall e_p\in B^c \quad ;\forall v_s\in U_2 \\ t_{\ddot{G}(e_q)}(v_s) & ;\forall e_q\in A^c \quad ;\forall v_s\in U_2 \\ max\left\{t_{\ddot{F}(e_p)}(v_s),t_{\ddot{G}(e_q)}(v_s)\right\} & ;\forall e_p,e_q\in(A\cap B);\ \forall v_s\in U_2 \end{cases}$$

$$1 - f_{\ddot{H}(e_g)}(v_s) = \begin{cases} 1 - f_{\ddot{F}(e_p)}(v_s) & ; \forall e_p \in B^c & ; \forall v_s \in U_2 \\ 1 - f_{\ddot{G}(e_q)}(v_s) & ; \forall e_q \in A^c & ; \forall v_s \in U_2 \\ max\left\{1 - f_{\ddot{F}(e_p)}(v_s), 1 - f_{\ddot{G}(e_q)}(v_s)\right\} & ; \forall e_p, e_q \in (A \cap B) & ; \forall v_s \in U_2 \end{cases}$$

Example 2.1.9.

Let $U_1 = \{b_1, b_2, b_3, b_4, b_5, b_6\}, U_2 = \{p_1, p_2, p_3, p_4\}$ *be the set of books and pencils respectively*

$$E = \begin{Bmatrix} e_1 = cheap & ; e_2 = expensive & ; e_3 = attractive \\ e_4 = small & ; \quad e_5 = long & ; \quad e_6 = colorful \end{Bmatrix} \text{ be the set of parameters}$$

Let A and B be two parameter sets formed, based on the parent parameter set E. Take, $A = \{e_3, e_4, e_6\}$ *and* $B = \{e_2, e_3, e_5\}$ *with* $A, B \subseteq E$. *Frame two VBSS's namely* $\left(\ddot{F}, A\right)$ *and* $\left(\ddot{G}, B\right)$ *over binary universe* (U_1, U_2) *based on A and B respectively.*

$\left(\ddot{F}, A\right) =$

$$\left\{\begin{array}{l} \left(e_3, \left(\left\langle \frac{[0.1,0.3]}{b_1}, \frac{[0.5,0.8]}{b_2}, \frac{[0.6,0.7]}{b_3}, \frac{[0.2,0.6]}{b_4}, \frac{[0.3,0.5]}{b_5} \right\rangle, \left\langle \frac{[0.4,0.8]}{p_1}, \frac{[0.2,0.9]}{p_2}, \frac{[0.2,0.7]}{p_3} \right\rangle\right)\right) \\ \left(e_4, \left(\left\langle \frac{[0.3,0.4]}{b_1}, \frac{[0.2,0.3]}{b_2}, \frac{[0.2,0.7]}{b_3}, \frac{[0.4,0.8]}{b_4}, \frac{[0.5,0.8]}{b_5} \right\rangle, \left\langle \frac{[0.2,0.7]}{p_1}, \frac{[0.3,0.5]}{p_2}, \frac{[0.4,0.5]}{p_3} \right\rangle\right)\right) \\ \left(e_6, \left(\left\langle \frac{[0.5,0.6]}{b_1}, \frac{[0.2,0.4]}{b_2}, \frac{[0.2,0.7]}{b_3}, \frac{[0.4,0.8]}{b_4}, \frac{[0.5,0.8]}{b_5} \right\rangle, \left\langle \frac{[0.2,0.7]}{p_1}, \frac{[0.3,0.5]}{p_2}, \frac{[0.4,0.5]}{p_3} \right\rangle\right)\right) \end{array}\right\}$$

It is clear that, in this context, $\ddot{F} : A \to V(U_1) \times V(U_2)$

$\left(\ddot{G}, B\right) =$

$$\left\{\begin{array}{l} \left(e_2, \left(\left\langle \frac{[0.4,0.6]}{b_1}, \frac{[0.2,0.5]}{b_2}, \frac{[0.4,0.7]}{b_3}, \frac{[0.2,0.9]}{b_4}, \frac{[0.2,0.3]}{b_5} \right\rangle, \left\langle \frac{[0.2,0.4]}{p_1}, \frac{[0.1,0.3]}{p_2}, \frac{[0.2,0.6]}{p_3} \right\rangle\right)\right) \\ \left(e_3, \left(\left\langle \frac{[0.5,0.6]}{b_1}, \frac{[0.4,0.5]}{b_2}, \frac{[0.6,0.8]}{b_3}, \frac{[0.3,0.6]}{b_4}, \frac{[0.3,0.5]}{b_5} \right\rangle, \left\langle \frac{[0.2,0.9]}{p_1}, \frac{[0.7,0.9]}{p_2}, \frac{[0.5,0.7]}{p_3} \right\rangle\right)\right) \\ \left(e_5, \left(\left\langle \frac{[0.2,0.9]}{b_1}, \frac{[0.2,0.4]}{b_2}, \frac{[0.2,0.5]}{b_3}, \frac{[0.2,0.6]}{b_4}, \frac{[0.5,0.7]}{b_5} \right\rangle, \left\langle \frac{[0.6,0.7]}{p_1}, \frac{[0.7,0.8]}{p_2}, \frac{[0.1,0.9]}{p_3} \right\rangle\right)\right) \end{array}\right\}$$

It is clear that, in this context, $\ddot{G} : B \to V(U_1) \times V(U_2)$

$$C = (A \cup B) = \{e_3, e_4, e_6\} \cup \{e_2, e_3, e_5\} = \{e_2, e_3, e_4, e_5, e_6\}$$

$\left(\ddot{F},A\right)\ddot{\cup}\left(\ddot{G},B\right)=\left(\ddot{K},C\right)=$

$$\left\{\begin{array}{l}\left(e_2,\left(\left\langle\frac{[0.4,0.6]}{b_1},\frac{[0.2,0.5]}{b_2},\frac{[0.4,0.7]}{b_3},\frac{[0.2,0.9]}{b_4},\frac{[0.2,0.3]}{b_5}\right\rangle,\left\langle\frac{[0.2,0.4]}{p_1},\frac{[0.1,0.3]}{p_2},\frac{[0.2,0.6]}{p_3}\right\rangle\right)\right)\\ \left(e_3,\left(\left\langle\frac{[0.5,0.6]}{b_1},\frac{[0.5,0.8]}{b_2},\frac{[0.6,0.8]}{b_3},\frac{[0.3,0.6]}{b_4},\frac{[0.3,0.5]}{b_5}\right\rangle,\left\langle\frac{[0.4,0.9]}{p_1},\frac{[0.7,0.9]}{p_2},\frac{[0.5,0.7]}{p_3}\right\rangle\right)\right)\\ \left(e_4,\left(\left\langle\frac{[0.3,0.4]}{b_1},\frac{[0.2,0.3]}{b_2},\frac{[0.2,0.7]}{b_3},\frac{[0.4,0.8]}{b_4},\frac{[0.5,0.8]}{b_5}\right\rangle,\left\langle\frac{[0.2,0.7]}{p_1},\frac{[0.3,0.5]}{p_2},\frac{[0.4,0.5]}{p_3}\right\rangle\right)\right)\\ \left(e_5,\left(\left\langle\frac{[0.2,0.9]}{b_1},\frac{[0.2,0.4]}{b_2},\frac{[0.2,0.5]}{b_3},\frac{[0.2,0.6]}{b_4},\frac{[0.5,0.7]}{b_5}\right\rangle,\left\langle\frac{[0.6,0.7]}{p_1},\frac{[0.7,0.8]}{p_2},\frac{[0.1,0.9]}{p_3}\right\rangle\right)\right)\\ \left(e_6,\left(\left\langle\frac{[0.5,0.6]}{b_1},\frac{[0.2,0.4]}{b_2},\frac{[0.2,0.7]}{b_3},\frac{[0.4,0.8]}{b_4},\frac{[0.5,0.8]}{b_5}\right\rangle,\left\langle\frac{[0.2,0.7]}{p_1},\frac{[0.3,0.5]}{p_2},\frac{[0.4,0.5]}{p_3}\right\rangle\right)\right)\end{array}\right\}$$

$\left(\ddot{F},A\right)\ddot{\cap}\left(\ddot{G},B\right)=\left(\ddot{H},C\right)=$

$$\left\{\begin{array}{l}\left(e_2,\left(\left\langle\frac{[0.4,0.6]}{b_1},\frac{[0.2,0.5]}{b_2},\frac{[0.4,0.7]}{b_3},\frac{[0.2,0.9]}{b_4},\frac{[0.2,0.3]}{b_5}\right\rangle,\left\langle\frac{[0.2,0.4]}{p_1},\frac{[0.1,0.3]}{p_2},\frac{[0.2,0.6]}{p_3}\right\rangle\right)\right)\\ \left(e_3,\left(\left\langle\frac{[0.1,0.3]}{b_1},\frac{[0.4,0.5]}{b_2},\frac{[0.6,0.7]}{b_3},\frac{[0.2,0.6]}{b_4},\frac{[0.3,0.5]}{b_5}\right\rangle,\left\langle\frac{[0.2,0.8]}{p_1},\frac{[0.2,0.9]}{p_2},\frac{[0.2,0.7]}{p_3}\right\rangle\right)\right)\\ \left(e_4,\left(\left\langle\frac{[0.3,0.4]}{b_1},\frac{[0.2,0.3]}{b_2},\frac{[0.2,0.7]}{b_3},\frac{[0.4,0.8]}{b_4},\frac{[0.5,0.8]}{b_5}\right\rangle,\left\langle\frac{[0.2,0.7]}{p_1},\frac{[0.3,0.5]}{p_2},\frac{[0.4,0.5]}{p_3}\right\rangle\right)\right)\\ \left(e_5,\left(\left\langle\frac{[0.2,0.9]}{b_1},\frac{[0.2,0.4]}{b_2},\frac{[0.2,0.5]}{b_3},\frac{[0.2,0.6]}{b_4},\frac{[0.5,0.7]}{b_5}\right\rangle,\left\langle\frac{[0.6,0.7]}{p_1},\frac{[0.7,0.8]}{p_2},\frac{[0.1,0.9]}{p_3}\right\rangle\right)\right)\\ \left(e_6,\left(\left\langle\frac{[0.5,0.6]}{b_1},\frac{[0.2,0.4]}{b_2},\frac{[0.2,0.7]}{b_3},\frac{[0.4,0.8]}{b_4},\frac{[0.5,0.8]}{b_5}\right\rangle,\left\langle\frac{[0.2,0.7]}{p_1},\frac{[0.3,0.5]}{p_2},\frac{[0.4,0.5]}{p_3}\right\rangle\right)\right)\end{array}\right\}$$

Definition 2.1.10. ***(Vague Binary Soft Subset)***

Let (U_1,U_2) be a binary universe.E be a fixed set of parameters

Take $U_1=\{u_1,u_2,--,u_r,--,u_i\}\,,U_2=\{v_1,v_2,--,v_s,--,v_j\}$ &

$E=\{e_1,e_2,--,e_t,--,e_k\}$ with $A,B\subseteq E$.

Also let $A=\{e_1,e_2,--,e_p,--,e_m\}\,,B=\{e_1,e_2,--,e_q,--,e_n\}$.

$\cup$ represents union operation in cantor set theory.

A Vague Binary Soft Set $\left(\ddot{F},A\right)$ is contained in another Vague Binary Soft Set $(\ddot{G},B)$ denoted as $\left(\ddot{F},A\right)\ddot{\subseteq}\left(\ddot{G},B\right)$ if

(1) $A\subseteq B$; $\subseteq$ denotes Subset Operation in Cantor Set Theory

(2) $\forall\quad e_p\in A,\exists$ some $e_q\in B$ such that $\ddot{F}(e_p)\subseteq\ddot{G}(e_q)$; $\subseteq$ denotes Subset Operation in Cantor Set Theory.

In this case,

$\left(\ddot{F},A\right)$ is called Vague Binary Soft SubSet of $\left(\ddot{G},B\right)$ denoted as $\left(\ddot{F},A\right)\ddot{\subseteq}\left(\ddot{G},B\right)$

&

$\left(\ddot{G},B\right)$ is called Vague Binary Soft Superset of $\left(\ddot{F},A\right)$ denoted as $\left(\ddot{G},B\right)\ddot{\supseteq}\left(\ddot{F},A\right)$

Remark 2.1.11.

Vague Binary Soft SubSet is written in short as VBSSS

Example 2.1.12.

Let $U_1 = \{b_1, b_2, b_3\}$; $U_2 = \{p_1, p_2, p_3\}$;
$E = \{e_1, e_2, e_3, e_4\}$; $A = \{e_3, e_4\} \subseteq E$ *and* $B = \{e_2, e_3, e_4\} \subseteq E$
Let $\left(\ddot{F}, A\right)$ *and* $\left(\ddot{G}, B\right)$ *be two VBSS's over binary universe* (U_1, U_2).
$\left(\ddot{F}, A\right) =$

$$\left\{\begin{array}{l} \left(e_3, \left(\left\langle \frac{[0.2,0.3]}{b_1}, \frac{[0.4,0.6]}{b_2}, \frac{[0.2,0.8]}{b_3}\right\rangle, \left\langle \frac{[0.5,0.6]}{p_1}, \frac{[0.2,0.6]}{p_2}, \frac{[0.8,0.9]}{p_3}\right\rangle\right)\right) \\ \left(e_4, \left(\left\langle \frac{[0.3,0.4]}{b_1}, \frac{[0.5,0.7]}{b_2}, \frac{[0.5,0.6]}{b_3}\right\rangle, \left\langle \frac{[0.7,0.8]}{p_1}, \frac{[0.4,0.7]}{p_2}, \frac{[0.2,0.8]}{p_3}\right\rangle\right)\right) \end{array}\right\}$$

$\left(\ddot{G}, B\right) =$

$$\left\{\begin{array}{l} \left(e_2, \left(\left\langle \frac{[0.4,0.5]}{b_1}, \frac{[0.5,0.6]}{b_2}, \frac{[0.3,0.5]}{b_3}\right\rangle, \left\langle \frac{[0.6,0.6]}{p_1}, \frac{[0.3,0.6]}{p_2}, \frac{[0.7,0.8]}{p_3}\right\rangle\right)\right) \\ \left(e_3, \left(\left\langle \frac{[0.2,0.3]}{b_1}, \frac{[0.4,0.6]}{b_2}, \frac{[0.2,0.8]}{b_3}\right\rangle, \left\langle \frac{[0.5,0.6]}{p_1}, \frac{[0.2,0.6]}{p_2}, \frac{[0.8,0.9]}{p_3}\right\rangle\right)\right) \\ \left(e_4, \left(\left\langle \frac{[0.3,0.4]}{b_1}, \frac{[0.5,0.7]}{b_2}, \frac{[0.5,0.6]}{b_3}\right\rangle, \left\langle \frac{[0.7,0.8]}{p_1}, \frac{[0.4,0.7]}{p_2}, \frac{[0.2,0.8]}{p_3}\right\rangle\right)\right) \end{array}\right\}$$

In this case, $\left(\ddot{F}, A\right) \ddot{\subseteq} (\ddot{G}, B)$.
i.e., $\left(\ddot{F}, A\right)$ *is a vague binary soft subset of* $\left(\ddot{G}, B\right)$.
As a consequence $\left(\ddot{G}, B\right) \ddot{\supseteq} \left(\ddot{F}, A\right)$.
i.e., $\left(\ddot{G}, B\right)$ *is a vague binary soft superset of* $(\ddot{F}, A)$

Definition 2.1.13. ***(Vague Binary Soft Equal Operation)***
Let $\left(\ddot{F}, A\right)$ *and* $\left(\ddot{G}, B\right)$ *are two VBSS's.*
These sets are said to be vague binary soft equal if,

$$\left(\ddot{F}, A\right) \text{ is a vague binary soft subset of } (\ddot{G}, B)$$
$$\text{and}$$
$$\left(\ddot{G}, B\right) \text{ is a vague binary soft superset of } \left(\ddot{F}, A\right).$$

It is denoted by $\left(\ddot{F}, A\right) \ddot{=} \left(\ddot{G}, B\right)$

$$\text{i.e., } \left(\ddot{F}, A\right) \ddot{\supseteq} \left(\ddot{G}, B\right) \text{ and } \left(\ddot{G}, B\right) \ddot{\supseteq} \left(\ddot{F}, A\right) \Rightarrow \left(\ddot{F}, A\right) \ddot{=} \left(\ddot{G}, B\right)$$

Remark 2.1.14.
*$\left(\ddot{F}, A\right) \ddot{\supseteq} \left(\ddot{G}, B\right)$ and $\left(\ddot{G}, B\right) \ddot{\supseteq} \left(\ddot{F}, A\right) \Leftrightarrow \left(\ddot{F}, A\right) \ddot{=} \left(\ddot{G}, B\right)$
will be true only if $A = B$*

Definition 2.1.15. ***(Vague Binary Soft Cartesian Product Operation)***
*Let (U_1, U_2) be a binary universe with
$U_1 = \{u_1, u_2, --, u_r, --, u_i\}, U_2 = \{v_1, v_2, --, v_s, --, v_j\}$.
$E = \{e_1, e_2, --, e_t, --, e_k\}$ be a fixed parameter set where
$A = \{e_1, e_2, --, e_p, --, e_m\}, B = \{e_1, e_2, --, e_q, --, e_n\}$ with $A, B \subseteq E$.
Vague Binary Soft Cartesian Product of $\left(\ddot{F}, A\right)$ and $\left(\ddot{G}, B\right)$ denoted as
$\left(\ddot{F}, A\right) \ddot{\times} \left(\ddot{G}, B\right)$ is a mapping $\ddot{M} : A \times B \to V(U_1 \times U_1) \times V(U_2 \times U_2)$
where*

$$P = (A \times B) = \{(e_1, e_1), (e_1, e_2), --, (e_p, e_q), --, (e_m, e_n)\}$$
$$(U_1 \times U_1) = \{(u_1, u_1), (u_1, u_2), --, (u_r, u_r), --, (u_i, u_i)\}$$
and
$$(U_2 \times U_2) = \{(v_1, v_1), (v_1, v_2), --, (v_s, v_s), --, (v_j, v_j)\}$$

Let the output is denoted as $\left(\ddot{M}, P\right)$.

$\forall \quad (e_p, e_q) \in P, \quad \forall \quad (u_r, u_r) \in V(U_1 \times U_1)$ *and* $\quad \forall \quad (v_s, v_s) \in V(U_2 \times U_2)$,
$\ddot{M}_{(e_p, e_q)}((u_r, u_r)(v_s, v_s)) = \ddot{F}_{(e_p, e_q)}(u_r, u_r) \times \ddot{G}_{(e_q, e_q)}(v_s, v_s)$

$$\ddot{M}_{(e_p, e_q)}((u_r, u_r)(v_s, v_s)) =$$
$$\left\{\left(\begin{array}{c} \left\langle \dfrac{\left[min\left(t_{\ddot{F}(e_p)}(u_r), t_{\ddot{G}(e_q)}(u_r)\right), max\left(1 - t_{\ddot{F}(e_p)}(u_r), 1 - t_{\ddot{G}(e_q)}(u_r)\right)\right]}{(u_r, u_r)} \right\rangle, \\ \left\langle \dfrac{\left[min\left(t_{\ddot{F}(e_p)}(v_s), t_{\ddot{G}(e_q)}(v_s)\right), max\left(1 - t_{\ddot{F}(e_p)}(v_s), 1 - t_{\ddot{G}(e_q)}(v_s)\right)\right]}{(v_s, v_s)} \right\rangle \end{array}\right)\right\}$$
$$\forall (e_p, e_q) \in P \quad ; \quad \forall (u_r, u_r) \in (U_1 \times U_1) \quad ; \quad \forall (v_s, v_s) \in (U_2 \times U_2)$$

Example 2.1.16.
*$U_1 = \{t_1, t_2\}$ be set T - shirts; $U_2 = \{k_1, k_2\}$ be set of kurthas.
$E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ be set of parameters. $A = \{e_1, e_2\} \subseteq E; B = \{e_3, e_4\} \subseteq E$;
Let $\left(\ddot{F}, A\right)$ and $\left(\ddot{G}, B\right)$ be two VBSS's over (U_1, U_2)*

$$\left(\ddot{F}, A\right) = \left\{\begin{array}{c} \left(e_1, \left(\left\langle \dfrac{[0.2, 0.3]}{t_1}, \dfrac{[0.4, 0.6]}{t_2} \right\rangle, \left\langle \dfrac{[0.5, 0.6]}{k_1}, \dfrac{[0.2, 0.6]}{k_2} \right\rangle\right)\right) \\ \left(e_2, \left(\left\langle \dfrac{[0.3, 0.4]}{t_1}, \dfrac{[0.5, 0.7]}{t_2} \right\rangle, \left\langle \dfrac{[0.7, 0.8]}{k_1}, \dfrac{[0.4, 0.7]}{k_2} \right\rangle\right)\right) \end{array}\right\}$$

$$\left(\ddot{G},B\right)=\left\{\begin{array}{l}\left(e_3,\left(\left\langle\frac{[0.4,0.5]}{t_1},\frac{[0.8,0.9]}{t_2}\right\rangle,\left\langle\frac{[0.4,0.6]}{k_1},\frac{[0.3,0.6]}{k_2}\right\rangle\right)\right)\\\left(e_4,\left(\left\langle\frac{[0.5,0.7]}{t_1},\frac{[0.4,0.8]}{t_2}\right\rangle,\left\langle\frac{[0.6,0.8]}{k_1},\frac{[0.3,0.7]}{k_2}\right\rangle\right)\right)\end{array}\right\}$$

$(U_1\times U_1)=\{t_1,t_2\}\times\{t_1,t_2\}=\{(t_1,t_1),(t_1,t_2),(t_2,t_1),(t_2,t_2)\}$

$(U_2\times U_2)=\{k_1,k_2\}\times\{k_1,k_2\}=\{(k_1,k_1),(k_1,k_2),(k_2,k_1),(k_2,k_2)\}$

$P=(A\times B)=\{(e_1,e_3),(e_1,e_4),(e_2,e_3),(e_2,e_4)\}$

$$\left(\ddot{F},A\right)\ddot{\times}\left(\ddot{G},B\right)=\left(\ddot{M},A\times B\right)$$

$$=\left(\ddot{M},P\right)=\left\{\begin{array}{ll}\left((e_1,e_3),\ddot{M}(e_1,e_3)\right), & \left((e_1,e_4),\ddot{M}(e_1,e_4)\right),\\\left((e_2,e_3),\ddot{M}(e_2,e_3)\right), & \left((e_2,e_4),\ddot{M}(e_2,e_4)\right)\end{array}\right\}$$

$$=\left\{\begin{array}{l}\left((e_1,e_3),\left(\left\langle\frac{[0.2,0.5]}{(t_1,t_1)},\frac{[0.2,0.9]}{(t_1,t_2)},\frac{[0.4,0.6]}{(t_2,t_1)},\frac{[0.4,0.9]}{(t_2,t_2)}\right\rangle,\left\langle\frac{[0.4,0.6]}{(k_1,k_1)},\frac{[0.3,0.6]}{(k_1,k_2)},\frac{[0.2,0.6]}{(k_2,k_1)},\frac{[0.2,0.6]}{(k_2,k_2)}\right\rangle\right)\right)\\\left((e_1,e_4),\left(\left\langle\frac{[0.2,0.7]}{(t_1,t_1)},\frac{[0.2,0.8]}{(t_1,t_2)},\frac{[0.4,0.7]}{(t_2,t_1)},\frac{[0.4,0.8]}{(t_2,t_2)}\right\rangle,\left\langle\frac{[0.5,0.8]}{(k_1,k_1)},\frac{[0.3,0.7]}{(k_1,k_2)},\frac{[0.2,0.8]}{(k_2,k_1)},\frac{[0.2,0.7]}{(k_2,k_2)}\right\rangle\right)\right)\\\left((e_2,e_3),\left(\left\langle\frac{[0.3,0.5]}{(t_1,t_1)},\frac{[0.3,0.9]}{(t_1,t_2)},\frac{[0.4,0.7]}{(t_2,t_1)},\frac{[0.5,0.9]}{(t_2,t_2)}\right\rangle,\left\langle\frac{[0.4,0.8]}{(k_1,k_1)},\frac{[0.3,0.8]}{(k_1,k_2)},\frac{[0.4,0.7]}{(k_2,k_1)},\frac{[0.3,0.7]}{(k_2,k_2)}\right\rangle\right)\right)\\\left((e_2,e_4),\left(\left\langle\frac{[0.3,0.7]}{(t_1,t_1)},\frac{[0.3,0.8]}{(t_1,t_2)},\frac{[0.5,0.7]}{(t_2,t_1)},\frac{[0.4,0.8]}{(t_2,t_2)}\right\rangle,\left\langle\frac{[0.6,0.8]}{(k_1,k_1)},\frac{[0.3,0.8]}{(k_1,k_2)},\frac{[0.4,0.8]}{(k_2,k_1)},\frac{[0.3,0.7]}{(k_2,k_2)}\right\rangle\right)\right)\end{array}\right\}$$

Definition 2.1.17.

(AND Operation & OR operation in Vague Binary Soft Sets)

Let (U_1,U_2) be a binary universe with

$U_1=\{u_1,u_2,--,u_r,--,u_i\},U_2=\{v_1,v_2,--,v_s,--,v_j\}$.

$E=\{e_1,e_2,--,e_t,--,e_k\}$ *be a fixed parameter set where*

$A=\{e_1,e_2,--,e_p,--,e_m\},B=\{e_1,e_2,--,e_q,--,e_n\}$ *with* $A,B\subseteq E$.

Let $P=(A\times B)=\{(e_1,e_1),--,(e_p,e_q),--,(e_m,e_n)\}$,

be the Cantor Set Cartesian Product of parameter sets A and B.

1. ***AND operation*** *between* $\left(\ddot{F},A\right)$ *and* $\left(\ddot{G},B\right)$ *is denoted as* $\left(\ddot{F},A\right)\ddot{\wedge}\left(\ddot{G},B\right)$. *Let the output be denoted as* $\left(\ddot{M},P\right)$, *where* $\ddot{M}:P=(A\times B)\rightarrow V(U_1)\times V(U_2)$

$$\ddot{M}_{(e_p,e_q)}(u_r,v_s)=\ddot{F}_{(e_p)}(u_r)\cap\ddot{G}_{(e_q)}(v_s);$$

$$\forall\quad(e_p,e_q)\in P,\quad\forall\quad(u_r,v_s)\in V(U_1)\times V(U_2)$$

Here $\cap$ denotes Cantor Set Intersection

2. ***OR operation*** *between* $\left(\ddot{F},A\right)$ *and* $\left(\ddot{G},B\right)$ *is denoted as* $\left(\ddot{F},A\right)\ddot{\vee}\left(\ddot{G},B\right)$. *Let the output be denoted as* $\left(\ddot{N},P\right)$, *where* $\ddot{N}:P=(A\times B)\rightarrow V(U_1)\times V(U_2)$

$$\begin{aligned}\ddot{N}_{(e_p,e_q)}(u_r,v_s) &= \ddot{F}_{(e_p)}(u_r)\cup\ddot{G}_{(e_q)}(v_s);\\ &\forall\quad(e_p,e_q)\in P,\quad\forall\quad(u_r,v_s)\in V(U_1)\times V(U_2)\\ &Here\ \cup\ denotes\ Cantor\ Set\ union\end{aligned}$$

Example 2.1.18.

Let $U_1=\{b_1,b_2,b_3,b_4,b_5,b_6\}$ *be the set of buses;* $U_2=\{r_1,r_2,r_3\}$ *be the set of routes;*

$$E=\left\{\begin{array}{l}e_1=rural\quad;e_2=urban;\quad e_3=inter-state\\ e_4=air-bus;\quad e_5=luxury;\quad e_6=sleeper\end{array}\right\}\text{ be the set of parameters}$$

Let $A=\{e_2,e_4\}\subseteq E$ *and* $B=\{e_3\}\subseteq E$ *; Also let* $(A\times B)$ *be* P *;*
$\therefore P=A\times B=\{(e_2,e_3),(e_4,e_3)\}$
Two VBSS's formed for the above context is as follows:

$$(\ddot{F},A)=\left\{\begin{array}{l}\left(e_2,\left(\left\langle\frac{[0.1,0.3]}{b_1},\frac{[0.5,0.8]}{b_2},\frac{[0.6,0.7]}{b_3},\frac{[0.2,0.6]}{b_4},\frac{[0.3,0.5]}{b_5}\right\rangle,\left\langle\frac{[0.4,0.8]}{r_1},\frac{[0.2,0.9]}{r_2},\frac{[0.2,0.7]}{r_3}\right\rangle\right)\right)\\ \left(e_4,\left(\left\langle\frac{[0.3,0.4]}{b_1},\frac{[0.2,0.3]}{b_2},\frac{[0.2,0.7]}{b_3},\frac{[0.4,0.8]}{b_4},\frac{[0.5,0.8]}{b_5}\right\rangle,\left\langle\frac{[0.2,0.7]}{r_1},\frac{[0.3,0.5]}{r_2},\frac{[0.4,0.5]}{r_3}\right\rangle\right)\right)\end{array}\right\}$$

$$(\ddot{G},B)=\left\{\left(e_3,\left(\left\langle\frac{[0.5,0.6]}{b_1},\frac{[0.4,0.5]}{b_2},\frac{[0.6,0.8]}{b_3},\frac{[0.3,0.6]}{b_4},\frac{[0.3,0.5]}{b_5}\right\rangle,\left\langle\frac{[0.2,0.9]}{r_1},\frac{[0.7,0.9]}{r_2},\frac{[0.5,0.7]}{r_3}\right\rangle\right)\right)\right\}$$

AND operation between $\left(\ddot{F},A\right)$ *and* $\left(\ddot{G},B\right)$ *is given as* $\left(\ddot{M},P\right)$
where $\ddot{M}:P=(A\times B)\rightarrow V(U_1)\times V(U_2)$
$(\ddot{M},P)=$

$$\left\{\begin{array}{l}\left((e_2,e_3),\left\langle\frac{[0.1,0.3]}{b_1},\frac{[0.4,0.5]}{b_2},\frac{[0.6,0.7]}{b_3},\frac{[0.2,0.6]}{b_4},\frac{[0.3,0.5]}{b_5}\right\rangle\left\langle\frac{[0.2,0.8]}{r_1},\frac{[0.2,0.9]}{r_2},\frac{[0.2,0.7]}{r_3}\right\rangle\right)\\ \left((e_4,e_3),\left\langle\frac{[0.3,0.4]}{b_1},\frac{[0.2,0.3]}{b_2},\frac{[0.2,0.7]}{b_3},\frac{[0.3,0.6]}{b_4},\frac{[0.3,0.5]}{b_5}\right\rangle\left\langle\frac{[0.2,0.7]}{r_1},\frac{[0.3,0.5]}{r_2},\frac{[0.4,0.5]}{r_3}\right\rangle\right)\end{array}\right\}$$

OR operation between $\left(\ddot{F},A\right)$ *and* $\left(\ddot{G},B\right)$ *is given as* $\left(\ddot{N},P\right)$
where $\ddot{N}:P=(A\times B)\rightarrow V(U_1)\times V(U_2)$
$(\ddot{N},P)=$

$$\left\{\begin{array}{l}\left((e_2,e_3),\left\langle\frac{[0.5,0.6]}{b_1},\frac{[0.5,0.8]}{b_2},\frac{[0.6,0.8]}{b_3},\frac{[0.3,0.6]}{b_4},\frac{[0.3,0.5]}{b_5}\right\rangle\left\langle\frac{[0.4,0.9]}{r_1},\frac{[0.7,0.9]}{r_2},\frac{[0.5,0.7]}{r_3}\right\rangle\right)\\ \left((e_4,e_3),\left\langle\frac{[0.5,0.6]}{b_1},\frac{[0.4,0.5]}{b_2},\frac{[0.6,0.8]}{b_3},\frac{[0.4,0.8]}{b_4},\frac{[0.5,0.8]}{b_5}\right\rangle\left\langle\frac{[0.2,0.9]}{r_1},\frac{[0.7,0.9]}{r_2},\frac{[0.5,0.7]}{r_3}\right\rangle\right)\end{array}\right\}$$

2.2 Algebraic Properties for VBSS Operations

Previous section discussed about VBSS's operations and concluded that they are not same as Cantor Set operations. A detailed discussion on the algebraic properties for Vague Binary Soft Set Operations are presented in this section.

Theorem 2.2.1.
Let (U_1, U_2) be a binary universe. E be a fixed set of parameters
Take $U_1 = \{u_1, u_2, --, u_r, --, u_i\}, U_2 = \{v_1, v_2, --, v_s, --, v_j\}$ &
$E = \{e_1, e_2, --, e_t, --, e_k\}$ *with* $A \subseteq E$, *where* $A = \{e_1, e_2, --, e_p, --, e_m\}$,
$\cup$ *represents cantor set union operation.*

1. **(Identity Laws)**
For any VBSS $\left(\ddot{F}, A\right)$ defined on the absolute VBSS $\left(\ddot{U}, A\right)$

   (a) $\left(\ddot{F}, A\right) \ddot{\cup} \left(\ddot{\Phi}, A\right) \ddot{=} \left(\ddot{F}, A\right)$

   (b) $\left(\ddot{F}, A\right) \ddot{\cap} \left(\ddot{U}, A\right) \ddot{=} \left(\ddot{F}, A\right)$

2. **(Domination Laws)**
For any VBSS $\left(\ddot{F}, A\right)$ defined on the absolute VBSS $\left(\ddot{U}, A\right)$

   (a) $\left(\ddot{F}, A\right) \ddot{\cap} \left(\ddot{\Phi}, A\right) = \left(\ddot{\Phi}, A\right)$

   (b) $\left(\ddot{F}, A\right) \ddot{\cup} \left(\ddot{U}, A\right) = \left(\ddot{U}, A\right)$

3. **(Complemental Laws)**
Null - VBSS $\left(\ddot{\Phi}, A\right)$ and Absolute - VBSS $\left(\ddot{U}, A\right)$ following are true:

   (a) $\left(\ddot{\Phi}, A\right)^c = \left(\ddot{U}, A\right)$

   (b) $\left(\ddot{U}, A\right)^c = (\Phi, A)$

Proof.
1. (a) Let $\left(\ddot{F}, A\right)$

$$= \left\{ \left(\begin{array}{l} \left\langle \dfrac{V_{\ddot{F}(e_p)}(u_r)}{u_r}; \quad \forall e_p \in A, \quad \forall u_r \in U_1 \right\rangle, \\ \left\langle \dfrac{V_{\ddot{F}(e_p)}(v_s)}{v_s}; \quad \forall e_p \in A, \quad \forall v_s \in U_2 \right\rangle \end{array} \right) \right\}$$

$$= \left\{ \left(\begin{array}{l} \left\langle \frac{\left[t_{\ddot{F}(e_p)}(u_r), 1 - f_{\ddot{F}(e_p)}(u_r)\right]}{u_r}; \forall e_p \in A, \forall u_r \in U_1 \right\rangle, \\ \left\langle \frac{\left[t_{\ddot{F}(e_p)}(v_s), 1 - f_{\ddot{F}(e_p)}(v_s)\right]}{v_s}; \forall e_p \in A, \forall v_s \in U_2 \right\rangle \end{array} \right) \right\}$$

Also, $\left(\ddot{\Phi}, A\right) = \left\{ \left(\left\langle \frac{[0,0]}{u_r}; \quad \forall e_p \in A, \quad \forall u_r \in U_1 \right\rangle, \left\langle \frac{[0,0]}{v_s}; \forall e_p \in A, \quad \forall v_s \in U_2 \right\rangle \right) \right\}$

$\therefore \left(\ddot{F}, A\right) \ddot{\cup} \left(\ddot{\Phi}, A\right)$

$$= \left\{ \left(\begin{array}{l} \left\langle \frac{max\left(\left[t_{\ddot{F}(e_p)}(u_r), [0,0]\right], max\left[1 - f_{\ddot{F}(e_p)}(u_r), [0,0]\right]\right)}{u_r}; \forall u_r \in U_1 \right\rangle, \\ \left\langle \frac{max\left(\left[t_{\ddot{F}(e_p)}(v_s), [0,0]\right], max\left[1 - f_{\ddot{F}(e_p)}(v_s), [0,0]\right]\right)}{v_s}; \forall v_s \in U_2 \right\rangle \end{array} \right) \right\}$$

$$= \left\{ \left(\begin{array}{l} \left\langle \frac{\left[t_{\ddot{F}(e_p)}(u_r), 1 - t_{\ddot{F}(e_p)}(u_r)\right]}{u_r}; \forall e_p \in A, \forall u_r \in U_1 \right\rangle, \\ \left\langle \frac{\left[t_{\ddot{F}(e_p)}(v_s), 1 - t_{\ddot{F}(e_p)}(v_s)\right]}{v_s}; \forall v_s \in U_2 \right\rangle \end{array} \right) \right\} = \left(\ddot{F}, A\right)$$

(b) Proof is similar to 1(a)

2. (a) Let $\left(\ddot{F}, A\right)$

$$= \left\{ \left(\begin{array}{l} \left\langle \frac{\left[t_{\ddot{F}(e_p)}(u_r), 1 - f_{\ddot{F}(e_p)}(u_r)\right]}{u_r}; \quad \forall e_p \in A, \quad \forall u_r \in U_1 \right\rangle, \\ \left\langle \frac{\left[t_{\ddot{F}(e_p)}(v_s), 1 - f_{\ddot{F}(e_p)}(v_s)\right]}{v_s}; \quad \forall e_p \in A, \quad \forall v_s \in U_2 \right\rangle \end{array} \right) \right\}$$

$\left(\ddot{F}, A\right) \ddot{\cap} \left(\ddot{F}, A\right)$

$$= \left\{ \left(\begin{array}{l} \left\langle \frac{\left[t_{\ddot{F}(e_p)}(u_r), 1 - f_{\ddot{F}(e_p)}(u_r)\right]}{u_r}; \quad \forall e_p \in A, \quad \forall u_r \in U_1 \right\rangle, \\ \left\langle \frac{\left[t_{\ddot{F}(e_p)}(v_s), 1 - f_{\ddot{F}(e_p)}(v_s)\right]}{v_s}; \quad \forall e_p \in A, \quad \forall v_s \in U_2 \right\rangle \end{array} \right) \right\}$$

$$\ddot{\cap}$$

$$\left\{\left(\begin{array}{l}\left\langle\dfrac{\left[t_{\ddot{F}(e_p)}(u_r), 1-f_{\ddot{F}(e_p)}(u_r)\right]}{u_r}; \quad \forall e_p \in A, \quad \forall u_r \in U_1\right\rangle, \\ \left\langle\dfrac{\left[t_{\ddot{F}(e_p)}(v_s), 1-f_{\ddot{F}(e_p)}(v_s)\right]}{v_s}; \quad \forall e_p \in A, \quad \forall v_s \in U_2\right\rangle\end{array}\right)\right\}$$

$$=\left\{\left(\begin{array}{l}\left\langle\dfrac{min\left(t_{\ddot{F}(e_p)}(u_r), t_{\ddot{F}(e_p)}(u_r)\right), min\left(1-f_{\ddot{F}(e_p)}(u_r), 1-f_{\ddot{F}(e_p)}(u_r)\right)}{u_r}\right\rangle, \\ \left\langle\dfrac{min\left(t_{\ddot{F}(e_p)}(v_s), t_{\ddot{F}(e_p)}(v_s)\right), min\left(1-f_{\ddot{F}(e_p)}(v_s), 1-f_{\ddot{F}(e_p)}(v_s)\right)}{v_s}\right\rangle\end{array}\right)\right\}$$

$$\forall e_p \in A, \qquad \forall u_r \in U_1 \qquad \forall v_s \in U_2$$

$$=\left\{\left(\begin{array}{l}\left\langle\dfrac{\left[t_{\ddot{F}(e_p)}(u_r), 1-f_{\ddot{F}(e_p)}(u_r)\right]}{u_r}; \quad \forall e_p \in A, \quad \forall u_r \in U_1\right\rangle, \\ \left\langle\dfrac{\left[t_{\ddot{F}(e_p)}(v_s), 1-f_{\ddot{F}(e_p)}(v_s)\right]}{v_s}; \quad \forall e_p \in A, \quad \forall v_s \in U_2\right\rangle\end{array}\right)\right\}$$

$$=\left(\ddot{F}, A\right)$$

(b) Proof is similar to 2(a)

3. (a) Let $\left(\ddot{F}, A\right)$ and $\left(\ddot{\Phi}, A\right)$ are as defined as below:
Also let $\left(\ddot{F}, A\right) \ddot{=} \left(\ddot{\Phi}, A\right)$. Then $\forall e_p \in A$, $\ddot{F}(e_p) = \ddot{\Phi}(e_p)$

$$\Rightarrow \left\{\left(\left\langle\frac{V_{\ddot{F}(e_p)}(u_r)}{u_r}; \forall u_r \in U_1\right\rangle, \left\langle\frac{V_{\ddot{F}(e_p)}(v_s)}{v_s}; \forall v_s \in U_2\right\rangle\right)\right\}$$

$$=\left\{\left(\left\langle\frac{[0,0]}{u_r}; \forall u_r \in U_1\right\rangle, \left\langle\frac{[0,0]}{v_s}; \forall v_s \in U_2\right\rangle\right)\right\}$$

$$\Rightarrow \ddot{F}(e_p)^c = \Phi(\ddot{e}_p)^c$$

$$\Rightarrow \ddot{F}(e_p)^c = \left\{\left(\left\langle\frac{[0,0]}{u_r}; \forall u_r \in U_1\right\rangle, \left\langle\frac{[0,0]}{v_s}; \forall v_s \in U_2\right\rangle\right)\right\}^c$$

$$=\left\{\left(\left\langle\frac{[1,1]}{u_r}; \forall u_r \in U_1\right\rangle, \left\langle\frac{[1,1]}{v_s}; \forall v_s \in U_2\right\rangle\right)\right\}^c = \ddot{U}(e_p)$$

$\therefore, \forall e_p \in A, \quad \left(\ddot{\Phi}, A\right)^c \ddot{=} \left(\ddot{U}, A\right)$

(b) Proof is similar to 3(a)

Theorem 2.2.2.

Let (U_1, U_2) be a binary universe. E be a fixed set of parameters
Take $U_1 = \{u_1, u_2, --, u_r, --, u_i\}, U_2 = \{v_1, v_2, --, v_s, --, v_j\}$ &
$E = \{e_1, e_2, --, e_t, --, e_k\}$ with $A, B \subseteq E$.
Also let $A = \{e_1, e_2, --, e_p, --, e_m\}, B = \{e_1, e_2, --, e_q, --, e_n\}$.
$\cup$ represents union operation in cantor set theory.

(a) ***(Commutative Laws)***
For any VBSS $\left(\ddot{F}, A\right)$ and $\left(\ddot{G}, B\right)$ over Absolute - VBSS $(\ddot{U}, E)$
   i. $\left(\ddot{F}, A\right) \ddot{\cap} \left(\ddot{G}, B\right) \ddot{=} \left(\ddot{G}, B\right) \ddot{\cap} \left(\ddot{F}, A\right)$
   ii. $\left(\ddot{F}, A\right) \ddot{\cup} \left(\ddot{G}, B\right) \ddot{=} \left(\ddot{G}, B\right) \ddot{\cup} \left(\ddot{F}, A\right)$

(b) ***(Associative Laws)***
For any VBSS $\left(\ddot{F}, A\right)$, $\left(\ddot{G}, B\right)$ and $\left(\ddot{H}, C\right)$ over Absolute - VBSS $(\ddot{U}, E)$
   i. $\left(\left(\ddot{F}, A\right) \ddot{\cap} \left(\ddot{G}, B\right)\right) \ddot{\cap} \left(\ddot{H}, C\right) \ddot{=} \left(\ddot{F}, A\right) \ddot{\cap} \left(\left(\ddot{G}, B\right) \ddot{\cap} \left(\ddot{H}, C\right)\right)$
   ii. $\left(\left(\ddot{F}, A\right) \ddot{\cup} \left(\ddot{G}, B\right)\right) \ddot{\cup} \left(\ddot{H}, C\right) \ddot{=} \left(\ddot{F}, A\right) \ddot{\cup} \left(\left(\ddot{G}, B\right) \ddot{\cup} \left(\ddot{H}, C\right)\right)$

(c) ***(Distributive Laws)***
For any VBSS $\left(\ddot{F}, A\right)$, $\left(\ddot{G}, B\right)$ and $\left(\ddot{H}, C\right)$ over Absolute - VBSS $(\ddot{U}, E)$
   i. $\left(\ddot{F}, A\right) \ddot{\cap} \left(\left(\ddot{G}, B\right) \ddot{\cup} \left(\ddot{H}, C\right)\right) \ddot{=} \left(\ddot{F}, A\right) \ddot{\cap} \left(\ddot{G}, B\right) \ddot{\cup} \left(\ddot{F}, A\right) \ddot{\cap} \left(\ddot{H}, C\right)$
   ii. $\left(\ddot{F}, A\right) \ddot{\cup} \left(\left(\ddot{G}, B\right) \ddot{\cap} \left(\ddot{H}, C\right)\right) \ddot{=} \left(\ddot{F}, A\right) \ddot{\cup} \left(\ddot{G}, B\right) \ddot{\cap} \left(\ddot{F}, A\right) \ddot{\cup} \left(\ddot{H}, C\right)$

Proof. Steps are Obvious

Remark 2.2.3.

Set of all VBSS's over the Binary Universe (U_1, U_2) with a fixed parameter set E is denoted as $VBSS\,(U_1, U_2)_E$

Theorem 2.2.4.

Let $\left(\ddot{F}, A\right)$ and $\left(\ddot{G}, A\right)$ be two VBSS's in a VBSS $(U_1, U_2)_A$.
Then following are true.

(a) $\left(\ddot{F}, A\right) \ddot{\subseteq} \left(\ddot{G}, A\right) \Leftrightarrow \left(\ddot{F}, A\right) \ddot{\cap} \left(\ddot{G}, A\right) \ddot{=} \left(\ddot{F}, A\right)$

(b) $\left(\ddot{F}, A\right) \ddot{\subseteq} \left(\ddot{G}, A\right) \Leftrightarrow \left(\ddot{F}, A\right) \ddot{\cup} \left(\ddot{G}, A\right) \ddot{=} \left(\ddot{G}, A\right)$

Proof.

(a) Assume $\left(\ddot{F}, A\right) \ddot{\subseteq} (\ddot{G}, A) \Rightarrow$

i. $A \subseteq A$

ii. $\ddot{F}(e_p) \subseteq \ddot{G}(e_p), \quad \forall e_p \in A$ [See definition 2.1.10]

Let $\left(\ddot{H}, A\right) \ddot{=} \left(\ddot{F}, A\right) \ddot{\cap} \left(\ddot{G}, A\right)$

$\Rightarrow \ddot{H}(e_p) = \left(\ddot{F}(e_p) \cap \ddot{G}(e_p)\right), \quad \forall e_p \in A$

$$\Rightarrow \ddot{H}(e_p) = \begin{cases} \ddot{F}(e_p) & ; \forall e_p \in A^c \\ \ddot{G}(e_p) & ; \forall e_p \in A^c \\ min\left(\ddot{F}(e_p), \ddot{G}(e_p)\right) & ; \forall e_p \in (A \cap A) \end{cases}$$

$\therefore$ By assumption, $\ddot{H}(e_p) = \left(\ddot{F}(e_p) \cap \ddot{G}(e_p)\right) = \ddot{F}(e_p) \quad ; \forall e_p \in A$

Moreover, $A \subseteq A$

$\therefore$ Vague Binary Soft Equality conditions got satisfied in this case [$By\ Definition$ 2.1.13]

$\Rightarrow \left(\ddot{H}, A\right) \ddot{=} \left(\ddot{F}, A\right)$

$\Rightarrow \left(\ddot{F}, A\right) \ddot{\cap} \left(\ddot{G}, A\right) \ddot{=} \left(\ddot{F}, A\right)$

Conversely, suppose, $\left(\ddot{F}, A\right) \ddot{\cap} \left(\ddot{G}, A\right) \ddot{=} \left(\ddot{F}, A\right)$.

Also, let $\left(\ddot{H}, A\right) \ddot{=} \left(\ddot{F}, A\right) \ddot{\cap} \left(\ddot{G}, A\right)$

$\Rightarrow \ddot{H}(e_p) = \left(\ddot{F}(e_p) \cap \ddot{G}(e_p)\right) \quad ; \forall e_p \in A$

$\Rightarrow \ddot{F}(e_p) = \left(\ddot{H}(e_p) \cap \ddot{G}(e_p)\right) \quad ; \forall e_p \in A$

$\Rightarrow \ddot{F}(e_p) \subseteq \ddot{G}(e_p) \quad ; \forall e_p \in A$

$\Rightarrow (\ddot{F}, A) \subseteq (\ddot{G}, A)$

(b) Proof is similar to (1)

Theorem 2.2.5.

Let (U_1, U_2) be a binary universe. E be a fixed set of parameters
Take $U_1 = \{u_1, u_2, --, u_r, --, u_i\}, U_2 = \{v_1, v_2, --, v_s, --, v_j\}$ &
$E = \{e_1, e_2, --, e_t, --, e_k\}$ with $A, B \subseteq E$.

Also let $A = \{e_1, e_2, --, e_p, --, e_m\}$, $B = \{e_1, e_2, --, e_q, --, e_n\}$
For any VBSS $\left(\ddot{F}, A\right)$ *and* $\left(\ddot{G}, B\right)$ *over Absolute - VBSS* $\left(\ddot{U}, E\right)$

(a) $\left(\left(\ddot{F}, A\right) \ddot{\cup} \left(\ddot{G}, B\right)\right)^c \ddot{\subseteq} \left(\ddot{F}, A\right)^c \ddot{\cup} \left(\ddot{G}, B\right)^c$

(b) $\left(\ddot{F}, A\right)^c \ddot{\cap} \left(\ddot{G}, B\right)^c \ddot{\subseteq} \left(\left(\ddot{F}, A\right) \ddot{\cap} \left(\ddot{G}, B\right)\right)^c$

Proof.

Let $C = (A \cup B) = \{e_1, e_2, --, e_g, --, e_h\}$

(a) Let $\left(\ddot{F}, A\right) \ddot{\cup} \left(\ddot{G}, B\right) = (\ddot{H}, C)$ where C $= (A \cup B)$
$\forall e_g \in C$, $t_{\ddot{H}(e_g)}(u_r)$, $f_{\ddot{H}(e_g)}(u_r)$, $t_{\ddot{H}(e_g)}(v_s)$, $f_{\ddot{H}(e_g)}(v_s)$ are defined as in definition 2.1.1. Here, $\left(\left(\ddot{F}, A\right) \ddot{\cup} \left(\ddot{G}, B\right)\right)^c = \left(\ddot{H}, C\right)^c$
Using definition 2.1.6, $\forall e_g \in C$,

$$\ddot{H}^c(e_g) = \begin{cases} \ddot{F}^c(e_p) & ; e_p \in B^c \\ \ddot{G}^c(e_q) & ; e_q \in A^c \\ \ddot{F}^c(e_p) \cup \ddot{G}^c(e_q); & e_p, e_q \in (A \cap B) \end{cases}$$

$$t_{\ddot{H}^c(e_g)}(u_r) = \begin{cases} f_{\ddot{F}(e_p)}(u_r) & ; \forall e_p \in B^c & ; \forall u_r \in U_1 \\ f_{\ddot{G}(e_q)}(u_r) & ; \forall e_q \in A^c & ; \forall u_r \in U_1 \\ max\left\{f_{\ddot{F}(e_p)}(u_r), f_{\ddot{G}(e_q)}(u_r)\right\} & ; \forall e_p, e_q \in (A \cap B) & ; \forall u_r \in U_1 \end{cases}$$

$$1 - f_{\ddot{H}^c(e_g)}(u_r) = \begin{cases} 1 - t_{\ddot{F}(e_p)}(u_r) & ; \forall e_p \in B^c & ; \forall u_r \in U_1 \\ 1 - t_{\ddot{G}(e_q)}(u_r) & ; \forall e_q \in A^c & ; \forall u_r \in U_1 \\ max\left\{1 - t_{\ddot{F}(e_p)}(u_r), 1 - t_{\ddot{G}(e_q)}(u_r)\right\} & ; \forall e_p, e_q \in (A \cap B) & ; \forall u_r \in U_1 \end{cases}$$

$$t_{\ddot{H}^c(e_g)}(v_s) = \begin{cases} f_{\ddot{F}(e_p)}(v_s) & ; \forall e_p \in B^c & ; \forall v_s \in U_2 \\ f_{\ddot{G}(e_q)}(v_s) & ; \forall e_q \in A^c & ; \forall v_s \in U_2 \\ max\left\{f_{\ddot{F}(e_p)}(v_s), f_{\ddot{G}(e_q)}(v_s)\right\} & ; \forall e_p, e_q \in (A \cap B) & ; \forall v_s \in U_2 \end{cases}$$

$$1 - f_{\ddot{H}^c(e_g)}(v_s) = \begin{cases} 1 - t_{\ddot{F}(e_p)}(v_s) & ; \forall e_p \in B^c & ; \forall v_s \in U_2 \\ 1 - t_{\ddot{G}(e_q)}(v_s) & ; \forall e_q \in A^c & ; \forall v_s \in U_2 \\ max\left\{1 - t_{\ddot{F}(e_p)}(v_s), 1 - t_{\ddot{G}(e_q)}(v_s)\right\} & ; \forall e_p, e_q \in (A \cap B) & ; \forall v_s \in U_2 \end{cases}$$

Let $\left(\ddot{F},A\right)^c \ddot{\cup} \left(\ddot{G},B\right)^c = (\ddot{M},C)$ where C = $(A \cup B)$. Then, $\forall e_p \in C$,

$$\ddot{M}(e_g) = \begin{cases} \ddot{F}^c(e_p) & ; e_p \in B^c \\ \ddot{G}^c(e_q) & ; e_q \in A^c \\ \ddot{F}^c(e_p) \cup \ddot{G}^c(e_q); e_p, e_q \in (A \cap B) \end{cases}$$

$$t_{\ddot{M}(e_g)}(u_r) = \begin{cases} f_{\ddot{F}(e_p)}(u_r) & ; \forall e_p \in B^c & ; \forall u_r \in U_1 \\ f_{\ddot{G}(e_q)}(u_r) & ; \forall e_q \in A^c & ; \forall u_r \in U_1 \\ max\left\{f_{\ddot{F}(e_p)}(u_r), f_{\ddot{G}(e_q)}(u_r)\right\} & ; \forall e_p, e_q \in (A \cap B) & ; \forall u_r \in U_1 \end{cases}$$

$$1 - f_{\ddot{M}(e_g)}(u_r) = \begin{cases} 1 - t_{\ddot{F}(e_p)}(u_r) & ; \forall e_p \in B^c & ; \forall u_r \in U_1 \\ 1 - t_{\ddot{G}(e_q)}(u_r) & ; \forall e_q \in A^c & ; \forall u_r \in U_1 \\ max\left\{1 - t_{\ddot{F}(e_p)}(u_r), 1 - t_{\ddot{G}(e_q)}(u_r)\right\} & ; \forall e_p, e_q \in (A \cap B) & ; \forall u_r \in U_1 \end{cases}$$

$$t_{\ddot{M}(e_g)}(v_s) = \begin{cases} f_{\ddot{F}(e_p)}(v_s) & ; \forall e_p \in B^c & ; \forall v_s \in U_2 \\ f_{\ddot{G}(e_q)}(v_s) & ; \forall e_q \in A^c & ; \forall v_s \in U_2 \\ max\left\{f_{\ddot{F}(e_p)}(v_s), f_{\ddot{G}(e_q)}(v_s)\right\} & ; \forall e_p, e_q \in (A \cap B) & ; \forall v_s \in U_2 \end{cases}$$

$$1 - f_{\ddot{M}(e_g)}(v_s) = \begin{cases} 1 - t_{\ddot{F}(e_p)}(v_s) & ; \forall e_p \in B^c & ; \forall v_s \in U_2 \\ 1 - t_{\ddot{G}(e_q)}(v_s) & ; \forall e_q \in A^c & ; \forall v_s \in U_2 \\ max\left\{1 - t_{\ddot{F}(e_p)}(v_s), 1 - t_{\ddot{G}(e_q)}(v_s)\right\} & ; \forall e_p, e_q \in (A \cap B) & ; \forall v_s \in U_2 \end{cases}$$

$\therefore \forall e_g \in C, \quad \ddot{H}^c(e_g) = \ddot{M}(e_g).$ Moreover, $C \subseteq C$

By definition 2.1.10. $\left(\ddot{H},C\right)^c$ is a vague binary soft subset of $\left(\ddot{M},C\right)$

Thus $\left(\left(\ddot{F},A\right) \ddot{\cup} \left(\ddot{G},B\right)\right)^c \ddot{\subseteq} \left(\ddot{F},A\right)^c \ddot{\cup} \left(\ddot{G},B\right)^c$

(b) Proof is similar to (1).

Theorem 2.2.6. ***(De-Morgan's Law)***

Let (U_1, U_2) be a binary universe. E be a fixed set of parameters

Take $U_1 = \{u_1, u_2, --, u_r, --, u_i\}$, $U_2 = \{v_1, v_2, --, v_s, --, v_j\}$ &

$E = \{e_1, e_2, --, e_t, --, e_k\}$ with $A \subseteq E$ with $A = \{e_1, e_2, --, e_p, --, e_m\}$

For any VBSS $\left(\ddot{F},A\right)$ and $\left(\ddot{G},A\right)$ over Absolute - VBSS $\left(\ddot{U},E\right)$

(a) $\left(\left(\ddot{F},A\right)\ddot{\cup}\left(\ddot{G},A\right)\right)^c=\left(\ddot{F},A\right)^c\ddot{\cap}\left(\ddot{G},A\right)^c$

(b) $\left(\left(\ddot{F},A\right)\ddot{\cap}\left(\ddot{G},A\right)\right)^c=\left(\ddot{F},A\right)^c\ddot{\cup}\left(\ddot{G},A\right)^c$

Proof.

(a) Let $\left(\ddot{H},A\right)=\left(\ddot{F},A\right)\ddot{\cup}\left(\ddot{G},A\right)$. Then, $\forall e_p\in A$, $\ddot{H}(e_p)=\ddot{F}(e_p)\ddot{\cup}\ddot{G}(e_p)$

$$=\left\{\left\langle\frac{t_{\ddot{F}(e_p)}(u_r),1-f_{\ddot{F}(e_p)}(u_r)]}{u_r};\forall u_r\in U_1\right\rangle,\left\langle\frac{[t_{\ddot{F}(e_p)}(v_s),1-f_{\ddot{F}(e_p)}v_s]}{v_s};\forall v_s\in U_2\right\rangle\right\}\ddot{\cup}$$

$$\left\{\left\langle\frac{t_{\ddot{G}(e_p)}(u_r),1-f_{\ddot{G}(e_p)}(u_r)]}{u_r};\forall u_r\in U_1\right\rangle,\left\langle\frac{[t_{\ddot{G}(e_p)}(v_s),1-f_{\ddot{G}(e_p)}v_s]}{v_s};\forall v_s\in U_2\right\rangle\right\}$$

$$=\left\{\left(\begin{array}{l}\left\langle\frac{max\left(t_{\ddot{F}(e_p)}(u_r),t_{\ddot{G}(e_p)}(u_r)\right),max\left(1-f_{\ddot{F}(e_p)}(u_r),1-f_{\ddot{G}(e_p)}(u_r)\right)}{u_r};\forall u_r\in U_1\right\rangle\\ \left\langle\frac{max\left(t_{\ddot{G}(e_p)}(v_s),t_{\ddot{G}(e_p)}(v_s)\right),max\left(1-f_{\ddot{G}(e_p)}(v_s),1-f_{\ddot{G}(e_p)}(v_s)\right)}{(v_s)};\forall v_s\in U_2\right\rangle\end{array}\right)\right\}$$

$$\therefore\left(\ddot{H}(e_p)\right)^c=$$

$$\left\{\left(\begin{array}{l}\left\langle\frac{min\left(t_{\ddot{F}(e_p)}(u_r),t_{\ddot{G}(e_p)}(u_r)\right),min\left(1-f_{\ddot{F}(e_p)}(u_r),1-f_{\ddot{G}(e_p)}(u_r)\right)}{u_r};\forall u_r\in U_1\right\rangle\\ \left\langle\frac{min\left(t_{\ddot{G}(e_p)}(v_s),t_{\ddot{G}(e_p)}(v_s)\right),min\left(1-f_{\ddot{G}(e_p)}(v_s),1-f_{\ddot{G}(e_p)}(v_s)\right)}{(v_s)};\forall v_s\in U_2\right\rangle\end{array}\right)\right\}$$

$$\left(\ddot{H},A\right)^c=$$

$$\left\{\left(e_p,\left(\begin{array}{l}\left\langle\frac{min\left(t_{\ddot{F}(e_p)}(u_r),t_{\ddot{G}(e_p)}(u_r)\right),min\left(1-f_{\ddot{F}(e_p)}(u_r),1-f_{\ddot{G}(e_p)}(u_r)\right)}{u_r};\forall u_r\in U_1\right\rangle\\ \left\langle\frac{min\left(t_{\ddot{G}(e_p)}(v_s),t_{\ddot{G}(e_p)}(v_s)\right),min\left(1-f_{\ddot{G}(e_p)}(v_s),1-f_{\ddot{G}(e_p)}(v_s)\right)}{(v_s)};\forall v_s\in U_2\right\rangle\end{array}\right)\right)\right\}$$

Let $\left(\ddot{M},A\right)=\left(\ddot{F},A\right)^c\ddot{\cap}\left(\ddot{G},A\right)^c\Rightarrow\forall e_p\in A,\quad\ddot{M}(e_p)=\ddot{F}^c(e_p)\ddot{\cap}\ddot{G}^c(e_p)$

$\therefore\ddot{M}(e_p)=$

$$\left\{\left\langle\frac{f_{\ddot{F}(e_p)}(u_r),1-t_{\ddot{F}(e_p)}(u_r)]}{u_r};\forall u_r\in U_1\right\rangle,\left\langle\frac{[f_{\ddot{F}(e_p)}(v_s),1-t_{\ddot{F}(e_p)}(v_s)]}{v_s};\forall v_s\in U_2\right\rangle\right\}\ddot{\cap}$$

$$\left\{\left\langle\frac{[f_{\ddot{G}(e_p)}(u_r),1-t_{\ddot{G}(e_p)}(u_r)]}{u_r};\forall u_r\in U_1\right\rangle,\left\langle\frac{[f_{\ddot{G}(e_r)}(v_s),1-t_{\ddot{G}(e_p)}(v_s)]}{v_s};\forall v_s\in U_2\right\rangle\right\}$$

$$= \left\{ \left(\begin{array}{l} \left\langle \dfrac{min\left(t_{\ddot{F}(e_p)}(u_r), t_{\ddot{G}(e_p)}(u_r)\right), min\left(1 - f_{\ddot{F}(e_p)}(u_r), 1 - f_{\ddot{G}(e_p)}(u_r)\right)}{u_r}; \forall u_r \in U_1 \right\rangle \\ \left\langle \dfrac{min\left(t_{\ddot{F}(e_p)}(v_s), t_{\ddot{G}(e_p)}(v_s)\right), min\left(1 - f_{\ddot{F}(e_p)}(v_s), 1 - f_{\ddot{G}(e_p)}(v_s)\right)}{v_s}; \forall v_s \in U_2 \right\rangle \end{array} \right) \right\}$$

$$\left(\ddot{H}(e_p)\right)^c =$$

$$\left\{ \left(\begin{array}{l} \left\langle \dfrac{min\left(t_{\ddot{F}(e_p)}(u_r), t_{\ddot{G}(e_p)}(u_r)\right), min\left(1 - f_{\ddot{F}(e_p)}(u_r), 1 - f_{\ddot{G}(e_p)}(u_r)\right)}{u_r}; \forall u_r \in U_1 \right\rangle \\ \left\langle \dfrac{min\left(t_{\ddot{G}(e_p)}(v_s), t_{\ddot{G}(e_p)}(v_s)\right), min\left(1 - f_{\ddot{G}(e_p)}(v_s), 1 - f_{\ddot{G}(e_p)}(v_s)\right)}{(v_s)}; \forall v_s \in U_2 \right\rangle \end{array} \right) \right\}$$

$= \ddot{M}(e_p), \quad \forall e_p \in A \Rightarrow \left(\ddot{H}, A\right)^c = (M, A)$

Proof follows.

(b) Proof is similar to (1)

Theorem 2.2.7. ***(Idempotent Property for AND & OR Operation)***
Let (U_1, U_2) be a binary universe. E be a fixed set of parameters
Take $U_1 = \{u_1, u_2, --, u_r, --, u_i\}, U_2 = \{v_1, v_2, --, v_s, --, v_j\}$ &
$E = \{e_1, e_2, --, e_t, --, e_k\}$ with $A \subseteq E$. Also let $A = \{e_1, e_2, --, e_p, --, e_m\}$
For any VBSS $\left(\ddot{F}, A\right)$ over Absolute- VBSS $\left(\ddot{U}, E\right)$ following is true:

(a) $\left(\ddot{F}, A\right) \ddot{\wedge} \left(\ddot{F}, A\right) = \left(\ddot{F}, A\right)$

(b) $\left(\ddot{F}, A\right) \ddot{\vee} \left(\ddot{F}, A\right) = \left(\ddot{F}, A\right)$

Proof.

(a) $\left(\ddot{F}, A\right) \ddot{\wedge} \left(\ddot{F}, A\right) = \left(\ddot{M}, C\right);$
Let $C = (A \times A)$. Then, $\forall (e_s, e_t) \in (A \times A) = C$,
$\ddot{M}(e_s, e_t) = \ddot{F}(e_s) \ddot{\cap} \ddot{F}(e_t)$

$$= \begin{cases} \ddot{F}(e_s) \\ \ddot{F}(e_t) & \text{Using definition 2.1.8 (1)} \\ min\left(\ddot{F}(e_s), \ddot{F}(e_t)\right) \end{cases}$$

$$\Rightarrow \left(\ddot{M}, C\right) = \left(\ddot{F}, A\right) \Rightarrow \left(\ddot{F}, A\right) \ddot{\wedge} \left(\ddot{F}, A\right) = \left(\ddot{F}, A\right)$$

(b) Use definition 2.1.8(2). Proof is similar to (1).

Theorem 2.2.8.

Let (U_1, U_2) be a binary universe. E be a fixed set of parameters
Take $U_1 = \{u_1, u_2, --, u_r, --, u_i\}, U_2 = \{v_1, v_2, --, v_s, --, v_j\}$ &
$E = \{e_1, e_2, --, e_t, --, e_k\}$ with $A \subseteq E$ with $A = \{e_1, e_2, --, e_p, --, e_m\}$
Let (Φ, B) and (U, B) represents the Null - VBSS and Absolute - VBSS respectively.

(a) $\left(\ddot{F}, A\right) \ddot{\cup} \left(\ddot{\Phi}, B\right) = \left(\ddot{F}, A\right) \Leftrightarrow B \subseteq A$

(b) $\left(\ddot{F}, A\right) \ddot{\cup} \left(\ddot{U}, B\right) = \left(\ddot{U}, A\right) \Leftrightarrow A \subseteq B$

Proof.

(a) $\left(\ddot{F}, A\right) = \left\{e_p \in A \diagup \left(e_p, \ddot{F}(e_p)\right)\right\}$

where $\ddot{F}(e_p) = \left(\left\langle \frac{V_{\ddot{F}(e_p)}(u_r)}{u_r}; \forall u_r \in U_1 \right\rangle, \left\langle \frac{V_{\ddot{F}(e_p)}(v_s)}{v_s}; \forall v_s \in U_2 \right\rangle\right)$

$$= \left(\begin{array}{c} \left\langle \frac{\left[t_{\ddot{F}(e_p)}(u_r), 1 - f_{\ddot{F}(e_p)}(u_r)\right]}{u_r}; \forall e_p \in A; \forall u_r \in U_1 \right\rangle, \\ \left\langle \frac{\left[t_{\ddot{F}(e_p)}(v_s), 1 - f_{\ddot{F}(e_p)}(v_s)\right]}{v_s}; \forall e_p \in A; \forall v_s \in U_2 \right\rangle \end{array} \right)$$

Also let $\left(\ddot{\Phi}, B\right) = \left(\ddot{G}, B\right)$

$\forall e_q \in B, \quad \ddot{G}(e_q) = \left\{\left\langle \frac{[0,0]}{u_r}; \forall u_r \in U_1 \right\rangle, \left\langle \frac{[0,0]}{v_s}; \forall v_s \in U_2 \right\rangle\right\}$

Let $\left(\ddot{F}, A\right) \ddot{\cup} \left(\ddot{G}, B\right) = \left(\ddot{N}, C\right)$ where $C = (A \cup B)$

$$\forall e_p \in C, \qquad \ddot{N}(e_p) = \begin{cases} \ddot{F}(e_p) & ; e_p \in B^c \\ \ddot{G}(e_p); & ; e_p \in A^c \\ \ddot{F}(e_p) \ddot{\cup} \ddot{G}(e_p); & e_p \in (A \cap B) \end{cases}$$

$$= \begin{cases} \left(\left\langle \frac{V_{\ddot{F}(e_p)}(u_r)}{u_r}; \forall u_r \in U_1 \right\rangle, \left\langle \frac{V_{\ddot{F}(e_p)}(v_s)}{v_s}; \forall v_s \in U_2 \right\rangle\right) ; e_p \in B^c \\ \left(\left\langle \frac{[0,0]}{u_r}; \forall u_r \in U_1 \right\rangle, \left\langle \frac{[0,0]}{v_s}; \quad \forall v_s \in U_2 \right\rangle\right) \qquad ; e_p \in A^c \\ \left(\left\langle \frac{max\left(V_{\ddot{F}(e_p)}(u_r), [0,0]\right)}{u_r}; \forall u_r \in U_1 \right\rangle, \left\langle \frac{max\left(V_{\ddot{F}(e_p)}(v_s), [0,0]\right)}{v_s}; \forall v_s \in U_2 \right\rangle\right) \\ \qquad ; e_p \in (A \cap B) \end{cases}$$

$$= \begin{cases} \left(\left\langle \frac{\left[t_{\ddot{F}(e_p)}(u_r), 1 - f_{\ddot{F}(e_p)}(u_r)\right]}{u_r}; \forall u_r \in U_1 \right\rangle, \left\langle \frac{\left[t_{\ddot{F}(e_p)}(v_s), 1 - f_{\ddot{F}(e_p)}(v_s)\right]}{v_s}; \forall v_s \in U_2 \right\rangle\right) ; e_p \in B^c \\ \left(\left\langle \frac{[0,0]}{u_r}; \forall u_r \in U_1 \right\rangle, \left\langle \frac{[0,0]}{v_s}; \forall v_s \in U_2 \right\rangle\right) \qquad ; e_p \in A^c \\ \left(\left\langle \frac{max\left(\left[t_{\ddot{F}(e_p)}(u_r), 1 - f_{\ddot{F}(e_p)}(u_r)\right], [0,0]\right)}{u_r}; \forall u_r \in U_1 \right\rangle, \left\langle \frac{max\left(\left[t_{\ddot{F}(e_p)}(v_s), 1 - f_{\ddot{F}(e_p)}(v_s)\right], [0,0]\right)}{v_s}; \forall v_s \in U_2 \right\rangle\right) \\ \qquad ; e_p \in (A \cap B) \end{cases}$$

$$= \begin{cases} \left(\left\langle \frac{\left[t_{\ddot{F}(e_p)}(u_r), 1 - f_{\ddot{F}(e_p)}(u_r)\right]}{u_r}; \forall u_r \in U_1 \right\rangle, \left\langle \frac{\left[t_{\ddot{F}(e_p)}(v_s), 1 - f_{\ddot{F}(e_p)}(v_s)\right]}{v_s}; \forall v_s \in U_2 \right\rangle\right) ; e_p \in B^c \\ \left(\left\langle \frac{[0,0]}{u_r}; \forall u_r \in U_1 \right\rangle, \left\langle \frac{[0,0]}{v_s}; \forall v_s \in U_2 \right\rangle\right) \qquad ; e_p \in A^c \\ \left(\left\langle \frac{\left[t_{\ddot{F}(e_p)}(u_r), 1 - f_{\ddot{F}(e_p)}(u_r)\right]}{u_r}; \forall u_r \in U_1 \right\rangle, \left\langle \frac{\left[t_{\ddot{F}(e_p)}(v_s), 1 - f_{\ddot{F}(e_p)}(v_s)\right]}{v_s}; \forall v_s \in U_2 \right\rangle\right) ; e_p \in (A \cap B) \end{cases}$$

$$= \begin{cases} \left(\left\langle \frac{V_{\ddot{F}(e_p)}(u_r)}{u_r}; \forall u_r \in U_1 \right\rangle, \left\langle \frac{V_{\ddot{F}(e_p)}(v_s)}{v_s}; \forall v_s \in U_2 \right\rangle\right) ; e_p \in B^c \\ \left(\left\langle \frac{[0,0]}{u_r}; \forall u_r \in U_1 \right\rangle, \left\langle \frac{[0,0]}{v_s}; \forall v_s \in U_2 \right\rangle\right) \qquad ; e_p \in A^c \\ \left(\left\langle \frac{V_{\ddot{F}(e_p)}(u_r)}{u_r}; \forall u_r \in U_1 \right\rangle, \left\langle \frac{V_{\ddot{F}(e_p)}(v_s)}{v_s}; \forall v_s \in U_2 \right\rangle\right) ; e_p \in (A \cap B) \end{cases}$$

Let $B \subseteq A \Rightarrow (A \cap B) = B$.

$\therefore$ Above becomes,

$\ddot{N}(e_p) = \left(\left\langle \frac{V_{\ddot{F}(e_p)}(u_r)}{u_r}; \forall u_r \in U_1 \right\rangle, \left\langle \frac{V_{\ddot{F}(e_p)}(v_s)}{v_s}; \forall v_s \in U_2 \right\rangle\right)$

$; \forall e_p \in B^c \quad \& \quad \forall e_p \in B$

$= \ddot{F}(e_p), \forall e_p \in A$

Conversely, $\left(\ddot{F}, A\right) \ddot{\cup} \left(\ddot{\Phi}, B\right) \ddot{=} \left(\ddot{F}, A\right) \Rightarrow (A \cup B) = A \Rightarrow B \subseteq A$

(b) Proof is similar to (1)

2.3 Vague Binary Soft Topology

Vague binary soft topological space over a Binary Universe with a fixed parameter set is developed in this section. Notions like interior, closure, exterior, boundary, neighborhood, separation axioms are also discussed with some of its properties

Definition 2.3.1. ***(Vague Binary Soft Topology)***
Let $\ddot{\tau}_\Delta$ be a collection of vague binary soft sets over an initial binary universe (U_1, U_2) and A be a non- empty subset of a fixed set of parameters E (i.e., $A \subseteq E$). Then the family $\ddot{\tau}_\Delta$ is said to be a Vague Binary Soft Topology (VBST, in short) over (U_1, U_2) if it satisfies the following axioms

1. *Null - VBSS, $\left(\ddot{\Phi}, A\right)$ and Absolute - VBSS, $\left(\ddot{U}, A\right)$ belongs to $\ddot{\tau}_\Delta$*
2. *Vague Binary Soft Intersection of finite collection of VBSS's in $\ddot{\tau}_\Delta$ belongs to $\ddot{\tau}_\Delta$. i.e., for any $\left(\ddot{G}_1, A\right), \left(\ddot{G}_2, A\right) \in \ddot{\tau}_\Delta$, $\left(\ddot{G}_1, A\right) \ddot{\cap} \left(\ddot{G}_2, A\right) \in \ddot{\tau}_\Delta$*
3. *Vague Binary Soft Union of arbitrary collection of VBSS's in $\ddot{\tau}_\Delta$ belongs to $\ddot{\tau}_\Delta$. i.e., for any family $\{\left(\ddot{G}_i, A\right) / i \in I\} \subseteq \ddot{\tau}_\Delta, \cup_{i\in I} \left(\ddot{G}_i, A\right) \in \ddot{\tau}_\Delta$*

In this case the 4-tuple $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$ is called a Vague Binary Soft Topological Space (VBSTS in short) and any VBSS in $\ddot{\tau}_\Delta$ is known as a Vague Binary Soft Open Set (VBSOS, in short) in (U_1, U_2)

Example 2.3.2.
Let $U_1 = \{b_1, b_2\}, U_2 = \{p_1, p_2\}$; and $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$.
Also let $A = \{e_2, e_4\}$. It is obvious that, $A \subseteq E$
Let $\left(\ddot{W}, A\right)$ and $\left(\ddot{H}, A\right)$ be two VBSS's in VBSTS $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$ defined as follows:

$$\left(\ddot{W}, A\right) = \left\{ \begin{array}{l} \left(e_2, \left(\left\langle \frac{[0.4, 0.6]}{b_1}, \frac{[0.2, 0.3]}{b_2} \right\rangle, \left\langle \frac{[0.5, 0.5]}{p_1}, \frac{[0.6, 0.8]}{p_2} \right\rangle\right)\right) \\ \left(e_4, \left(\left\langle \frac{[0.8, 0.9]}{b_1}, \frac{[0.4, 0.5]}{b_2} \right\rangle, \left\langle \frac{[0.3, 0.6]}{p_1}, \frac{[0.6, 0.7]}{p_2} \right\rangle\right)\right) \end{array} \right\}$$

$$\left(\ddot{H}, A\right) = \left\{ \begin{array}{l} \left(e_2, \left(\left\langle \frac{[0.2, 0.6]}{b_1}, \frac{[0.6, 0.7]}{b_2} \right\rangle, \left\langle \frac{[0.5, 0.6]}{p_1}, \frac{[0.4, 0.6]}{p_2} \right\rangle\right)\right) \\ \left(e_4, \left(\left\langle \frac{[0.4, 0.5]}{b_1}, \frac{[0.3, 0.6]}{b_2} \right\rangle, \left\langle \frac{[0.2, 0.5]}{p_1}, \frac{[0.1, 0.3]}{p_2} \right\rangle\right)\right) \end{array} \right\}$$

In this case,

$$\textbf{\textit{Null VBSS,}}\ \left(\ddot{\Phi}, A\right) = \left\{ \begin{array}{l} \left(e_2, \left(\left\langle \frac{[0,0]}{b_1}, \frac{[0,0]}{b_2} \right\rangle, \left\langle \frac{[0,0]}{p_1}, \frac{[0,0]}{p_2} \right\rangle\right)\right) \\ \left(e_4, \left(\left\langle \frac{[0,0]}{b_1}, \frac{[0,0]}{b_2} \right\rangle, \left\langle \frac{[0,0]}{p_1}, \frac{[0,0]}{p_2} \right\rangle\right)\right) \end{array} \right\}$$

$$\left(\ddot{W}, A\right) \ddot{\cap} \left(\ddot{H}, A\right) = \left(\ddot{M}, A\right) = \left\{ \begin{array}{l} \left(e_2, \left(\left\langle \frac{[0.2,0.6]}{b_1}, \frac{[0.2,0.3]}{b_2} \right\rangle, \left\langle \frac{[0.5,0.5]}{p_1}, \frac{[0.4,0.6]}{p_2} \right\rangle\right)\right) \\ \left(e_4, \left(\left\langle \frac{[0.4,0.5]}{b_1}, \frac{[0.3,0.5]}{b_2} \right\rangle, \left\langle \frac{[0.2,0.5]}{p_1}, \frac{[0.1,0.3]}{p_2} \right\rangle\right)\right) \end{array} \right\}$$

$$\left(\ddot{W}, A\right) \ddot{\cup} \left(\ddot{H}, A\right) = \left(\ddot{N}, A\right) = \left\{ \begin{array}{l} \left(e_2, \left(\left\langle \frac{[0.4,0.6]}{b_1}, \frac{[0.6,0.7]}{b_2} \right\rangle, \left\langle \frac{[0.5,0.6]}{p_1}, \frac{[0.6,0.8]}{p_2} \right\rangle\right)\right) \\ \left(e_4, \left(\left\langle \frac{[0.8,0.9]}{b_1}, \frac{[0.4,0.6]}{b_2} \right\rangle, \left\langle \frac{[0.3,0.6]}{p_1}, \frac{[0.6,0.7]}{p_2} \right\rangle\right)\right) \end{array} \right\}$$

$$\textbf{\textit{Absolute VBSS}}\ ,\ \left(\ddot{U}, A\right) = \left\{ \begin{array}{l} \left(e_2, \left(\left\langle \frac{[1,1]}{b_1}, \frac{[1,1]}{b_2} \right\rangle, \left\langle \frac{[1,1]}{p_1}, \frac{[1,1]}{p_2} \right\rangle\right)\right) \\ \left(e_4, \left(\left\langle \frac{[1,1]}{b_1}, \frac{[1,1]}{b_2} \right\rangle, \left\langle \frac{[1,1]}{p_1}, \frac{[1,1]}{p_2} \right\rangle\right)\right) \end{array} \right\}$$

Then, clearly, the collection $\ddot{\tau}_\Delta = \left\{\left(\ddot{\Phi}, A\right), \left(\ddot{W}, A\right), \left(\ddot{H}, A\right), \left(\ddot{M}, A\right), \left(\ddot{N}, A\right), \left(\ddot{U}, A\right)\right\}$ is a VBST. Also,all the elements of $\ddot{\tau}_\Delta$ are VBSOS's

Notions in Vague Binary Soft Topology

Definition 2.3.3. ***(Vague Binary Soft Closed Set)***:
The relative complement $\left(\ddot{F}, A\right)^c$ *of a VBSOS* $\left(\ddot{F}, A\right)$ *in a VBSTS* $(U_1, U_2, \ddot{\tau}_\Delta, E)_A$ *is called a Vague Binary Soft Closed Set (VBSCS in short) over* (U_1, U_2)

Example 2.3.4.
In example 2.3.2. $\left(\ddot{M}, A\right)$ *is a VBSOS.* *Let* $\left(\ddot{M}, A\right)^c = \left(\ddot{B}, A\right)$

$$\therefore \left(\ddot{B}, A\right) = \left\{ \begin{array}{l} \left(e_2, \left(\left\langle \frac{[0.4,0.8]}{b_1}, \frac{[0.7,0.8]}{b_2} \right\rangle, \left\langle \frac{[0.5,0.5]}{p_1}, \frac{[0.4,0.6]}{p_2} \right\rangle\right)\right) \\ \left(e_4, \left(\left\langle \frac{[0.5,0.6]}{b_1}, \frac{[0.5,0.7]}{b_2} \right\rangle, \left\langle \frac{[0.5,0.8]}{p_1}, \frac{[0.7,0.9]}{p_2} \right\rangle\right)\right) \end{array} \right\} \textit{ is a VBSCS}$$

Theorem 2.3.5.

$\ddot{\tau}_\Delta$ *and* $\ddot{\sigma}_\Delta$ *are two VBST's over an initial binary universe* (U_1, U_2) *under a fixed parameter set* E *with* $A \subseteq E$. *Then*

1. $(\ddot{\tau}_\Delta \ \ddot{\cap}\ \ddot{\sigma}_\Delta)$ *is also a VBST over* (U_1, U_2)

2. $(\ddot{\tau}_\Delta \ \ddot{\cup} \ \ddot{\sigma}_\Delta)$ *need not be a VBST over* (U_1, U_2)

Proof.

1. $\ddot{\tau}_\Delta$ and $\ddot{\sigma}_\Delta$ be two VBST's over (U_1, U_2) under a parameter set E with $\Lambda \subseteq E$.

   (a) $\left(\ddot{U}, A\right) \in \ddot{\tau}_\Delta$ and $\left(\ddot{U}, A\right) \in \ddot{\sigma}_\Delta \Rightarrow \left(\ddot{U}, A\right) \in (\ddot{\tau}_\Delta \ddot{\cap} \ddot{\sigma}_\Delta)$;
   $\left(\ddot{\Phi}, A\right) \in \ddot{\tau}_\Delta$ and $\left(\ddot{\Phi}, A\right) \in \ddot{\sigma}_\Delta \Rightarrow \left(\ddot{\Phi}, A\right) \in (\ddot{\tau}_\Delta \ddot{\cap} \ddot{\sigma}_\Delta)$

   (b) Let $(\ddot{N}, A), (\ddot{L}, A) \in (\ddot{\tau}_\Delta \ddot{\cap} \ddot{\sigma}_\Delta)$
   $\Rightarrow \left(\ddot{N}, A\right), \left(\ddot{L}, A\right) \in \ddot{\tau}_\Delta$ and $\left(\ddot{N}, A\right), \left(\ddot{L}, A\right) \in \ddot{\sigma}_\Delta$
   But $\ddot{\tau}_\Delta$ is a VBST. So $\left(\ddot{N}, A\right) \ddot{\cap} \left(\ddot{L}, A\right) \in \ddot{\tau}_\Delta$.
   Similarly, $\left(\ddot{N}, A\right) \ddot{\cap} \left(\ddot{L}, A\right) \in \ddot{\sigma}_\Delta$.
   Thus, $\left(\ddot{N}, A\right) \ddot{\cap} \left(\ddot{L}, A\right) \in (\ddot{\tau}_\Delta \ddot{\cap} \ddot{\sigma}_\Delta)$

   (c) Let $\left(\ddot{N}, A\right)_{\Lambda_\alpha} \in (\ddot{\tau}_\Delta \ddot{\cap} \ddot{\sigma}_\Delta)$; $\forall \alpha \in \Lambda$.
   Then $\left(\ddot{N}, A\right)_{\Lambda_\alpha} \in \ddot{\tau}_\Delta, \forall \alpha \in \Lambda$ and $\left(\ddot{N}, A\right)_{\Lambda_\alpha} \in \ddot{\sigma}_\Delta; \forall \alpha \in \Lambda$;
   $\ddot{\tau}_\Delta$ and $\ddot{\sigma}_\Delta$ are VBST's over (U_1, U_2)
   $\Rightarrow \ddot{\cup}_{\Lambda_\alpha} \left(\ddot{N}, A\right)_{\Lambda_\alpha} \in \ddot{\tau}_\Delta$ & $\ddot{\cup}_{\Lambda_\alpha} \left(\ddot{N}, A\right)_{\Lambda_\alpha} \in \ddot{\sigma}_\Delta$
   $\Rightarrow \ddot{\cup}_{\Lambda_\alpha} \left(\ddot{N}, A\right)_{\Lambda_\alpha} \in (\ddot{\tau}_\Delta \ddot{\cap} \ddot{\sigma}_\Delta)$

   $\therefore (\ddot{\tau}_\Delta \ddot{\cap} \ddot{\sigma}_\Delta)$ is a VBST over (U_1, U_2)

2. Proof is given through a counter example. Consider example 2.3.2
   $\ddot{\tau}_\Delta = \left\{\left(\ddot{\Phi}, A\right), \left(\ddot{W}, A\right), \left(\ddot{H}, A\right), \left(\ddot{M}, A\right), \left(\ddot{N}, A\right), \left(\ddot{U}, A\right)\right\}$
   $\ddot{\sigma}_\Delta = \left\{(\ddot{\Phi}, A), (\ddot{O}, A), (\ddot{Y}, A), (\ddot{R}, A), (\ddot{Q}, A), (\ddot{U}, A)\right\}$ is another VBST under same conditions

$$\left(\ddot{O}, A\right) = \left\{ \begin{array}{l} \left(e_2, \left(\left\langle \frac{[0.5, 0.9]}{b_1}, \frac{[0.4, 0.5]}{b_2} \right\rangle, \left\langle \frac{[0.2, 0.3]}{p_1}, \frac{[0.6, 0.7]}{p_2} \right\rangle\right)\right) \\ \left(e_4, \left(\left\langle \frac{[0.1, 0.8]}{b_1}, \frac{[0.3, 0.4]}{b_2} \right\rangle, \left\langle \frac{[0.8, 0.9]}{p_1}, \frac{[0.2, 0.5]}{p_2} \right\rangle\right)\right) \end{array} \right\}$$

$$\left(\ddot{Y}, A\right) = \left\{ \begin{array}{l} \left(e_2, \left(\left\langle \frac{[0.4, 0.8]}{b_1}, \frac{[0.3, 0.5]}{b_2} \right\rangle, \left\langle \frac{[0.4, 0.5]}{p_1}, \frac{[0.7, 0.9]}{p_2} \right\rangle\right)\right) \\ \left(e_4, \left(\left\langle \frac{[0.2, 0.4]}{b_1}, \frac{[0.1, 0.5]}{b_2} \right\rangle, \left\langle \frac{[0.3, 0.6]}{p_1}, \frac{[0.4, 0.6]}{p_2} \right\rangle\right)\right) \end{array} \right\}$$

$$\left(\ddot{R}, A\right) = \left\{ \begin{array}{l} \left(e_2, \left(\left\langle \frac{[0.4, 0.8]}{b_1}, \frac{[0.3, 0.5]}{b_2} \right\rangle, \left\langle \frac{[0.2, 0.3]}{p_1}, \frac{[0.6, 0.7]}{p_2} \right\rangle\right)\right) \\ \left(e_4, \left(\left\langle \frac{[0.1, 0.4]}{p_1}, \frac{[0.1, 0.4]}{p_2} \right\rangle, \left\langle \frac{[0.3, 0.6]}{p_1}, \frac{[0.2, 0.5]}{p_2} \right\rangle\right)\right) \end{array} \right\}$$

$$\left(\ddot{Q},A\right)=\left\{\begin{array}{c}\left(e_2,\left(\left\langle\frac{[0.5,0.9]}{b_1},\frac{[0.4,0.5]}{b_2}\right\rangle,\left\langle\frac{[0.4,0.5]}{p_1},\frac{[0.7,0.9]}{p_2}\right\rangle\right)\right)\\ \left(e_4,\left(\left\langle\frac{[0.2,0.8]}{b_1},\frac{[0.3,0.5]}{b_2}\right\rangle,\left\langle\frac{[0.8,0.9]}{p_1},\frac{[0.4,0.6]}{p_2}\right\rangle\right)\right)\end{array}\right\}$$

$$(\ddot{\tau}_\Delta\ddot{\cup}\ddot{\sigma}_\Delta)=\left\{\begin{array}{l}\left(\ddot{\Phi},A\right),\left(\ddot{W},A\right),\left(\ddot{H},A\right),\left(\ddot{M},A\right),\left(\ddot{N},A\right),\\ \left(\ddot{O},A\right),\left(\ddot{Y},A\right),\left(\ddot{R},A\right),\left(\ddot{Q},A\right),\left(\ddot{U},A\right)\end{array}\right\}$$

$$\left(\ddot{M},A\right)\ddot{\cup}\left(\ddot{O},A\right)=\left\{\begin{array}{c}\left(e_2,\left(\left\langle\frac{[0.5,0.9}{b_1},\frac{[0.4,0.5}{b_2}\right\rangle,\left\langle\frac{[0.5,0.5}{p_1},\frac{[0.6,0.7}{p_2}\right\rangle\right)\right)\\ \left(e_4,\left(\left\langle\frac{[0.4,0.8}{b_1},\frac{[0.3,0.5}{b_2}\right\rangle,\left\langle\frac{[0.8,0.9}{p_1},\frac{[0.2,0.5}{p_2}\right\rangle\right)\right)\end{array}\right\}$$

$$\left(\ddot{M},A\right)\in(\ddot{\tau}_\Delta\ddot{\cup}\ddot{\sigma}_\Delta)\text{ and }\left(\ddot{O},A\right)\in(\ddot{\tau}_\Delta\ddot{\cup}\ddot{\sigma}_\Delta)\nRightarrow\left(\ddot{M},A\right)\ddot{\cup}\left(\ddot{O},A\right)\in(\ddot{\tau}_\Delta\ddot{\cup}\ddot{\sigma}_\Delta)$$

Definition 2.3.6. ***(Vague Binary Soft Discrete Topological Space)***
Let (U_1,U_2) be a binary universe and E be a set of fixed parameters with $A\subseteq E$ and let $\ddot{\tau}_\Delta$ be the collection of all vague binary soft subsets which can be defined over (U_1,U_2) . Then $\ddot{\tau}_\Delta$ is called vague binary soft discrete topology on $(U_1,U_2)$and $(U_1,U_2,\ddot{\tau}_\Delta,E)_A$ is said to be vague binary soft discrete topological space over (U_1,U_2)

Definition 2.3.7. ***(Vague Binary Soft Indiscrete Topological Space)***
Let (U_1,U_2) be a binary universe and E be a set of fixed parameters with $A\subseteq E$ and let $\ddot{\tau}_\Delta=\left\{\left(\ddot{\Phi},A\right),\left(\ddot{U},A\right)\right\}$. Then $\ddot{\tau}_\Delta$ is called vague binary soft indiscrete topology on (U_1,U_2) and $(U_1,U_2,\ddot{\tau}_\Delta,E)_A$ is said to be vague binary soft indiscrete topological space over (U_1,U_2)

Definition 2.3.8. ***(Vague Binary Soft Point)***
A VBSS $\left(\ddot{F},A\right)$ is called a VBSP if

$$\begin{cases}\ddot{F}(e_p)=\ddot{\Phi}(e_p), & \forall e_p\in A\\ \ddot{F}(e_p)\neq\ddot{\Phi}(e_p), & \forall e_p'\in A-\{e_p\}\end{cases}$$

Definition 2.3.9. ***(Vague Binary Soft Interior point)***
Let $(U_1,U_2,\ddot{\tau}_\Delta,E)_A$ be a VBSTS over (U_1,U_2) and $\left(\ddot{L},A\right)$ be a VBSS over a binary universe (U_1,U_2). Then $\ddot{F}(e_p)$ is said to be vague binary soft interior point of $\left(\ddot{L},A\right)$ if there exists a VBSOS $\left(\ddot{Q},A\right)$ such that $\ddot{F}(e_p)\in\left(\ddot{Q},A\right)\ddot{\subseteq}\left(\ddot{L},A\right)$

Definition 2.3.10. ***(Vague Binary Soft Interior of a VBSS)***

Let $(U_1, U_2, \ddot{\tau}_\Delta, E)_A$ be a VBSTS over (U_1, U_2) and $\left(\ddot{I}, A\right)$ be a VBSS over the binary universe (U_1, U_2). Then vague binary soft interior of $\left(\ddot{I}, A\right)$ is denoted by $\ddot{vbs}$ int $\left(\ddot{I}, A\right)$ or $\left(\ddot{I}, A\right)^0$ and is defined by

$$\ddot{vbs}\ int\left(\ddot{I}, A\right) = \ddot{\cup}\left\{\left(\ddot{K}, A\right) \in \ddot{\tau}_\Delta \quad ; \quad \left(\ddot{K}, A\right) \ddot{\subseteq} \left(\ddot{I}, A\right)\right\}$$

$\ddot{vbs}$ int $\left(\ddot{I}, A\right)$ is the largest VBSOS in $\left(\ddot{I}, A\right)$

Example 2.3.11.

Let $V_1 = \{m_1, m_2\}, V_2 = \{n_1, n_2\}$ be a binary universe;

Also let $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ with $B = \{e_3, e_6\} \subseteq E$

A VBST defined over (V_1, V_2) with a fixed parameter set B is given as,

$\ddot{\tau}_{\Delta_B} = \left\{\left(\ddot{\Phi}, B\right), \left(\ddot{M}, B\right), \left(\ddot{E}, B\right), \left(\ddot{S}, B\right), \left(\ddot{U}, B\right)\right\}$

$$\left(\ddot{\Phi}, B\right) = \left\{\begin{array}{c} \left(e_3, \left(\left\langle \frac{[0,0]}{m_1}, \frac{[0,0]}{m_2}\right\rangle, \left\langle \frac{[0,0]}{n_1}, \frac{[0,0]}{n_2}\right\rangle\right)\right), \\ \left(e_6, \left(\left\langle \frac{[0,0]}{m_1}, \frac{[0,0]}{m_2}\right\rangle, \left\langle \frac{[0,0]}{n_1}, \frac{[0,0]}{n_2}\right\rangle\right)\right) \end{array}\right\}$$

$$\left(\ddot{M}, B\right) = \left\{\begin{array}{c} \left(e_3, \left(\left\langle \frac{[0.4,0.6]}{m_1}, \frac{[0.2,0.3]}{m_2}\right\rangle, \left\langle \frac{[0.5,0.5]}{n_1}, \frac{[0.6,0.8]}{n_2}\right\rangle\right)\right), \\ \left(e_6, \left(\left\langle \frac{[0.8,0.9]}{m_1}, \frac{[0.4,0.5]}{m_2}\right\rangle, \left\langle \frac{[0.3,0.6]}{n_1}, \frac{[0.6,0.7]}{n_2}\right\rangle\right)\right) \end{array}\right\}$$

$$\left(\ddot{E}, B\right) = \left\{\begin{array}{c} \left(e_3, \left(\left\langle \frac{[0.4,0.6]}{m_1}, \frac{[0.2,0.3]}{m_2}\right\rangle, \left\langle \frac{[0.5,0.5]}{n_1}, \frac{[0.6,0.8]}{n_2}\right\rangle\right)\right), \\ \left(e_6, \left(\left\langle \frac{[0,0]}{m_1}, \frac{[0,0]}{m_2}\right\rangle, \left\langle \frac{[0,0]}{n_1}, \frac{[0,0]}{n_2}\right\rangle\right)\right) \end{array}\right\}$$

$$\left(\ddot{S}, B\right) = \left\{\begin{array}{c} \left(e_3, \left(\left\langle \frac{[0,0]}{m_1}, \frac{[0,0]}{m_2}\right\rangle, \left\langle \frac{[0,0]}{n_1}, \frac{[0,0]}{n_2}\right\rangle\right)\right), \\ \left(e_6, \left(\left\langle \frac{[0.8,0.9]}{m_1}, \frac{[0.4,0.5]}{m_2}\right\rangle, \left\langle \frac{[0.3,0.6]}{n_1}, \frac{[0.6,0.7]}{n_2}\right\rangle\right)\right) \end{array}\right\}$$

$$\left(\ddot{U}, B\right) = \left\{\begin{array}{c} \left(e_3, \left(\left\langle \frac{[1,1]}{m_1}, \frac{[1,1]}{m_2}\right\rangle, \left\langle \frac{[1,1]}{n_1}, \frac{[1,1]}{n_2}\right\rangle\right)\right), \\ \left(e_6, \left(\left\langle \frac{[1,1]}{m_1}, \frac{[1,1]}{m_2}\right\rangle, \left\langle \frac{[1,1]}{n_1}, \frac{[1,1]}{n_2}\right\rangle\right)\right) \end{array}\right\}$$

$$\left(\ddot{M}, B\right)^0 = \left\{\begin{array}{c} \left(e_3, \left(\left\langle \frac{[0.4,0.6]}{m_1}, \frac{[0.2,0.3]}{m_2}\right\rangle, \left\langle \frac{[0.5,0.5]}{n_1}, \frac{[0.6,0.8]}{n_2}\right\rangle\right)\right), \\ \left(e_6, \left(\left\langle \frac{[0.8,0.9]}{m_1}, \frac{[0.4,0.5]}{m_2}\right\rangle, \left\langle \frac{[0.3,0.6]}{n_1}, \frac{[0.6,0.7]}{n_2}\right\rangle\right)\right) \end{array}\right\}$$

Definition 2.3.12. ***(Vague Binary Soft Closure of a VBSS)***
Let $(U_1, U_2, \ddot{\tau}_\Delta, E)_A$ *be a VBSTS over* (U_1, U_2) *and* $\left(\ddot{G}, A\right)$ *be a VBSS over* (U_1, U_2). *Then vague binary soft closure of* $\left(\ddot{G}, A\right)$ *is denoted by* $\ddot{vbs}\ cl\left(\ddot{G}, A\right)$ *or* $\overline{\left(\ddot{G}, A\right)}$ *is the intersection of all vague binary soft closed super sets of* $\left(\ddot{G}, A\right)$.
$\ddot{vbs}\ cl\left(\ddot{G}, A\right)$ *is the smallest VBSCS containing* $\left(\ddot{G}, A\right)$ *over* (U_1, U_2) *and it is a VBSCS*

Example 2.3.13.
Using the data in example 2.3.11, find some VBSCS's :

$$\left(\ddot{\Phi}, B\right)^c = \left\{ \begin{array}{c} \left(e_3, \left(\left\langle \frac{[1,1]}{m_1}, \frac{[1,1]}{m_2} \right\rangle, \left\langle \frac{[1,1]}{n_1}, \frac{[1,1]}{n_2} \right\rangle\right)\right), \\ \left(e_6, \left(\left\langle \frac{[1,1]}{m_1}, \frac{[1,1]}{m_2} \right\rangle, \left\langle \frac{[1,1]}{n_1}, \frac{[1,1]}{n_2} \right\rangle\right)\right) \end{array} \right\}$$

$$\left(\ddot{M}, B\right)^c = \left\{ \begin{array}{c} \left(e_3, \left(\left\langle \frac{[0.4,0.6]}{m_1}, \frac{[0.7,0.8]}{m_2} \right\rangle, \left\langle \frac{[0.5,0.5]}{n_1}, \frac{[0.2,0.4]}{n_2} \right\rangle\right)\right), \\ \left(e_6, \left(\left\langle \frac{[0.1,0.2]}{m_1}, \frac{[0.5,0.6]}{m_2} \right\rangle, \left\langle \frac{[0.4,0.7]}{n_1}, \frac{[0.3,0.4]}{n_2} \right\rangle\right)\right) \end{array} \right\}$$

$$\left(\ddot{E}, B\right)^c = \left\{ \begin{array}{l} \left(e_3, \left(\left\langle \frac{[0.4,0.6]}{m_1}, \frac{[0.7,0.8]}{m_2} \right\rangle, \left\langle \frac{[0.5,0.5]}{n_1}, \frac{[0.2,0.4]}{n_2} \right\rangle\right)\right), \\ \left(e_6, \left(\left\langle \frac{[1,1]}{m_1}, \frac{[1,1]}{m_2} \right\rangle, \left\langle \frac{[1,1]}{n_1}, \frac{[1,1]}{n_2} \right\rangle\right)\right) \end{array} \right\}$$

$$\left(\ddot{S}, B\right)^c = \left\{ \begin{array}{l} \left(e_3, \left(\left\langle \frac{[1,1]}{m_1}, \frac{[1,1]}{m_2} \right\rangle, \left\langle \frac{[1,1]}{n_1}, \frac{[1,1]}{n_2} \right\rangle\right)\right), \\ \left(e_6, \left(\left\langle \frac{[0.1,0.2]}{m_1}, \frac{[0.5,0.6]}{m_2} \right\rangle, \left\langle \frac{[0.4,0.7]}{n_1}, \frac{[0.3,0.4]}{n_2} \right\rangle\right)\right) \end{array} \right\}$$

$$\left(\ddot{U}, B\right)^c = \left\{ \begin{array}{l} \left(e_3, \left(\left\langle \frac{[0,0]}{m_1}, \frac{[0,0]}{m_2} \right\rangle, \left\langle \frac{[0,0]}{n_1}, \frac{[0,0]}{n_2} \right\rangle\right)\right), \\ \left(e_6, \left(\left\langle \frac{[0,0]}{m_1}, \frac{[0,0]}{m_2} \right\rangle, \left\langle \frac{[0,0]}{n_1}, \frac{[0,0]}{n_2} \right\rangle\right)\right) \end{array} \right\}$$

All these sets are VBSCS's ; *Let* $\left(\ddot{Z}, B\right) \ddot{=} \left(\ddot{E}, B\right)^c$;
$\overline{\left(\ddot{Z}, B\right)} \ddot{=} \overline{\left(\ddot{E}, B\right)^c}$

$$= \left\{ \begin{array}{l} \left(e_3, \left(\left\langle \frac{[0.4,0.6]}{m_1}, \frac{[0.7,0.8]}{m_2} \right\rangle, \left\langle \frac{[0.5,0.5]}{n_1}, \frac{[0.2,0.4]}{n_2} \right\rangle\right)\right), \\ \left(e_6, \left(\left\langle \frac{[1,1]}{m_1}, \frac{[1,1]}{m_2} \right\rangle, \left\langle \frac{[1,1]}{n_1}, \frac{[1,1]}{n_2} \right\rangle\right)\right) \end{array} \right\}$$

Definition 2.3.14. ***(Vague Binary Soft Exterior of a VBSS)***
Let $(U_1, U_2, \ddot{\tau}_\Delta, E)_A$ *be a VBSTS over* (U_1, U_2) *and* $\left(\ddot{E}, A\right)$ *be a VBSS over* (U_1, U_2). *Vague Binary Soft Exterior of* $\left(\ddot{E}, A\right)$ *is denoted by* $\ddot{vbs}$ *ext* $\left(\ddot{E}, A\right)$ *or* $\left(\ddot{E}, A\right)_0$ *and is defined as* $\left(\ddot{E}, A\right)_0 \ddot{=} \left[\left(\ddot{E}, A\right)^c\right]^0$

$$\left(\ddot{E}, A\right)_0 \text{ is the largest VBSOS contained in } \left(\ddot{E}, A\right)^c$$

Remark 2.3.15.

1. *VBSS's* $\left(\ddot{H}, A\right)$ *and* $\left(\ddot{H}, A\right)^c$ *have the same* $\ddot{vbs}$ *bd*

2. $\underline{\left(\ddot{H}, A\right)}$ *is a smallest VBSS over* U_1, U_2 *containing* $\left(\ddot{H}, A\right)$

Definition 2.3.16. ***(Vague Binary Soft Neighborhood)***
Let $(U_1, U_2, \ddot{\tau}_\Delta, E)_A$ *be a VBSTS over* (U_1, U_2). $\left(\ddot{F}, A\right)$ *be a VBSS and* $\ddot{F}_{e_p} \in \left(\ddot{F}, A\right)$. *Then* $\left(\ddot{F}, A\right)$ *is said to be a vague binary soft neighborhood of* $\ddot{F}_{e_p}$ *denoted by* $\ddot{vbs}$ *nbd, if there exists a VBSOS* $\left(\ddot{K}, A\right)$ *such that* $\ddot{F}_{e_p} \in \left(\ddot{K}, A\right) \ddot{\subseteq} \left(\ddot{F}, A\right)$

Remark 2.3.17.
Collection $\ddot{N}_{\ddot{F}_{e_p}}$ *of all* $\ddot{vbs}$ *nbd's of* $\ddot{F}_{e_p}$ *is called the vague binary soft neighborhood system of* $\ddot{F}_{e_p}$

Theorem 2.3.18.
Let $\left(\ddot{F}, A\right)$ *be a VBSSS of a VBSTS* $(U_1, U_2, \ddot{\tau}_\Delta, E)_A$, $\left(\ddot{F}, A\right)$ *is a VBSOS, each VBSP* $\ddot{F}_e$ *of* $\left(\ddot{F}, A\right)$ *has a* $\ddot{vbs}$ *nbd contained in* $\left(\ddot{F}, A\right)$. *Consequently,* $\left(\ddot{F}, A\right)$ *is a VBSOS* $\Leftrightarrow \left(\ddot{F}, A\right)$ *is a* $\ddot{vbs}$ *nbd of each of its VBSP's.*

Proof.
Let $\left(\ddot{F}, A\right)$ is a VBSOS. Then for each VBSP $\ddot{F}(e_p)$ of $\left(\ddot{F}, A\right)$, there is a $\ddot{vbs}$ nbd of $\ddot{F}(e_p)$, namely, $\left(\ddot{F}, A\right)$ itself, contained in $\left(\ddot{F}, A\right)$. Conversely, suppose that, each VBSP of $\left(\ddot{F}, A\right)$ has a $\ddot{vbs}$ nbd contained in $\left(\ddot{F}, A\right)$. Let $\ddot{F}(e_p) \in \left(\ddot{F}, A\right)$. Then by hypothesis,$\ddot{F}(e_p)$ has a $\ddot{vbs}$ nbd $\left(\ddot{K}, A\right) \ddot{\subseteq} \left(\ddot{F}, A\right)$. Hence $\ddot{F}(e_p) \in \left(\ddot{L}, A\right) \ddot{\subseteq} \left(\ddot{K}, A\right)$ for some VBSOS $\left(\ddot{L}, A\right)$. Thus for each $\ddot{F}(e_p) \in \left(\ddot{F}, A\right)$; $\ddot{F}(e_p) \in \left(\ddot{L}, A\right) \ddot{\subseteq} \left(\ddot{K}, A\right)$ and so $\cup_{i\in I} \left(\ddot{L}, A\right) \ddot{\subseteq} \left(\ddot{F}, A\right)$. Also $\left(\ddot{F}, A\right) \ddot{\subseteq} \cup_{i\in I} \left(\ddot{L}, A\right)$. Hence $\left(\ddot{F}, A\right) \ddot{=} \cup_{i\in I} \left(\ddot{L}, A\right)$ is a VBSOS, so their union $\left(\ddot{F}, A\right)$ is also VBSOS

Theorem 2.3.19.

Let $\ddot{N}_{\ddot{F}_{e_p}}$ be the $\ddot{vbs}$ nbd system of a VBSP $\ddot{F}(e_p)$ in the VBSTS $(U_1, U_2, \ddot{\tau}_\Delta, E)_A$. Then,

1. $\left(\ddot{K}, A\right) \ddot{\subseteq} \left(\ddot{M}, A\right)$ *and* $\left(\ddot{K}, A\right) \in \ddot{N}_{\ddot{F}_{e_p}} \Rightarrow \left(\ddot{M}, A\right) \in \ddot{N}_{\ddot{F}_{e_p}}$
2. $\left(\ddot{K}, A\right), \left(\ddot{M}, A\right) \in \ddot{N}_{\ddot{F}_{e_p}} \Rightarrow \left(\ddot{K}, A\right) \ddot{\cap} \left(\ddot{M}, A\right) \in \ddot{N}_{\ddot{F}_{e_p}}$

Proof.

1. $\left(\ddot{K}, A\right) \in \ddot{N}_{\ddot{F}_{e_p}} \Rightarrow \ddot{F}(e_p) \in \left(\ddot{L}, A\right) \ddot{\subseteq} \left(\ddot{K}, A\right)$ for some VBSOS $\left(\ddot{L}, A\right)$.$\ddot{F}(e_p) \in \left(\ddot{L}, A\right) \ddot{\subseteq} \left(\ddot{K}, A\right)$ and $\left(\ddot{K}, A\right)$ is a VBSOS. But $\left(\ddot{K}, A\right) \ddot{\subseteq} \left(\ddot{M}, A\right)$.
   So $\ddot{F}(e_p) \in \left(\ddot{L}, A\right) \ddot{\subseteq} \left(\ddot{K}, A\right) \ddot{\subseteq} \left(\ddot{M}, A\right) \Rightarrow \ddot{F}(e_p) \in \left(\ddot{L}, A\right) \ddot{\subseteq} \left(\ddot{M}, A\right)$ and $\left(\ddot{L}, A\right)$ is a VBSOS $\Rightarrow \left(\ddot{M}, A\right)$ is a $\ddot{vbs}$ nbd of $\ddot{F}(e_p) \Rightarrow \left(\ddot{M}, A\right) \in \ddot{N}_{\ddot{F}_{e_p}}$ proves (i)
2. Let $\left(\ddot{K}, A\right), \left(\ddot{M}, A\right) \in \ddot{N}_{\ddot{F}_{e_p}} \Rightarrow \left(\ddot{K}, A\right)$ is a $\ddot{vbs}$ nbd of $\ddot{F}(e_p) \Rightarrow \ddot{F}(e_p) \in \left(\ddot{G}, A\right) \ddot{\subseteq} \left(\ddot{K}, A\right)$ for some VBSOS $\left(\ddot{G}, A\right)$ and $\ddot{F}(e_p) \in \left(\ddot{H}, A\right) \ddot{\subseteq} \left(\ddot{M}, A\right)$, for some VBSOS $\left(\ddot{H}, A\right) \Rightarrow \ddot{F}(e_p) \left(\ddot{G}, A\right) \ddot{\cap} \left(\ddot{H}, A\right) \ddot{\subseteq} \left(\ddot{K}, A\right) \ddot{\cap} \left(\ddot{M}, A\right) \Rightarrow \left(\ddot{K}, A\right) \ddot{\cap} \left(\ddot{M}, A\right)$ is a $\ddot{vbs}$ nbd of $\ddot{F}(e_p)$, since $\left(\ddot{G}, A\right) \ddot{\cap} \left(\ddot{H}, A\right)$ is a VBSOS $\Rightarrow \left(\ddot{K}, A\right) \ddot{\cap} \left(\ddot{M}, A\right) \in \ddot{N}_{\ddot{F}_{e_p}}$. Proves(ii)

Definition 2.3.20. ***(Vague Binary Soft Accumulation Point or Limit Point)*** *$\left(\ddot{K}, A\right)$ be a VBSSS of a VBSTS $(U_1, U_2, \ddot{\tau}_\Delta, E)_A$. Then a VBSP $\ddot{K}(e_p)$ of $(U_1, U_2, \ddot{\tau}_\Delta, E)_A$ is called a vague binary soft accumulation point (or limit point)of $\left(\ddot{K}, A\right)$, if every $\ddot{vbs}$ nbd $\left(\ddot{N}, A\right)$ of $\ddot{K}(e_p)$ contains a point $\ddot{N}(e_p)$ of $\left(\ddot{K}, A\right)$ with $\ddot{K}(e_p) \ddot{\neq} \ddot{N}(e_p)$. Set of all vague binary soft accumulation points of $\left(\ddot{K}, A\right)$ is known as the* ***vague binary soft derived set*** *of $\left(\ddot{K}, A\right)$ and is denoted by $\ddot{D}\left(\ddot{K}, A\right)$*

Theorem 2.3.21.

Let $\left(\ddot{P}, A\right)$ be a VBSSS of $\left(\ddot{U}, A\right)$. Then

1. $\left(\ddot{P}, A\right)$ *is the largest VBSOSS of* $\left(\ddot{P}, A\right)$
2. $\left(\ddot{P}, A\right)$ *is a VBSOS.* $\left(\ddot{P}, A\right)^0 \ddot{=} \left(\ddot{P}, A\right)$

Proof.

1. Let $\left(\ddot{P},A\right)$ be a VBSSS of $\left(\ddot{U},A\right)$. For any VBSP in $\left(\ddot{P},A\right)^0$ there exists a VBSOS $\left(\ddot{H},A\right)$ in $\left(\ddot{U},A\right)$ which contains that VBSP and contained in the VBSS $\left(\ddot{P},A\right)$. Clearly follows that every VBSP belongs to $\left(\ddot{H},A\right)$ also belongs to $\left(\ddot{P},A\right)$, indicates that $\left(\ddot{P},A\right)$ is a $v\ddot{b}s$ nbd of that VBSP. Now let $\left(\ddot{H},A\right)$ is VBSOS. Obviously, every VBSP in $\left(\ddot{H},A\right)$ also belongs to $\left(\ddot{P},A\right)^0$ follows $\left(\ddot{H},A\right)\ddot{\subseteq}\left(\ddot{P},A\right)^0$. Thus $\left(\ddot{P},A\right)^0$ is a $v\ddot{b}s$ nbd of any VBSP in it, Follows that $\left(\ddot{P},A\right)^0$ is a VBSOS. Let $\left(\ddot{T},A\right)$ be the vague binary soft union of all the VBSOS's of $\left(\ddot{P},A\right)$. So $\left(\ddot{P},A\right)^0\ddot{\subseteq}\left(\ddot{T},A\right)$. On the other hand if some VBSP belongs to $\left(\ddot{T},A\right)$ implies it belongs to some VBSOSS of $\left(\ddot{P},A\right)$ and so that VBSP belongs to $\left(\ddot{P},A\right)^0$. Therefore $\left(\ddot{T},A\right)\ddot{\subseteq}\left(\ddot{P},A\right)^0$. Hence $\left(\ddot{P},A\right)^0\ddot{=}\left(\ddot{T},A\right)$, the union of all VBSOSS's of $\left(\ddot{P},A\right)$. Hence $\left(\ddot{P},A\right)^0$ is the largest VBSOSS of $\left(\ddot{P},A\right)$

2. If $\left(\ddot{P},A\right)$ is VBSOS, then $\left(\ddot{P},A\right)$ is a VBSOSS $\left(\ddot{P},A\right)$. Since $\left(\ddot{P},A\right)^0$ is the largest VBSOSS of $\left(\ddot{P},A\right)$; $\left(\ddot{P},A\right)\ddot{\subseteq}\left(\ddot{P},A\right)^0$. Also it is obvious that $\left(\ddot{P},A\right)^0\ddot{\subseteq}\left(\ddot{P},A\right)$. Hence vague binary soft equality holds. Conversely, let $\left(\ddot{P},A\right)\ddot{=}\left(\ddot{P},A\right)^0$. Since $\left(\ddot{P},A\right)^0$ is a VBSOS indicates that $\left(\ddot{P},A\right)$ is also VBSOS

Theorem 2.3.22.
A VBSSS $\left(\ddot{C},A\right)$ *of* $\left(\ddot{U},A\right)$ *is a VBSCS* $\Leftrightarrow\left(\ddot{C},A\right)\ddot{=}\overline{\left(\ddot{C},A\right)}$

Proof.
Assume $\left(\ddot{C},A\right)$ is a VBSCS. Then $\left(\ddot{C},A\right)$ is vague binary soft closed superset of $\left(\ddot{C},A\right)$. But as per definition, $\overline{\left(\ddot{C},A\right)}$ is the smallest vague binary soft closed superset of $\left(\ddot{C},A\right)$ and therefore $\overline{\left(\ddot{C},A\right)}\ddot{\subseteq}\left(\ddot{C},A\right)$. But $\left(\ddot{C},A\right)\ddot{\subseteq}\overline{\left(\ddot{C},A\right)}$, always. Result follows.

Theorem 2.3.23. ***(Properties of vague binary soft interior)***
For VBSS's $\left(\ddot{D},A\right),\left(\ddot{S},A\right)$ *in a VBSTS* $\left(U_1,U_2,\ddot{\tau}_\Delta,E\right)_A$

1. $\left(\ddot{U},A\right)^{0} \ddot{=} \left(\ddot{U},A\right)$ *and* $\left(\ddot{\Phi},A\right)^{0} \ddot{=} \left(\ddot{\Phi},A\right)$
2. $\left(\ddot{D},A\right)^{0} \ddot{\subseteq} \left(\ddot{D},A\right)$
3. $\left[\left(\ddot{D},A\right)^{0}\right]^{0} \ddot{=} \left(\ddot{D},A\right)^{0}$
4. $\left(\ddot{D},A\right) \ddot{\subseteq} \left(\ddot{S},A\right) \Rightarrow \left(\ddot{D},A\right)^{0} \ddot{\subseteq} \left(\ddot{S},A\right)^{0}$
5. $\left[\left(\ddot{D},A\right) \ddot{\cap} \left(\ddot{S},A\right)\right]^{0} \ddot{=} \left(\ddot{D},A\right)^{0} \ddot{\cap} \left(\ddot{S},A\right)^{0}$

Proof.

1. $\left(\ddot{U},A\right)$ and $\left(\ddot{\Phi},A\right)$ are VBSOS's.
   Using second part of theorem 2.3.21. proof follows

2. Choose an arbitrary VBSP of $\left(\ddot{D},A\right)^{0}$. Then there exists a VBSOS $\left(\ddot{Y},A\right)$ in $\left(\ddot{X},A\right)$ which contains that VBSP and contained in $\left(\ddot{D},A\right)$. Proof follows.

3. $\left(\ddot{D},A\right)^{0}$ is a VBSOS. Using second part of theorem 2.3.21. proof follows.

4. Let $\left(\ddot{D},A\right) \ddot{\subseteq} \left(\ddot{S},A\right)$. Choose an arbitrary VBSP from $\left(\ddot{D},A\right)^{0}$. Clearly there exists a VBSOS say $\left(\ddot{G},A\right)$ such that it contains that VBSP and which is contained in $\left(\ddot{D},A\right)$, so in $\left(\ddot{S},A\right)$ also, using,our assumption. So $\left(\ddot{S},A\right)$ is a $v\ddot{b}s$ nbd of the arbitrary selected VBSP. Thus that VBSP belongs to $\left(\ddot{S},A\right)^{0}$ also. Since VBSP is selected as arbitrary the proof follows.

5. $\left[\left(\ddot{D},A\right) \ddot{\cap} \left(\ddot{S},A\right)\right] \ddot{\subseteq} \left(\ddot{D},A\right), \left[\left(\ddot{D},A\right) \ddot{\cap} \left(\ddot{S},A\right)\right] \ddot{\subseteq} \left(\ddot{S},A\right)$
   $\Rightarrow \left[\left(\ddot{D},A\right) \ddot{\cap} \left(\ddot{S},A\right)\right]^{0} \ddot{\subseteq} \left(\ddot{D},A\right)^{0}, \left[\left(\ddot{D},A\right) \ddot{\cap} \left(\ddot{S},A\right)\right]^{0} \ddot{\subseteq} \left(\ddot{S},A\right)^{0}$.
   Also $\left(\ddot{D},A\right)^{0} \ddot{\subseteq} \left(\ddot{D},A\right)$ and $\left[\left(\ddot{D},A\right)\right]^{0} \ddot{\subseteq} \left(\ddot{D},A\right)$ and $\left[\left(\ddot{S},A\right)\right]^{0} \ddot{\subseteq} \left(\ddot{S},A\right)$, using second part of current theorem.
   Therefore, $\left[\left(\ddot{D},A\right)^{0} \ddot{\cap} \left(\ddot{S},A\right)\right]^{0} \ddot{\subseteq} \left(\ddot{D},A\right) \ddot{\cap} \left(\ddot{S},A\right)$ *.Clearly,* $\left(\ddot{D},A\right)^{0} \ddot{\cap} \left(\ddot{S},A\right)^{0}$ is a VBSOS contained in $\left(\ddot{D},A\right) \ddot{\cap} \left(\ddot{S},A\right)$. But $\left[\left(\ddot{D},A\right) \ddot{\cap} \left(\ddot{S},A\right)\right]^{0}$ is the largest VBSOS contained in $\left(\ddot{D},A\right) \ddot{\cap} \left(\ddot{S},A\right)$. But $\left[\left(\ddot{D},A\right) \ddot{\cap} \left(\ddot{S},A\right)\right]^{0}$ is the largest VBSOS contained in $\left(\ddot{D},A\right) \ddot{\cap} \left(\ddot{S},A\right)$.
   So $\left(\ddot{D},A\right)^{0} \ddot{\cap} \left(\ddot{S},A\right)^{0} \ddot{\subseteq} \left[\left(\ddot{D},A\right) \ddot{\cap} \left(\ddot{S},A\right)\right]^{0}$. Proof follows.

Theorem 2.3.24. ***(Properties of vague binary soft closure)***
For VBSS's $\left(\ddot{W},A\right),\left(\ddot{T},A\right)\in\left(\ddot{U},A\right)$

1. $\overline{\left(\ddot{\Phi},A\right)}\ddot{=}\left(\ddot{\Phi},A\right)$
2. $\left(\ddot{W},A\right)\ddot{\subseteq}\overline{\left(\ddot{W},A\right)}$
3. $\overline{\overline{\left(\ddot{W},A\right)}}\ddot{=}\overline{\left(\ddot{W},A\right)}$
4. $\overline{\left(\ddot{W},A\right)\ddot{\cup}\left(\ddot{T},A\right)}\ddot{=}\overline{\left(\ddot{W},A\right)}\ddot{\cup}\overline{\left(\ddot{T},A\right)}$
5. $\overline{\left(\ddot{W},A\right)\ddot{\cap}\left(\ddot{T},A\right)}\ddot{\subseteq}\overline{\left(\ddot{W},A\right)}\ddot{\cap}\overline{\left(\ddot{T},A\right)}$

Proof.

1. $\left(\ddot{\Phi},A\right)$ is a vague binary soft clopen set [a VBSS which is both VBSOS and VBSCS]. By theorem 2.3.22. proof follows.

2. Clearly follows from the definition of $v\ddot{b}s$ cl

3. $\overline{\left(\ddot{W},A\right)}$ is obviously a VBSCS. Result follows from theorem 2.3.22.

4. Observe that $\left(\ddot{W},A\right)\ddot{\subseteq}\overline{\left(\ddot{W},A\right)}$; $\left(\ddot{T},A\right)\ddot{\subseteq}\overline{\left(\ddot{T},A\right)}$, from (2)of the current theorem. So $\left(\left(\ddot{W},A\right)\ddot{\cup}\left(\ddot{T},A\right)\right)\ddot{\subseteq}\overline{\left(\ddot{W},A\right)}\ddot{\cup}\overline{\left(\ddot{T},A\right)}$, which is a vague binary soft closed superset of $\left(\left(\ddot{W},A\right)\ddot{\cup}\left(\ddot{T},A\right)\right)$. But $\overline{\left(\ddot{W},A\right)\ddot{\cup}\left(\ddot{T},A\right)}$ is the smallest vague binary soft closed superset of $\left(\ddot{W},A\right)\ddot{\cup}\left(\ddot{T},A\right)$.
   Hence $\overline{\left(\ddot{W},A\right)\ddot{\cup}\left(\ddot{T},A\right)}\ddot{\subseteq}\overline{\left(\ddot{W},A\right)}\ddot{\cup}\overline{\left(\ddot{T},A\right)}$.
   Again $\left(\ddot{W},A\right)\ddot{\subseteq}\left(\left(\ddot{W},A\right)\ddot{\cup}\left(\ddot{T},A\right)\right)$ and $\left(\ddot{T},A\right)\ddot{\subseteq}\left(\left(\ddot{W},A\right)\ddot{\cup}\left(\ddot{T},A\right)\right)$.
   Hence $\overline{\left(\ddot{W},A\right)}\ddot{\cup}\overline{\left(\ddot{T},A\right)}\ddot{\subseteq}\overline{\left(\left(\ddot{W},A\right)\ddot{\cup}\left(\ddot{T},A\right)\right)}$.Proof follows.

5. $\left(\left(\ddot{W},A\right)\ddot{\cap}\left(\ddot{T},A\right)\right)\ddot{\subseteq}\left(\ddot{W},A\right)$ and $\left(\left(\ddot{W},A\right)\right)\ddot{\cap}\left(\ddot{T},A\right)\ddot{\subseteq}\left(\ddot{T},A\right)$
   $\Rightarrow\overline{\left(\left(\ddot{W},A\right)\ddot{\cap}\left(\ddot{T},A\right)\right)}\ddot{\subseteq}\overline{\left(\ddot{W},A\right)}$ and $\overline{\left(\left(\ddot{W},A\right)\ddot{\cap}\left(\ddot{T},A\right)\right)}\ddot{\subseteq}\overline{\left(\ddot{T},A\right)}$
   $\Rightarrow\overline{\left(\left(\ddot{W},A\right)\ddot{\cap}\left(\ddot{T},A\right)\right)}\ddot{\subseteq}\overline{\left(\ddot{W},A\right)}\ddot{\cap}\overline{\left(\ddot{T},A\right)}$

Separation Axioms for Vague Binary Soft Topological Spaces

Developing *vague binary soft separation axioms* is the main target in this section. For that, separation axioms for vague soft sets are essential.

Separation Axioms for Vague Binary Soft Set's

Definition 2.3.25. ***(Vague Binary Soft** T_0- **axiom)***
A VBSTS $(U_1, U_2, \ddot{\tau}_\Delta, E)_A$ *over* $V(U_1) \times V(U_2)$ *is said to be vague binary soft* T_0- *space if and only if for each pair of distinct vague binary soft points* $\ddot{F}(e_p)$ *and* $\ddot{G}(e_q)$ *in* $(U_1, U_2, \ddot{\tau}_\Delta, E)_A$*, there exists a VBSOS* $\left(\ddot{I}, A\right)$ *such that either* $\ddot{F}(e_p) \in \left(\ddot{I}, A\right)$ *and* $\ddot{G}(e_q) \notin (\ddot{I}, A)$ *or* $\ddot{G}(e_q) \in \left(\ddot{I}, A\right)$ *and* $\ddot{F}(e_p) \notin \left(\ddot{I}, A\right)$

Definition 2.3.26. ***(Vague Binary Soft** T_1- **axiom)***
A VBSTS $(U_1, U_2, \ddot{\tau}_\Delta, E)_A$ *over* $V(U_1) \times V(U_2)$ *is said to be vague binary soft* T_1 *space if and only if for each pair of distinct vague binary soft points* $\ddot{F}(e_p)$ *and* $\ddot{G}(e_q)$ *in* $(U_1, U_2, \ddot{\tau}_\Delta, E)_A$*, there exists VBSOS's* $\left(\ddot{I}, A\right)$ *and* $\left(\ddot{J}, A\right)$ *such that* $\ddot{F}(e_p) \in \left(\ddot{I}, A\right)$ *and* $\ddot{G}(e_q) \notin \left(\ddot{I}, A\right)$ *and* $\ddot{G}(e_q) \in \left(\ddot{J}, A\right)$ *and* $\ddot{F}(e_p) \notin \left(\ddot{J}, A\right)$

Definition 2.3.27.
(Vague Binary Soft Hausdorff Space or Vague Binary Soft** T_2- **space)
A VBSTS $(U_1, U_2, \ddot{\tau}_\Delta, E)_A$ *over* $V(U_1) \times V(U_2)$ *is said to be a vague binary soft* T_2- *space or vague binary soft hausdorff space if and only if for each pair of distinct VBSP's* $\ddot{F}(e_p)$ *and* $\ddot{G}(e_q)$ *in* $(U_1, U_2, \ddot{\tau}_\Delta, E)_A$ *there exists a* $\ddot{vbs}$ *nbd* $\left(\ddot{L}, A\right)$ *of* $\ddot{F}(e_p)$ *and a* $\ddot{vbs}$ *nbd* $\left(\ddot{M}, A\right)$ *of* $\ddot{G}(e_q)$ *such that* $\left(\left(\ddot{L}, A\right) \ddot{\cap} \left(\ddot{M}, A\right)\right) = \left(\ddot{\Phi}, A\right)$

Definition 2.3.28. ***(Vague Binary Soft Regular Space)***
A VBSTS $(U_1, U_2, \ddot{\tau}_\Delta, E)_A$ *over* $V(U_1) \times V(U_2)$ *is said to be a vague binary soft regular space at a VBSP* $\ddot{F}(e_p) \in (U_1, U_2, \ddot{\tau}_\Delta, E)_A$ *if for every VBSCS* $(\ddot{C}, A)$ *of* $(U_1, U_2, \ddot{\tau}_\Delta, E)_A$ *not containing* $\ddot{F}(e_p)$ *there exists disjoint VBSOS's* $\left(\ddot{S}, A\right)$ *and* $\left(\ddot{V}, A\right)$ *such that* $\ddot{F}(e_p) \in \left(\ddot{S}, A\right)$ *and* $\left(\ddot{C}, A\right) \ddot{\subseteq} \left(\ddot{V}, A\right)$. $(U_1, U_2, \ddot{\tau}_\Delta, E)_A$ *is said to be vague binary soft regular space if it is vague binary soft regular at each of it's vague binary soft points.*

Definition 2.3.29. ***(Vague Binary Soft** T_3 **Space)***
A VBSTS $(U_1, U_2, \ddot{\tau}_\Delta, E)_A$ *over* $V(U_1) \times V(U_2)$ *is said to be a vague binary soft* T_3 *space if it is vague binary soft regular and vague binary soft* T_1

Definition 2.3.30. ***(Vague Binary Soft Normal Space)***
A VBSTS $(U_1, U_2, \ddot{\tau}_\Delta, E)_A$ over $V(U_1) \times V(U_2)$ is said to be a vague binary soft normal if for every two disjoint VBSCS's $\left(\ddot{C}, A\right)$ and $\left(\ddot{D}, A\right)$ of $(U_1, U_2, \ddot{\tau}_\Delta, E)_A$ there exists two disjoint VBSOS's $\left(\ddot{S}, A\right)$ and $\left(\ddot{V}, A\right)$ such that $\left(\ddot{C}, A\right) \ddot{\subseteq} \left(\ddot{S}, A\right)$ and $\left(\ddot{D}, A\right) \ddot{\subseteq} \left(\ddot{S}, A\right)$, where $V(U_1), V(U_2)$ denotes power set of vague sets on (U_1, U_2) respectively.

Definition 2.3.31. ***(Vague Binary Soft T_4- Space)***
A VBSTS $(U_1, U_2, \ddot{\tau}_\Delta, E)_A$ over $V(U_1) \times V(U_2)$ is said to be a vague binary soft T_4- space if it is vague binary soft normal and vague binary soft T_1

Theorem 2.3.32.

1. *Every $\ddot{\tau}_{\Delta_1}$ space is a $\ddot{\tau}_{\Delta_0}$ space*

2. *Every $\ddot{\tau}_{\Delta_2}$ space is a $\ddot{\tau}_{\Delta_1}$ space*

Proof.

1. Proof is straight forward.

2. Let $(U_1, U_2, \ddot{\tau}_\Delta, E)_A$ be a $\ddot{\tau}_{\Delta_2}$ space. For two disjoint VBSP's $\ddot{R}(e_p)$ and $\ddot{S}(e_p)$ from $\left(\ddot{U}, A\right)$ there exists atleast one VBSOS's $\left(\ddot{B}, A\right)$ and $\left(\ddot{C}, A\right)$ such that $\ddot{R}(e_p) \in \left(\ddot{B}, A\right)$ and $\ddot{S}(e_p) \in \left(\ddot{C}, A\right)$ and $\left(\ddot{B}, A\right) \ddot{\cap} \left(\ddot{C}, A\right) = \left(\ddot{\Phi}, A\right)$ clearly implies $\ddot{R}(e_p) \notin \left(\ddot{C}, A\right)$ and $\ddot{S}(e_p) \notin \left(\ddot{B}, A\right)$ indicates vague binary soft space $(U_1, U_2, \ddot{\tau}_\Delta, E)_A$ is $\ddot{\tau}_{\Delta_1}$ also. In general, converse is not true.

2.4 Vague Binary Soft Continuity

In this section continuity is developed for vague binary soft set.

Definition 2.4.1. ***(Vague Binary Soft Continuity)***
Let $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$ and $(W_1, W_2, E^*, \ddot{\sigma}_\Delta)_B$ be two Vague Binary Soft Topological Space's over $\{U_1, U_2\}$ and $\{W_1, W_2\}$ respectively, where $A \subseteq E$, $B \subseteq E^*$. Let $\zeta_1 : U_1 \to W_1$, $\zeta_2 : U_2 \to W_2$ and $\Psi : A \to B$ be mappings. Then vague binary soft mapping $g = ((\zeta_1, \zeta_2), \Psi) : (U_1, U_2, E, \ddot{\tau}_\Delta)_A \to (W_1, W_2, E^*, \ddot{\sigma}_\Delta)_B$ is said to be vague binary soft continuous if the inverse image under g of any vague binary soft open

set $\left(\ddot{H}, B\right) \in \ddot{\sigma}_\Delta$ *with* $H : B \to V(W_1) \times V(W_2)$*, there exists some vague binary soft open set* $\left(\ddot{G}, A\right) \in \ddot{\tau}_\Delta$ *with* $G : A \to V(U_1) \times V(U_2)$

Remark 2.4.2.

Vague Binary Soft Classes are collections of vague binary soft sets. Vague Binary Soft Class is denoted by $\left(\overset{\bowtie}{W}, A\right)$.

Example 2.4.3.

Let $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$ *and* $(W_1, W_2, E^*, \ddot{\sigma}_\Delta)_B$ *be two VBSTS's. Let* $U_1 = \{p, q, r\}$, $U_2 = \{x, y\}$ *be a common universe with a fixed parameter set* $A = \{e_2, e_4\} \subseteq E = \{e_1, e_2, e_3, e_4, e_5\}$*. Let* $V_1 = \{m, n, d\}$, $V_2 = \{k, l\}$ *be another common universe with fixed parameter set* $B = \left\{e_1^{'}, e_3^{'}, e_4^{'}\right\} \subseteq E^* = \left\{e_1^{'}, e_2^{'}, e_3^{'}, e_4^{'}, e_5^{'}, e_6^{'}\right\}$*. Let* $\zeta_1 : U_1 \to V_1$ *is defined as* $\zeta_1(p) = d, \zeta_1(q) = n, \zeta_1(r) = m, \zeta_2 : U_2 \to V_2$ *is defined as* $\zeta_2(x) = l, \zeta_2(y) = k$ *and* $\Psi : E \to E^*$ *is defined as* $\Psi(e_2) = e_3^{'}, \Psi(e_4) = e_1^{'}$

Let $\ddot{\tau}_\Delta = \left\{\left(\ddot{\Phi}, A\right), \left(\ddot{F}_1, A\right), \left(\ddot{F}_2, A\right), \left(\ddot{F}_3, A\right), \left(\ddot{F}_4, A\right), \left(\ddot{U}, A\right)\right\}$ *be a VBST,*

$$(\ddot{F}_1, A) = \left\{\begin{array}{l} \left(e_2, \left(\left\langle \frac{[0.3, 0.4]}{p}, \frac{[0.1, 0.5]}{q}, \frac{[0.6, 0.7]}{r} \right\rangle, \left\langle \frac{[0.3, 0.6]}{x}, \frac{[0.7, 0.8]}{y} \right\rangle\right)\right) \\ \left(e_4, \left(\left\langle \frac{[0.1, 0.8]}{p}, \frac{[0.2, 0.4]}{q}, \frac{[0.1, 0.7]}{r} \right\rangle, \left\langle \frac{[0.2, 0.7]}{x}, \frac{[0.5, 0.7]}{y} \right\rangle\right)\right) \end{array}\right\}$$

$$\left(\ddot{F}_2, A\right) = \left\{\begin{array}{l} \left(e_2, \left(\left\langle \frac{[0.4, 0.5]}{p}, \frac{[0.1, 0.3]}{q}, \frac{[0.2, 0.6]}{r} \right\rangle, \left\langle \frac{[0.1, 0.7]}{x}, \frac{[0.2, 0.9]}{y} \right\rangle\right)\right) \\ \left(e_4, \left(\left\langle \frac{[0.1, 0.8]}{p}, \frac{[0.2, 0.4]}{q}, \frac{[0.1, 0.7]}{r} \right\rangle, \left\langle \frac{[0.2, 0.7]}{x}, \frac{[0.5, 0.7]}{y} \right\rangle\right)\right) \end{array}\right\}$$

$$\left(\ddot{F}_3, A\right) = \left\{\begin{array}{l} \left(e_2, \left(\left\langle \frac{[0.3, 0.4]}{p}, \frac{[0.1, 0.3]}{q}, \frac{[0.2, 0.6]}{r} \right\rangle, \left\langle \frac{[0.1, 0.6]}{x}, \frac{[0.2, 0.8]}{y} \right\rangle\right)\right) \\ \left(e_4, \left(\left\langle \frac{[0.1, 0.6]}{p}, \frac{[0.2, 0.3]}{q}, \frac{[0.1, 0.6]}{r} \right\rangle, \left\langle \frac{[0.2, 0.6]}{x}, \frac{[0.5, 0.7]}{y} \right\rangle\right)\right) \end{array}\right\}$$

$$\left(\ddot{F}_4, A\right) = \left\{\begin{array}{l} \left(e_2, \left(\left\langle \frac{[0.4, 0.5]}{p}, \frac{[0.1, 0.5]}{q}, \frac{[0.6, 0.7]}{r} \right\rangle, \left\langle \frac{[0.3, 0.7]}{x}, \frac{[0.7, 0.9]}{y} \right\rangle\right)\right) \\ \left(e_4, \left(\left\langle \frac{[0.1, 0.8]}{p}, \frac{[0.2, 0.4]}{q}, \frac{[0.4, 0.7]}{r} \right\rangle, \left\langle \frac{[0.5, 0.7]}{x}, \frac{[0.6, 0.7]}{y} \right\rangle\right)\right) \end{array}\right\}$$

Let $\ddot{\sigma}_\Delta = \left\{\left(\ddot{\Phi}, B\right), \left(\ddot{G}_1, B\right), \left(\ddot{G}_2, B\right), \left(\ddot{G}_3, B\right), \left(\ddot{G}_4, B\right), \left(\ddot{U}, B\right)\right\}$ *be a VBST*

$$\left(\ddot{G}_1, B\right) = \left\{\begin{array}{l} \left(e_1^{'}, \left(\left\langle \frac{[0.1, 0.7]}{m}, \frac{[0.2, 0.4]}{n}, \frac{[0.1, 0.8]}{d} \right\rangle, \left\langle \frac{[0.5, 0.7]}{k}, \frac{[0.2, 0.7]}{l} \right\rangle\right)\right) \\ \left(e_3^{'}, \left(\left\langle \frac{[0.2, 0.6]}{p}, \frac{[0.1, 0.3]}{q}, \frac{[0.4, 0.5]}{r} \right\rangle, \left\langle \frac{[0.2, 0.9]}{x}, \frac{[0.1, 0.7]}{y} \right\rangle\right)\right) \end{array}\right\}$$

$$\left(\ddot{G}_2, B\right) = \left\{ \begin{array}{l} \left(e_1^{'}, \left(\left\langle \frac{[0.4, 0.6]}{m}, \frac{[0.2, 0.3]}{n}, \frac{[0.1, 0.6]}{d} \right\rangle, \left\langle \frac{[0.6, 0.7]}{k}, \frac{[0.5, 0.6]}{l} \right\rangle\right)\right) \\ \left(e_3^{'}, \left(\left\langle \frac{[0.6, 0.7]}{m}, \frac{[0.1, 0.5]}{n}, \frac{[0.3, 0.4]}{d} \right\rangle, \left\langle \frac{[0.7, 0.8]}{k}, \frac{[0.3, 0.6]}{l} \right\rangle\right)\right) \end{array} \right\}$$

$$\left(\ddot{G}_3, B\right) = \left\{ \begin{array}{l} \left(e_1^{'}, \left(\left\langle \frac{[0.1, 0.6]}{m}, \frac{[0.2, 0.3]}{n}, \frac{[0.1, 0.6]}{d} \right\rangle, \left\langle \frac{[0.5, 0.7]}{k}, \frac{[0.2, 0.6]}{l} \right\rangle\right)\right) \\ \left(e_3^{'}, \left(\left\langle \frac{[0.2, 0.6]}{m}, \frac{[0.1, 0.3]}{n}, \frac{[0.3, 0.4]}{d} \right\rangle, \left\langle \frac{[0.2, 0.8]}{k}, \frac{[0.1, 0.6]}{l} \right\rangle\right)\right) \end{array} \right\}$$

$$\left(\ddot{G}_4, B\right) = \left\{ \begin{array}{l} \left(e_1^{'}, \left(\left\langle \frac{[0.4, 0.7]}{m}, \frac{[0.2, 0.4]}{n}, \frac{[0.1, 0.8]}{d} \right\rangle, \left\langle \frac{[0.6, 0.7]}{k}, \frac{[0.5, 0.7]}{l} \right\rangle\right)\right) \\ \left(e_3^{'}, \left(\left\langle \frac{[0.6, 0.7]}{m}, \frac{[0.1, 0.5]}{n}, \frac{[0.4, 0.5]}{d} \right\rangle, \left\langle \frac{[0.7, 0.9]}{k}, \frac{[0.3, 0.7]}{l} \right\rangle\right)\right) \end{array} \right\}$$

$h^{-1}\left(\ddot{\Phi}, B\right) = \left(\ddot{\Phi}, A\right)$, $h^{-1}\left(\ddot{G}_1, B\right) = \left(\ddot{F}_2, A\right)$, $h^{-1}\left(\ddot{G}_2, B\right) = \left(\ddot{F}_1, A\right)$,
$h^{-1}\left(\ddot{G}_3, B\right) = \left(\ddot{F}_3, A\right)$, $h^{-1}\left(\ddot{G}_4, B\right) = \left(\ddot{F}_4, A\right)$, $h^{-1}\left(\ddot{U}, B\right) = \left(\ddot{U}, A\right)$

So h is vague binary soft continuous mapping.

Definition 2.4.4. ***(Vague Binary Soft Open Map)***
Let $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$ and $(V_1, V_2, E, \ddot{\sigma}_\Delta)_B$ be two VBSTS's. A vague binary soft map $g : (U_1, U_2, E, \ddot{\tau}_\Delta)_A \to (V_1, V_2, E, \ddot{\sigma}_\Delta)_B$ is called a vague binary soft open map if $g\left(\ddot{k}, A\right)$ is vague binary soft open in $(V_1, V_2, E, \ddot{\sigma}_\Delta)_B$ for each vague binary soft open set $\left(\ddot{K}, A\right)$ in $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$.

Theorem 2.4.5.
Let $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$ and $(V_1, V_2, E, \ddot{\sigma}_\Delta)_B$ be two VBSTS's. $g : (U_1, U_2, E, \ddot{\tau}_\Delta)_A \to (V_1, V_2, E, \ddot{\sigma}_\Delta)_B$ be a mapping. Then the following conditions are equivalent:

1. *$g : (U_1, U_2, E, \ddot{\tau}_\Delta)_A \to (V_1, V_2, E, \ddot{\sigma}_\Delta)_B$ is a vague binary soft continuous mapping*
2. *$g^{-1}\left(\ddot{H}, A\right)$ is VBSCS in $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$ whenever $\left(\ddot{H}, A\right)$ is VBSCS in $(V_1, V_2, E, \ddot{\sigma}_\Delta)_B$*

Proof.

1. Let g be a vague binary soft continuous mapping. $\left(\ddot{H}, A\right)$ be any VBSCS of $(V_1, V_2, E, \ddot{\sigma}_\Delta)_B$. Then $\left(\ddot{H}, A\right)^c$ is VBSOS in $(V_1, V_2, E, \ddot{\sigma}_\Delta)_B$. Since g is vague binary soft continuous, $g^{-1}\left(\ddot{H}, A\right)^c$ is VBSOS in $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$. Therefore $g^{-1}\left(\ddot{H}, A\right)$ is VBSCS in $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$

2. Let $\left(\ddot{H}, A\right)$ be a VBSCS of $(V_1, V_2, E, \ddot{\sigma}_\Delta)_B$. Then using assumption $g^{-1}\left(\ddot{H}, A\right)$ is VBSCS in $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$. Therefore g is vague binary soft continuous mapping

Theorem 2.4.6.

A vague binary soft mapping $g : (U_1, U_2, E, \ddot{\tau}_\Delta)_A \to (V_1, V_2, E, \ddot{\sigma}_\Delta)_B$ *is vague binary soft continuous if and only if* $g\overline{\left(\ddot{H}, A\right)} \subset \overline{g\left(\ddot{H}, A\right)}$ *for every* $\left(\ddot{H}, A\right) \subset (U_1, U_2, E, \ddot{\tau}_\Delta)_A$

Proof.

Let g be a vague binary soft continuous and $\left(\ddot{H}, A\right)$ be any VBSSS of $\left(\overset{\bowtie}{U}, A\right)$. Let $g\left(\ddot{H}, A\right) = \left(\ddot{N}, A\right)$. Therefore $\left(\ddot{N}, A\right) \subset \left(\overset{\bowtie}{V}, A\right)$. But $\overline{\left(\ddot{N}, A\right)}$ is VBSCS of $\left(\overset{\bowtie}{U}, A\right) \Rightarrow g^{-1}\overline{g\left(\ddot{H}, A\right)}$ is a VBSCS of $\left(\overset{\bowtie}{U}, A\right) \Rightarrow \overline{g^{-1}\overline{g\left(\ddot{H}, A\right)}} = g^{-1}\overline{g\left(\ddot{H}, A\right)}$. Now $g\left(\ddot{H}, A\right) \subseteq \overline{g\left(\ddot{H}, A\right)} \Rightarrow \left(\ddot{H}, A\right) \subseteq g^{-1}\overline{g\left(\ddot{H}, A\right)} \Rightarrow \overline{\left(\ddot{H}, A\right)} \subseteq \overline{g^{-1}\overline{g\left(\ddot{H}, A\right)}} = g^{-1}\overline{g\left(\ddot{H}, A\right)} \Rightarrow g\overline{\left(\ddot{H}, A\right)} \subseteq \overline{g\left(\ddot{H}, A\right)}$. Conversely, let $g : (U_1, U_2, E, \ddot{\tau}_\Delta)_A \to (V_1, V_2, E, \ddot{\sigma}_\Delta)_A$ is vague binary soft mapping with $g\overline{\left(\ddot{H}, A\right)} \subseteq \overline{g\left(\ddot{H}, A\right)}$ for any VBSS $\left(\ddot{H}, A\right)$ of $\left(\overset{\bowtie}{U}, A\right)$. Let $\left(\ddot{M}, A\right)$ be an arbitrary VBSCS in $\left(\overset{\bowtie}{V}, A\right) \Rightarrow \overline{\left(\ddot{M}, A\right)} = \left(\ddot{M}, A\right)$. Now $\overline{g^{-1}\left(\ddot{M}, A\right)}$ is a VBSSS of $\left(\overset{\bowtie}{U}, A\right) \Rightarrow \overline{g^{-1}\left(\ddot{M}, A\right)} \subseteq g^{-1}\left(\ddot{M}, A\right)$. But $g^{-1}\left(\ddot{M}, A\right) \subseteq \overline{g^{-1}\left(\ddot{M}, A\right)} \Rightarrow g^{-1}\left(\ddot{M}, A\right)$ is a VBSCS in $\left(\overset{\bowtie}{U}, A\right) \Rightarrow$ g is vague binary soft continuous mapping.

Theorem 2.4.7.

Let $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$ *and* $(V_1, V_2, E, \ddot{\sigma}_\Delta)_A$ *be VBSTS's.*
$f : (U_1, U_2, E, \ddot{\tau}_\Delta)_A \to (V_1, V_2, E, \ddot{\sigma}_\Delta)_A$. *Then the following are equivalent:*

1. *f is vague binary soft continuous*
2. *For every VBSSS* $\left(\ddot{J}, A\right)$ *of* $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$. $f\overline{\left(\ddot{J}, A\right)} \subset \overline{f\left(\ddot{J}, A\right)}$
3. *For every VBSCS* $\left(\ddot{T}, A\right)$ *of* $(V_1, V_2, E, \ddot{\sigma}_\Delta)_A$, *the VBSS* $f^{-1}\left(\ddot{T}, A\right)$ *is VBSCS in* $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$

4. *For each* $\ddot{F}_e \in \left(\overset{\bowtie}{U}, A\right)$ *and each* $v\ddot{b}s$ *nbd* $\left(\ddot{T}, A\right)$ *of* $f(\ddot{F}_e)$ *there is a* $v\ddot{b}s$ *nbd* $\left(\ddot{J}, A\right)$ *of* $\ddot{F}_e$ *such that* $f\left(\ddot{J}, A\right) \subset \left(\ddot{T}, A\right)$. *In this case f is vague binary soft continuous at the point* $\ddot{F}_e \in \left(\overset{\bowtie}{U}, A\right)$

Proof.
Here (i) $\Longrightarrow$ (ii)$\Longrightarrow$ (i) and (i)$\Longrightarrow$ (iv)$\Longrightarrow$ (i). From theorem 2.4.6 and 2.4.7 (i)$\Longrightarrow$ (ii) and (iii)$\Longrightarrow$ (i)are clear. To prove (ii)$\Rightarrow$ (iii). Let $\left(\ddot{J}, A\right)$ be VBSCS in $\left(\overset{\bowtie}{V}, A\right)$ and let $\left(\ddot{T}, A\right) = f^{-1}\left(\ddot{J}, A\right)$. To prove $\left(\ddot{T}, A\right)$ is VBSCS $\left(\overset{\bowtie}{U}, A\right)$. By elementary set theory, $f\left(\ddot{T}, A\right) = f\left(f^{-1}\left(\ddot{J}, A\right)\right) \subset \left(\ddot{V}, A\right)$. Hence if $\ddot{F}_e \in \overline{\left(\ddot{T}, A\right)}$, $f\left(\ddot{F}_e\right) \in f\overline{\left(\ddot{T}, A\right)} \subset \overline{f\left(\ddot{T}, A\right)} \subset \overline{\left(\ddot{J}, A\right)} = \left(\ddot{J}, A\right)$. Therefore $\ddot{F}_e \in f^{-1}\left(\ddot{J}, A\right) = \left(\ddot{T}, A\right)$. Thus $\overline{\left(\ddot{T}, A\right)} \subset \left(\ddot{T}, A\right)$ and $\left(\ddot{T}, A\right) \subset \overline{\left(\ddot{T}, A\right)}$. Combining $\overline{\left(\ddot{T}, A\right)} = \left(\ddot{T}, A\right) \Rightarrow \overline{\left(\ddot{T}, A\right)}$ is VBSCS in $\left(\overset{\bowtie}{U}, A\right) \Rightarrow f^{-1}\left(\ddot{J}, A\right)$ is VBSCS in ($\overset{\bowtie}{U}$, A)as required. To prove (i)$\Rightarrow$ (iv), let $\ddot{F}_e \in \left(\overset{\bowtie}{U}, A\right)$ and $\left(\ddot{T}, A\right)$ be a v$\ddot{b}$s nbd of $f(\ddot{F}_e)$. Then the set $\left(\ddot{J}, A\right) = f^{-1}(\ddot{T}, A)$ is a v$\ddot{b}$s nbd of $\ddot{F}_e$ such that $f(\ddot{J}, A) \subset (\ddot{T}, A)$. To prove (iv) $\Rightarrow$ (i), let $\left(\ddot{T}, A\right)$ be a VBSOS of $\left(\overset{\bowtie}{V}, A\right)$. Let $\ddot{F}_e$ be a VBSP of $f^{-1}\left(\ddot{T}, A\right) \Rightarrow f\left(\ddot{F}_e\right) \in \left(\ddot{T}, A\right)$. Therefore by hypotheses there exists a v$\ddot{b}$s nbd $\left(\ddot{R}, A\right)$ of $\ddot{F}_e$ such that $f\left(\ddot{R}, A\right) \subset \left(\ddot{T}, A\right)$. It follows that $f^{-1}\left(\ddot{T}, A\right)$ can be written as the vague binary soft union of VBSOS's , hence it is VBSOS $\Rightarrow$ f is a vague binary soft continuous mapping.

Theorem 2.4.8.
Let $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$ *and* $(V_1, V_2, E, \ddot{\sigma}_\Delta)_A$ *and* $\left(W_1, W_2, E, \ddot{\vartheta}_\Delta\right)_A$ *be three VBSTS's. If* $g : (U_1, U_2, E, \ddot{\tau}_\Delta) \to (V_1, V_2, E, \ddot{\sigma}_\Delta)_A$ *and* $h : (V_1, V_2, E, \ddot{\sigma}_\Delta)_A$ $\to \left(W_1, W_2, E, \ddot{\vartheta}_\Delta\right)_A$ *are both vague binary soft continuous then their composition* $h \circ g : (U_1, U_2, E, \ddot{\tau}_\Delta)_A \to \left(W_1, W_2, E, \ddot{\vartheta}_\Delta\right)_A$ *are both vague binary soft continuous*

Proof.
Let $\left(\ddot{V}, A\right)$ be a VBSS in $\left(\overset{\bowtie}{V}, A\right)$. Then $\left((h \circ g)^{-1}\left(\ddot{V}, A\right)\right) = (g^{-1}.h^{-1})\left(\ddot{V}, A\right)$. Since g is vague binary soft continuous, $\left((g^{-1}.h^{-1})\left(\ddot{V}, A\right)\right)$ is Vague Binary Soft Open Set in $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$.

Theorem 2.4.9.

A vague binary soft mapping $k : (U_1, U_2, E, \ddot{\tau}_\Delta)_A \to (V_1, V_2, E, \ddot{\sigma}_\Delta)_A$ *is vague binary soft continuous if and only if* $k^{-1}\left[\ddot{vbs}\ int\left(\ddot{P}, A\right) \subset \ddot{vbs}\ int\left[k^{-1}\left(\ddot{P}, A\right)\right]\right]$ *for every* $\left(\ddot{P}, A\right) \subset \left(\overset{\bowtie}{V}, A\right)$

Proof.

Let $\left(\ddot{P}, A\right)$ be a VBSS in $\left(\overset{\bowtie}{V}, A\right)$ and also let k be vague soft continuous mapping. Clearly $\ddot{vbs}\ int\left(\ddot{P}, A\right)$ is VBSOS of $\left(\overset{\bowtie}{U}, A\right)$. But $\ddot{vbs}\ int\left(\ddot{P}, A\right) \subset \left(\ddot{P}, A\right) \implies k^{-1}\left[\left(\ddot{P}, A\right)\right] \subset k^{-1}\left(\ddot{P}, A\right) \implies \ddot{vbs}\ int\left[k^{-1}\left[\ddot{vbs}\ int\left(\ddot{P}, A\right)\right]\right] = \ddot{vbs}\ int\left[k^{-1}\left(\ddot{P}, A\right)\right] = k^{-1}\left(\ddot{P}, A\right)$. Conversely, let $\left(\ddot{P}, A\right)$ be an arbitrary VBSOS in $\left(\overset{\bowtie}{V}, A\right)$so that $\ddot{vbs}\ int\left(\ddot{P}, A\right) = \left(\ddot{P}, A\right)$. By hypotheses, $k^{-1}\left[\ddot{vbs}\ int\left(\ddot{P}, A\right)\right] \subset \ddot{vbs}\ int\left[k^{-1}\left(\ddot{P}, A\right)\right] \implies k^{-1}\left(\ddot{P}, A\right) \subset \ddot{vbs}\ int\left[k^{-1}\left(\ddot{P}, A\right)\right]$. But $\ddot{vbs}\ int\ \left[k^{-1}\left(\ddot{P}, A\right)\right] \subset k^{-1}\left(\ddot{P}, A\right)$ always $\implies \ddot{vbs}\ int\left[k^{-1}\left(\ddot{P}, A\right)\right] = k^{-1}\left(\ddot{P}, A\right) \implies k^{-1}\left(\ddot{P}, A\right)$ is VBSOS of $\left(\overset{\bowtie}{U}, A\right)$ $\implies$ k is a vague binary soft continuous mapping

Theorem 2.4.10.

If f and g are vague binary soft continuous functions on a VBSTS $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$ *with values in a vague binary soft hausdorff space* $(V_1, V_2, E, \ddot{\sigma}_\Delta)_A$ *then* $\left\{\ddot{G}_e \in (U_1, U_2, E, \ddot{\tau}_\Delta)_A : f\left(\ddot{G}_e\right) = g\left(\ddot{G}_e\right)\right\}$ *is a VBSCS*

Proof.

Let $T = \left\{\ddot{G}_e \in (U_1, U_2, E, \ddot{\tau}_\Delta)_A : f\left(\ddot{G}_e\right) = g\left(\ddot{G}_e\right)\right\}$. To prove T is vague binary soft closed, it is enough to prove that $T^c = \left\{\ddot{G}_e \in (U_1, U_2, E, \ddot{\tau}_\Delta)_A : f\left(\ddot{G}_e\right) \neq g\left(\ddot{G}_e\right)\right\}$ is VBSOS. Let $\ddot{R}_e$ be a VBSP of T^c. Then $f\left(\ddot{R}_e\right) \neq g\left(\ddot{R}_e\right)$ with $f\left(\ddot{R}_e\right)$. $g\left(\ddot{R}_e\right) \in (V_1, V_2, E, \ddot{\sigma}_\Delta)_A$. Since $(V_1, V_2, E, \ddot{\sigma}_\Delta)_A$ is vague binary soft hausdorff, there exists VBSOS's $\left(\ddot{M}, A\right)$ and $\left(\ddot{N}, A\right) \in (V_1, V_2, E, \ddot{\sigma}_\Delta)_A$ such that $f\left(\ddot{R}_e\right) \in \left(\ddot{M}, A\right)$, $g\left(\ddot{R}_e\right) \in \left(\ddot{N}, A\right)$ and $\left(\ddot{M}, A\right) \cap \left(\ddot{N}, A\right) = \left(\ddot{\Phi}, A\right)$. Since f and g are vague binary soft continuous, f^{-1}, g^{-1} are VBSOS's in $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$ such that $\ddot{R}_e \in f^{-1}\left(\ddot{M}, A\right)$ and $\ddot{R}_e \in g^{-1}\left(\ddot{N}, A\right)$ and so $\ddot{R}_e \in \left(\ddot{M}, A\right) \cap \left(\ddot{N}, A\right) = \left(\ddot{I}, A\right)$ also. To prove that $\left(\ddot{I}, A\right) \subset T^c$. Assume $\left(\ddot{I}, A\right) \not\subseteq T^c$. Then there exists at least one point in $\left(\ddot{I}, A\right)$, say $\ddot{D}_e$ such that $\ddot{D}_e \neq T^c$ with respect to $(V_1, V_2, E, \ddot{\sigma}_\Delta)_A$. Now if $\ddot{D}_e \in \left(\ddot{I}, A\right) \implies \ddot{D}_e \in f^{-1}\left(\ddot{M}, A\right)$ and $\ddot{D}_e \in g^{-1}\left(\ddot{N}, A\right) \implies f\left(\ddot{D}_e\right) \in \left(\ddot{M}, A\right)$ and $g\left(\ddot{D}_e\right) \in \left(\ddot{N}, A\right)$. Also $\ddot{D}_e \notin T^c \implies \ddot{D}_e \in T \implies f\left(\ddot{D}_e\right) = g\left(\ddot{D}_e\right)$ so

$f\left(\ddot{D}_e\right) \in \left(\ddot{M}, A\right) \cap \left(\ddot{N}, A\right)$ is a contradiction. Thus $\ddot{D}_e \in \left(\ddot{I}, A\right) \subset T^c$. Since $\ddot{D}_e$ is taken as arbitrary T^c contains a $\ddot{vbsnbd}$ of each of its points. Therefore T^c is VBSOS $\implies$ T is VBSCS $\implies \left\{\ddot{G}_e \in (U_1, U_2, E, \ddot{\tau}_\Delta)_A : f\left(\ddot{G}_e\right) = g\left(\ddot{G}_e\right)\right\}$ is VBSCS.

Theorem 2.4.11. ***(Pasting Lemma in VBSTS's)***

Let $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$ and $(V_1, V_2, E, \ddot{\sigma}_\Delta)_A$ be two Vague Binary Soft Topological Space's with $\left(\overset{\bowtie}{U}, A\right) = \left(\ddot{F}, A\right) \ddot{\cup} \left(\ddot{G}, A\right)$ where $\left(\ddot{F}, A\right)$ and $\left(\ddot{G}, A\right)$ are VBSCS in $\left(\overset{\bowtie}{U}, A\right)$. Let $f : \left(\ddot{F}, A\right) \to (V_1, V_2, E, \ddot{\sigma}_\Delta)_A$ and $g : \left(\ddot{G}, A\right) \to (V_1, V_2, E, \ddot{\sigma}_\Delta)_A$ be vague binary soft continuous. If $f\left(\ddot{N}_e\right) = g\left(\ddot{N}_e\right)$ for every $\ddot{N}_e \in \left(\ddot{N}, A\right) \ddot{\cap} \left(\ddot{G}, A\right)$ then f and g combine to give a vague binary soft continuous function $h : (U_1, U_2, E, \ddot{\tau}_\Delta)_A \to (V_1, V_2, E, \ddot{\sigma}_\Delta)_A$ defined by setting $h\left(\ddot{F}_e\right) = f\left(\ddot{N}_e\right)$ if $\ddot{N}_e \in \left(\ddot{F}, A\right)$ and $h\left(\ddot{N}_e\right) = g\left(\ddot{N}_e\right)$ if $\ddot{N}_e \in \left(\ddot{G}, A\right)$

Proof.

Let $\left(\ddot{C}, A\right)$ be a VBSCS in $(V_1, V_2, E, \ddot{\sigma}_\Delta)_A$, f is vague binary soft continuous $\implies f^{-1}\left(\ddot{C}, A\right)$ is VBSCS in $\left(\ddot{F}, A\right)$ in turn which is VBSCS in $\left(\overset{\bowtie}{U}, A\right)$. g is vague binary soft continuous $\implies g^{-1}\left(\ddot{C}, A\right)$ is VBSCS $\left(\ddot{G}, A\right)$, in turn which is VBSCS in $\left(\overset{\bowtie}{U}, A\right)$. Hence their vague binary soft union $f^{-1}\left(\ddot{C}, A\right) \ddot{\cup} g^{-1}\left(\ddot{C}, A\right)$ is VBSCS in $\left(\overset{\bowtie}{U}, A\right)$. Let $h^{-1}\left(\ddot{C}, A\right) = f^{-1}\left(\ddot{C}, A\right) \ddot{\cup} g^{-1}\left(\ddot{C}, A\right) \implies h : (U_1, U_2, E, \ddot{\tau}_\Delta)_A \to (V_1, V_2, E, \ddot{\sigma}_\Delta)_A$ is vague binary soft continuous.

Theorem 2.4.12.

If $\left(\ddot{D}, A\right)$ is a VBSCSS of a VBSTS $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$ and if any vague binary soft continuous extension $k : \left(\ddot{D}, A\right) \to [-1, 1]$ has a vague binary soft continuous extension to $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$ then $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$ is vague binary soft normal space

Proof.

Let $\left(\ddot{R}, A\right)$ and $\left(\ddot{T}, A\right)$ be disjoint VBSCS's in a VBSTS $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$. Then $\left(\ddot{R}, A\right) \ddot{\cup} \left(\ddot{T}, A\right)$ is a VBSCS in $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$. Define a vague binary soft continuous function $k : \left(\ddot{R}, A\right) \ddot{\cup} \left(\ddot{T}, A\right) \to [-1, 1]$ by $k\left(\ddot{M}_e\right) = 1$ if $\ddot{M}_e \in \left(\ddot{T}, A\right)$. By hypothesis, k has a vague binary soft extension $h : (U_1, U_2, E, \ddot{\tau}_\Delta)_A \to [-1, 1]$. Since k is vague binary soft continuous, $[-1, 0)$ and $(0, 1]$. Also $\left(\ddot{R}, A\right) \subseteq \left(\ddot{S}, A\right)$, VBSOS's in $[-1, 1]$. Therefore $\left(\ddot{S}, A\right) = k^{-1}[-1, 0)$ and $\left(\ddot{J}, A\right) = k^{-1}(0, -1]$.

Also $\left(\ddot{R},A\right) \subseteq \left(\ddot{S},A\right)$, $\left(\ddot{T},A\right) \subseteq \left(\ddot{J},A\right)$, $\left(\ddot{S},A\right) \ddot{\cap} \left(\ddot{J},A\right) = \left(\ddot{\Phi},A\right)$. Hence $(U_1,U_2,E,\ddot{\tau}_\Delta)_A$ is vague binary soft normal space.

Theorem 2.4.13.

Homeomorphic image of a vague binary soft normal space is vague binary soft normal space

Proof.

Let $(U_1,U_2,E,\ddot{\tau}_\Delta)_A$ and $(V_1,V_2,E,\ddot{\sigma}_\Delta)_A$ be two VBSTS's. Let $f:(U_1,U_2,E,\ddot{\tau}_\Delta)_A \to (V_1,V_2,E,\ddot{\sigma}_\Delta)_A$ be a vague binary soft homeomorphism. Let $(U_1,U_2,E,\ddot{\tau}_\Delta)_A$ be vague binary soft normal space. Let $\left(\ddot{E},A\right)$ and $\left(\ddot{F},A\right)$ be any pair of disjoint Vague Binary Soft Closed Set's in $(V_1,V_2,E,\ddot{\sigma}_\Delta)_A$. $f^{-1}\left(\ddot{E},A\right)$ and $f^{-1}\left(\ddot{F},A\right)$ are Vague Binary Soft Closed Set's in $(U_1,U_2,E,\ddot{\tau}_\Delta)_A$ using vague binary soft continuity of f. Now $f^{-1}\left(\ddot{E},A\right)\ddot{\cap}f^{-1}\left(\ddot{F},A\right) = f^{-1}\left[\left(\ddot{E},A\right)\ddot{\cap}\left(\ddot{F},A\right)\right] = f^{-1}\left(\ddot{\Phi},A\right) = \left(\ddot{\Phi},A\right)$. Since $(U_1,U_2,E,\ddot{\tau}_\Delta)_A$ is vague binary soft normal space, there exists VBSOS's $\left(\ddot{G},A\right)$ and $\left(\ddot{H},A\right)$ such that $f^{-1}\left(\ddot{E},A\right) \subseteq \left(\ddot{G},A\right)$, $f^{-1}\left(\ddot{F},A\right) \subseteq \left(\ddot{H},A\right)$ and $\left(\ddot{G},A\right)\ddot{\cap}\left(\ddot{H},A\right) = \left(\ddot{\Phi},A\right)$. Thus $f\left[f^{-1}\left(\ddot{F},A\right)\right] \subseteq f\left(\ddot{G},A\right)$, $f\left[f^{-1}\left(\ddot{F},A\right)\right] \subseteq f\left(\ddot{H},A\right)$ and $f\left[\left(\ddot{G},A\right)\ddot{\cap}\left(\ddot{H},A\right)\right] = \left(\ddot{\Phi},A\right)$. Using vague binary soft homeomorphism, f is vague binary soft open mapping $\Longrightarrow f\left(\ddot{G},A\right)$ and $f\left(\ddot{H},A\right)$ are VBSOS's in $(V_1,V_2,E,\ddot{\sigma}_\Delta)_A$. Therefore there exists disjoint VBSOS's $f\left(\ddot{G},A\right)$ and $f\left(\ddot{H},A\right)$ such that $\left(\ddot{E},A\right) \subseteq f\left(\ddot{G},A\right)$, $\left(\ddot{F},A\right) \subseteq f\left(\ddot{H},A\right)$. Hence $(V_1,V_2,E,\ddot{\sigma}_\Delta)_A$ is vague binary soft normal space.

Theorem 2.4.14.

Let $h:(U_1,U_2,E,\ddot{\tau}_\Delta)_A \to (V_1,V_2,E,\ddot{\sigma}_\Delta)_A$ be a one-one, vague binary soft continuous mapping with $(V_1,V_2,E,\ddot{\sigma}_\Delta)_A$ is a vague binary soft hausdorff space and then $(U_1,U_2,E,\ddot{\tau}_\Delta)_A$ is also a vague binary soft hausdorff space

Proof.

Let $\ddot{B}_e$ and $\ddot{K}_e$ be two distinct VBSP's of $(U_1,U_2,E,\ddot{\tau}_\Delta)_A$. Since h is one-one, $h\left(\ddot{B}_e\right) = \ddot{T}_e$ and $h\left(\ddot{K}_e\right) = \ddot{P}_e$ with $\ddot{T}_e \neq \ddot{P}_e$. Using vague binary soft hausdorff property of $(V_1,V_2,E,\ddot{\sigma}_\Delta)_A$ there exists disjoint VBSOS's $\left(\ddot{G},A\right)$ and $\left(\ddot{H},A\right)$ such that $\ddot{T}_e \in \left(\ddot{G},A\right)$ and $\ddot{P}_e \in \left(\ddot{H},A\right)$. Therefore $\ddot{B}_e \in h^{-1}\left(\ddot{G},A\right)$, $\ddot{K}_e \in h^{-1}\left(\ddot{H},A\right)$. Also $\left(\ddot{G},A\right)\ddot{\cap}\left(\ddot{H},A\right) = \left(\ddot{\Phi},A\right) \Longrightarrow h\left[\left(\ddot{G},A\right)\ddot{\cap}\left(\ddot{H},A\right)\right] = h\left(\ddot{\Phi},A\right) \Longrightarrow h\left(\ddot{G},A\right)$ $\ddot{\cap}h\left(\ddot{H},A\right) = \left(\ddot{\Phi},A\right)$. Therefore there exists distinct VBSP's $\ddot{B}_e$ and $\ddot{K}_e$ contained in disjoint VBSOS's $\left(\ddot{G}, A\right)$ and $\left(\ddot{H}, A\right)$ in $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$. Hence $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$ is also vague binary soft hausdorff space.

Theorem 2.4.15.

The property of being a vague binary soft hausdorff space is preserved by bijective, vague binary soft open mappings and hence is a vague binary soft topological property

Proof.
Let $k : (U_1, U_2, E, \ddot{\tau}_\Delta)_A \to (V_1, V_2, E, \ddot{\sigma}_\Delta)_A$ be a bijective, vague binary soft open mapping where $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$ is a vague binary soft hausdorff space. It will be established that $(V_1, V_2, E, \ddot{\sigma}_\Delta)_A$ is also a vague binary soft hausdorff space. Let $\ddot{P}_{e_1}, \ddot{P}_{e_2} \in (U_1, U_2, E, \ddot{\tau}_\Delta)_A$ are two VBSP's with $\ddot{P}_{e_1} \neq \ddot{P}_{e_2}$. Using vague binary soft hausdorff property of $(U_1, U_2, E, \ddot{\tau}_\Delta)_A$ there exists disjoint VBSOS's $\left(\ddot{P}_1, A\right)$ and $\left(\ddot{P}_2, A\right)$ with $\ddot{P}_{e_1} \in \left(\ddot{P}_1, A\right)$ and $\ddot{P}_{e_2} \in \left(\ddot{P}_2, A\right)$. Now using bijectiveness of k there exists VBSP's $\ddot{T}_{e_1}$ and $\ddot{T}_{e_2}$ such that $k\left(\ddot{P}_{e_1}\right) = \ddot{T}_{e_1} \in k\left(\ddot{P}_1, A\right)$ and $k\left(\ddot{P}_{e_2}\right) = \ddot{T}_{e_2} \in k\left(\ddot{P}_2, A\right)$. Also $\left(\ddot{P}_1, A\right) \ddot{\cap} \left(\ddot{P}_2, A\right) = \left(\ddot{\Phi}, A\right) \implies k\left[\left(\ddot{P}_1, A\right) \ddot{\cap} \left(\ddot{P}_2, A\right)\right] = k\left[\left(\ddot{\Phi}, A\right)\right] \implies k\left[\left(\ddot{P}_1, A\right) \ddot{\cap} \left(\ddot{P}_2, A\right)\right] = \left(\ddot{\Phi}, A\right)$. Since k is vague binary soft open mapping, $k\left(\ddot{P}_1, A\right)$ and $k\left(\ddot{P}_2, A\right)$ are VBSOS's. Therefore $k\left(\ddot{P}_1, A\right)$ and $k\left(\ddot{P}_2, A\right)$ are disjoint VBSOS's in $(V_1, V_2, E, \ddot{\sigma}_\Delta)_A$ containing $\ddot{T}_{e_1}$ and $\ddot{T}_{e_2}$ respectively, with $\ddot{T}_{e_1} \neq \ddot{T}_{e_2}$. Hence $(V_1, V_2, E, \ddot{\sigma}_\Delta)_A$ is a vague binary soft hausdorff space. So property of being a vague binary soft hausdorff space is preserved under bijective, vague binary soft open mapping $\implies$ it is also preserved under vague binary soft homeomorphism $\implies$ vague binary soft hausdorff property is a vague binary soft topological property.

Conclusion

Vague binary soft Sets, it's operations, properties, topology and continuity are developed in this chapter. Topological properties for vague binary soft sets are found invariant under vague binary soft homeomorphisms in vague binary soft hausdorff spaces. Pasting lemma holds good for vague binary soft sets too.

Chapter 3

Chapter 3

Various Measures of Vague Binary Soft Sets

This chapter aims to develop different measures namely, *Distance, Similarity and Entropy* measures for VBSS's. Distance Measures & Similarity Measures are dual to each other. These are two widely used tools in uncertainty theories. *Similarity Measure* between two sets will project *how much akin* two sets are. Similarly, *Distance Measure & Entropy Measure* will act as good comparing weapons in uncertainty set theory. VBSS has been developed in *chapter 2*. This chapter aims to mould Distance Measure & Similarity Measure for this Set. Moreover, *Pythagorean* version of VBSS (in short, PVBSS) with its operations, distance measure & entropy measure have also been designed. Pythagorean Vague Binary Soft Set is a special type of VBSS. It is developed in this chapter with prime importance.

Chapter Scheme:

Section 3.1 : Distance Measure of Vague Binary Soft Set's
Section 3.2 : Similarity Measure of Vague Binary Soft Set's
Section 3.3 : Distance Measure of Pythagorean Vague Binary Soft Set's
Section 3.4 : Entropy Measure of Pythagorean Vague Binary Soft Set's

3.1 Distance Measure of VBSS's

Distance measures and Similarity Measures are very useful in decision making problems. Being duals, comparison between them is also found beneficial in certain real-life situations. In this section, firstly developed distance measures for VBSS's. Based on this, a special distance formulae for VBSS namely, **Trigonometric Normalised *Hamming/Euclidean* Similarity Measures** have also been constructed for VBSS's. *Szmidt and Kacprzyk's Distance formulae* [88] (which includes uncertainty also and hence will draw a reliable output), is used as the fundamental stone for this work. It can be further extended to correlation like similarity measures and to different set theoretical approaches. In this section, firstly, various distance measures are developed for vague soft sets using Szmidt and Kacprzyk's [23] Distance formulae. It is further extended to vague binary soft sets with an example and verified all the conditions to become a distance measure.

Distance Measure of Vague Binary Soft Set's

(Using Szmidt and Kacprzyk's Distance formulae)

Definition 3.1.1. ***(Axioms for Distance Measure of VBSS's)***
Let (U_1, U_2) *be a binary universe with* $U_1 = \{u_1, u_2, --, u_r, -- u_i\}$ *and* $U_2 = \{v_1, v_2, --, v_s, --, v_j\}$. *Also let* $E = \{e_1, e_2, --, e_t, --, e_k\}$ *be a fixed parameter set with* $A = \{e_1, e_2, --, e_p, --, e_m\}$ *where* $A \subseteq E$. *Let* $\left(\ddot{F}, A\right)$ *and* $\left(\ddot{G}, A\right)$ *be two VBSS's such that* $\left(\ddot{F}, A\right), \left(\ddot{G}, A\right) \in VBSS\,(U_1, U_2)$.
Let $d : VBSS\,(U_1, U_2) \times VBSS\,(U_1, U_2) \to [0, 1]$ *be a mapping.*
Then $d\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right)$ *is called the distance measure of these VBSS's if it satisfies the following conditions:*

1. $d\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right) = d\left(\left(\ddot{G}, A\right), \left(\ddot{F}, A\right)\right)$

2. $d\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right) \in [0, 1]$

3. $d\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right) = 0 \Leftrightarrow \left(\ddot{F}, A\right) = \left(\ddot{G}, A\right)$

4. $d\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right) = 1 \Leftrightarrow$

$\forall e_p \in A, \quad \forall u_r \in U_1$

$$\begin{cases} t_{\ddot{F}(e_p)}(u_r) = 0, & t_{\ddot{G}(e_p)}(u_r) = 1 \quad or \quad t_{\ddot{F}(e_p)}(u_r) = 1, \quad t_{\ddot{G}(e_p)}(u_r) = 0; \quad \& \\ f_{\ddot{F}(e_p)}(u_r) = 1, & f_{\ddot{G}(e_p)}(u_r) = 0 \quad or \quad f_{\ddot{F}(e_p)}(u_r) = 0, \quad f_{\ddot{G}(e_p)}(u_r) = 1; \end{cases}$$

and $\forall e_p \in A, \quad \forall v_s \in U_2$

$$\begin{cases} t_{\ddot{F}(e_p)}(v_s) = 0, & t_{\ddot{G}(e_p)}(v_s) = 1 \quad or \quad t_{\ddot{F}(e_p)}(v_s) = 1, \quad t_{\ddot{G}(e_p)}(v_s) = 0; \quad \& \\ f_{\ddot{F}(e_p)}(v_s) = 1, & f_{\ddot{G}(e_p)}(v_s) = 0 \quad or \quad f_{\ddot{F}(e_p)}(v_s) = 0, \quad f_{\ddot{G}(e_p)}(v_s) = 1; \end{cases}$$

5. $\left(\ddot{F}, A\right) \subseteq \left(\ddot{G}, A\right) \subseteq \left(\ddot{P}, A\right)$

$$\Rightarrow d\left(\left(\ddot{F}, A\right), \left(\ddot{P}, A\right)\right) \geq max\left(d\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right), d\left(\left(\ddot{G}, A\right), \left(\ddot{P}, A\right)\right)\right)$$

$$where \quad \left(\ddot{F}, A\right), \left(\ddot{G}, A\right), \left(\ddot{P}, A\right) \in VBSS(U_1, U_2) \quad \& \quad A \subseteq E$$

Definition 3.1.2. ***(Distance Measure of Vague Binary Soft Set's)***
Let (U_1, U_2) be a binary universe with $U_1 = \{u_1, u_2, --, u_r, -- u_i\}$ and $U_2 = \{v_1, v_2, --, v_s, --, v_j\}$. Also let $E = \{e_1, e_2, --, e_t, --, e_k\}$ be a fixed parameter set where $A = \{e_1, e_2, --, e_p, --, e_m\}$ with $A \subseteq E$. Let,
Cardinality of universal set $U_1 = \#(U_1) = i$; Cardinality of universal set $U_2 = \#(U_2) = j$
Cardinality of parameter set $E = \#(E) = k$; Cardinality of parameter set $A = \#(A) = m$
Let $\left(\ddot{F}, A\right)$ and $\left(\ddot{G}, A\right)$ be two VBSS's over binary universe (U_1, U_2)
(1) Hamming Distance:
Hamming Distance Measure of Vague Binary Soft Set's $\left(\ddot{F}, A\right)$ and $\left(\ddot{G}, A\right)$ is denoted as $d^H_{VBSS}\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right)$ and it is given as follows :

$$d^H_{VBSS}\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right) =$$

$$\frac{1}{4m}\sum_{p=1}^{m}\sum_{r=1}^{i}\begin{bmatrix} \left|t_{\ddot{F}(e_p)}(u_r) - t_{\ddot{G}(e_p)}(u_r)\right| \\ + \\ \left|f_{\ddot{F}(e_p)}(u_r) - f_{\ddot{G}(e_p)}(u_r)\right| \\ + \\ \left|\Pi_{\ddot{F}(e_p)}(u_r) - \Pi_{\ddot{G}(e_p)}(u_r)\right| \end{bmatrix} + \frac{1}{4m}\sum_{p=1}^{m}\sum_{s=1}^{j}\begin{bmatrix} \left|t_{\ddot{F}(e_p)}(v_s) - t_{\ddot{G}(e_p)}(v_s)\right| \\ + \\ \left|f_{\ddot{F}(e_p)}(v_s) - f_{\ddot{G}(e_p)}(v_s)\right| \\ + \\ \left|\Pi_{\ddot{F}(e_p)}(v_s) - \Pi_{\ddot{G}(e_p)}(v_s)\right| \end{bmatrix}$$

(2) Normalized Hamming Distance:

Normalized Hamming Distance Measure of Vague Binary Soft Set's $(\ddot{F}, A)$ *and* $(\ddot{G}, A)$ *is denoted as* $d^{nH}_{VBSS}\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right)$ *and it is given as follows :*

$d^{nH}_{VBSS}\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right) =$

$$\frac{1}{4mi}\sum_{p=1}^{m}\sum_{r=1}^{i}\begin{bmatrix}\left|t_{\ddot{F}(e_p)}(u_r) - t_{\ddot{G}(e_p)}(u_r)\right| \\ + \\ \left|f_{\ddot{F}(e_p)}(u_r) - f_{\ddot{G}(e_p)}(u_r)\right| \\ + \\ \left|\Pi_{\ddot{F}(e_p)}(u_r) - \Pi_{\ddot{G}(e_p)}(u_r)\right|\end{bmatrix} + \frac{1}{4mj}\sum_{p=1}^{m}\sum_{s=1}^{j}\begin{bmatrix}\left|t_{\ddot{F}(e_p)}(v_s) - t_{\ddot{G}(e_p)}(v_s)\right| \\ + \\ \left|f_{\ddot{F}(e_p)}(v_s) - f_{\ddot{G}(e_p)}(v_s)\right| \\ + \\ \left|\Pi_{\ddot{F}(e_p)}(v_s) - \Pi_{\ddot{G}(e_p)}(v_s)\right|\end{bmatrix}$$

(3) Euclidean Distance:

Euclidean Distance Measure of Vague Binary Soft Set's $\left(\ddot{F}, A\right)$ *and* $\left(\ddot{G}, A\right)$ *is denoted as* $d^{E}_{VBSS}\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right)$ *and it is given as follows :*

$d^{E}_{VBSS}\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right) =$

$$\sqrt{\frac{1}{4m}\sum_{p=1}^{m}\sum_{r=1}^{i}\begin{bmatrix}\left|t_{\ddot{F}(e_p)}(u_r) - t_{\ddot{G}(e_p)}(u_r)\right|^2 \\ + \\ \left|f_{\ddot{F}(e_p)}(u_r) - f_{\ddot{G}(e_p)}(u_r)\right|^2 \\ + \\ \left|\Pi_{\ddot{F}(e_p)}(u_r) - \Pi_{\ddot{G}(e_p)}(u_r)\right|^2\end{bmatrix} + \frac{1}{4m}\sum_{p=1}^{m}\sum_{s=1}^{j}\begin{bmatrix}\left|t_{\ddot{F}(e_p)}(v_s) - t_{\ddot{G}(e_p)}(v_s)\right|^2 \\ + \\ \left|f_{\ddot{F}(e_p)}(v_s) - f_{\ddot{G}(e_p)}(v_s)\right|^2 \\ + \\ \left|\Pi_{\ddot{F}(e_p)}(v_s) - \Pi_{\ddot{G}(e_p)}(v_s)\right|^2\end{bmatrix}}$$

(4) Normalized Euclidean Distance:

Euclidean Distance Measure of Vague Binary Soft Set's $\left(\ddot{F}, A\right)$ *and* $\left(\ddot{G}, A\right)$ *is denoted as* $d^{nE}_{VBSS}\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right)$ *and it is given as follows :*

$$d^{nE}_{VBSS}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right)=$$

$$\sqrt{\frac{1}{4mi}\sum_{p=1}^{m}\sum_{r=1}^{i}\begin{bmatrix}\left|t_{\ddot{F}(e_p)}(u_r)-t_{\ddot{G}(e_p)}(u_r)\right|^2\\+\\\left|f_{\ddot{F}(e_p)}(u_r)-f_{\ddot{G}(e_p)}(u_r)\right|^2\\+\\\left|\Pi_{\ddot{F}(e_p)}(u_r)-\Pi_{\ddot{G}(e_p)}(u_r)\right|^2\end{bmatrix}+\frac{1}{4mj}\sum_{p=1}^{m}\sum_{s=1}^{j}\begin{bmatrix}\left|t_{\ddot{F}(e_p)}(v_s)-t_{\ddot{G}(e_p)}(v_s)\right|^2\\+\\\left|f_{\ddot{F}(e_p)}(v_s)-f_{\ddot{G}(e_p)}(v_s)\right|^2\\+\\\left|\Pi_{\ddot{F}(e_p)}(v_s)-\Pi_{\ddot{G}(e_p)}(v_s)\right|^2\end{bmatrix}}$$

Proof.

Method of proof:

Verification of vague binary soft distance measure axioms given in definition 3.1.1.

Axiom (1) :

$$d^{H}_{VBSS}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right)=$$

$$\frac{1}{4m}\sum_{p=1}^{m}\sum_{r=1}^{i}\begin{bmatrix}\left|t_{\ddot{F}(e_p)}(u_r)-t_{\ddot{G}(e_p)}(u_r)\right|\\+\\\left|f_{\ddot{F}(e_p)}(u_r)-f_{\ddot{G}(e_p)}(u_r)\right|\\+\\\left|\Pi_{\ddot{F}(e_p)}(u_r)-\Pi_{\ddot{G}(e_p)}(u_r)\right|\end{bmatrix}+\frac{1}{4m}\sum_{p=1}^{m}\sum_{s=1}^{j}\begin{bmatrix}\left|t_{\ddot{F}(e_p)}(v_s)-t_{\ddot{G}(e_p)}(v_s)\right|\\+\\\left|f_{\ddot{F}(e_p)}(v_s)-f_{\ddot{G}(e_p)}(v_s)\right|\\+\\\left|\Pi_{\ddot{F}(e_p)}(v_s)-\Pi_{\ddot{G}(e_p)}(v_s)\right|\end{bmatrix}$$

$$=$$

$$\frac{1}{4m}\sum_{p=1}^{m}\sum_{r=1}^{i}\begin{bmatrix}\left|t_{\ddot{G}(e_p)}(u_r)-t_{\ddot{F}(e_p)}(u_r)\right|\\+\\\left|f_{\ddot{G}(e_p)}(u_r)-f_{\ddot{F}(e_p)}(u_r)\right|\\+\\\left|\Pi_{\ddot{G}(e_p)}(u_r)-\Pi_{\ddot{F}(e_p)}(u_r)\right|\end{bmatrix}+\frac{1}{4m}\sum_{p=1}^{m}\sum_{s=1}^{j}\begin{bmatrix}\left|t_{\ddot{G}(e_p)}(v_s)-t_{\ddot{F}(e_p)}(v_s)\right|\\+\\\left|f_{\ddot{G}(e_p)}(v_s)-f_{\ddot{F}(e_p)}(v_s)\right|\\+\\\left|\Pi_{\ddot{G}(e_p)}(v_s)-\Pi_{\ddot{F}(e_p)}(v_s)\right|\end{bmatrix}$$

$$=d^{H}_{VBSS}\left(\left(\ddot{G},A\right),\left(\ddot{F},A\right)\right)$$

Axiom (2):

$$\left.\begin{matrix}t_{\ddot{F}(e_p)}(u_r),f_{\ddot{F}(e_p)}(u_r),\Pi_{\ddot{F}(e_p)}(u_r),t_{\ddot{G}(e_p)}(u_r),f_{\ddot{G}(e_p)}(u_r),\Pi_{\ddot{G}(e_p)}(u_r)\\t_{\ddot{F}(e_p)}(v_s),f_{\ddot{F}(e_p)}(v_s),\Pi_{\ddot{F}(e_p)}(v_s),t_{\ddot{G}(e_p)}(v_s),f_{\ddot{G}(e_p)}(v_s),\Pi_{\ddot{G}(e_p)}(v_s)\end{matrix}\right\}\text{ all belong to }[0,\ 1]$$

Hence $d^{H}_{VBSS}\left(\left(\ddot{F},A\right)\left(\ddot{G},A\right)\right)\in[0,1]$

Axiom (3) :

$\forall e_p\in A$, $\forall u_r\in U_1$ and $\forall v_s\in U_2$,

$$d^{H}_{VBSS}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right)=0$$

$$\Leftrightarrow \frac{1}{4m}\sum_{p=1}^{m}\sum_{r=1}^{i}\begin{bmatrix}\left|t_{\ddot{F}(e_p)}(u_r)-t_{\ddot{G}(e_p)}(u_r)\right|\\+\\\left|f_{\ddot{F}(e_p)}(u_r)-f_{\ddot{G}(e_p)}(u_r)\right|\\+\\\left|\Pi_{\ddot{F}(e_p)}(u_r)-\Pi_{\ddot{G}(e_p)}(u_r)\right|\end{bmatrix}+\frac{1}{4m}\sum_{p=1}^{m}\sum_{s=1}^{j}\begin{bmatrix}\left|t_{\ddot{F}(e_p)}(v_s)-t_{\ddot{G}(e_p)}(v_s)\right|\\+\\\left|f_{\ddot{F}(e_p)}(v_s)-f_{\ddot{G}(e_p)}(v_s)\right|\\+\\\left|\Pi_{\ddot{F}(e_p)}(v_s)-\Pi_{\ddot{G}(e_p)}(v_s)\right|\end{bmatrix}=0$$

$$\Leftrightarrow \left|t_{\ddot{F}(e_p)}(u_r)-t_{\ddot{G}(e_p)}(u_r)\right|=0,\left|f_{\ddot{F}(e_p)}(u_r)-f_{\ddot{G}(e_p)}(u_r)\right|=0,\left|\Pi_{\ddot{F}(e_p)}(u_r)-\Pi_{\ddot{G}(e_p)}(u_r)\right|=0$$

$$\left|t_{\ddot{F}(e_p)}(v_s)-t_{\ddot{G}(e_p)}(v_s)\right|=0,\left|f_{\ddot{F}(e_p)}(v_s)-f_{\ddot{G}(e_p)}(v_s)\right|=0,\left|\Pi_{\ddot{F}(e_p)}(v_s)-\Pi_{\ddot{G}(e_p)}(v_s)\right|=0$$

$$\Leftrightarrow t_{\ddot{F}(e_p)}(u_r)=t_{\ddot{G}(e_p)}(u_r),\ f_{\ddot{F}(e_p)}(u_r)=f_{\ddot{G}(e_p)}(u_r),\ \Pi_{\ddot{F}(e_p)}(u_r)=\Pi_{\ddot{G}(e_p)}(u_r)$$

$$t_{\ddot{F}(e_p)}(v_s)=t_{\ddot{G}(e_p)}(v_s),\ f_{\ddot{F}(e_p)}(v_s)=f_{\ddot{G}(e_p)}(v_s),\ \Pi_{\ddot{F}(e_p)}(v_s)=\Pi_{\ddot{G}(e_p)}(v_s)$$

$\Leftrightarrow\left(\ddot{F},A\right)=\left(\ddot{G},A\right)$

Axiom (4):

$\forall e_p\in A,\forall u_r\in U_1$ and $\forall v_s\in U_2$,

$d^{H}_{VBSS}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right)=1$

$$\Leftrightarrow \frac{1}{4m}\sum_{p=1}^{m}\sum_{r=1}^{i}\begin{bmatrix}\left|t_{\ddot{F}(e_p)}(u_r)-t_{\ddot{G}(e_p)}(u_r)\right|\\+\\\left|f_{\ddot{F}(e_p)}(u_r)-f_{\ddot{G}(e_p)}(u_r)\right|\\+\\\left|\Pi_{\ddot{F}(e_p)}(u_r)-\Pi_{\ddot{G}(e_p)}(u_r)\right|\end{bmatrix}+\frac{1}{4m}\sum_{p=1}^{m}\sum_{s=1}^{j}\begin{bmatrix}\left|t_{\ddot{F}(e_p)}(v_s)-t_{\ddot{G}(e_p)}(v_s)\right|\\+\\\left|f_{\ddot{F}(e_p)}(v_s)-f_{\ddot{G}(e_p)}(v_s)\right|\\+\\\left|\Pi_{\ddot{F}(e_p)}(v_s)-\Pi_{\ddot{G}(e_p)}(v_s)\right|\end{bmatrix}=1$$

$\Leftrightarrow$ Either

$$\left|t_{\ddot{F}(e_p)}(u_r)-t_{\ddot{G}(e_p)}(u_r)\right|=0,\left|f_{\ddot{F}(e_p)}(u_r)-f_{\ddot{G}(e_p)}(u_r)\right|=0,\left|\Pi_{\ddot{F}(e_p)}(u_r)-\Pi_{\ddot{G}(e_p)}(u_r)\right|=0$$

$$\left|t_{\ddot{F}(e_q)}(v_s)-t_{\ddot{G}(e_q)}(v_s)\right|=1,\ \left|f_{\ddot{F}(e_q)}(v_s)-f_{\ddot{G}(e_q)}(v_s)\right|=1,\ \left|\Pi_{\ddot{F}(e_q)}(v_s)-\Pi_{\ddot{G}(e_q)}(v_s)\right|=1$$

Or

$$\left|t_{\ddot{F}(e_p)}(u_r)-t_{\ddot{G}(e_p)}(u_r)\right|=1,\left|f_{\ddot{F}(e_p)}(u_r)-f_{\ddot{G}(e_p)}(u_r)\right|=1,\left|\Pi_{\ddot{F}(e_p)}(u_r)-\Pi_{\ddot{G}(e_p)}(u_r)\right|=1$$

$$\left|t_{\ddot{F}(e_q)}(v_s)-t_{\ddot{G}(e_q)}(v_s)\right|=0,\left|f_{\ddot{F}(e_q)}(v_s)-f_{\ddot{G}(e_p)}(v_s)\right|=0,\left|\Pi_{\ddot{F}(e_q)}(v_s)-\Pi_{\ddot{G}(e_q)}(v_s)\right|=0$$

$\Leftrightarrow$ Either

$$t_{\ddot{F}(e_p)}(u_r)=t_{\ddot{G}(e_p)}(u_r),\quad f_{\ddot{F}(e_p)}(u_r)=f_{\ddot{G}(e_p)}(u_r),\quad \Pi_{\ddot{F}(e_p)}(u_r)=\Pi_{\ddot{G}(e_p)}(u_r)$$

$$t_{\ddot{F}(e_q)}(v_s)=t_{\ddot{G}(e_q)}(v_s),\quad f_{\ddot{F}(e_q)}(v_s)=f_{\ddot{G}(e_q)}(v_s),\quad \Pi_{\ddot{F}(e_q)}(v_s)=\Pi_{\ddot{G}(e_q)}(v_s)$$

Or

$$t_{\ddot{F}(e_p)}(u_r)=t_{\ddot{G}(e_p)}(u_r),f_{\ddot{F}(e_p)}(u_r)=f_{\ddot{G}(e_p)}(u_r),\Pi_{\ddot{F}(e_p)}(u_r)=\Pi_{\ddot{G}(e_p)}(u_r)$$

$$t_{\ddot{F}(e_q)}(v_s)=t_{\ddot{G}(e_q)}(v_s),f_{\ddot{F}(e_q)}(v_s)=f_{\ddot{G}(e_q)}(v_s),\Pi_{\ddot{F}(e_q)}(v_s)=\Pi_{\ddot{G}(e_q)}(v_s)$$

$\Leftrightarrow$ Either

$$t_{\ddot{F}(e_p)}(u_r)=0,f_{\ddot{F}(e_p)}(u_r)=1\quad;\quad t_{\ddot{G}(e_p)}(u_r)=1,f_{\ddot{G}(e_p)}(u_r)=0$$

$$t_{\ddot{F}(e_q)}(v_s)=0,f_{\ddot{F}(e_q)}(v_s)=1\quad;\quad t_{\ddot{G}(e_q)}(v_s)=1,f_{\ddot{G}(e_q)}(v_s)=0$$

Or

$$t_{\ddot{F}(e_p)}(u_r)=1,f_{\ddot{F}(e_p)}(u_r)=0\quad;\quad t_{\ddot{G}(e_p)}(u_r)=0,f_{\ddot{G}(e_p)}(u_r)=1$$

$$t_{\ddot{F}(e_q)}(v_s)=1,f_{\ddot{F}(e_q)}(v_s)=0\quad;\quad t_{\ddot{G}(e_q)}(v_s)=0,f_{\ddot{G}(e_q)}(v_s)=1$$

Axiom (5):
Let $\left(\ddot{F},A\right)\in VBSS(U_1,U_2)$. Also let $\left(\ddot{F},A\right)\ddot{\subseteq}\left(\ddot{G},A\right)\ddot{\subseteq}\left(\ddot{P},A\right)$
$\Rightarrow \forall e_p\in A,\quad \forall u_r\in U_1\quad$ and $\quad\forall v_s\in U_2;$

$$\begin{cases} t_{\ddot{F}(e_p)}(u_r)\leqslant t_{\ddot{G}(e_p)}(u_r)\leqslant t_{\ddot{P}(e_p)}(u_r) \\ f_{\ddot{F}(e_p)}(u_r)\geqslant f_{\ddot{G}(e_p)}(u_r)\geqslant f_{\ddot{P}(e_p)}(u_r)\end{cases}$$

$$\Rightarrow\begin{cases} \left|t_{\ddot{F}(e_p)}(u_r)-t_{\ddot{P}(e_p)}(u_r)\right|\geqslant\left|t_{\ddot{F}(e_p)}(u_r)-t_{\ddot{G}(e_p)}(u_r)\right| \\ \left|f_{\ddot{F}(e_p)}(u_r)-f_{\ddot{P}(e_p)}(u_r)\right|\geqslant\left|f_{\ddot{F}(e_p)}(u_r)-f_{\ddot{G}(e_p)}(u_r)\right|\end{cases}$$

$$\Rightarrow d^{H}_{VBSS}\left(\left(\ddot{F},A\right)\left(\ddot{P},A\right)\right) \geqslant d^{H}_{VBSS}\left(\left(\ddot{F},A\right)\left(\ddot{G},A\right)\right)$$

Similarly, it can be proved that,

$$\Rightarrow d^{H}_{VBSS}\left(\left(\ddot{F},A\right)\left(\ddot{P},A\right)\right) \geqslant d^{H}_{VBSS}\left(\left(\ddot{G},A\right)\left(\ddot{P},A\right)\right)$$

Combining,

$$d^{H}_{VBSS}\left(\left(\ddot{F},A\right)\left(\ddot{P},A\right)\right) \geqslant \max\left[d^{H}_{VBSS}\left(\left(\ddot{F},A\right)\left(\ddot{G},A\right)\right), d^{H}_{VBSS}\left(\left(\ddot{G},A\right)\left(\ddot{P},A\right)\right)\right]$$

All the axioms for Vague Binary Soft Distance Measure given by, definition 3.1.1. got satisfied by the newly defined Hamming Distance Measure $d^{H}_{VBSS}\left(\left(\ddot{F},A\right)\left(\ddot{G},A\right)\right)$. So $d^{H}_{VBSS}\left(\left(\ddot{F},A\right)\left(\ddot{G},A\right)\right)$ is clearly a distance measure for VBSS's. *Similarly, it could be proved for other three newly developed distance measures* $d^{nH}_{VBSS}\left(\left(\ddot{F},A\right)\left(\ddot{G},A\right)\right)$, $d^{E}_{VBSS}\left(\left(\ddot{F},A\right)\left(\ddot{G},A\right)\right)$ *and* $d^{nE}_{VBSS}\left(\left(\ddot{F},A\right)\left(\ddot{G},A\right)\right)$ *for VBSS's.* So all the above defined formulae clearly indicate a Distance Measure for *Vague Binary Soft Sets.*

Example 3.1.3.

Let $U_1 = \{u_x^1, u_x^2, u_x^3\}, U_2 = \{u_y^1, u_y^2\}$ *be a binary universe and* $E = \{e_1, e_2, e_3, e_4, e_5\}$ *be a fixed parameter set. Consider* $A \subseteq E$ *with* $A = \{e_1, e_3, e_5\}$. *Two VBSS's framed based on the above data are given as follows:*

$$\left(\ddot{F},A\right) = \left\{\begin{array}{l} \left(e_1, \left(\left\langle \frac{[0.2,0.4}{u_x^1}, \frac{[0.6,0.7}{u_x^2}, \frac{[0.8,0.9}{u_x^3} \right\rangle, \left\langle \frac{[0.5,0.6]}{u_y^1}, \frac{[0.7,0.9]}{u_y^2} \right\rangle\right)\right) \\ \left(e_3, \left(\left\langle \frac{[0.6,0.9}{u_x^1}, \frac{[0.4,0.5}{u_x^2}, \frac{[0.6,0.7}{u_x^3} \right\rangle, \left\langle \frac{[0.4,0.7]}{u_y^1}, \frac{[0.3,0.6]}{u_y^2} \right\rangle\right)\right) \\ \left(e_5, \left(\left\langle \frac{[0.1,0.3}{u_x^1}, \frac{[0.2,0.6}{u_x^2}, \frac{[0.7,0.8}{u_x^3} \right\rangle, \left\langle \frac{[0.2,0.7]}{u_y^1}, \frac{[0.5,0.9]}{u_y^2} \right\rangle\right)\right) \end{array}\right\}$$

$$\left(\ddot{G},A\right) = \left\{\begin{array}{l} \left(e_1, \left(\left\langle \frac{[0.4,0.6}{u_x^1}, \frac{[0.7,0.8}{u_x^2}, \frac{[0.3,0.5}{u_x^3} \right\rangle, \left\langle \frac{[0.6,0.7]}{u_y^1}, \frac{[0.8,0.9]}{u_y^2} \right\rangle\right)\right) \\ \left(e_3, \left(\left\langle \frac{[0.5,0.8}{u_x^1}, \frac{[0.1,0.4}{u_x^2}, \frac{[0.3,0.5}{u_x^3} \right\rangle, \left\langle \frac{[0.6,0.8]}{u_y^1}, \frac{[0.7,0.9]}{u_y^2} \right\rangle\right)\right) \\ \left(e_5, \left(\left\langle \frac{[0.2,0.6}{u_x^1}, \frac{[0.4,0.7}{u_x^2}, \frac{[0.5,0.7}{u_x^3} \right\rangle, \left\langle \frac{[0.4,0.9]}{u_y^1}, \frac{[0.7,0.8]}{u_y^2} \right\rangle\right)\right) \end{array}\right\}$$

For above defined VBSS's various distance measures are calculated as follows:

- $d^{H}_{VBSS}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right) = 0.4 \in [0,1]$
- $d^{nH}_{VBSS}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right) = 0.16 \in [0,1]$
- $d^{E}_{VBSS}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right) = 0.1 \in [0,1]$
- $d^{nE}_{VBSS}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right) = 0.0407 \in [0,1]$

3.2 Similarity Measure of VBSS's

Similarity Measure (in short, SM) indicates resemblance of two objects. It takes value *zero* if objects are totally different and takes value *one* if objects are same. Chang Wang and Anjing Qu [17] defined a new similarity measure for vague soft sets based on the notion *core or support*. In this section, this idea is made use to formulate a new similarity measure for VBSS's.

Definition 3.2.1. ***(Axioms for Similarity Measure of VBSS's)***
Let $M : VBSS(U_1, U_2) \times VBSS(U_1, U_2) \to [0,1]$ *be a mapping.*
Let $\left(\ddot{F}, A\right)$ *and* $\left(\ddot{G}, A\right)$ *be two VBSS's such that,* $\left(\ddot{F}, A\right), \left(\ddot{G}, A\right) \in VBSS(U_1, U_2)$.
Then $M\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right)$ *is called a similarity measure of VBSS's if it satisfies the following axioms:*

1. $M\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right) = M\left(\left(\ddot{G}, A\right), \left(\ddot{F}, A\right)\right)$
2. $M\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right) \in [0,1]$
3. $M\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right) = 1 \Leftrightarrow \left(\ddot{F}, A\right) = \left(\ddot{G}, A\right)$
4. $M\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right) = 0 \Leftrightarrow$

$\forall e_p \in A, \forall u_r \in U_1$

$$\begin{cases} t_{\ddot{F}(e_p)}(u_r) = 0, \quad t_{\ddot{G}(e_p)}(u_r) = 0 \quad or \quad t_{\ddot{F}(e_p)}(u_r) = 1, \quad t_{\ddot{G}(e)}(u_r) = 0 \quad \& \\ f_{\ddot{F}(ep)}(u_r) = 0, \quad f_{\ddot{G}(e_p)}(u_r) = 0 \quad or \quad f_{\ddot{F}(e_p)}(u_r) = 1, \quad f_{\ddot{G}(e_p)}(u_r) = 0 \end{cases}$$

and

$\forall e_p \in A, \quad \forall v_s \in U_2$

$$\begin{cases} t_{\ddot{F}(e_p)}(v_s) = 0, \quad t_{\ddot{G}(e_p)}(v_s) = 0 \quad or \quad t_{\ddot{F}(e_p)}(v_s) = 1, \quad t_{\ddot{G}(e_p)}(v_s) = 0 \quad \& \\ f_{\ddot{F}(e_p)}(v_s) = 0, \quad f_{\ddot{G}(e_p)}(v_s) = 0 \quad or \quad f_{\ddot{F}(e_p)}(v_s) = 1, \quad f_{\ddot{G}(e_p)}(v_s) = 0 \end{cases}$$

5. $\left(\ddot{F}, A\right) \subseteq \left(\ddot{G}, A\right) \subseteq \left(\ddot{P}, A\right)$

$$\Rightarrow M\left(\left(\ddot{F}, A\right), \left(\ddot{P}, A\right)\right) = min\left(M\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right), M\left(\left(\ddot{G}, A\right), \left(\ddot{P}, A\right)\right)\right)$$

where $\left(\ddot{F}, A\right), \left(\ddot{G}, A\right), \left(\ddot{P}, A\right) \in VBSS(U_1, U_2)$ *and* $A \subseteq E$

Definition 3.2.2. ***(Formula for Similarity Measure of two VBSS's)***
Let $U_1 = \{u_1, u_2, --, u_r, --, u_i\}, U_2 = \{v_1, v_2, --, v_s, --, v_j\}$ *be a binary universe*

and $E = \{e_1, e_2, --, e_t, --, e_k\}$ *be a fixed set of parameters. Hence* $\left(\ddot{F}, A\right) = \left\{\left(e_p, \ddot{F}(e_p)\right) / p = 1, 2, --, m\right\}$ & $\left(\ddot{G}, A\right) = \left\{\left(e_p, \ddot{G}(e_p)\right) / p = 1, 2, --, m\right\}$ *are two families of VBSS's.*

A Similarity Measure of $\left(\ddot{F}, A\right)$ *and* $\left(\ddot{G}, A\right)$ *is defined as follows :*

$M\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right) = \frac{\sum_{p=1}^{m} M_p\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right)}{p}$

where $M_p\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right)$ *is defined as follows:*

$M_p\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right)$

$$= 1 - \frac{1}{4i}\sum_{r=1}^{i}\left|S_{\ddot{F}(e_p)}(u_r) - S_{\ddot{G}(e_p)}(u_r)\right| + \left|t_{\ddot{F}(e_p)}(u_r) - t_{\ddot{G}(e_p)}(u_r)\right| + \left|f_{\ddot{F}(e_p)}(u_r) - f_{\ddot{G}(e_p)}(u_r)\right|$$

$$- \frac{1}{4j}\sum_{s=1}^{j}\left|S_{\ddot{F}(e_p)}(v_s) - S_{\ddot{G}(e_p)}(v_s)\right| + \left|t_{\ddot{F}(e_p)}(v_s) - t_{\ddot{G}(e_p)}(v_s)\right| + \left|f_{\ddot{F}(e_p)}(v_s) - f_{\ddot{G}(e_p)}(v_s)\right|$$

$$\forall p \in A, \quad \forall u_r \in U_1 \text{ and } \quad \forall v_s \in U_2$$

where $S_{\ddot{F}(e_p)}(u_r) = \left(t_{\ddot{F}(e_p)}(u_r) - f_{\ddot{F}(e_p)}(u_r)\right)$ *is called core of* $\ddot{F}(e_p)$ *in universe* U_1

$S_{\ddot{F}(e_p)}(v_s) = \left(t_{\ddot{F}(e_p)}(v_s) - f_{\ddot{G}(e_p)}(v_s)\right)$ *is called core of* $\ddot{F}(e_p)$ *in universe* U_2

$S_{\ddot{G}(e_p)}(u_r) = \left(t_{\ddot{G}(e_p)}(u_r) - f_{\ddot{G}(e_p)}(u_r)\right)$ *is called core of* $\ddot{G}(e_p)$ *in universe* U_1

$S_{\ddot{G}(e_p)}(v_s) = \left(t_{\ddot{G}(e_p)}(v_s) - f_{\ddot{G}(e_p)}(v_s)\right)$ *is called core of* $\ddot{G}(e_p)$ *in universe* U_2

&

$$S_{\ddot{F}(e_p)}(u_r), \quad S_{\ddot{F}(e_p)}(v_s), \quad S_{\ddot{G}(e_p)}(u_r), \quad S_{\ddot{G}(e_p)}(v_s) \in [-1, 1]$$

Proof.

Proof is straight forward verification of the axioms in Definition 3.2.1.

Example 3.2.3. ***(With fixed parameter set)***

Let $U_1 = \{u_a, u_b\}, U_2 = \{u_x, u_y\}$ *be a binary universe with a fixed set of parameters* $E = \{e_1, e_2, e_3, e_4\}$ *and* $A = \{e_1, e_3\}$*. Let* $\left(\ddot{F}, A\right)$ *and* $\left(\ddot{G}, A\right)$ *be two vague binary soft sets defined as follows :*

$$\left(\ddot{F}, A\right) = \left\{ \begin{array}{l} \left(e_1, \left(\left\langle \frac{[0.1, 0.4}{u_a}, \frac{[0.3, 0.6}{u_b} \right\rangle, \left\langle \frac{[0.6, 0.8]}{u_x}, \frac{[0.5, 0.7}{u_y} \right\rangle\right)\right) \\ \left(e_3, \left(\left\langle \frac{[0.5, 0.7}{u_a}, \frac{[0.8, 0.9}{u_b} \right\rangle, \left\langle \frac{[0.7, 0.8]}{u_x}, \frac{[0.3, 0.5}{u_y} \right\rangle\right)\right) \end{array} \right\}$$

$$\left(\ddot{G}, A\right) = \left\{ \begin{array}{l} \left(e_1, \left(\left\langle \frac{[0.2, 0.6}{u_a}, \frac{[0.6, 0.7}{u_b} \right\rangle, \left\langle \frac{[0.3, 0.7]}{u_x}, \frac{[0.6, 0.7}{u_y} \right\rangle\right)\right) \\ \left(e_3, \left(\left\langle \frac{[0.4, 0.7}{u_a}, \frac{[0.8, 0.9}{u_b} \right\rangle, \left\langle \frac{[0.7, 0.8]}{u_x}, \frac{[0.5, 0.6}{u_y} \right\rangle\right)\right) \end{array} \right\}$$

Using above method Similarity Measure of these sets is found as
$M\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right)=0.45\in[0,1]$

Example 3.2.4. ***(With different parameter set)***
Let $U_1=\{u_a,u_b\}\,,U_2=\{u_x,u_y\}$ *be a binary universe with a fixed set of parameters* $E=\{e_1,e_2\}$ *and* $A=\{e_1,e_3,e_4\}\,,B=\{e_3,e_4,e_5\}$. *Let* $\left(\ddot{F},A\right)$ *and* $\left(\ddot{G},B\right)$ *be two VBSS's defined as follows:*

$$\left(\ddot{F},A\right)=\left\{\begin{array}{l}\left(e_1,\left(\left\langle\frac{[0.2,0.4]}{u_a},\frac{[0.1,0.5]}{u_b}\right\rangle,\left\langle\frac{[0.5,0.6]}{u_x},\frac{[0.4,0.8]}{u_y}\right\rangle\right)\right)\\ \left(e_3,\left(\left\langle\frac{[0.5,0.9]}{u_a},\frac{[0.4,0.8]}{u_b}\right\rangle,\left\langle\frac{[0.7,0.9]}{u_x},\frac{[0.3,0.4]}{u_y}\right\rangle\right)\right)\\ \left(e_4,\left(\left\langle\frac{[0.1,0.3]}{u_a},\frac{[0.3,0.6]}{u_b}\right\rangle,\left\langle\frac{[0.2,0.4]}{u_x},\frac{[0.6,0.7]}{u_y}\right\rangle\right)\right)\end{array}\right\}$$

$$\left(\ddot{G},B\right)=\left\{\begin{array}{l}\left(e_3,\left(\left\langle\frac{[0.3,0.7]}{u_a},\frac{[0.5,0.9]}{u_b}\right\rangle,\left\langle\frac{[0.2,0.9]}{u_x},\frac{[0.4,0.8]}{u_y}\right\rangle\right)\right)\\ \left(e_4,\left(\left\langle\frac{[0.2,0.8]}{u_a},\frac{[0.1,0.5]}{u_b}\right\rangle,\left\langle\frac{[0.6,0.8]}{u_x},\frac{[0.4,0.9]}{u_y}\right\rangle\right)\right)\\ \left(e_5,\left(\left\langle\frac{[0.4,0.6]}{u_a},\frac{[0.3,0.7]}{u_b}\right\rangle,\left\langle\frac{[0.6,0.7]}{u_x},\frac{[0.2,0.9]}{u_y}\right\rangle\right)\right)\end{array}\right\}$$

Using above mentioned method Similarity Measure of these sets is
$M\left(\left(\ddot{F},A\right),\left(\ddot{G},B\right)\right)=0.26\in[0,1]$

Remark 3.2.5.
A and B are both empty or any one of them takes empty value, Similarity Measure between these sets will be always zero.

Some Special Kind of Similarity Measure's of VBSS's

In this section, two special kinds of Similarity Measure's are developed for VBSS's. *Hamming / Euclidean* in its *"Normalised Zone"* is *combined with trigonometric functions* to measure 'similarities' of VBSS's. One application algorithm, *for both* these tools has also been provided with proper examples to convey the result properly. Trigonometric Functions discussed in this section are *COSine, Sine* & *COTangent.*

Definition 3.2.6.
Let $U_1=\{u_1,u_2,--,u_r,--,u_i\}$ *and* $U_2=\{v_1,v_2,--,v_s,--,v_j\}$ *be a binary universe under consideration. Parameter set under concern, related to this binary*

universe is $E = \{e_1, e_2, --, e_t, --, e_k\}$ *with* $A \subseteq E$ *where* $A = \{e_1, e_2, --, e_p, --, e_m\}$. *Let* $\left(\ddot{F}, A\right)$ *and* $\left(\ddot{G}, A\right)$ *be two VBSS's. Let*

Cardinality of universe $U_1 = \#(U_1) = i$; *Cardinality of universe* $U_2 = \#(U_2) = j$; *Cardinality of parameter set* $E = \#(E) = k$; *Cardinality of parameter set* $A = \#(A) = m$

(1): Trigonometric Normalized Hamming Similarity Measure

Let $\left(\ddot{F}, A\right)$ *and* $(\ddot{F}, A)$ *be two VBSS's. Various Trigonometric Normalized Euclidean Similarity Measure of* $\left(\ddot{F}, A\right)$ *and* $(\ddot{G}, A)$ *is defined as follows:*

1. ***Cosine*** *Normalised Hamming Similarity Measure :*

   $Cos_{VBSS}^{nH-SM} =$

$$\frac{1}{4mi}\sum_{p=1}^{m}\sum_{r=1}^{i} Cos\frac{\Pi}{4}\begin{pmatrix} |t_{\ddot{F}(e_p)}(u_r) - t_{\ddot{G}(e_p)}(u_r)| \\ + |f_{\ddot{F}(e_p)}(u_r) - f_{\ddot{G}(e_p)}(u_r)| \\ + |\Pi_{\ddot{F}(e_p)}(u_r) - \Pi_{\ddot{G}(e_p)}(u_r)| \end{pmatrix} +$$

$$\frac{1}{4mj}\sum_{p=1}^{m}\sum_{s=1}^{j} Cos\frac{\Pi}{4}\begin{pmatrix} \left|t_{\ddot{F}(e_p)}(v_s) - t_{\ddot{G}(e_p)}(v_s)\right| \\ + \left|f_{\ddot{F}(e_p)}(v_s) - f_{\ddot{G}(e_p)}(v_s)\right| \\ + \left|\Pi_{\ddot{F}(e_p)}(v_s) - \Pi_{\ddot{G}(e_p)}(v_s)\right| \end{pmatrix}$$

2. ***Sine*** *Normalised Hamming Similarity Measure :*

   $Sin_{VBSS}^{nH-SM} =$

$$1 - \frac{1}{4mi}\sum_{p=1}^{m}\sum_{r=1}^{i} Sin\frac{\pi}{4}\begin{pmatrix} |t_{\ddot{F}(e_p)}(u_r) - t_{\ddot{G}(e_p)}(u_r)| \\ + |f_{\ddot{F}(e_p)}(u_r) - f_{\ddot{G}(e_p)}(u_r)| \\ + |\Pi_{\ddot{F}(e_p)}(u_r) - \Pi_{\ddot{G}(e_p)}(u_r)| \end{pmatrix}$$

$$-\frac{1}{4mj}\sum_{p=1}^{m}\sum_{s=1}^{j} Sin\frac{\Pi}{4}\begin{pmatrix} \left|t_{\ddot{F}(e_p)}(v_s) - t_{\ddot{G}(e_p)}(v_s)\right| \\ + \left|f_{\ddot{F}(e_p)}(v_s) - f_{\ddot{G}(e_p)}(v_s)\right| \\ + \left|\Pi_{\ddot{F}(e_p)}(v_s) - \Pi_{\ddot{G}(e_p)}(v_s)\right| \end{pmatrix}$$

3. ***COTangent*** *Normalised Hamming Similarity Measure :*

   $Cot_{VBSS}^{nH-SM} =$

$$\frac{1}{4mi}\sum_{p=1}^{m}\sum_{r=1}^{i} Cot\left(\frac{\Pi}{4} + \frac{\Pi}{8}\begin{pmatrix} |t_{\ddot{F}(e_p)}(u_r) - t_{\ddot{G}(e_p)}(u_r)| \\ + |f_{\ddot{F}(e_p)}(u_r) - f_{\ddot{G}(e_p)}(u_r)| \\ + |\Pi_{\ddot{F}(e_p)}(u_r) - \Pi_{\ddot{G}(e_p)}(u_r)| \end{pmatrix}\right)$$

$$+$$

$$\frac{1}{4mj}\sum_{p=1}^{m}\sum_{s=1}^{j} Cot\left(\frac{\Pi}{4}+\frac{\Pi}{8}\left(\begin{array}{c}\left|t_{\ddot{F}(e_p)}(v_s)-t_{\ddot{G}(e_p)}(v_s)\right| \\ +\left|f_{\ddot{F}(e_p)}(v_s)-f_{\ddot{G}(e_p)}(v_s)\right| \\ +\left|\Pi_{\ddot{F}(e_p)}(v_s)-\Pi_{\ddot{G}(e_p)}(v_s)\right|\end{array}\right)\right)$$

(2): Trigonometric Normalized Euclidean Similarity Measure

Let $\left(\ddot{S},A\right)$ and $(\ddot{T},A)$ be two VBSS's. Various Trigonometric Normalized Euclidean Similarity Measure of VBSS's $\left(\ddot{S},A\right)$ and $(\ddot{T},A)$ is defined as follows:

1. ***COSine** Normalised Euclidean Similarity Measure,* $Cos_{VBSS}^{nE-SM}=$

$$\sqrt{\begin{array}{l}\frac{1}{4mi}\sum_{p=1}^{m}\sum_{r=1}^{i} Cos\frac{\Pi}{4}\left(\left|t_{\ddot{S}(e_p)}(u_r)-t_{\ddot{T}(e_p)}(u_r)\right|^2+\left|f_{\ddot{S}(e_p)}(u_r)-f_{\ddot{T}(e_p)}(u_r)\right|^2+\left|\Pi_{\ddot{S}(e_p)}(u_r)-\Pi_{\ddot{T}(e_p)}(u_r)\right|^2\right) \\ +\frac{1}{4mj}\sum_{p=1}^{m}\sum_{s=1}^{j} Cos\frac{\Pi}{4}\left(\left|t_{\ddot{S}(e_p)}(v_s)-t_{\ddot{T}(e_p)}(v_s)\right|^2+\left|f_{\ddot{S}(e_p)}(v_s)-f_{\ddot{T}(e_p)}(v_s)\right|^2+\left|\Pi_{\ddot{S}(e_p)}(v_s)-\Pi_{\ddot{T}(e_p)}(v_s)\right|^2\right)\end{array}}$$

2. ***Sine** Normalised Euclidean Similarity Measure,* $Sin_{VBSS}^{nE-SM}=$

$$\sqrt{\begin{array}{l}1-\frac{1}{4mi}\sum_{p=1}^{m}\sum_{r=1}^{i} Sin\frac{\Pi}{4}\left(\left|t_{\ddot{S}(e_p)}(u_r)-t_{\ddot{T}(e_p)}(u_r)\right|^2+\left|f_{\ddot{S}(e_p)}(u_r)-f_{\ddot{T}(e_p)}(u_r)\right|^2+\left|\Pi_{\ddot{S}(e_p)}(u_r)-\Pi_{\ddot{T}(e_p)}(u_r)\right|^2\right) \\ -\frac{1}{4mj}\sum_{p=1}^{m}\sum_{s=1}^{j} Sin\frac{\Pi}{4}\left(\left|t_{\ddot{S}(e_p)}(v_s)-t_{\ddot{T}(e_p)}(v_s)\right|^2+\left|f_{\ddot{S}(e_p)}(v_s)-f_{\ddot{T}(e_p)}(v_s)\right|^2+\left|\Pi_{\ddot{S}(e_p)}(v_s)-\Pi_{\ddot{T}(e_p)}(v_s)\right|^2\right)\end{array}}$$

3. ***COTangent** Normalised Euclidean Similarity Measure,* $Cot_{VBSS}^{nE-SM}=$

$$\sqrt{\begin{array}{l}\frac{1}{4mi}\sum_{p=1}^{m}\sum_{r=1}^{i} Cot\left(\frac{\Pi}{4}+\frac{\Pi}{8}\left(\left|t_{\ddot{S}(e_p)}(u_r)-t_{\ddot{T}(e_p)}(u_r)\right|^2+\left|f_{\ddot{S}(e_p)}(u_r)-f_{\ddot{T}(e_p)}(u_r)\right|^2+\left|\Pi_{\ddot{S}(e_p)}(u_r)-\Pi_{\ddot{T}(e_p)}(u_r)\right|^2\right)\right) \\ +\frac{1}{4mj}\sum_{p=1}^{m}\sum_{s=1}^{j} Cot\left(\frac{\Pi}{4}+\frac{\Pi}{8}\left(\left|t_{\ddot{S}(e_p)}(v_s)-t_{\ddot{T}(e_p)}(v_s)\right|^2+\left|f_{\ddot{S}(e_p)}(v_s)-f_{\ddot{T}(e_p)}(v_s)\right|^2+\left|\Pi_{\ddot{S}(e_p)}(v_s)-\Pi_{\ddot{T}(e_p)}(v_s)\right|^2\right)\right)\end{array}}$$

Theorem 3.2.7.

Cosine Normalised Hamming Similarity Measure of VBSS's satisfies following properties

1. $0\leq Cos_{VBSS}^{nH-SM}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right)\leq 1$
2. $0\leq Cos_{VBSS}^{nH-SM}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right)=1\Leftrightarrow\left(\ddot{F},A\right)=\left(\ddot{G},A\right)$
3. $Cos_{VBSS}^{nH-SM}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right)=Cos_{VBSS}^{nH-SM}\left(\left(\ddot{G},A\right),\left(\ddot{F},A\right)\right)$

Proof.

1. $t_{\ddot{F}(e_p)}(u_r)$, $t_{\ddot{G}(e_p)}(u_r)$, $f_{\ddot{F}(e_p)}(u_r)$, $f_{\ddot{G}(e_p)}(u_r)$, $\Pi_{\ddot{F}(e_p)}(u_r)$, $\Pi_{\ddot{G}(e_p)}(u_r)$, $t_{\ddot{F}(e_p)}(v_s)$, $t_{\ddot{G}(e_p)}(v_s)$, $f_{\ddot{F}(e_p)}(v_s)$, $f_{\ddot{G}(e_p)}(v_s)$, $\Pi_{\ddot{F}(e_p)}(v_s)$, $\Pi_{\ddot{G}(e_p)}(v_s)$ and the value of Cosine Function are within $[0,1]$. So Cosine Normalised Hamming Similarity Measure based on VBSS's also lies in $[0,1]$

2. For any two VBSS's $\left(\ddot{F}, A\right)$ and $\left(\ddot{G}, A\right)$ if $\left(\ddot{F}, A\right) = \left(\ddot{G}, A\right)$ gives

$$\begin{cases} t_{\ddot{F}(e_p)}(u_r) = t_{\ddot{G}(e_p)}(u_r) & \Rightarrow \quad \left|t_{\ddot{F}(e_p)}(u_r) - t_{\ddot{G}(e_p)}(u_r)\right| = 0 \\ f_{\ddot{F}(e_p)}(u_r) = f_{\ddot{G}(e_p)}(u_r) & \Rightarrow \quad \left|f_{\ddot{F}(e_p)}(u_r) - f_{\ddot{G}(e_p)}(u_r)\right| = 0 \\ \Pi_{\ddot{F}(e_p)}(u_r) = \Pi_{\ddot{G}(e_p)}(u_r) & \Rightarrow \quad \left|\Pi_{\ddot{F}(e_p)}(u_r) - \Pi_{\ddot{G}(e_p)}(u_r)\right| = 0 \end{cases}$$

$$\begin{cases} t_{\ddot{F}(e_p)}(v_s) = t_{\ddot{G}(e_p)}(v_s) & \Rightarrow \quad \left|t_{\ddot{F}(e_p)}(v_s) - t_{\ddot{G}(e_p)}(v_s)\right| = 0 \\ f_{\ddot{F}(e_p)}(v_s) = f_{\ddot{G}(e_p)}(v_s) & \Rightarrow \quad \left|f_{\ddot{F}(e_p)}(v_s) - f_{\ddot{G}(e_p)}(v_s)\right| = 0 \\ \Pi_{\ddot{F}(e_p)}(v_s) = \Pi_{\ddot{G}(e_p)}(v_s) & \Rightarrow \quad \left|\Pi_{\ddot{F}(e_p)}(v_s) - \Pi_{\ddot{G}(e_p)}(v_s)\right| = 0 \end{cases}$$

Hence $Cos^{nH-SM}_{VBSS}\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right) = 1$.

Conversely, if $Cos^{nH-SM}_{VBSS}\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right) = 1$, then

$$\begin{cases} \left|t_{\ddot{F}(e_p)}(u_r) - t_{\ddot{G}(e_p)}(u_r)\right| = 0; \left|f_{\ddot{F}(e_p)}(u_r) - f_{\ddot{G}(e_p)}(u_r)\right| = 0; \left|\Pi_{\ddot{F}(e_p)}(u_r) - \Pi_{\ddot{G}(e_p)}(u_r)\right| = 0 \\ \Rightarrow t_{\ddot{F}(e_p)}(u_r) = t_{\ddot{G}(e_p)}(u_r) \quad ; \quad f_{\ddot{F}(e_p)}(u_r) = f_{\ddot{G}(e_p)}(u_r) \quad ; \quad \Pi_{\ddot{F}(e_p)}(u_r) = \Pi_{\ddot{G}(e_p)}(u_r) = 0 \end{cases}$$

$$\begin{cases} \left|t_{\ddot{F}(e_p)}(v_s) - t_{\ddot{G}(e_p)}(v_s)\right| = 0; \left|f_{\ddot{F}(e_p)}(v_s) - f_{\ddot{G}(e_p)}(v_s)\right| = 0; \left|\Pi_{\ddot{F}(e_p)}(v_s) - \Pi_{\ddot{G}(e_p)}(v_s)\right| = 0 \\ \Rightarrow t_{\ddot{F}(e_p)}(v_s) = t_{\ddot{G}(e_p)}(v_s) \quad ; \quad f_{\ddot{F}(e_p)}(v_s) = f_{\ddot{G}(e_p)}(v_s) \quad ; \quad \Pi_{\ddot{F}(e_p)}(v_s) = \Pi_{\ddot{G}(e_p)}(v_s) = 0 \end{cases}$$

$$\Rightarrow \left(\ddot{F}, A\right) = \left(\ddot{G}, A\right)$$

3. Proof is trivial

Theorem 3.2.8.

Sine Normalised Hamming Similarity Measure of VBSS's satisfies following properties :

1. $0 \leq Sin^{nH-SM}_{VBSS}\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right) \leq 1$
2. $Sin^{nH-SM}_{VBSS}\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right) = 1 \Leftrightarrow \left(\ddot{F}, A\right) = \left(\ddot{G}, A\right)$
3. $Sin^{nH-SM}_{VBSS}\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right) = Sin^{SM}_{VBSS}\left(\left(\ddot{G}, A\right), \left(\ddot{F}, A\right)\right)$

Proof.

Similar to theorem 3.2.7.

Theorem 3.2.9.

Cotangent Normalised Hamming Similarity Measure of VBSS's satisfies following properties

1. $0 \leqslant Cot^{nH-SM}_{VBSS}\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right) \leqslant 1$
2. $Cot^{nH-SM}_{VBSS}\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right) = 1 \Leftrightarrow \left(\ddot{F}, A\right) = \left(\ddot{G}, A\right)$
3. $Cot^{nH-SM}_{VBSS}\left(\left(\ddot{F}, A\right), \left(\ddot{G}, A\right)\right) = Cot^{nH-SM}_{VBSS}\left(\left(\ddot{G}, A\right), \left(\ddot{F}, A\right)\right)$

Proof.

Similar to theorem 3.2.7.

Applicational Algorithm and Numerical Example for VBSS's

In this section a decision - making method using Trigonometric Normalised Hamming/Euclidean Similarity Measure for VBSS's are developed and it is used in a numerical example.

(i) Algorithm

Step 1 : Find binary universe and parameter set for the problem under concern. Let it be (U_1, U_2) and $A \subseteq E$, where U_1 and U_2 are universes and E be a fixed parameter set.

Step 2 : Depending on this, frame VBSS $\left(\ddot{F}, A\right)$ for the problem under consideration

Step 3 : Determine the benefit type attribute and cost type attribute from the given parameter set. Benefit type attribute will be obtained by taking maximum and Cost type attribute, by taking minimum

Step 4 : Fix the ideal VBSS $\left(\ddot{G}, A\right)$ using benefit type attribute and cost type attribute

Step 5 : Using the newly developed formulae, *Hamming/Euclidean* for VBSS's, find similarity measures to each of the parameters with ideal VBSS

Step 6 : After fixing the parameter, using step 5, we do one more filtration to find best choice among the universal sets depending upon the context. i.e., simultaneous selection or independent selection among the alternatives.

(i) (Numerical Example - Hamming)

$$\text{Let} \quad U_1 = \left\{ \begin{array}{l} w_1 = portable \quad mini \quad washing \quad machine, \\ w_2 = fully \quad automatic, \\ w_3 = handy \quad bucket \quad washing \quad machine \end{array} \right\}$$

$$U_2 = \left\{ \begin{array}{l} f_1 = single \quad door \quad refrigerator, \\ f_2 = double \quad door \quad refrigerator \end{array} \right\}$$

be a binary universe of domestic utensils under consideration.

Let the parameter set under consideration be

$$A = \left\{ \begin{array}{ll} e_1 = service \quad after \quad sale, & e_2 = reasonable \quad prize, \\ e_3 = customer \quad care, & e_4 = discount \end{array} \right\}$$

Let VBSS $\left(\ddot{F}, A\right)$ be the VBSS framed for the corresponding binary universe and to fixed parameter set. For above problem, let $\left(\ddot{G}, A\right)$ be the constructed Ideal VBSS

$$\left(\ddot{F}, A\right) = \left\{ \begin{array}{l} \left(e_1, \left(\left\langle \frac{[0.2, 0.6]}{w_1}, \frac{[0.3, 0.5]}{w_2}, \frac{[0.1, 0.8]}{w_3} \right\rangle, \left\langle \frac{[0.2, 0.6]}{f_1}, \frac{[0.3, 0.5]}{f_2} \right\rangle\right)\right) \\ \left(e_2, \left(\left\langle \frac{[0.2, 0.6]}{w_1}, \frac{[0.3, 0.5]}{w_2}, \frac{[0.1, 0.8]}{w_3} \right\rangle, \left\langle \frac{[0.2, 0.6]}{f_1}, \frac{[0.3, 0.5]}{f_2} \right\rangle\right)\right) \\ \left(e_3, \left(\left\langle \frac{[0.2, 0.6]}{w_1}, \frac{[0.3, 0.5]}{w_2}, \frac{[0.1, 0.8]}{w_3} \right\rangle, \left\langle \frac{[0.2, 0.6]}{f_1}, \frac{[0.3, 0.5]}{f_2} \right\rangle\right)\right) \\ \left(e_4, \left(\left\langle \frac{[0.2, 0.6]}{w_1}, \frac{[0.3, 0.5]}{w_2}, \frac{[0.1, 0.8]}{w_3} \right\rangle, \left\langle \frac{[0.2, 0.6]}{f_1}, \frac{[0.3, 0.5]}{f_2} \right\rangle\right)\right) \end{array} \right\}$$

$$\left(\ddot{G}, A\right) = \left\{ \begin{array}{l} \left(e_1, \left(\left\langle \frac{[0.1, 0.5]}{w_1}, \frac{[0.1, 0.5]}{w_2}, \frac{[0.1, 0.5]}{w_3} \right\rangle, \left\langle \frac{[0.6, 0.7]}{f_1}, \frac{[0.6, 0.7]}{f_2} \right\rangle\right)\right) \\ \left(e_2, \left(\left\langle \frac{[0.7, 0.8]}{w_1}, \frac{[0.7, 0.8]}{w_2}, \frac{[0.7, 0.8]}{w_3} \right\rangle, \left\langle \frac{[0.7, 0.9]}{f_1}, \frac{[0.7, 0.9]}{f_2} \right\rangle\right)\right) \\ \left(e_3, \left(\left\langle \frac{[0.3, 0.4]}{w_1}, \frac{[0.3, 0.4]}{w_2}, \frac{[0.3, 0.4]}{w_3} \right\rangle, \left\langle \frac{[0.7, 0.8]}{f_1}, \frac{[0.7, 0.8]}{f_2} \right\rangle\right)\right) \\ \left(e_4, \left(\left\langle \frac{[0.7, 0.9]}{w_1}, \frac{[0.7, 0.9]}{w_2}, \frac{[0.7, 0.9]}{w_3} \right\rangle, \left\langle \frac{[0.5, 0.9]}{f_1}, \frac{[0.5, 0.9]}{f_2} \right\rangle\right)\right) \end{array} \right\}$$

First iteration for the ideal parameter gives the following result:

$$Cos_{VBSS}^{nH-SM}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right)=\begin{cases}0.03 & for \quad e_1;\\ 0.04 & for \quad e_2;\\ 0.02 & for \quad e_3;\\ 0.04 & for \quad e_4\end{cases}$$

$$Clearly, \qquad 0.04 \geq 0.04 \geq 0.03 \geq 0.02$$

$$\Rightarrow \quad \left[Cos_{VBSS}^{nH-SM}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right)\right]_{e_4} \geqslant \left[Cos_{VBSS}^{nH-SM}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right)\right]_{e_2}$$
$$\geqslant \left[Cos_{VBSS}^{nH-SM}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right)\right]_{e_1} \geqslant \left[Cos_{VBSS}^{nH-SM}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right)\right]_{e_3}$$

Calculations showed that, prime care, should be assigned to risk factor .

In next stage, second iterations have done depending upon e_4 only to both of the universal sets independently. Results showed that,

$$Cos_{VBSS}^{nH-SM}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right)=\begin{cases}0.22275 & for \quad w_1;\\ 0.23775 & for \quad w_2;\\ 0.25 & for \quad w_3;\\ 0.23775 & for \quad f_1;\\ 0.25 & for \quad f_2\end{cases}$$

Selection is in two ways :

Case (i): (Simultaneous selection of universes)

$$\left[Cos_{VBSS}^{nH-SM}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right)\right]_{w_3} \geqslant \left[Cos_{VBSS}^{nH-SM}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right)\right]_{f_2}$$
$$\geqslant \left[Cos_{VBSS}^{nH-SM}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right)\right]_{w_2} \geqslant \left[Cos_{VBSS}^{nH-SM}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right)\right]_{f_1}$$
$$\geqslant \left[Cos_{VBSS}^{nH-SM}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right)\right]_{w_1}$$

Case (ii): (Independent selection of universes)

$$\left[Cos_{VBSS}^{nH-SM}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right)\right]_{w_3} \geqslant \left[Cos_{VBSS}^{nH-SM}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right)\right]_{w_2}$$
$$\geqslant \left[Cos_{VBSS}^{nH-SM}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right)\right]_{w_1} \geqslant \left[Cos_{VBSS}^{nH-SM}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right)\right]_{f_2}$$
$$\geqslant \left[Cos_{VBSS}^{nH-SM}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right)\right]_{f_1}$$

Case (i) indicates, w_3 will be good choice.

Case (ii) indicates, w_3 will be good choice for first set of universes and for the second set of universes f_2 will be the good choice for first set of universes.

Remark 3.2.10.

Above method could be applied to other two trigonometric similarity measures and the result can be verified.

(ii) (Numerical Example - Euclidean)

A large family of viruses transmitted from animals (civet cats, camels, etc) to human are called Corona viruses. It's a very dangerous situation and has become fatal. It is spreading in an uncontrollable way and it has become a great issue for different countries and their governments. Its outbreak is from China and recorded death rate is large. Let (U_1, U_2) be a binary universe which describes the people returned to their home countries.

$$\text{Let } U_1 = \Big\{ w_1 = wuhan, \quad w_2 = beijing, \quad w_3 = shanghai \Big\}$$

$$U_2 = \Big\{ f_1 = xiantao, \quad f_2 = hongkong \Big\}$$

These people are kept under observation in two different army hospitals of a country. Let the parameter set under consideration be

$$A = \left\{ \begin{array}{l} e_1 = fever \quad and \quad cough, \\ e_2 = shortness \quad of \quad breath \quad and \quad breathing \quad difficulties, \\ e_3 = pneumonia, \quad e_4 = kidney \quad failure \end{array} \right\}$$

Let the VBSS $\left(\ddot{S}, A\right)$ give the vague binary soft values of the corresponding universal sets and parameter set under consideration. Ideal VBSS constructed for the above set is given as

$$\left(\ddot{S}, A\right) = \left\{ \begin{array}{l} \left(e_1, \left(\left\langle \frac{[0.2,0.6]}{w_1}, \frac{[0.3,0.5]}{w_2}, \frac{[0.1,0.8]}{w_3} \right\rangle, \left\langle \frac{[0.2,0.6]}{f_1}, \frac{[0.3,0.5]}{f_2} \right\rangle\right)\right) \\ \left(e_2, \left(\left\langle \frac{[0.2,0.6]}{w_1}, \frac{[0.3,0.5]}{w_2}, \frac{[0.1,0.8]}{w_3} \right\rangle, \left\langle \frac{[0.2,0.6]}{f_1}, \frac{[0.3,0.5]}{f_2} \right\rangle\right)\right) \\ \left(e_3, \left(\left\langle \frac{[0.2,0.6]}{w_1}, \frac{[0.3,0.5]}{w_2}, \frac{[0.1,0.8]}{w_3} \right\rangle, \left\langle \frac{[0.2,0.6]}{f_1}, \frac{[0.3,0.5]}{f_2} \right\rangle\right)\right) \\ \left(e_4, \left(\left\langle \frac{[0.2,0.6]}{w_1}, \frac{[0.3,0.5]}{w_2}, \frac{[0.1,0.8]}{w_3} \right\rangle, \left\langle \frac{[0.2,0.6]}{f_1}, \frac{[0.3,0.5]}{f_2} \right\rangle\right)\right) \end{array} \right\}$$

$$\left(\ddot{T},A\right)=\left\{\begin{array}{l}\left(e_1,\left(\left\langle\frac{[0.1,0.5]}{w_1},\frac{[0.1,0.5]}{w_2},\frac{[0.1,0.5]}{w_3}\right\rangle,\left\langle\frac{[0.6,0.7]}{f_1},\frac{[0.6,0.7]}{f_2}\right\rangle\right)\right)\\\left(e_2,\left(\left\langle\frac{[0.7,0.8]}{w_1},\frac{[0.7,0.8]}{w_2},\frac{[0.7,0.8]}{w_3}\right\rangle,\left\langle\frac{[0.7,0.9]}{f_1},\frac{[0.7,0.9]}{f_2}\right\rangle\right)\right)\\\left(e_3,\left(\left\langle\frac{[0.3,0.4]}{w_1},\frac{[0.3,0.4]}{w_2},\frac{[0.3,0.4]}{w_3}\right\rangle,\left\langle\frac{[0.7,0.8]}{f_1},\frac{[0.7,0.8]}{f_2}\right\rangle\right)\right)\\\left(e_4,\left(\left\langle\frac{[0.7,0.9]}{w_1},\frac{[0.7,0.9]}{w_2},\frac{[0.7,0.9]}{w_3}\right\rangle,\left\langle\frac{[0.5,0.9]}{f_1},\frac{[0.5,0.9]}{f_2}\right\rangle\right)\right)\end{array}\right\}$$

First iteration for ideal parameter gives following result:

$$Cos_{VBSS}^{nE-SM}\left(\left(\ddot{S},A\right),\left(\ddot{T},A\right)\right)=\begin{cases}0.6299 & for \quad e_1,\\0.5268 & for \quad e_2,\\0.5175 & for \quad e_3,\\0.5498 & for \quad e_4\end{cases}$$

$$Sin_{VBSS}^{nE-SM}\left(\left(\ddot{S},A\right),\left(\ddot{T},A\right)\right)=\begin{cases}0.6686 & for \quad e_1;\\0.4742 & for \quad e_2;\\0.4606 & for \quad e_3;\\0.5309 & for \quad e_4\end{cases}$$

$$Cot_{VBSS}^{nE-SM}\left(\left(\ddot{S},A\right),\left(\ddot{T},A\right)\right)=\begin{cases}0.6118 & for \quad e_1;\\0.4108 & for \quad e_2;\\0.3989 & for \quad e_3;\\0.4498 & for \quad e_4\end{cases}$$

Comparing above results, it is clear that,

$$\begin{aligned}\left[Cos_{VBSS}^{nE-SM}\left(\left(\ddot{S},A\right),\left(\ddot{T},A\right)\right)\right]_{e_1} &\geqslant \left[Cos_{VBSS}^{nE-SM}\left(\left(\ddot{S},A\right),\left(\ddot{T},A\right)\right)\right]_{e_4}\\ \geqslant \left[Cos_{VBSS}^{nE-SM}\left(\left(\ddot{S},A\right),\left(\ddot{T},A\right)\right)\right]_{e_2} &\geqslant \left[Cos_{VBSS}^{nE-SM}\left(\left(\ddot{S},A\right),\left(\ddot{T},A\right)\right)\right]_{e_3}\end{aligned}$$

Similarly,

$$\begin{aligned}\left[Sin_{VBSS}^{nE-SM}\left(\left(\ddot{S},A\right),\left(\ddot{T},A\right)\right)\right]_{e_1} &\geqslant \left[Sin_{VBSS}^{nE-SM}\left(\left(\ddot{S},A\right),\left(\ddot{T},A\right)\right)\right]_{e_4}\\ \geqslant \left[Sin_{VBSS}^{nE-SM}\left(\left(\ddot{S},A\right),\left(\ddot{T},A\right)\right)\right]_{e_2} &\geqslant \left[Sin_{VBSS}^{nE-SM}\left(\left(\ddot{S},A\right),\left(\ddot{T},A\right)\right)\right]_{e_3}\end{aligned}$$

$$\begin{aligned}\left[Cot_{VBSS}^{nE-SM}\left(\left(\ddot{S},A\right),\left(\ddot{T},A\right)\right)\right]_{e_1} &\geqslant \left[Cot_{VBSS}^{nE-SM}\left(\left(\ddot{S},A\right),\left(\ddot{T},A\right)\right)\right]_{e_4}\\ \geqslant \left[Cot_{VBSS}^{nE-SM}\left(\left(\ddot{S},A\right),\left(\ddot{T},A\right)\right)\right]_{e_2} &\geqslant \left[Cot_{VBSS}^{nE-SM}\left(\left(\ddot{S},A\right),\left(\ddot{T},A\right)\right)\right]_{e_3}\end{aligned}$$

Each tool of measurement, indicates that similarity is greater for parameter e_1. So risk factor is high for e_1. i.e., for fever and cough. COVID - 2019 test [COrona VIrus Disease - 2019] should be conducted for fever and cough-affected people with much caution.

Theorem 3.2.11.

Cosine Normalised Euclidean Similarity of VBSS's satisfies following properties:

1. $0 \leq Cos_{VBSS}^{nE-SM}\left(\left(\ddot{S},A\right),\left(\ddot{T},A\right)\right) \leq 1$
2. $0 \leq Cos_{VBSS}^{nE-SM}\left(\left(\ddot{S},A\right),\left(\ddot{T},A\right)\right) = 1 \Leftrightarrow \left(\ddot{S},A\right) = \left(\ddot{T},A\right)$
3. $Cos_{VBSS}^{nE-SM}\left(\left(\ddot{S},A\right),\left(\ddot{T},A\right)\right) = Cos_{VBSS}^{nE-SM}\left(\left(\ddot{T},A\right),\left(\ddot{S},A\right)\right)$

Proof.

1. $t_{\ddot{S}(e_p)}(u_r)$, $t_{\ddot{T}(e_p)}(u_r)$, $f_{\ddot{S}(e_p)}(u_r)$, $f_{\ddot{T}(e_p)}(u_r)$, $\Pi_{\ddot{S}(e_p)}(u_r), \Pi_{\ddot{T}(e_p)}(u_r)$, $t_{\ddot{S}(e_p)}(v_s)$, $t_{\ddot{T}(e_p)}(v_s)$, $f_{\ddot{S}(e_p)}(v_s)$, $f_{\ddot{T}(e_p)}(v_s)$, $\Pi_{\ddot{S}(e_p)}(v_s)$, $\Pi_{\ddot{T}(e_p)}(v_s)$ and the value of the cosine function are within $[0,1]$. So cosine normalised euclidean similarity measure based on VBSS's also lies in $[0,1]$

2. For any two VBSS's $\left(\ddot{S},A\right)$ and $\left(\ddot{T},A\right)$ if $\left(\ddot{S},A\right) = \left(\ddot{T},A\right)$ then

$$\begin{cases} t_{\ddot{S}(e_p)}(u_r) = t_{\ddot{T}(e_p)}(u_r) \Rightarrow \left|t_{\ddot{S}(e_p)}(u_r) - t_{\ddot{T}(e_p)}(u_r)\right| = 0 \Rightarrow \left|t_{\ddot{S}(e_p)}(u_r) - t_{\ddot{T}(e_p)}(u_r)\right|^2 = 0\\ f_{\ddot{S}(e_p)}(u_r) = f_{\ddot{T}(e_p)}(u_r) \Rightarrow \left|f_{\ddot{S}(e_p)}(u_r) - f_{\ddot{T}(e_p)}(u_r)\right| = 0 \Rightarrow \left|f_{\ddot{S}(e_p)}(u_r) - f_{\ddot{T}(e_p)}(u_r)\right|^2 = 0\\ \Pi_{\ddot{S}(e_p)}(u_r) = \Pi_{\ddot{T}(e_p)}(u_r) \Rightarrow \left|\Pi_{\ddot{S}(e_p)}(u_r) - \Pi_{\ddot{T}(e_p)}(u_r)\right| = 0 \Rightarrow \left|\Pi_{\ddot{S}(e_p)}(u_r) - \Pi_{\ddot{T}(e_p)}(u_r)\right|^2 = 0 \end{cases}$$

$$\begin{cases} t_{\ddot{S}(e_p)}(v_s) = t_{\ddot{T}(e_p)}(v_s) \Rightarrow \left|t_{\ddot{S}(e_p)}(v_s) - t_{\ddot{T}(e_p)}(v_s)\right| = 0 \Rightarrow \left|t_{\ddot{S}(e_p)}(v_s) - t_{\ddot{T}(e_p)}(v_s)\right|^2 = 0\\ f_{\ddot{S}(e_p)}(v_s) = f_{\ddot{T}(e_p)}(v_s) \Rightarrow \left|f_{\ddot{S}(e_p)}(v_s) - f_{\ddot{T}(e_p)}(v_s)\right| = 0 \Rightarrow \left|f_{\ddot{S}(e_p)}(v_s) - f_{\ddot{T}(e_p)}(v_s)\right|^2 = 0\\ \Pi_{\ddot{S}(e_p)}(v_s) = \Pi_{\ddot{T}(e_p)}(v_s) \Rightarrow \left|\Pi_{\ddot{S}(e_p)}(v_s) - \Pi_{\ddot{T}(e_p)}(v_s)\right| = 0 \Rightarrow \left|\Pi_{\ddot{S}(e_p)}(v_s) - \Pi_{\ddot{T}(e_p)}(v_s)\right|^2 = 0 \end{cases}$$

$$\text{Hence } Cos_{VBSS}^{nE-SM}\left(\left(\ddot{F},A\right),\left(\ddot{G},A\right)\right) = 1.$$

Conversely, if $Cos^{nE-SM}_{VBSS}\left(\left(\ddot{S},A\right),\left(\ddot{T},A\right)\right)=1$, then

$$\left|t_{\ddot{S}(e_p)}(u_r)-t_{\ddot{T}(e_p)}(u_r)\right|^2=0\Longrightarrow \quad \left|t_{\ddot{S}(e_i)}(u_r)-t_{\ddot{T}(e_p)}(u_r)\right|=0$$
$$\left|t_{\ddot{S}(e_p)}(v_s)-t_{\ddot{T}(e_p)}(v_s)\right|^2=0\Longrightarrow \quad \left|t_{\ddot{S}(e_p)}(v_s)-t_{\ddot{T}(e_p)}(v_s)\right|=0$$

$Cos\frac{\Pi}{4}=1$ gives

$$\begin{cases} t_{\ddot{S}(e_p)}(u_r)=t_{\ddot{T}(e_p)}(u_r) & \Rightarrow \quad \left|t_{\ddot{S}(e_p)}(u_r)-t_{\ddot{T}(e_p)}(u_r)\right|^2=0 \\ t_{\ddot{S}(e_p)}(v_s)=t_{\ddot{T}(e_p)}(v_s) & \Rightarrow \quad \left|t_{\ddot{S}(e_p)}(v_s)-t_{\ddot{T}(e_p)}(v_s)\right|^2=0 \end{cases}$$

$$\begin{cases} f_{\ddot{S}(e_p)}(u_r)=f_{\ddot{T}(e_p)}(u_r) & \Rightarrow \quad \left|f_{\ddot{S}(e_p)}(u_r)-f_{\ddot{T}(e_p)}(u_r)\right|^2=0 \\ f_{\ddot{S}(e_p)}(v_s)=f_{\ddot{T}(e_p)}(v_s) & \Rightarrow \quad \left|f_{\ddot{S}(e_p)}(v_s)-f_{\ddot{T}(e_p)}(v_s)\right|^2=0 \end{cases}$$

$$\begin{cases} \Pi_{\ddot{S}(e_p)}(u_r)=\Pi_{\ddot{T}(e_p)}(u_r) & \Rightarrow \quad \left|\Pi_{\ddot{S}(e_p)}(u_r)-\Pi_{\ddot{T}(e_p)}(u_r)\right|^2=0 \\ \Pi_{\ddot{S}(e_p)}(v_s)=\Pi_{\ddot{T}(e_p)}(v_s) & \Rightarrow \quad \left|\Pi_{\ddot{S}(e_p)}(v_s)-\Pi_{\ddot{T}(e_p)}(v_s)\right|^2=0 \end{cases}$$

3. Proof is Obvious

Remark 3.2.12.

If the parameter sets are different then the intersected region or common elements are considered for calculations.

Theorem 3.2.13.

Sine Normalised Euclidean Similarity of VBSS's satisfies following properties :

1. $0\leq Sin^{nE-SM}_{VBSS}((\ddot{S},A),(\ddot{T},A))\leq 1$
2. $Sin^{nE-SM}_{VBSS}((\ddot{S},A),(\ddot{T},A))=1 \quad\Leftrightarrow\quad (\ddot{S},A)=(\ddot{T},A)$
3. $Sin^{nE-SM}_{VBSS}((\ddot{S},A),(\ddot{T},A)) \quad = \quad Sin^{nE-SM}_{VBSS}((\ddot{T},A),(\ddot{S},A))$

Proof.

Similar to theorem 3.2.11.

Theorem 3.2.14.

Cotangent Normalised Euclidean Similarity Measure of VBSS's satisfies following properties

1. $0 \leq Cot^{nE-SM}_{VBSS}((\ddot{S}, A), (\ddot{T}, A)) \leq 1$

2. $Cot^{nE-SM}_{VBSS}((\ddot{S}, A), (\ddot{T}, A)) = 1 \quad \Leftrightarrow \quad (\ddot{S}, A) = (\ddot{T}, A)$

3. $Cot^{nE-SM}_{VBSS}((\ddot{S}, A), (\ddot{T}, A)) \quad = \quad Cot^{nE-SM}_{VBSS}((\ddot{T}, A), (\ddot{S}, A))$

Proof.
Similar to theorem 3.2.11.

Pythagorean Vague Binary Soft Set & Properties

Pythagorean Vague Binary Soft Set is a special kind of Vague Binary Soft Set. In this section, this special version of VBSS is developed with some of its basic notions.

Definition 3.2.15. ***(Pythagorean vague binary soft set)***
Let (U_1, U_2) be a binary universe and E be a fixed parameter set with $A \subseteq E$.
Let $PV(U_1)$ and $PV(U_2)$ denote the power set of Pythagorean Vague subsets on U_1, U_2 respectively.
A Pythagorean Vague Binary Soft Set (in short, PVBSS) over a binary universe (U_1, U_2) is a pair $\left(\ddot{F}_{PVB}, A\right)$ where $\ddot{F}_{PVB}$ is a mapping given by
$\ddot{F}_{PVB} : A \to PV(U_1) \times PV(U_2)$
$\left(\ddot{F}_{PVB}, A\right) = \left\{e_p \in A / \left(e_p, \ddot{F}_{PVB}(e_p)\right)\right\}$

$$\text{where,} \quad \ddot{F}_{PVB}(e_p) = \left\{ \begin{array}{c} \left\langle \dfrac{V_{\ddot{F}_{PVB}(e_p)}(u_r)}{u_r}; \forall e_p \in A, \forall u_r \in U_1 \right\rangle, \\ \left\langle \dfrac{V_{\ddot{F}_{PVB}(e_p)}(v_s)}{v_s}; \forall e_p \in A, \forall v_s \in U_2 \right\rangle \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left\langle \dfrac{\left[t_{\ddot{F}_{PVB}(e_p)}(u_r), 1 - f_{\ddot{F}_{PVB}(e_p)}(u_r)\right]}{u_r}; \quad \forall e_p \in A, \quad \forall u_r \in U_1 \right\rangle, \\ \left\langle \dfrac{\left[t_{\ddot{F}_{PVB}(e_p)}(v_s), 1 - f_{\ddot{F}_{PVB}(e_p)}(v_s)\right]}{v_s}; \quad \forall e_p \in A, \quad \forall v_s \in U_2 \right\rangle \end{array} \right\}$$

where

$$t_{\ddot{F}_{PVB}(e_p)}(u_r) : U_1 \to [0,1]; \qquad 1 - f_{\ddot{F}_{PVB}(e_p)}(u_r) : U_1 \to [0,1]; \quad \&$$
$$t_{\ddot{F}_{PVB}(e_p)}(v_s) : U_2 \to [0,1]; \qquad 1 - f_{\ddot{F}_{PVB}(e_p)}(v_s) : U_2 \to [0,1];$$

$$0 \leq \left[t_{\ddot{F}_{PVB(e_p)}}(u_r)\right]^2 + \left[1 - f_{\ddot{F}_{PVB(e_p)}}(u_r)\right]^2 \leq 1$$
$$0 \leq \left[t_{\ddot{F}_{PVB(e_p)}}(v_s)\right]^2 + \left[1 - f_{\ddot{F}_{PVB(e_p)}}(v_s)\right]^2 \leq 1$$

Example 3.2.16. ***(Example for Pythagorean Vague Binary Soft Set)***
Let $U_1 = \{f_1, f_2, f_3\}$, $U_2 = \{b_1, b_2, b_3\}$ be the set of flights and trains from Kochi to Bangalore respectively, and $A = \{e_1 = first\ class, e_2 = second\ class, e_3 = third\ class\}$ be the set of parameters with $A \subseteq E$. Let $\left(\ddot{F}_{PVB}, A\right)$ be a PVBSS which describes the availability of tickets as follows:
$\left(\ddot{F}_{PVB}, A\right) =$

$$\left\{\begin{array}{l} \left(e_1, \left(\left\langle \frac{[0.2, 0.3]}{f_1}, \frac{[0.4, 0.2]}{f_2}, \frac{[0.7, 0.8]}{f_3} \right\rangle, \left\langle \frac{[0.3, 0.8]}{b_1}, \frac{[0.6, 0.7]}{b_2}, \frac{[0.4, 0.1]}{b_3} \right\rangle\right)\right) \\ \left(e_2, \left(\left\langle \frac{[0.2, 0.4]}{f_1}, \frac{[0.4, 0.3]}{f_2}, \frac{[0.7, 0.8]}{f_3} \right\rangle, \left\langle \frac{[0.3, 0.2]}{b_1}, \frac{[0.6, 0.7]}{b_2}, \frac{[0.4, 0.6]}{b_3} \right\rangle\right)\right) \\ \left(e_3, \left(\left\langle \frac{[0.2, 0.1]}{f_1}, \frac{[0.4, 0.7]}{f_2}, \frac{[0.7, 0.8]}{f_3} \right\rangle, \left\langle \frac{[0.3, 0.8]}{b_1}, \frac{[0.6, 0.7]}{b_2}, \frac{[0.4, 0.6]}{b_3} \right\rangle\right)\right) \end{array}\right\}$$

Remark 3.2.17.
Set of all Pythagorean vague binary soft sets over a binary universe (U_1, U_2) is denoted by $PVBSS(U_1, U_2)$.

Definition 3.2.18. ***(Operations on Pythagorean Vague binary soft set's)***
Consider two Pythagorean Vague Binary Soft Set's as follows:
Let, $\forall e_p \in A, \quad \forall u_r \in U_1 \quad, \forall v_s \in U_2$

$$\left(\ddot{F}_{PVB}, A\right) = \left\{\left(e_p, \left(\begin{array}{c} \left\langle \frac{\left[t_{\ddot{F}_{PVB(e_p)}}(u_r), 1 - f_{\ddot{F}_{PVB(e_p)}}(u_r)\right]}{u_r} \right\rangle, \\ \left\langle \frac{\left[t_{\ddot{F}_{PVB(e_p)}}(v_s), 1 - f_{\ddot{F}_{PVB(e_p)}}(v_s)\right]}{v_s} \right\rangle \end{array}\right)\right)\right\} \text{ and}$$

$$\left(\ddot{G}_{PVB}, A\right) = \left\{\left(e_p, \left(\begin{array}{c} \left\langle \frac{\left[t_{\ddot{G}_{PVB(e_p)}}(u_r), 1 - f_{\ddot{G}_{PVB(e_p)}}(u_r)\right]}{u_r} \right\rangle, \\ \left\langle \frac{\left[t_{\ddot{G}_{PVB(e_p)}}(v_s), 1 - f_{\ddot{G}_{PVB(e_p)}}(v_s)\right]}{v_s} \right\rangle \end{array}\right)\right)\right\}$$

1. ***(Pythagorean Vague Binary Soft Union)***
$\left(\ddot{F}_{PVB}, A\right) \ddot{\cup} \left(\ddot{G}_{PVB}, A\right) =$

$$\left\{\left(e_p, \left(\begin{array}{c} \left\langle \dfrac{max\left(t_{\ddot{F}(e_p)}(u_r), t_{\ddot{G}(e_p)}(u_r)\right), max\left(1 - f_{\ddot{F}(e_p)}(u_r)\right), \left(1 - f_{\ddot{F}(e_p)}(u_r)\right)}{u_r} \right\rangle \\ \left\langle \dfrac{max\left(t_{\ddot{F}(e_p)}(v_s), t_{\ddot{G}(e_p)}(v_s)\right), max\left(1 - f_{\ddot{F}(e_p)}(v_s)\right), \left(1 - f_{\ddot{F}(e_p)}(v_s)\right)}{v_s} \right\rangle \end{array}\right)\right)\right\}$$

2. ***(Pythagorean Vague Binary Soft Intersection)***
$\left(\ddot{F}_{PVB}, A\right) \ddot{\cap} \left(\ddot{G}_{PVB}, A\right) =$

$$\left\{\left(e_p, \left(\begin{array}{c} \left\langle \dfrac{min\left(t_{\ddot{F}(e_p)}(u_r), t_{\ddot{G}(e_p)}(u_r)\right), min\left(\left(1 - f_{\ddot{F}(e_p)}(u_r)\right), \left(1 - f_{\ddot{F}(e_p)}(u_r)\right)\right)}{u_r} \right\rangle \\ \left\langle \dfrac{min\left(t_{\ddot{F}(e_p)}(v_s), t_{\ddot{G}(e_p)}(v_s)\right), min\left(\left(1 - f_{\ddot{F}(e_p)}(v_s)\right), \left(1 - f_{\ddot{F}(e_p)}(v_s)\right)\right)}{v_s} \right\rangle \end{array}\right)\right)\right\}$$

3. ***(Pythagorean Vague Binary Soft Complement)***
$\left(\ddot{F}_{PVB}, A\right)^c =$

$$\left\{\left(e_p, \left(\begin{array}{c} \left\langle \dfrac{\left(1 - f_{\ddot{F}(e_p)}(u_r), t_{\ddot{F}(e_p)}(u_r)\right)}{u_r} / e_p \in A, u_r \in U_1 \right\rangle \\ \left\langle \dfrac{\left(1 - f_{\ddot{F}(e_p)}(v_s), t_{\ddot{F}(e_p)}(v_s)\right)}{v_s} / e_p \in A, v_s \in U_2 \right\rangle \end{array}\right)\right)\right\}$$

4. ***(Pythagorean Vague Binary Soft Sum)***
$\left(\ddot{F}_{PVB}, A\right) \oplus \left(\ddot{G}_{PVB}, A\right) =$

$$\left\{\left(e_p, \left(\begin{array}{c} \left\langle \dfrac{\sqrt{t_{\ddot{F}(e_p)}(u_r)^2 + t_{\ddot{G}(e_p)}(u_r)^2 - \left(t_{\ddot{F}(e_p)}(u_r)\right)^2 \cdot \left(t_{\ddot{G}(e_p)}(u_r)\right)^2}, \left(1 - f_{\ddot{F}(e_p)}(u_r)\right) \cdot \left(1 - f_{\ddot{G}(e_p)}(u_r)\right)}{u_r} \right\rangle \\ \left\langle \dfrac{\sqrt{t_{\ddot{F}(e_p)}(v_s)^2 + t_{\ddot{G}(e_p)}(v_s)^2 - \left(t_{\ddot{F}(e_p)}(v_s)\right)^2 \cdot \left(t_{\ddot{G}(e_p)}(v_s)\right)^2}, \left(1 - f_{\ddot{F}(e_i)}(v_s)\right) \cdot \left(1 - f_{\ddot{G}(e_p)}(v_s)\right)}{v_s} \right\rangle \end{array}\right)\right)\right\}$$

5. ***(Pythagorean Vague Binary Soft Product)***
$\left(\ddot{F}_{PVB}, A\right) \otimes \left(\ddot{G}_{PVB}, A\right) =$

$$\left\{\left(e_p, \left(\begin{array}{c} \left\langle \dfrac{\sqrt{\left(1 - f_{\ddot{F}(e_p)}(u_r)\right)^2 + \left(1 - f_{\ddot{G}(e_p)}(u_r)\right)^2 - \left(1 - f_{\ddot{F}(e_i)}(u_r)\right)^2 \cdot \left(1 - f_{\ddot{G}(e_p)}(u_r)\right)^2}, \left(t_{\ddot{F}(e_p)}(u_r)\right) \cdot \left(t_{\ddot{G}(e_p)}(u_r)\right)}{u_r} \right\rangle \\ \left\langle \dfrac{\sqrt{\left(1 - f_{\ddot{F}(e_p)}(v_s)\right)^2 + \left(1 - f_{\ddot{G}(e_p)}(v_s)\right)^2 - \left(1 - f_{\ddot{F}(e_p)}(v_s)\right)^2 \cdot \left(1 - f_{\ddot{G}(e_p)}(v_s)\right)^2}, \left(t_{\ddot{F}(e_p)}(v_s)\right) \cdot \left(t_{\ddot{G}(e_p)}(v_s)\right)}{v_s} \right\rangle \end{array}\right)\right)\right\}$$

Definition 3.2.19. ***(Properties of PVBSS's)***

Let $\left(\ddot{F}_{PVB}, A\right), \left(\ddot{G}_{PVB}, A\right) \in PVBSS\,(U_1, U_2)$*. Following properties found true:*

(i) Idempotent Laws

Operations union, intersection, sum and product follows idempotent laws for PVBSS's

1. $\left(\ddot{F}_{PVB}, A\right) \ddot{\cap} \left(\ddot{F}_{PVB}, A\right) = \left(\ddot{F}_{PVB}, A\right)$

2. $\left(\ddot{F}_{PVB}, A\right) \ddot{\cup} \left(\ddot{F}_{PVB}, A\right) = \left(\ddot{F}_{PVB}, A\right)$

3. $\left(\ddot{F}_{PVB}, A\right) \ddot{\oplus} \left(\ddot{F}_{PVB}, A\right) = \left(\ddot{F}_{PVB}, A\right)$

4. $\left(\ddot{F}_{PVB}, A\right) \ddot{\otimes} \left(\ddot{F}_{PVB}, A\right) = \left(\ddot{F}_{PVB}, A\right)$

(ii) Commutativity

Operations union, intersection, sum and product follows commutativity for PVBSS's

1. $\left(\ddot{F}_{PVB}, A\right) \ddot{\cap} \left(\ddot{G}_{PVB}, A\right) = \left(\ddot{G}_{PVB}, A\right) \ddot{\cap} \left(\ddot{F}_{PVB}, A\right)$

2. $\left(\ddot{F}_{PVB}, A\right) \ddot{\cup} \left(\ddot{G}_{PVB}, A\right) = \left(\ddot{G}_{PVB}, A\right) \ddot{\cup} \left(\ddot{F}_{PVB}, A\right)$

3. $\left(\ddot{F}_{PVB}, A\right) \ddot{\oplus} \left(\ddot{G}_{PVB}, A\right) = \left(\ddot{G}_{PVB}, A\right) \ddot{\oplus} \left(\ddot{F}_{PVB}, A\right)$

4. $\left(\ddot{F}_{PVB}, A\right) \ddot{\otimes} \left(\ddot{G}_{PVB}, A\right) = \left(\ddot{G}_{PVB}, A\right) \otimes \left(\ddot{F}_{PVB}, A\right)$

(iii) De Morgan's Laws

Union and intersection violates De Morgan's laws. But sum and product operations follows

1. $\left(\left(\ddot{F}_{PVB}, A\right) \ddot{\cap} \left(\ddot{G}_{PVB}, A\right)\right)^c \neq \left(\ddot{F}_{PVB}, A\right)^c \ddot{\cup} \left(\ddot{G}_{PVB}, A\right)^c$
   But $\left(\left(\ddot{F}_{PVB}, A\right) \ddot{\cap} \left(\ddot{G}_{PVB}, A\right)\right)^c = \left(\ddot{F}_{PVB}, A\right)^c \ddot{\cap} \left(\ddot{G}_{PVB}, A\right)^c$

2. $\left(\left(\ddot{F}_{PVB}, A\right) \ddot{\cup} \left(\ddot{G}_{PVB}, A\right)\right)^c \neq \left(\ddot{F}_{PVB}, A\right)^c \ddot{\cap} \left(\ddot{G}_{PVB}, A\right)^c.$
   But $\left(\left(\ddot{F}_{PVB}, A\right) \ddot{\cup} \left(\ddot{G}_{PVB}, A\right)\right)^c = \left(\ddot{F}_{PVB}, A\right)^c \ddot{\cup} \left(\ddot{G}_{PVB}, A\right)^c$

3. $\left(\left(\ddot{F}_{PVB}, A\right) \ddot{\oplus} \left(\ddot{G}_{PVB}, A\right)\right)^c = \left(\ddot{F}_{PVB}, A\right)^c \ddot{\otimes} \left(\ddot{G}_{PVB}, A\right)^c$

4. $\left(\left(\ddot{F}_{PVB}, A\right) \ddot{\otimes} \left(\ddot{G}_{PVB}, A\right)\right)^c = \left(\ddot{F}_{PVB}, A\right)^c \ddot{\oplus} \left(\ddot{G}_{PVB}, A\right)^c$

3.3 Distance Measure of PVBSS's

In this section various distance measures of Pythagorean Vague Binary Soft Set's are developed. For that consider two Pythagorean Vague Binary Soft Set's as follows: $\forall e_p \in A, \quad \forall u_r \in U_1, \quad \forall v_s \in U_2$

$$\left(\ddot{F}_{PVB}, A\right) = \left\{ \left(e_p, \left(\begin{array}{c} \left\langle \dfrac{\left[t_{\ddot{F}_{PVB(e_p)}}(u_r), 1 - f_{\ddot{F}_{PVB(e_p)}}(u_r)\right]}{u_r} \right\rangle, \\ \left\langle \dfrac{\left[t_{\ddot{F}_{PVB(e_p)}}(v_s), 1 - f_{\ddot{F}_{PVB(e_p)}}(v_s)\right]}{v_s} \right\rangle \end{array} \right) \right) \right\}$$

$$\left(\ddot{G}_{PVB}, A\right) = \left\{ \left(e_p, \left(\begin{array}{c} \left\langle \dfrac{\left[t_{\ddot{G}_{PVB(e_p)}}(u_r), 1 - f_{\ddot{G}_{PVB(e_p)}}(u_r)\right]}{u_r} \right\rangle, \\ \left\langle \dfrac{\left[t_{\ddot{G}_{PVB(e_p)}}(v_s), 1 - f_{\ddot{G}_{PVB(e_p)}}(v_s)\right]}{v_s}; \right\rangle \end{array} \right) \right) \right\}$$

Definition 3.3.1.
Let $\left(\ddot{F}_{PVB}, A\right)$ and $\left(\ddot{G}_{PVB}, A\right)$ be two PVBSS's over a binary universe (U_1, U_2) and let $A \subseteq E$ be the set of parameters. Following formulae will give various distance measures under pythagorean enviornment.

1. $d_{PVBSS}\left(\left(\ddot{F}_{PVB}, A\right), \left(\ddot{G}_{PVB}, A\right)\right) =$

$$\left(\frac{1}{4m} \Sigma_{p=1}^{m} \Sigma_{r=1}^{i} \left(\begin{array}{c} \left|(t_{\ddot{F}_{PVB(e_p)}}(u_r) - t_{\ddot{G}_{PVB(e_p)}}(u_r))\right|^q \\ + \left|(1 - f_{\ddot{F}_{PVB(e_p)}}(u_r)) - (1 - f_{\ddot{G}_{PVB(e_p)}}(u_r))\right|^q \\ + \left|\Pi_{\ddot{F}_{PVB(e_p)}}(u_r) - \Pi_{\ddot{G}_{PVB(e_p)}}(u_r)\right|^q \end{array} \right) \right)$$
$$+$$
$$\left(\frac{1}{4m} \sum_{p=1}^{m} \sum_{s=1}^{j} \left(\begin{array}{c} \left|t_{\ddot{F}_{PVB(e_p)}}(v_s) - t_{\ddot{G}_{PVB(e_p)}}(v_s)\right|^q \\ + \left|\left(1 - f_{\ddot{F}_{PVB(e_p)}}(v_s)\right) - \left(1 - f_{\ddot{G}_{PVB(e_p)}}(v_s)\right)\right|^q \\ + \left|\Pi_{\ddot{F}_{PVB(e_p)}}(v_s) - \Pi_{\ddot{G}_{PVB(e_p)}}(v_s)\right|^q \end{array} \right) \right)$$

2. $d_{PVBSS}\left(\left(\ddot{F}_{PVB}, A\right), \left(\ddot{G}_{PVB}, A\right)\right) =$

$$\sqrt{\begin{array}{l} \frac{1}{4mi} \Sigma_{i=1}^{m} \Sigma_{j=1}^{n} \left[\left|t_{\ddot{F}(e_p)}(u_r) - t_{\ddot{G}(e_p)}(u_r)\right|^q + \left|\left(1 - f_{\ddot{F}(e_p)}(u_r)\right) - \left(1 - f_{\ddot{G}(e_p)}(u_r)\right)\right|^q + \left|\Pi_{\ddot{F}(e_p)}(u_r) - \Pi_{\ddot{G}(e_p)}(u_r)\right|^q \right] \\ + \frac{1}{4mj} \Sigma_{i=1}^{m} \Sigma_{j=1}^{n} \left[\left|t_{\ddot{F}(e_p)}(v_s) - t_{\ddot{G}(e_p)}(v_s)\right|^q + \left|\left(1 - f_{\ddot{F}(e_p)}(v_s)\right) - \left(1 - f_{\ddot{G}(e_p)}(v_s)\right)\right|^q + \left|\Pi_{\ddot{F}(e_p)}(v_s) - \Pi_{\ddot{G}(e_p)}(v_s)\right|^q \right] \end{array}}$$

where q is always a positive integer. In this case, $t_{\ddot{F}}, f_{\ddot{F}}, \Pi_{\ddot{F}}, t_{\ddot{G}}, f_{\ddot{G}}, \Pi_{\ddot{G}}$ *respectively indicate* $t_{\ddot{F}_{PNVB}}, f_{\ddot{F}_{PNVB}}, \Pi_{\ddot{F}_{PNVB}}, t_{\ddot{G}_{PNVB}}, f_{\ddot{G}_{PNVB}}, \Pi_{\ddot{G}_{PNVB}}$

1. *If* $q = 1$ *then* (1) *and* (2) *reduced to hamming distance and normalised hamming distance respectively and it is denoted as* $d^{H}_{PVBSS}\left(\left(\ddot{F}_{PVB}, A\right), \left(\ddot{G}_{PVB}, A\right)\right)$ *and* $d^{nH}_{PVBSS}\left(\left(\ddot{F}_{PVB}, A\right), \left(\ddot{G}_{PVB}, A\right)\right)$

2. *If* $q = 2$ *then* (1) *and* (2) *reduced to euclidean distance and normalised euclidean distance respectively and it is denoted as* $d^{E}_{PVBSS}\left(\left(\ddot{F}_{PVB}, A\right), \left(\ddot{G}_{PVB}, A\right)\right)$ *and* $d^{nE}_{PVBSS}\left(\left(\ddot{F}_{PVB}, A\right), \left(\ddot{G}_{PVB}, A\right)\right)$

Definition 3.3.2.

Distance measure of Pythagorean Vague Binary Soft Set's satisfies the following:

1. $0 \leq d^{H}_{PVBSS} \leq 2\left(\#(U_1) + \#(U_2)\right)$

2. $0 \leq d^{NH}_{PVBSS} \leq 2$

3. $0 \leq d^{E}_{PVBSS} \leq \sqrt{2\left(\#(U_1) + \#(U_2)\right)}$

4. $0 \leq d^{NE}_{PVBSS} \leq 2$

 # denotes the cardinality of the set

Remark 3.3.3.

It is observed that,

1. $d^{NE}_{PVBSS} \leq d^{NH}_{PVBSS} \leq d^{E}_{PVBSS} \leq d^{H}_{PVBSS}$ *and*

2. *Calculations showed that* d^{NH}_{PVBSS} *and* d^{NE}_{PVBSS} *are more accurate.*

Application in decision making

In this section newly introduced Pythagorean Vague Binary Soft Distance Measures are used in a decision making problem and the values are compared. An algorithm is given below.

Step 1: Construct Pythagorean Vague Binary Soft Set's $\left(\ddot{F}_{PVB}, A\right)$ and $\left(\ddot{G}_{PVB}, A\right)$ based on the given real life situations.

Step 2: Calculate Pythagorean Vague Binary Soft Distances between these sets

Step 3: *Shortest distance* indicates the result

A real life example is given below:

Mr. X. wants to purchase a lap-top. Shops under consideration are $U_1 = \{s_1, s_2\}$ for H.P and $U_2 = \{s_1^*, s_2^*\}$ for DELL. In the selection procedure the parameters under consideration are processor, RAM and Hard drive. He went to these shops and collected the details which are given as PVBSS's.

Let the following PVBSS's give data related to qualities wise brands

$$\left(\ddot{P}_{PVB_1}, A\right) = \left\{ \begin{array}{l} \left(e_1, \left(\left\langle \frac{[0.4,0.6]}{s_1}, \frac{[0.7,0.1]}{s_2} \right\rangle, \left\langle \frac{[0.7,0.8]}{s_1^*}, \frac{[0.3,0.4]}{s_2^*} \right\rangle\right)\right) \\ \left(e_2, \left(\left\langle \frac{[0.5,0.7]}{s_1}, \frac{[0.6,0.4]}{s_2} \right\rangle, \left\langle \frac{[0.8,0.9]}{s_1^*}, \frac{[0.4,0.3]}{s_2^*} \right\rangle\right)\right) \\ \left(e_3, \left(\left\langle \frac{[0.6,0.4]}{s_1}, \frac{[0.3,0.6]}{s_2} \right\rangle, \left\langle \frac{[0.7,0.9]}{s_1^*}, \frac{[0.8,0.4]}{s_2^*} \right\rangle\right)\right) \end{array} \right\}$$

$$\left(\ddot{P}_{PVB_2}, A\right) = \left\{ \begin{array}{l} \left(e_1, \left(\left\langle \frac{[0.7,0.8]}{s_1}, \frac{[0.4,0.1]}{s_2} \right\rangle, \left\langle \frac{[0.7,0.5]}{s_1^*}, \frac{[0.6,0.1]}{s_2^*} \right\rangle\right)\right) \\ \left(e_2, \left(\left\langle \frac{[0.8,0.1]}{s_1}, \frac{[0.4,0.7]}{s_2} \right\rangle, \left\langle \frac{[0.8,0.9]}{s_1^*}, \frac{[0.4,0.5]}{s_2^*} \right\rangle\right)\right) \\ \left(e_3, \left(\left\langle \frac{[0.6,0.4]}{s_1}, \frac{[0.3,0.2]}{s_2} \right\rangle, \left\langle \frac{[0.7,0.8]}{s_1^*}, \frac{[0.8,0.6]}{s_2^*} \right\rangle\right)\right) \end{array} \right\}$$

$$\left(\ddot{P}_{PVB_3}, A\right) = \left\{ \begin{array}{l} \left(e_1, \left(\left\langle \frac{[0.7,0.8]}{s_1}, \frac{[0.4,0.1]}{s_2} \right\rangle, \left\langle \frac{[0.8,0.3]}{s_1^*}, \frac{[0.9,0.5]}{s_2^*} \right\rangle\right)\right) \\ \left(e_2, \left(\left\langle \frac{[0.8,0.1]}{s_1}, \frac{[0.6,0.5]}{s_2} \right\rangle, \left\langle \frac{[0.6,0.9]}{s_1^*}, \frac{[0.3,0.4]}{s_2^*} \right\rangle\right)\right) \\ \left(e_3, \left(\left\langle \frac{[0.6,0.7]}{s_1}, \frac{[0.8,0.9]}{s_2} \right\rangle, \left\langle \frac{[0.3,0.1]}{s_1^*}, \frac{[0.6,0.7]}{s_2^*} \right\rangle\right)\right) \end{array} \right\}$$

$$\left(\ddot{P}_{PVB_4}, A\right) = \left\{ \begin{array}{l} \left(e_1, \left(\left\langle \frac{[0.5,0.1]}{s_1}, \frac{[0.2,0.5]}{s_2} \right\rangle, \left\langle \frac{[0.6,0.4]}{s_1^*}, \frac{[0.7,0.9]}{s_2^*} \right\rangle\right)\right) \\ \left(e_2, \left(\left\langle \frac{[0.8,0.1]}{s_1}, \frac{[0.6,0.5]}{s_2} \right\rangle, \left\langle \frac{[0.8,0.3]}{s_1^*}, \frac{[0.8,0.7]}{s_2^*} \right\rangle\right)\right) \\ \left(e_3, \left(\left\langle \frac{[0.8,0.7]}{s_1}, \frac{[0.6,0.4]}{s_2} \right\rangle, \left\langle \frac{[0.1,0.6]}{s_1^*}, \frac{[0.4,0.7]}{s_2^*} \right\rangle\right)\right) \end{array} \right\}$$

Let the following PVBSS's give data related to qualities wise rates

$$\left(\ddot{Q}_{PVB_1}, A\right) = \left\{ \begin{array}{l} \left(e_1, \left(\left\langle \frac{[0.7,0.8]}{s_1}, \frac{[0.9,0.2]}{s_2} \right\rangle, \left\langle \frac{[0.8,0.9]}{s_1^*}, \frac{[0.4,0.3]}{s_2^*} \right\rangle\right)\right) \\ \left(e_2, \left(\left\langle \frac{[0.4,0.6]}{s_1}, \frac{[0.5,0.6]}{s_2} \right\rangle, \left\langle \frac{[0.9,0.4]}{s_1^*}, \frac{[0.8,0.1]}{s_2^*} \right\rangle\right)\right) \\ \left(e_3, \left(\left\langle \frac{[0.7,0.4]}{s_1}, \frac{[0.8,0.4]}{s_2} \right\rangle, \left\langle \frac{[0.4,0.5]}{s_1^*}, \frac{[0.6,0.7]}{s_2^*} \right\rangle\right)\right) \end{array} \right\}$$

$$\left(\ddot{Q}_{PVB_2}, A\right) = \left\{ \begin{array}{l} \left(e_1, \left(\left\langle \frac{[0.9, 0.1]}{s_1}, \frac{[0.7, 0.2]}{s_2} \right\rangle, \left\langle \frac{[0.7, 0.4]}{s_1^*}, \frac{[0.6, 0.4]}{s_2^*} \right\rangle\right)\right) \\ \left(e_2, \left(\left\langle \frac{[0.7, 0.1]}{s_1}, \frac{[0.5, 0.3]}{s_2} \right\rangle, \left\langle \frac{[0.7, 0.2]}{s_1^*}, \frac{[0.8, 0.3]}{s_2^*} \right\rangle\right)\right) \\ \left(e_3, \left(\left\langle \frac{[0.4, 0.1]}{s_1}, \frac{[0.8, 0.4]}{s_2} \right\rangle, \left\langle \frac{[0.8, 0.1]}{s_1^*}, \frac{[0.4, 0.5]}{s_2^*} \right\rangle\right)\right) \end{array} \right\}$$

$$\left(\ddot{Q}_{PVB_3}, A\right) = \left\{ \begin{array}{l} \left(e_1, \left(\left\langle \frac{[0.4, 0.7]}{s_1}, \frac{[0.5, 0.3]}{s_2} \right\rangle, \left\langle \frac{[0.9, 0.6]}{s_1^*}, \frac{[0.7, 0.4]}{s_2^*} \right\rangle\right)\right) \\ \left(e_2, \left(\left\langle \frac{[0.7, 0.3]}{s_1}, \frac{[0.6, 0.2]}{s_2} \right\rangle, \left\langle \frac{[0.8, 0.9]}{s_1^*}, \frac{[0.7, 0.4]}{s_2^*} \right\rangle\right)\right) \\ \left(e_3, \left(\left\langle \frac{[0.4, 0.5]}{s_1}, \frac{[0.5, 0.2]}{s_2} \right\rangle, \left\langle \frac{[0.7, 0.3]}{s_1^*}, \frac{[0.9, 0.6]}{s_2^*} \right\rangle\right)\right) \end{array} \right\}$$

$$\left(\ddot{Q}_{PVB_4}, A\right) = \left\{ \begin{array}{l} \left(e_1, \left(\left\langle \frac{[0.6, 0.5]}{s_1}, \frac{[0.8, 0.9]}{s_2} \right\rangle, \left\langle \frac{[0.7, 0.4]}{s_1^*}, \frac{[0.6, 0.1]}{s_2^*} \right\rangle\right)\right) \\ \left(e_2, \left(\left\langle \frac{[0.7, 0.5]}{s_1}, \frac{[0.6, 0.1]}{s_2} \right\rangle, \left\langle \frac{[0.5, 0.9]}{s_1^*}, \frac{[0.6, 0.5]}{s_2^*} \right\rangle\right)\right) \\ \left(e_3, \left(\left\langle \frac{[0.8, 0.6]}{s_1}, \frac{[0.4, 0.9]}{s_2} \right\rangle, \left\langle \frac{[0.6, 0.7]}{s_1^*}, \frac{[0.8, 0.7]}{s_2^*} \right\rangle\right)\right) \end{array} \right\}$$

The shortest distance from table indicates qualities wise rates based on brands Calculations showed that shortest Pythagorean Vague Binary Soft Distance is for

Table 3.1:

Universal Sets	S_1	S_2	S_1^*	S_2^*
d^{nE}_{PVBSS}	0.47	0.666	0.5882	0.707

show room S_1. So better choice for Mr. X is to buy H.P from show-room S_1

3.4 Entropy Measure of PVBSS's

In this section entropy measure of PVBSS's with it's definition and properties are discussed. $PVBSS(U_1, U_2)$ denotes set of all Pythagorean Vague Binary Soft Set's over the binary universe (U_1, U_2)

Definition 3.4.1. ***(Axioms for Entropy Measure of PVBSS)***
Let (U_1, U_2) be a binary universe with $U_1 = \{u_1, u_2, --, u_r, --, u_i\}$ and $U_2 =$

$\{v_1, v_2, --, v_s, --, v_j\}$. *Let* $E = \{e_1, e_2, --, e_t, --, e_k\}$ *be a fixed parameter set with* $A = \{e_1, e_2, --, e_p, --, e_m\} \subseteq E$. *Let* $H : PVBSS(U_1, U_2) \times PVBSS(U_1, U_2) \to [0, 1]$ *be a mapping. Also let* $\left(\ddot{P}_{PVB}, A\right) \in PVBSS(U_1, U_2)$. $H\left(\ddot{P}_{PVB}, A\right)$ *is called entropy of* $\left(\ddot{P}_{PVB}, A\right)$ *if it satisfies the following axioms:*
$\forall e_p \in A, \quad \forall u_r \in U_1, \quad \forall v_s \in U_2$

1. $H\left(\ddot{P}_{PVB}, A\right) = 0 \Leftrightarrow$

$$\begin{cases} \left(t_{\ddot{P}_{PVB(e_p)}}(u_r)\right) = 0, \left(1 - f_{\ddot{F}_{PVB(e_p)}}(u_r)\right) = 1 \text{ or } \left(t_{\ddot{P}_{PVB(e_p)}}(u_r)\right) = 1, \left(1 - f_{\ddot{F}_{PVB(e_p)}}(u_r)\right) = 0 \\ \left(t_{\ddot{P}_{PVB(e_p)}}(v_s)\right) = 0, \left(1 - f_{\ddot{F}_{PVB(e_p)}}(v_s)\right) = 1 \text{ or } \left(t_{\ddot{P}_{PVB(e_p)}}(v_s)\right) = 1, \left(1 - f_{\ddot{F}_{PVB(e_p)}}(v_s)\right) = 0 \end{cases}$$

2. $H\left(\ddot{P}_{PVB}, A\right) = 1 \Leftrightarrow n = p$ *and*

$$\forall e_p \in A, \forall u_r \in U_1, \ t_{\ddot{P}_{PVB(e_p)}}(u_r) = \left(1 - f_{\ddot{P}_{PVB(e_p)}}(u_r)\right),$$
$$\forall e_p \in A, \forall v_s \in U_2, \ t_{\ddot{P}_{PVB(e_p)}}(v_s) = \left(1 - f_{\ddot{P}_{PVB(e_p)}}(v_s)\right)$$

3. $H\left(\ddot{P}_{PVB}, A\right) = H\left(\ddot{P}_{PVB}, A\right)^c$

4. *If,* $\forall e_p \in A, \quad \forall u_r \in U_1,$ *and* $\quad \forall v_s \in U_2,$

   (a) $\left(\ddot{P}_{PVB}, A\right) \ddot{\subseteq} \left(\ddot{R}_{PVB}, A\right)$ *and*

   (b) $t_{\ddot{R}_{PVB(e_p)}}(u_r) \leq \left(1 - f_{\ddot{R}_{PVB(e_p)}}(u_r)\right) \& t_{\ddot{R}_{PVB(e_p)}}(v_s) \leq \left(1 - f_{\ddot{R}_{PVB(e_p)}}(v_s)\right)$ *or*

   (a) $\left(\ddot{P}_{PVB}, A\right) \ddot{\supseteq} \left(\ddot{R}_{PVB}, A\right)$ *and*

   (b) $t_{\ddot{R}_{PVB(e_p)}}(u_r) \geq \left(1 - f_{\ddot{R}_{PVB(e_p)}}(u_r)\right) \ \& \ t_{\ddot{R}_{PVB(e_p)}}(v_s) \geq \left(1 - f_{\ddot{R}_{PVB(e_p)}}(v_s)\right)$

   then $H\left(\ddot{P}_{PVB}, A\right) \leq H\left(\ddot{R}_{PVB}, A\right)$

Theorem 3.4.2.

Let (U_1, U_2) *be a binary universe with* $U_1 = \{u_1, u_2, --, u_r, --, u_i\}$ *and* $U_2 = \{v_1, v_2, --, v_s, --, v_j\}$. *Let* $A = \{e_1, e_2, --, e_p, --, e_m\}$ *be a parameter set. Let* $\left(\ddot{P}_{PVB}, A\right) = \{P_{PVB}(e_k) \quad / \quad 0 \leq k \leq m\}$ *be a family of PVBSS's.* $H_t\left(\ddot{P}_{NVB}, A\right)$ *is a family of pythagorean vague binary soft entropy's . Define* $H\left(\ddot{P}_{NVB}, A\right)$ *as follows:*

$$H\left(\ddot{P}_{NVB}, A\right) = \left(\frac{\Sigma_{t=1}^{w} H_t\left(\ddot{P}_{NVB}, A\right)}{w}\right) \quad \text{where}$$

$$H_t\left(\ddot{P}_{NVB},A\right)=\frac{1}{4mi}\Sigma_{p=1}^{m}\Sigma_{r=1}^{i}\left(\frac{1-\left|t^2_{\ddot{F}_{PVB(e_p)}}(u_r)-\left(1-f^2_{\ddot{F}_{PVB(e_p)}}(u_r)\right)\right|}{1+\left|t^2_{\ddot{F}_{PVB(e_p)}}(u_r)-\left(1-f^2_{\ddot{F}_{PVB(e_p)}}(u_r)\right)\right|}\right)$$
$$+$$
$$\frac{1}{4mj}\Sigma_{p=1}^{m}\Sigma_{s=1}^{j}\left(\frac{1-\left|t^2_{\ddot{F}_{PVB(e_p)}}(v_s)-\left(1-f^2_{\ddot{F}_{PVB(e_p)}}(v_s)\right)\right|}{1+\left|t^2_{\ddot{F}_{PVB(e_p)}}(v_s)-\left(1-f^2_{\ddot{F}_{PVB(e_p)}}(v_s)\right)\right|}\right)$$

Then $H\left(\ddot{P}_{PVB},A\right)$ is a pythagorean vague binary soft entropy .

Proof.

1. $H\left(\ddot{P}_{PVB},A\right)=0$

$$\Leftrightarrow\quad \Sigma_{t=1}^{w}H_t\left(\ddot{P}_{PVB},A\right)=0$$
$$\Leftrightarrow 1-\left|t^2_{\ddot{F}_{PVB(e_p)}}(u_r)-\left(1-f^2_{\ddot{F}_{PVB(e_p)}}(u_r)\right)\right|=0$$
$$\text{for } 0\le t^2_{\ddot{F}(e_p)}(u_r)\le 1,\quad 0\le\left(1-f^2_{\ddot{F}_{PVB(e_p)}}(u_r)\right)\le 1\ \&$$
$$1-\left|t^2_{\ddot{F}_{PVB(e_p)}}(v_s)-\left(1-f^2_{\ddot{F}_{PVB(e_p)}}(v_s)\right)\right|=0$$
$$\text{for } 0\le t^2_{\ddot{F}_{PVB(e_p)}}(v_s)\le 1,\quad 0\le\left(1-f^2_{\ddot{F}(e_p)}(v_s)\right)\le 1$$
$$\therefore \forall e_p\in A,\quad \forall u_r\in U_1,\quad \forall v_s\in U_2,$$
$$\left|t^2_{\ddot{F}_{PVB(e_p)}}(u_r)-\left(1-f^2_{\ddot{F}_{PVB(e_p)}}(u_r)\right)\right|\le 1\Rightarrow\left|t^2_{\ddot{F}_{PVB(e_p)}}(u_r)-\left(1-f^2_{\ddot{F}_{PVB(e_p)}}(u_r)\right)\right|=1.$$
$$\left|t^2_{\ddot{F}_{PVB(e_p)}}(v_s)-\left(1-f^2_{\ddot{F}_{PVB(e_p)}}(v_s)\right)\right|\le 1\Rightarrow\left|t^2_{\ddot{F}_{PVB(e_p)}}(v_s)-\left(1-f^2_{\ddot{F}_{PVB(e_p)}}(v_s)\right)\right|=1.$$
$$\Rightarrow t_{\ddot{F}_{PVB(e_p)}}(u_r)=0,\left(1-f_{\ddot{F}_{PVB(e_p)}}(u_r)\right)=1\text{ or }t_{\ddot{F}_{PVB(e_p)}}(u_r)=1,\left(1-f_{\ddot{F}_{PVB(e_p)}}(u_r)\right)=0$$
$$t_{\ddot{F}_{PVB(e_p)}}(v_s)=0,\left(1-f_{\ddot{F}_{PVB(e_p)}}(v_s)\right)=1\text{ or }t_{\ddot{F}_{PVB(e_p)}}(v_s)=1,\left(1-f_{\ddot{F}_{PVB(e_p)}}(v_s)\right)=0$$

2. $H\left(\ddot{P}_{PVB},A\right)=1 \qquad \Leftrightarrow \Sigma_{t=1}^{w}H_t\left(\ddot{P}_{PVB},A\right)=w$

$$\Leftrightarrow\quad H_t\left(\ddot{P}_{PVB},A\right)=1$$

$$\Leftrightarrow\frac{1}{4mi}\Sigma_{p=1}^{m}\Sigma_{r=1}^{i}\left(\frac{1-\left|t^2_{\ddot{F}_{PVB(e_p)}}(u_r)-\left(1-f^2_{\ddot{F}_{PVB(e_p)}}(u_r)\right)\right|}{1+\left|t^2_{\ddot{F}_{PVB(e_p)}}(u_r)-\left(1-f^2_{\ddot{F}_{PVB(e_p)}}(u_r)\right)\right|}\right)+$$
$$\frac{1}{4mj}\Sigma_{p=1}^{m}\Sigma_{s=1}^{j}\left(\frac{1-\left|t^2_{\ddot{F}_{PVB(e_p)}}(v_s)-\left(1-f^2_{\ddot{F}_{PVB(e_p)}}(v_s)\right)\right|}{1+\left|t^2_{\ddot{F}_{PVB(e_p)}}(v_s)-\left(1-f^2_{\ddot{F}_{PVB(e_p)}}(v_s)\right)\right|}\right)=1$$
$$\Leftrightarrow\frac{1}{4mi}\Sigma_{p=1}^{m}\Sigma_{r=1}^{i}\left(\frac{1-\left|t^2_{\ddot{F}_{PVB(e_p)}}(u_r)-\left(1-f^2_{\ddot{F}_{PVB(e_p)}}(u_r)\right)\right|}{1+\left|t^2_{\ddot{F}_{PVB(e_p)}}(u_r)-\left(1-f^2_{\ddot{F}_{PVB(e_p)}}(u_r)\right)\right|}\right)$$
$$=\frac{1}{2}=\frac{1}{4mj}\Sigma_{p=1}^{m}\Sigma_{s=1}^{j}\left(\frac{1-\left|t^2_{\ddot{F}_{PVB(e_p)}}(v_s)-\left(1-f^2_{\ddot{F}_{PVB(e_p)}}(v_s)\right)\right|}{1+\left|t^2_{\ddot{F}_{PVB(e_p)}}(v_s)-\left(1-f^2_{\ddot{F}_{PVB(e_p)}}(v_s)\right)\right|}\right)$$

$\Leftrightarrow i = j$ and

$$1-\left|t^2_{\ddot{F}_{PVB(e_p)}}(u_r) - \left(1 - f^2_{\ddot{F}_{PVB(e_p)}}(u_r)\right)\right| = 1+\left|t^2_{\ddot{F}_{PVB(e_p)}}(u_r) - \left(1 - f^2_{\ddot{F}_{PVB(e_p)}}(u_r)\right)\right|$$

and

$$1-\left|t^2_{\ddot{F}_{PVB(e_p)}}(v_s) - \left(1 - f^2_{\ddot{F}_{PVB(e_p)}}(v_s)\right)\right| = 1+\left|t^2_{\ddot{F}_{PVB(e_p)}}(v_s) - \left(1 - f^2_{\ddot{F}_{PVB(e_p)}}(v_s)\right)\right|$$

$\Leftrightarrow i = j$ and

$$\left|t^2_{\ddot{F}_{PVB(e_p)}}(u_r) - \left(1 - f^2_{\ddot{F}_{PVB(e_p)}}(u_r)\right)\right| = 0;\ \forall e_p \in A,\ \forall u_r \in U_1 \text{ and}$$
$$\left|t^2_{\ddot{F}_{PVB(e_p)}}(v_s) - \left(1 - f^2_{\ddot{F}_{PVB(e_p)}}(v_s)\right)\right| = 0;\ \forall e_p \in A,\ \forall v_s \in U_2$$

$\Leftrightarrow i = j$ and

$$\begin{aligned} t_{\ddot{F}_{PVB(e_p)}}(u_r) &= \left(1 - f_{\ddot{F}_{PVB(e_p)}}(u_r)\right);\ \forall e_p \in A,\ \forall u_r \in U_1 \quad \text{and} \\ t_{\ddot{F}_{PVB(e_p)}}(v_s) &= \left(1 - f_{\ddot{F}_{PVB(e_p)}}(v_s)\right);\ \forall e_p \in A,\ \forall v_s \in U_2 \end{aligned}$$

3. $H_t\left(\ddot{P}_{PVB}, A\right) =$

$$\frac{1}{4mi}\Sigma_{p=1}^{m}\Sigma_{r=1}^{i}\left(\frac{1 - \left|t^2_{\ddot{F}_{PVB(e_p)}}(u_r) - \left(1 - f^2_{\ddot{F}_{PVB(e_p)}}(u_r)\right)\right|}{1 + \left|t^2_{\ddot{F}_{PVB(e_p)}}(u_r) - \left(1 - f^2_{\ddot{F}_{PVB(e_p)}}(u_r)\right)\right|}\right)$$
$$+$$
$$\frac{1}{4mj}\Sigma_{p=1}^{m}\Sigma_{s=1}^{j}\left(\frac{1 - \left|t^2_{\ddot{F}_{PVB(e_p)}}(v_s) - \left(1 - f^2_{\ddot{F}_{PVB(e_p)}}(v_s)\right)\right|}{1 + \left|t^2_{\ddot{F}_{PVB(e_p)}}(v_s) - \left(1 - f^2_{\ddot{F}_{PVB(e_p)}}(v_s)\right)\right|}\right)$$

$$= \frac{1}{4mi}\Sigma_{p=1}^{m}\Sigma_{r=1}^{i}\left(\frac{1 - \left|\left(1 - f^2_{\ddot{F}^c_{PVB(\neg e_p)}}(u_r)\right) - t^2_{\ddot{F}^c_{PVB(\neg e_p)}}(u_r)\right|}{1 + \left|\left(1 - f^2_{\ddot{F}^c_{PVB(\neg e_p)}}(u_r)\right) - t^2_{\ddot{F}^c_{PVB(\neg e_p)}}(u_r)\right|}\right)$$
$$+$$
$$\frac{1}{4mj}\Sigma_{p=1}^{m}\Sigma_{s=1}^{j}\left(\frac{1 - \left|\left(1 - f^2_{\ddot{F}^c_{PVB(\neg e_p)}}(v_s)\right) - t^2_{\ddot{F}^c_{PVB(\neg e_p)}}(v_s)\right|}{1 + \left|\left(1 - f^2_{\ddot{F}^c_{PVB(\neg e_p)}}(v_s)\right) - t^2_{\ddot{F}^c_{PVB(\neg e_p)}}(v_s)\right|}\right)$$

$$= \frac{1}{4mi}\Sigma_{p=1}^{m}\Sigma_{r=1}^{i}\left(\frac{1 - \left|t^2_{\ddot{F}^c_{PVB(\neg e_p)}}(u_r) - \left(1 - f^2_{\ddot{F}^c_{PVB(\neg e_p)}}(u_r)\right)\right|}{1 + \left|t^2_{\ddot{F}^c_{PVB(\neg e_p)}}(u_r) - \left(1 - f^2_{\ddot{F}^c_{PVB(\neg e_p)}}(u_r)\right)\right|}\right)$$
$$+$$
$$\frac{1}{4mj}\Sigma_{p=1}^{m}\Sigma_{s=1}^{j}\left(\frac{1 - \left|t^2_{\ddot{F}^c_{PVB(\neg e_p)}}(v_s) - \left(1 - f^2_{\ddot{F}^c_{PVB(\neg e_p)}}(v_s)\right)\right|}{1 + \left|t^2_{\ddot{F}^c_{PVB(\neg e_p)}}(v_s) - \left(1 - f^2_{\ddot{F}^c_{PVB(\neg e_p)}}(v_s)\right)\right|}\right)$$

$= H_t\left(\ddot{P}_{NVB}, A\right)^c$

$\therefore H\left(\ddot{P}_{NVB}, A\right) = H\left(\ddot{P}_{PVB}, A\right)^c$

4. (a) $\left(\ddot{P}_{PVB}, A\right) \subseteq \left(\ddot{R}_{PVB}, A\right)$ &
$t_{\ddot{R}_{PVB(e_p)}}(u_r) \leq \left(1 - f_{\ddot{R}_{PVB(e_p)}}(u_r)\right)$ for $\forall e_p \in A,\ \forall u_r \in U_1$ &
$t_{\ddot{R}_{PVB(e_p)}}(v_s) \leq \left(1 - f_{\ddot{R}_{PVB(e_p)}}(v_s)\right)$ for $\forall e_p \in A,\ \forall v_s \in U_2$

$$\Rightarrow 0 \leq t_{\ddot{P}_{PVB(e_p)}}(u_r) \leq t_{\ddot{R}_{PVB(e_p)}}(u_r) \leq \left(1 - f_{\ddot{R}_{PVB(e_p)}}(u_r)\right) \leq \left(1 - f_{\ddot{P}_{PVB(e_p)}}(u_r)\right) \leq 1$$

&

$$0 \leq t_{\ddot{P}_{PVB(e_p)}}(v_s) \leq t_{\ddot{R}_{PVB(e_p)}}(v_s) \leq \left(1 - f_{\ddot{R}_{PVB(e_p)}}(v_s)\right) \leq \left(1 - f_{\ddot{P}_{PVB(e_p)}}(v_s)\right) \leq 1$$

$$\Rightarrow \left|t^2_{\ddot{P}_{PVB(e_p)}}(u_r) - \left(1 - f^2_{\ddot{P}_{PVB(e_p)}}(u_r)\right)\right| \geq \left|t^2_{\ddot{R}_{PVB(e_p)}}(u_r) - \left(1 - f^2_{\ddot{R}_{PVB(e_p)}}(u_r)\right)\right|$$

and

$$\left|t^2_{\ddot{P}_{PVB(e_p)}}(v_s) - \left(1 - f^2_{\ddot{P}_{PVB(e_p)}}(v_s)\right)\right| \geq \left|t^2_{\ddot{R}_{PVB(e_p)}}(v_s) - \left(1 - f^2_{\ddot{R}_{PVB(e_p)}}(v_s)\right)\right|$$

$$\Rightarrow 1 - \left|t^2_{\ddot{P}_{PVB(e_p)}}(u_r) - \left(1 - f^2_{\ddot{P}_{PVB(e_p)}}(u_r)\right)\right| \leq 1 - \left|t^2_{\ddot{R}_{PVB(e_p)}}(u_r) - \left(1 - f^2_{\ddot{R}_{PVB(e_p)}}(u_r)\right)\right|,$$

$$1 + \left|t^2_{\ddot{P}_{PVB(e_p)}}(u_r) - \left(1 - f^2_{\ddot{P}_{PVB(e_p)}}(u_r)\right)\right| \geq 1 + \left|t^2_{\ddot{R}_{PVB(e_p)}}(u_r) - \left(1 - f^2_{\ddot{R}_{PVB(e_p)}}(u_r)\right)\right|$$

&

$$1 - \left|t^2_{\ddot{P}_{PVB(e_p)}}(v_s) - \left(1 - f^2_{\ddot{P}_{PVB(e_p)}}(v_s)\right)\right| \leq 1 - \left|t^2_{\ddot{R}_{PVB(e_p)}}(v_s) - \left(1 - f^2_{\ddot{R}_{PVB(e_p)}}(v_s)\right)\right|,$$

$$1 + \left|t^2_{\ddot{P}_{PVB(e_p)}}(v_s) - \left(1 - f^2_{\ddot{P}_{PVB(e_p)}}(v_s)\right)\right| \geq 1 + \left|t^2_{\ddot{R}_{PVB(e_p)}}(v_s) - \left(1 - f^2_{\ddot{R}_{PVB(e_p)}}(v_s)\right)\right|.$$

Thus $H_t\left(\ddot{P}_{PVB}, A\right) \leq H_t\left(\ddot{R}_{PVB}, A\right) \Rightarrow H\left(\ddot{P}_{PVB}, A\right) \leq H\left(\ddot{R}_{PVB}, A\right)$

(b) Similarly when $\left(\ddot{P}_{PVB}, A\right) \supseteq \left(\ddot{R}_{PVB}, A\right)$ &
$t_{\ddot{R}_{PVB(e_p)}}(u_r) \geq \left(1 - f_{\ddot{R}_{PVB(e_p)}}(u_r)\right)$; $\forall e_p \in A, \quad \forall u_r \in U_1$ and
$t_{\ddot{R}_{PVB(e_p)}}(v_s) \geq \left(1 - f_{\ddot{R}_{PVB(e_p)}}(v_s)\right)$; $\forall e_p \in A, \quad \forall v_s \in U_2$

$$H_t\left(\ddot{P}_{PVB}, A\right) \geq H_t\left(\ddot{R}_{PVB}, A\right) \Rightarrow H\left(\ddot{P}_{PVB}, A\right) \geq H\left(\ddot{R}_{PVB}, A\right)$$

Conclusion

Similarity measuring tools for fuzzy sets found get failed in many situations. This enforced researchers for developing new tools independently to vague sets. Shyi - Ming Chen [83] and Chang Wang and Anjing Qu [17] succeded in this direction. This chapter developed mainly based on their ideas. In this section different measures are developed for VBSS's and one of its special type, PVBSS's. Application of Pythagorean concepts on VBSS's will handle application field in a more robust and consistent manner. Validity of basic properties in Classical Set Theory are also verified for Pythagorean Vague Binary Soft Set's.

Chapter 4

Chapter 4

Neutrosophic Vague Binary Sets

By applying *vague* concepts to *neutrosophic* notions Shawkath Alkhazalekh [80] designed *Neutrosophic Vague* sets under the scheme of single universe. It is clear that binary stands for two. This chapter concentrates the extension of single universe concept in *neutrosophic vague set theory* to binary universe and hence developed to *neutrosophic vague binary set theory.* New theory is effective to pull out the positive effects of binary. Method of binary concept developed in this chapter is different from existing binary notion.

Chapter Scheme:

Section 4.1. Neutrosophic Vague Binary Set & It's properties
Section 4.2. Neutrosophic Vague Binary Topology
Section 4.3. Neutrosophic Vague Binary Continuity
Section 4.4. Neutrosophic Vague Binary Distance Measure

Novel Model 'Binary Set' In Uncertain Set's

In this section notions like Binary Set, Fuzzy Binary Set, Vague Binary Set, Neutrosophic Binary Set are developed first to make a plat-form to launch Neutrosophic Vague Binary Set. Novel sets are presented with proper examples.

Definition 4.0.3. ***(Binary Set)***
Binary Set (in short, BS) denoted as A_B *over a binary universe* (U_1, U_2)
where, $U_1 = \{u_1, u_2, --, u_r, --, u_i\} = \{u_r \mid r \in [1, i]\} = \{u_r \quad | \quad 1 \leq r \leq i\}$

and $U_2 = \{v_1, v_2, --, v_s, --, v_j\} = \{v_s \mid s \in [1, j]\} = \{v_s \mid 1 \le s \le j\}$
is a set defined by a mapping,

$A_B : (U_1, U_2) \to (U_1, U_2)$ *and is an object of the form*

$$A_B = \Big\{ \left(\langle u_r \, ; \, \forall u_r \in U_1 \rangle \, , \langle v_s \, ; \, \forall v_s \in U_2 \rangle \right) \Big\}$$

Example 4.0.4.
Let $U_1 = \left\{ h_1^N, h_2^N, h_3^N \right\}$ *; be the collection of rental houses in the north side of a city and* $U_2 = \left\{ h_1^S, h_2^S \right\}$ *gives the collection of rental houses in the south side of a city. Combined representation of the availability of rental houses in north and south zone in that city in binary set outlook is given as follows:*
$A_B = \left\{ \left(\langle h_1^N, h_2^N, h_3^N \rangle \, \langle h_1^S, h_2^S \rangle \right) \right\}$

Definition 4.0.5. ***(Fuzzy Binary Set)***
Fuzzy Binary Set (in short, FBS)denoted as μ_{A_B} *over a binary universe* (U_1, U_2)
where, $U_1 = \{u_1, u_2, --, u_r, --, u_i\} = \{u_r \mid r \in [1, i]\} = \{u_r \mid 1 \le r \le i\}$
and $U_2 = \{v_1, v_2, --, v_s, --, v_j\} = \{v_s \mid s \in [1, j]\} = \{v_s \mid 1 \le s \le j\}$
is a set defined by a mapping,

$\mu_{A_B} : (U_1, U_2) \to [0, 1]$ *and is an object of the form*

$$\mu_{A_B} = \left\{ \left\langle \frac{\mu_{A_B}(u_r)}{u_r} \quad ; \quad \forall u_r \in U_1 \right\rangle, \left\langle \frac{\mu_{A_B}(v_s)}{v_s} \quad ; \quad \forall v_s \in U_2 \right\rangle \right\}$$

where
$\mu_{A_B}(u_r) : U_1 \to [0, 1]$ *gives truth membership values of the elements* u_r *in* U_1
$\mu_{A_B}(v_s) : U_2 \to [0, 1]$ *gives truth membership values of the elements* v_s *in* U_2

Example 4.0.6.
Example given in 4.0.4 in fuzzy binary set outlook is given as follows:

$$\mu_{A_B} = \left\{ \left\{ \left\langle \frac{0.2}{h_1^N}, \frac{0.4}{h_2^N}, \frac{0.1}{h_3^N} \right\rangle, \left\langle \frac{0.6}{h_1^S}, \frac{0.3}{h_2^S} \right\rangle \right\} \right\}$$

Definition 4.0.7. ***(Vague Binary Set)***
Vague Binary Set (in short, VBS) denoted as A_{VB}*, over a binary universe* (U_1, U_2)
with $U_1 = \{u_1, u_2, --, u_r, --, u_i\} = \{u_r \mid r \in [1, i]\} = \{u_r \mid 1 \le r \le i\}$
and $U_2 = \{v_1, v_2, --, v_s, --, v_j\} = \{v_s \mid s \in [1, j]\} = \{v_s \mid 1 \le s \le j\}$
is a set defined by a mapping,

$V_{A_B} : (U_1, U_2) \to [0, 1]$ *and is an object of the form*

$$A_{VB} = \left\{ \left(\left\langle \frac{V_{A_{VB}}(u_r)}{u_r}; \forall u_r \in U_1 \right\rangle, \left\langle \frac{V_{A_{VB}}(v_s)}{v_s}; \forall v_s \in U_2 \right\rangle \right) \right\}$$

$$= \left\{ \left(\begin{array}{c} \left\langle \frac{[t_{A_{VB}}(u_r), 1 - f_{A_{VB}}(u_r)]}{u_r}; \forall u_r \in U_1 \right\rangle \\ \left\langle \frac{[t_{A_{VB}}(v_s), 1 - f_{A_{VB}}(v_s)]}{v_s}; \forall v_s \in U_2 \right\rangle \end{array} \right) \right\}$$

$t_{A_{VB}}(u_r)$ is the truth membership degree of the element $u_r \in U_1$ to the vague binary set A_{NB}. $f_{A_{VB}}(u_r)$ denotes the degree of non-membership of the element $u_r \in U_1$ to the vague binary set A_{VB}, with the conditions :

$$t_{A_{VB}}(u_r) : U_1 \to [0,1] \;; \quad f_{A_{VB}}(u_r) : U_1 \to [0,1] \;; \quad 0 \leq t_{A_{VB}}(u_r) + f_{A_{VB}}(u_r) \leqslant 1$$

Here, $t_{A_{VB}}(u_r)$ is a lower bound on the grade of membership of u_r to A_{VB} derived from the evidence for u_r and $f_{A_{VB}}$ is a lower bound on the negation of u_r derived from the evidence against u_r. Similarly, for $t_{A_{VB}}(v_s)$ and $f_{A_{VB}}(v_s)$.

Remark 4.0.8.

$$V_{A_{VB}}(u_r) = [t_{A_{VB}}(u_r), 1 - f_{A_{VB}}(u_r)] \;; \quad V_{A_{VB}}(v_s) = [t_{A_{VB}}(v_s), 1 - f_{A_{VB}}(v_s)]$$

$V_{A_{VB}}(u_r)$ denotes vague binary value of u_r from first universe U_1.
$V_{A_{VB}}(v_s)$ denotes the vague binary value of v_s from second universe U_2

Example 4.0.9.
Let $U_1 = \{b_1, b_2, b_3\}$ and $U_2 = \{l_1, l_2, l_3\}$ be food varieties for break -fast and lunch respectively. Availability of these items in a particular hotel can be given using a vague binary set, say A_{VB} as follows :

$$A_{VB} = \left\{ \left(\left\langle \frac{[0.3, 0.5]}{b_1}, \frac{[0.5, 0.7]}{b_2}, \frac{[0.7, 0.9]}{b_3} \right\rangle, \left\langle \frac{[0.4, 0.5]}{l_1}, \frac{[0.6, 0.8]}{l_2}, \frac{[0.8, 0.9]}{l_3} \right\rangle \right) \right\}$$

Definition 4.0.10. ***(Neutrosophic Binary Set)***
Neutrosophic Binary Set A_{NB} over a binary universe (U_1, U_2) where
$U_1 = \{u_r \quad | \quad 1 \leqslant r \leqslant n\} \quad ; \quad U_2 = \{v_s \quad | \quad 1 \leqslant s \leqslant p\}$
is a set given by a mapping,

$$A_{NB} : (U_1, U_2) \to [0,1] \qquad \textit{and is an object of the form}$$

$$A_{NB} = \left\{ \begin{array}{l} \left\langle \frac{(T_{A_{NB}}(u_r), I_{A_{NB}}(u_r), F_{A_{NB}}(u_r))}{u_r} \mid u_r \in U_1 \right\rangle, \\ \left\langle \frac{(T_{A_{NB}}(v_s), I_{A_{NB}}(v_s), F_{A_{NB}}(v_s))}{v_s} \mid v_s \in U_2 \right\rangle \end{array} \right\}$$

$T_{A_{NB}}(u_r), I_{A_{NB}}(u_r), F_{A_{NB}}(u_r) : U_1 \to [0,1]$ gives the truth, indeterminacy and false membership values of the elements u_r in U_1 &
$T_{A_{NB}}(v_s), I_{A_{NB}}(v_s), F_{A_{NB}}(v_s) : U_2 \to [0,1]$ gives the truth, indeterminacy and false membership values of the elements v_s in U_2

Example 4.0.11.

Example given in 4.0.9 in a neutrosophic binary set outlook is given as follows:
$A_{NB} =$

$$\left\{ \left\langle \frac{(0.2,0.3,0.4)}{b_1}, \frac{(0.4,0.1,0.3)}{b_2}, \frac{(0.1,0.3,0.1)}{b_3} \right\rangle, \left\langle \frac{[0.6,0.2,0.1]}{l_1}, \frac{(0.3,0.5,0.6)}{l_2}, \frac{(0.2,0.4,0.1)}{l_3} \right\rangle \right\}$$

4.1 Neutrosophic Vague Binary Set & It's Operations

Using previous section neutrosophic vague binary set is developed in this section with example and with some inevitable and basic operations.

Definition 4.1.1. ***(Neutrosophic Vague Binary Set)***
A Neutrosophic Vague Binary Set (NVBS in short), say M_{NVB}, over a binary universe (U_1, U_2) where $U_1 = \{u_r / 1 \leqslant r \leqslant i\}; U_2 = \{v_s / 1 \leqslant s \leqslant j\}$ is given by

a mapping $M_{NVB} : (U_1, U_2) \to [0,1]$ *and is an object of the form*

$$M_{NVB} = \left\{ \begin{array}{l} \left\langle \dfrac{\hat{T}_{M_{NVB}}(u_r), \hat{I}_{M_{NVB}}(u_r), \hat{F}_{M_{NVB}}(u_r)}{u_r} \quad ; \quad \forall u_r \in U_1 \right\rangle, \\ \left\langle \dfrac{\hat{T}_{M_{NVB}}(v_s), \hat{I}_{M_{NVB}}(v_s), \hat{F}_{M_{NVB}}(v_s)}{v_s} \quad ; \quad \forall v_s \in U_2 \right\rangle \end{array} \right\}$$

is defined as

$$\begin{cases} \hat{T}_{M_{NVB}(u_r)} = \left[T^-(u_r), T^+(u_r)\right] \\ \hat{I}_{M_{NVB}(u_r)} = \left[I^-(u_r), I^+(u_r)\right] & ; \forall u_r \in U_1 \\ \hat{F}_{M_{NVB}(u_r)} = \left[F^-(u_r), F^+(u_r)\right] \end{cases}$$

$$\begin{cases} \hat{T}_{M_{NVB}(v_s)} = \left[T^-(v_s), T^+(v_s)\right] \\ \hat{I}_{M_{NVB}(v_s)} = \left[I^-(v_s), I^+(v_s)\right] & ; \forall v_s \in U_2 \\ \hat{F}_{M_{NVB}(v_s)} = \left[F^-(v_s), F^+(v_s)\right] \end{cases}$$

$$where \begin{cases} T^{+}(u_r) = 1 - F^{-}(u_r) \quad ; \quad F^{+}(u_r) = 1 - T^{-}(u_r) \quad ; \forall u_r \in U_1 \quad and \\ T^{+}(v_s) = 1 - F^{-}(v_s) \quad ; \quad F^{+}(v_s) = 1 - T^{-}(v_s) \quad ; \forall v_s \in U_2 \end{cases}$$

$$\begin{cases} {}^{-}0 \leq T^{-}(u_r) + I^{-}(u_r) + F^{-}(u_r) \leq 2^{+}; \ {}^{-}0 \leq T^{-}(v_s) + I^{-}(v_s) + F^{-}(v_s) \leq 2^{+} \quad or \\ {}^{-}0 \leq T^{-}(u_r) + I^{-}(u_r) + F^{-}(u_r) + T^{-}(v_s) + I^{-}(v_s) + F^{-}(v_s) \leq 4^{+} \quad and \\ {}^{-}0 \leq T^{+}(u_r) + I^{-}(u_r) + F^{+}(u_r) \leq 2^{+}; \ {}^{-}0 \leq T^{+}(v_s) + I^{-}(v_s) + F^{+}(v_s) \leq 2^{+} \quad or \\ {}^{-}0 \leq T^{+}(u_r) + I^{-}(u_r) + F^{+}(u_r) + T^{+}(v_s) + I^{+}(v_s) + F^{+}(v_s) \leq 4^{+} \end{cases}$$

$$\begin{cases} T^{-}(u_r), I^{-}(u_r), F^{-}(u_r) : V(U_1) \to [0,1] \ and \ T^{-}(v_s), I^{-}(v_s), F^{-}(v_s) : V(U_2) \to [0,1] \\ T^{+}(u_r), I^{+}(u_r), F^{+}(u_r) : V(U_1) \to [0,1] \ and \ T^{+}(v_s), I^{+}(v_s), F^{+}(v_s) : V(U_2) \to [0,1] \end{cases}$$

Here $V(U_1), V(U_2)$ denotes power set of vague sets on U_1, U_2 respectively.

Example 4.1.2.

Let $U_1 = \{u_1, u_2, u_3\}, U_2 = \{v_1, v_2\}$ be a binary universe under consideration.

A NVBS is given below :

$M_{NVB} =$

$$\left\{ \begin{array}{l} \left\langle \frac{[0.2,0.3],[0.6,0.7],[0.7,0.8]}{u_1}, \quad \frac{[0.3,0.7],[0.5,0.6],[0.3,0.7]}{u_2}, \quad \frac{[0.1,0.9],[0.4,0.8],[0.1,0.9]}{u_3} \right\rangle \\ \left\langle \frac{[0.6,0.8],[0.5,0.7],[0.2,0.4]}{v_1}, \quad \frac{[0.2,0.7],[0.6,0.9],[0.3,0.8]}{v_2} \right\rangle \end{array} \right\}$$

Definition 4.1.3.

(Zero Neutrosophic Vague Binary Set & Unit Neutrosophic Vague Binary Set)

Let (U_1, U_2) be a binary universe with $U_1 = \{u_r \mid 1 \leq r \leq i\}; U_2 = \{v_s \mid 1 \leq s \leq j\}$.

(i) A zero NVBS denoted as Φ_{NVB} over (U_1, U_2) is given by,

$$\Phi_{NVB} = \left\{ \left\langle \frac{[1,1],[0,0],[0,0]}{u_r} ; \forall u_r \in U_1 \right\rangle, \quad \left\langle \frac{[1,1],[0,0],[0,0]}{v_s} ; \forall v_s \in U_2 \right\rangle \right\}$$

Operations On Neutrosophic Vague Binary Sets

In this section some Cantor Set theoretical operations with prime importance are developed in NVBS's and found that they differ from classical set theory.

Definition 4.1.4. ***(Subset of Neutrosophic Vague Binary Sets)***

Let M_{NVB} and P_{NVB} be two NVBS's over a binary universe (U_1, U_2). Then M_{NVB}

is included by P_{NVB} denoted by $M_{NVB} \subseteq P_{NVB}$ if following conditions found true:

$$\left.\begin{array}{l}\hat{T}_{M_{NVB}}(u_r) \leq \hat{T}_{P_{NVB}}(u_r) \\ \hat{I}_{M_{NVB}}(v_s) \geq \hat{I}_{P_{NVB}}(u_r) \\ \hat{F}_{M_{NVB}}(u_r) \geq \hat{F}_{P_{NVB}}(u_r)\end{array}\right\} \forall u_r \in U_1 \text{ with } 1 \leq r \leq i \qquad \text{and}$$

$$\left.\begin{array}{l}\hat{T}_{M_{NVB}}(v_s) \leq \hat{T}_{P_{NVB}}(v_s) \\ \hat{I}_{M_{NVB}}(v_s) \geq \hat{I}_{P_{NVB}}(v_s) \\ \hat{F}_{M_{NVB}}(v_s) \geq \hat{F}_{P_{NVB}}(v_s)\end{array}\right\} \forall v_s \in U_2 \text{ with } 1 \leq s \leq j$$

Example 4.1.5.

Let $U_1 = \{u_1, u_2\}$, $U_2 = \{v_1\}$ be a common universe. Let,

$$M_{NVB} = \left\{ \begin{array}{l} \left\langle \dfrac{[0.1, 0.2], [0.6, 0.7], [0.8, 0.9]}{u_1}, \dfrac{[0.2, 0.6], [0.5, 0.6], [0.4, 0.8]}{u_2} \right\rangle, \\ \left\langle \dfrac{[0.1, 0.3], [0.6, 0.7], [0.7, 0.9]}{v_1} \right\rangle \end{array} \right\}$$

$$P_{NVB} = \left\{ \begin{array}{l} \left\langle \dfrac{[0.2, 0.3], [0.5, 0.6], [0.7, 0.8]}{u_1}, \dfrac{[0.3, 0.7], [0.4, 0.5], [0.3, 0.7]}{u_2} \right\rangle, \\ \left\langle \dfrac{[0.2, 0.4], [0.5, 0.6], [0.6, 0.8]}{v_1} \right\rangle \end{array} \right\}$$

Clearly, $M_{NVB} \subseteq P_{NVB}$

Definition 4.1.6. ***(Union of two neutrosophic vague binary sets)***

Let M_{NVB} and P_{NVB} are two NVBS's

Union of two NVBS's. M_{NVB} and P_{NVB} is a NVBS, given as

$(M_{NVB} \cup P_{NVB}) = S_{NVB}$

$$= \left\{ \begin{array}{l} \left\langle \dfrac{\hat{T}_{S_{NVB}}(u_r), \hat{I}_{S_{NVB}}(u_r), \hat{F}_{S_{NVB}}(u_r)}{u_r}\,;\, \forall u_r \in U_1 \right\rangle, \\ \left\langle \dfrac{\hat{T}_{S_{NVB}}(v_s), \hat{I}_{S_{NVB}(v_s)}, \hat{F}_{S_{NVB}}(v_s)}{v_s}\,;\, \forall v_s \in U_2 \right\rangle \end{array} \right\}$$

whose truth- membership, indeterminacy-membership and false-membership functions are related to those of M_{NVB} and P_{NVB} is given by

$$\hat{T}_{S_{NVB}}(u_r) = \left[max\left(T^-_{M_{NVB}}(u_r), T^-_{P_{NVB}}(u_r)\right), max\left(T^+_{M_{NVB}}(u_r), T^+_{P_{NVB}}(u_r)\right)\right]$$

$$\hat{I}_{S_{NVB}}(u_r) = \left[min\left(I^-_{M_{NVB}}(u_r), I^-_{P_{NVB}}(u_r)\right), min\left(I^+_{M_{NVB}}(u_r), I^+_{P_{NVB}}(u_r)\right)\right]$$

$$\hat{F}_{S_{NVB}}(u_r) = \left[min\left(F^-_{M_{NVB}}(u_r), F^-_{P_{NVB}}(u_r)\right), min\left(F^+_{M_{NVB}}(u_r), F^+_{P_{NVB}}(u_r)\right)\right]$$

and

$$\hat{T}_{S_{NVB}}(v_s) = \left[max\left(T^-_{M_{NVB}}(v_s), T^-_{P_{NVB}}(v_s)\right), max\left(T^+_{M_{NVB}}(v_s), T^+_{P_{NVB}}(v_s)\right)\right]$$
$$\hat{I}_{S_{NVB}}(v_s) = \left[min\left(I^-_{M_{NVB}}(v_s), I^-_{P_{NVB}}(v_s)\right), min\left(I^+_{M_{NVB}}(v_s), I^+_{P_{NVB}}(v_s)\right)\right]$$
$$\hat{F}_{S_{NVB}}(v_s) = \left[min\left(F^-_{M_{NVB}}(v_s), F^-_{P_{NVB}}(v_s)\right), min\left(F^+_{M_{NVB}}(v_s), F^+_{P_{NVB}}(v_s)\right)\right]$$

Example 4.1.7.

Consider example 4.1.5.

$$(M_{NVB} \cup P_{NVB}) = S_{NVB} = \left\{ \begin{array}{l} \left\langle \frac{[0.2,0.3],[0.5,0.6],[0.7,0.8]}{u_1}, \frac{[0.3,0.7],[0.4,0.5],[0.3,0.7]}{u_2} \right\rangle \\ \left\langle \frac{[0.2,0.4],[0.5,0.6],[0.6,0.8]}{v_1} \right\rangle \end{array} \right\}$$

Definition 4.1.8.

(Intersection of two Neutrosophic Vague Binary Sets)

Let M_{NVB} and P_{NVB} are two NVBS's.

(i) Intersection of two NVBS's, M_{NVB} and P_{NVB} is a NVBS, given as

$$(M_{NVB} \cap P_{NVB}) = R_{NVB} = \left\{ \begin{array}{l} \left\langle \frac{\hat{T}_{R_{NVB}}(u_r), \hat{I}_{R_{NVB}}(u_r), \hat{F}_{R_{NVB}}(u_r)}{u_r}; \forall u_r \in U_1 \right\rangle \\ \left\langle \frac{\hat{T}_{R_{NVB}}(v_s), \hat{I}_{R_{NVB}}(v_s), \hat{F}_{R_{NVB}}(v_s)}{v_s}; \forall v_s \in U_2 \right\rangle \end{array} \right\}$$

whose truth-membership, indeterminacy - membership and false - membership functions are related to those of M_{NVB} and P_{NVB} is given by

$$\hat{T}_{R_{NVB}}(u_r) = \left[min\left(T^-_{M_{NVB}}(u_r), T^-_{P_{NVB}}(u_r)\right), min\left(T^+_{M_{NVB}}(u_r), T^+_{P_{NVB}}(u_r)\right)\right]$$
$$\hat{I}_{R_{NVB}}(u_r) = \left[max\left(I^-_{M_{NVB}}(u_r), I^-_{P_{NVB}}(u_r)\right), max\left(I^+_{M_{NVB}}(u_r), I^+_{P_{NVB}}(u_r)\right)\right]$$
$$\hat{F}_{R_{NVB}}(u_r) = \left[max\left(F^-_{M_{NVB}}(u_r), F^-_{P_{NVB}}(u_r)\right), max\left(F^+_{M_{NVB}}(u_r), F^+_{P_{NVB}}(u_r)\right)\right] \&$$
$$\hat{T}_{R_{NVB}}(v_s) = \left[min\left(T^-_{M_{NVB}}(v_s), T^-_{P_{NVB}}(v_s)\right), min\left(T^+_{M_{NVB}}(v_s), T^+_{P_{NVB}}(v_s)\right)\right]$$
$$\hat{I}_{R_{NVB}}(v_s) = \left[max\left(I^-_{M_{NVB}}(v_s), I^-_{P_{NVB}}(v_s)\right), max\left(I^+_{M_{NVB}}(v_s), I^+_{P_{NVB}}(v_s)\right)\right]$$
$$\hat{F}_{R_{NVB}}(v_s) = \left[max\left(F^-_{M_{NVB}}(v_s), F^-_{P_{NVB}}(v_s)\right), max\left(F^+_{M_{NVB}}(v_s), F^+_{P_{NVB}}(v_s)\right)\right]$$

Example 4.1.9.

Consider example 4.1.5.

$$(M_{NVB} \cap P_{NVB}) = R_{NVB} = \left\{ \begin{array}{l} \left\langle \frac{[0.1,0.2],[0.6,0.7],[0.8,0.9]}{u_1}, \frac{[0.2,0.6],[0.5,0.6],[0.4,0.8]}{u_2} \right\rangle \\ \left\langle \frac{[0.1,0.3],[0.6,0.7],[0.7,0.9]}{v_1} \right\rangle \end{array} \right\}$$

Definition 4.1.10. ***(Complement of a NVBS)***

Let M_{NVB} is defined as in definition 4.1.1. It's complement is denoted by M^c_{NVB} and is given by,

$$M^c_{NVB} = \left\{ \begin{array}{l} \left\langle \dfrac{\hat{T}^c_{M_{NVB}}(u_r), \hat{I}^c_{M_{NVB}}(u_r), \hat{F}^c_{M_{NVB}}(u_r)}{u_r}; \forall u_r \in U_1 \right\rangle, \\ \left\langle \dfrac{\hat{T}^c_{M_{NVB}}(v_s), \hat{I}^c_{M_{NVB}}(v_s), \hat{F}^c_{M_{NVB}}(v_s)}{v_s}; \forall v_s \in U_2 \right\rangle \end{array} \right\} \text{ is defined as,}$$

$$\begin{cases} \hat{T}^c_{M_{NVB}}(u_r) = \left[1 - T^+(u_r), 1 - T^-(u_r)\right] \\ \hat{I}^c_{M_{NVB}}(u_r) = \left[1 - I^+(u_r), 1 - I^-(u_r)\right] \quad ; \quad \forall u_r \in U_1 \quad \text{and} \\ \hat{F}^c_{M_{NVB}}(u_r) = \left[1 - F^+(u_r), 1 - F^-(u_r)\right] \end{cases}$$

$$\begin{cases} \hat{T}^c_{M_{NVB}}(v_s) = \left[1 - T^+(v_s), 1 - T^-(v_s)\right] \\ \hat{I}^c_{M_{NVB}}(v_s) = \left[1 - I^+(v_s), 1 - I^-(v_s)\right] \quad ; \quad \forall v_s \in U_2 \\ \hat{F}^c_{M_{NVB}}(v_s) = \left[1 - F^+(v_s), 1 - F^-(v_s)\right] \end{cases}$$

Example 4.1.11.

Let M_{NVB} is defined as in example 4.1.2. It's complement is given by,

$M^c_{NVB} =$

$$\left\{ \begin{array}{l} \left\langle \dfrac{[0.7, 0.8], [0.3, 0.4], [0.2, 0.3]}{u_1}, \dfrac{[0.3, 0.7], [0.4, 0.5], [0.3, 0.7]}{u_2}, \dfrac{[0.1, 0.9], [0.2, 0.6], [0.1, 0.9]}{u_3} \right\rangle \\ \left\langle \dfrac{[0.2, 0.4], [0.3, 0.5], [0.6, 0.8]}{v_1}, \dfrac{[0.3, 0.8], [0.1, 0.4], [0.2, 0.7]}{v_2} \right\rangle \end{array} \right\}$$

4.2 Neutrosophic Vague Binary Topology

In this section neutrosophic vague binary topology (NVBT in short) is developed for NVBS's. It's various concepts are also discussed.

Definition 4.2.1. ***(Neutrosophic Vague Binary Topology)***

A neutrosophic vague binary topology over a binary universe (U_1, U_2) is a family τ^{NVB}_Δ of neutrosophic vague binary sets in (U_1, U_2) satisfying the following axioms:

1. $\Phi_{NVB}, U_{NVB} \in \tau^{NVB}_\Delta$
2. *For any $M_{NVB}, P_{NVB} \in \tau^{NVB}_\Delta, (M_{NVB} \cap P_{NVB}) \in \tau^{NVB}_\Delta$*
   i.e., finite intersection of NVBS's of τ^{NVB}_Δ is again a member of τ^{NVB}_Δ

3. *Let* $\{M^i_{NVB}; i \in I\} \subseteq \tau^{NVB}_{\Delta}$ *then* $\cup_{i\in I}\tau^{NVB}_{\Delta} \subseteq \tau^{NVB}_{\Delta}$
   i.e., arbitrary union of neutrosophic vague binary sets in τ^{NVB}_{Δ} *is again a member of* τ^{NVB}_{Δ}

Example 4.2.2.

Let $U_1 = \{u_1, u_2\}; U_2 = \{v_1\}$. *Following is a neutrosophic vague binary topology,* $\tau^{NVB}_{\Delta} = \{\Phi_{NVB}, M_{NVB}, P_{NVB}, K_{NVB}, H_{NVB}, U_{NVB}\}$, *where*

$$\Phi_{NVB} = \left\{ \left\langle \frac{[0,0],[1,1],[1,1]}{u_1}, \frac{[0,0],[1,1],[1,1]}{u_2} \right\rangle, \left\langle \frac{[0,0],[1,1],[1,1]}{v_1} \right\rangle \right\}$$

$$M_{NVB} = \left\{ \begin{array}{l} \left\langle \frac{[0.2,0.4],[0.6,0.8],[0.6,0.8]}{u_1}, \frac{[0.3,0.6],[0.7,0.8],[0.4,0.7]}{u_2} \right\rangle, \\ \left\langle \frac{[0.6,0.8],[0.7,0.9],[0.2,0.4]}{v_1} \right\rangle \end{array} \right\}$$

$$P_{NVB} = \left\{ \begin{array}{l} \left\langle \frac{[0.6,0.7],[0.1,0.9],[0.3,0.4]}{u_1}, \frac{[0.7,0.8],[0.3,0.7],[0.2,0.3]}{u_2} \right\rangle, \\ \left\langle \frac{[0.6,0.7],[0.2,0.5],[0.3,0.4]}{v_1} \right\rangle \end{array} \right\}$$

$K_{NVB} = (M_{NVB} \cap P_{NVB}) =$

$$\left\{ \left\langle \frac{[0.2,0.4],[0.6,0.9],[0.6,0.8]}{u_1}, \frac{[0.3,0.6],[0.7,0.8],[0.4,0.7]}{u_2} \right\rangle, \left\langle \frac{[0.6,0.7],[0.7,0.9],[0.3,0.4]}{v_1} \right\rangle \right\}$$

$H_{NVB} = (M_{NVB} \cup P_{NVB}) =$

$$\left\{ \left\langle \frac{[0.6,0.7],[0.1,0.8],[0.3,0.4]}{u_1}, \frac{[0.7,0.8],[0.7,0.8],[0.2,0.3]}{u_2} \right\rangle, \left\langle \frac{[0.6,0.8],[0.2,0.5],[0.2,0.4]}{v_1} \right\rangle \right\}$$

$$U_{NVB} = \left\{ \left\langle \frac{[1,1],[0,0],[0,0]}{u_1}, \frac{[1,1],[0,0],[0,0]}{u_2} \right\rangle, \left\langle \frac{[1,1],[0,0],[0,0]}{v_1} \right\rangle \right\}$$

Definition 4.2.3. ***(Neutrosophic Vague Binary Open Set)***

Every elements of a NVBT is known as a Neutrosophic Vague Binary Open Set (NVBOS, in short)

Example 4.2.4.

In example 4.2.2, $\Phi_{NVB}, M_{NVB}, P_{NVB}, K_{NVB}, H_{NVB}, U_{NVB}$ *are all NVBOS's*

Definition 4.2.5. ***(Neutrosophic Vague Binary Closed Set)***

Complement of a NVBOS is known as a Neutrosophic Vague Binary Closed Set (NVBCS, in short)

Example 4.2.6.

In example 4.2.2., $\Phi^c_{NVB}, M^c_{NVB}, P^c_{NVB}, K^c_{NVB}, H^c_{NVB}, U^c_{NVB}$ *are all NVBCS's, where*

$$\Phi^c_{NVB} = \left\{ \left(\begin{array}{l} \left\langle \frac{[1,1],[0,0],[0,0]}{u_1}, \frac{[1,1],[0,0],[0,0]}{u_2} \right\rangle, \\ \left\langle \frac{[1,1],[0,0],[0,0]}{v_1} \right\rangle \end{array} \right) \right\} = U_{NVB}$$

$$M^c_{NVB} = \left\{ \left(\begin{array}{l} \left\langle \frac{[0.6,0.8],[0.2,0.4],[0.2,0.4]}{u_1}, \frac{[0.4,0.7],[0.2,0.3],[0.3,0.6]}{u_2} \right\rangle, \\ \left\langle \frac{[0.2,0.4],[0.1,0.3],[0.6,0.8]}{v_1} \right\rangle \end{array} \right) \right\}$$

$$P^c_{NVB} = \left\{ \left(\begin{array}{l} \left\langle \frac{[0.3,0.4],[0.1,0.9],[0.6,0.7]}{u_1}, \frac{[0.2,0.3],[0.3,0.7],[0.7,0.8]}{u_2} \right\rangle, \\ \left\langle \frac{[0.3,0.4],[0.1,0.3],[0.6,0.7]}{v_1} \right\rangle \end{array} \right) \right\}$$

$$K^c_{NVB} = \left\{ \left(\begin{array}{l} \left\langle \frac{[0.6,0.8],[0.1,0.4],[0.2,0.4]}{u_1}, \frac{[0.4,0.7],[0.2,0.3],[0.3,0.6]}{u_2} \right\rangle, \\ \left\langle \frac{[0.2,0.4],[0.5,0.8],[0.6,0.8]}{v_1} \right\rangle \end{array} \right) \right\}$$

$$U^c_{NVB} = \left\{ \left(\begin{array}{l} \left\langle \frac{[0,0],[1,1],[1,1]}{u_1}, \frac{[0,0],[1,1],[1,1]}{u_2} \right\rangle, \\ \left\langle \frac{[0,0],[1,1],[1,1]}{v_1} \right\rangle \end{array} \right) \right\} = \Phi_{NVB}$$

Remark 4.2.7.

Φ_{NVB}, U_{NVB} *will act both as NVBOS and NVBCS*

Definition 4.2.8. ***(Neutrosophic Vague Binary Topological Space)***

The triplet $\left(U_1, U_2, \tau^{NVB}_{\Delta}\right)$ *is known as a Neutrosophic Vague Binary Topological Space (NVBTS in short), where* τ^{NVB}_{Δ} *is a Neutrosophic Vague Binary Topology defined as in definition 4.2.1.*

Example 4.2.9.

If $U_1 = \{u_1, u_2\}; U_2 = \{v_1\};$ $\tau^{NVB}_{\Delta} = \{\Phi_{NVB}, M_{NVB}, P_{NVB}, K_{NVB}, H_{NVB}, U_{NVB}\}$ *are defined as in example 4.2.2., then the triplet* $\left(U_1, U_2, \tau^{NVB}_{\Delta}\right)$ *is clearly a NVBTS.*

Definition 4.2.10.

(Neutrosophic Vague Binary Discrete Topology &

Neutrosophic Vague Binary Discrete Topological Space)

A topology consisting of only empty and unit NVBS's is known as a Neutrosophic

Vague Binary Discrete Topology (NVBDT in short) and the corresponding neutrosophic vague binary topological space is known as a Neutrosophic Vague Binary Discrete Topological Space(NVBDTS, in short)
i.e., $\tau_{\Delta}^{NVB} = \{\Phi_{NVB}, U_{NVB}\}$

Example 4.2.11.
In example 4.2.2,

$$\tau_{\Delta}^{NVB} = \{\Phi_{NVB}, U_{NVB}\} = \left\{ \begin{array}{l} \left(\left\langle \frac{[0,0],[1,1],[1,1]}{u_1}, \frac{[0,0],[1,1],[1,1]}{u_2} \right\rangle, \left\langle \frac{[0,0],[1,1],[1,1]}{v_1} \right\rangle\right) \\ \left(\left\langle \frac{[1,1],[0,0],[0,0]}{u_1}, \frac{[1,1],[0,0],[0,0]}{u_2} \right\rangle, \left\langle \frac{[1,1],[0,0],[0,0]}{v_1} \right\rangle\right) \end{array} \right\}$$

is clearly a NVBDT and the corresponding neutrosophic vague topological space is a NVBDTS

Definition 4.2.12.
***(Neutrosophic Vague Binary Indiscrete Topology** &*
Neutrosophic Vague Binary Indiscrete Topological Space)
A NVBT defined by it's power set is known as Neutrosophic Vague Binary Indiscrete Topology (NVBIDT, in short) and the corresponding neutrosophic vague binary topological space is known as a Neutrosophic Vague Binary Indiscrete Topological Space (NVBIDTS, in short)

Definition 4.2.13.
(Neutrosophic Vague Binary Interior** & **Neutrosophic Vague Binary Closure)
Let $(U_1, U_2, \tau_{\Delta}^{NVB})$ *be a NVBTS and also let*

$$M_{NVB} = \left\{ \begin{array}{l} \left\langle \frac{\hat{T}_{M_{NVB}}(u_r), \hat{I}_{M_{NVB}}(u_r), \hat{F}_{M_{NVB}}(u_r)}{u_r} \quad ; \quad \forall u_r \in U_1 \right\rangle, \\ \left\langle \frac{\hat{T}_{M_{NVB}}(v_s), \hat{I}_{M_{NVB}}(v_s), \hat{F}_{M_{NVB}}(v_s)}{v_s} \quad ; \quad \forall v_s \in U_2 \right\rangle \end{array} \right\}$$

is a NVBS over a binary universe (U_1, U_2) *defined as in definition 4.1.1. Then it's neutrosophic vague binary interior (denoted as* M_{NVB}^0*) and neutrosophic vague binary closure (denoted as* $\overline{M_{NVB}}$*) are defined as follows:*

$M_{NVB}^0 =$
$\cup \{M_{NVB}^i; i \in I / M_{NVB}^i \text{ is a } NVBOS \text{ over } (U_1, U_2) \text{ with } M_{NVB}^i \subseteq M_{NVB}; \forall i\}$
$\overline{M_{NVB}} =$
$\cap \{M_{NVB}^i; i \in I / M_{NVB}^i \text{ is a } NVBCS \text{ over } (U_1, U_2) \text{ with } M_{NVB} \subseteq M_{NVB}^i; \forall i\}$

Definition 4.2.14.

In example 4.2.2,

$$H^0_{NVB} = \left\{ \begin{array}{l} \left\langle \frac{[0.6,0.7],[0.1,0.8],[0.3,0.4]}{u_1}, \frac{[0.7,0.8],[0.7,0.8],[0.2,0.3]}{u_2} \right\rangle, \\ \left\langle \frac{[0.6,0.8],[0.2,0.5],[0.2,0.4]}{v_1} \right\rangle \end{array} \right\} = H_{NVB}$$

From example 4.2.6,

$$\overline{M^c_{NVB}} = \left\{ \begin{array}{l} \left\langle \frac{[0.6,0.8],[0.2,0.4],[0.2,0.4]}{u_1}, \frac{[0.4,0.7],[0.2,0.3],[0.3,0.6]}{u_2} \right\rangle, \\ \left\langle \frac{[0.2,0.4],[0.1,0.3],[0.6,0.8]}{v_1} \right\rangle \end{array} \right\} = M^c_{NVB}$$

Theorem 4.2.15.

$(i) M_{NVB}$ *is a NVBOS* $\Leftrightarrow M^0_{NVB} = M_{NVB}$
$(ii) M_{NVB}$ *is a NVBCS* $\Leftrightarrow \overline{M_{NVB}} = M_{NVB}$

Proof.

Steps are Obvious

Theorem 4.2.16.

1. $M^1_{NVB} \subseteq M^2_{NVB}$ *and* $P^1_{NVB} \subseteq P^2_{NVB}$
   $\Rightarrow (M^1_{NVB} \cup P^1_{NVB}) \subseteq (M^2_{NVB} \cup P^2_{NVB})$ *and* $(M^1_{NVB} \cap P^1_{NVB}) \subseteq (M^2_{NVB} \cap P^2_{NVB})$
2. $M_{NVB} \subseteq M^1_{NVB}$ *and* $M_{NVB} \subseteq M^2_{NVB}$
   $\Rightarrow M_{NVB} \subseteq (M^1_{NVB} \cup M^2_{NVB}) \subseteq M_{NVB}$
3. $\overline{\overline{M_{NVB}}} = M_{NVB}$
4. $M_{NVB} \subseteq P_{NVB} \Rightarrow \overline{P_{NVB}} \subseteq \overline{M_{NVB}}$
5. $M_{NVB} \subseteq P_{NVB} \Rightarrow \overline{P_{NVB}} \subseteq \overline{M_{NVB}}$
6. $\overline{\Phi}_{NVB} = U_{NVB}$
7. $\overline{U}_{NVB} = \Phi_{NVB}$

Proof.

Proof is Obvious

4.3 Neutrosophic Vague Binary Continuity

Continuity plays vital role in any topology. In this section image, pre-image and continuity are developed for NVBS's.

Definition 4.3.1.

(Image and Pre- image of neutrosophic vague binary sets)

Let M_{NVB} and P_{NVB} be two non - empty NVBS's defined over two binary universes (U_1, U_2) and (V_1, V_2) respectively. Define a function $f : M_{NVB} \to P_{NVB}$, then the following statements hold:

1. *If*

$$D_{NVB} = \left\{ \begin{array}{c} \left\langle \dfrac{\hat{T}_{D_{NVB}}(s_i), \hat{I}_{D_{NVB}}(s_i), \hat{F}_{D_{NVB}}(s_i)}{s_i}, s_i \in V_1 \right\rangle, \\ \left\langle \dfrac{\hat{T}_{D_{NVB}}(t_r), \hat{I}_{D_{NVB}}(t_r), \hat{F}_{D_{NVB}}(t_r)}{t_r}, t_r \in V_2 \right\rangle \end{array} \right\} \text{ is a NVBS in } P_{NVB},$$

then the preimage of D_{NVB}, under f, denoted by $f^{-1}(D_{NVB})$, is a NVBS in M_{NVB} defined by

$$f^{-1}(D_{NVB}) = \left\{ \begin{array}{c} \left\langle \dfrac{f^{-1}\left(\hat{T}_{D_{NVB}}\right)(s_i), f^{-1}\left(\hat{I}_{D_{NVB}}\right)(s_i), f^{-1}\left(\hat{F}_{D_{NVB}}\right)(s_i)}{s_i}, \forall s_i \in V_1 \right\rangle, \\ \left\langle \dfrac{f^{-1}\left(\hat{T}_{D_{NVB}}\right)(t_r), f^{-1}\left(\hat{I}_{D_{NVB}}\right)(t_r), f^{-1}\left(\hat{F}_{D_{NVB}}\right)(t_r)}{t_r}; \forall t_r \in V_2 \right\rangle \end{array} \right\}$$

2. *If*

$$A_{NVB} = \left\{ \begin{array}{c} \left\langle \dfrac{\hat{T}_{A_{NVB}}(x_j), \hat{I}_{A_{NVB}}(x_j), \hat{F}_{A_{NVB}}(x_j)}{x_j}; \forall x_j \in U_1 \right\rangle, \\ \left\langle \dfrac{\hat{T}_{A_{NVB}}(y_k), \hat{I}_{A_{NVB}}(y_k), \hat{F}_{A_{NVB}}(y_k)}{y_k}; \forall y_k \in U_2 \right\rangle \end{array} \right\}$$

is a NVBS in M_{NVB}, then the image of A_{NVB} under f, denoted by $f(A_{NVB})$, is a NVBS in P_{NVB} defined by

$$f(A_{NVB}) = \left\{ \begin{array}{c} \left\langle \dfrac{f_{sup}\left(\hat{T}_{A_{NVB}}(s_i)\right); f_{inf}\left(\hat{I}_{A_{NVB}}(s_i)\right); f_{inf}\left(\hat{F}_{A_{NVB}}(s_i)\right)}{s_i}; \forall s_i \in U_1 \right\rangle, \\ \left\langle \dfrac{f_{sup}\left(\hat{T}_{A_{NVB}}(t_r)\right); f_{inf}\left(\hat{I}_{A_{NVB}}(t_r)\right); f_{inf}\left(\hat{F}_{A_{NVB}}(t_r)\right)}{t_r}; \forall t_r \in U_2 \right\rangle \end{array} \right\}$$

where

$$f_{sup}\left(\hat{T}_{A_{NVB}(s_i)}\right)=\begin{cases} sup_{x_j\in f^{-1}(s_i)}\hat{T}_{A_{NVB}}\left(x_j\right); iff^{-1}\left(s_i\right)\neq\Phi \\ 0 \quad ; \quad otherwise \end{cases}$$

$$f_{sup}\left(\hat{T}_{A_{NVB}(t_r)}\right)=\begin{cases} sup_{t_r\in f^{-1}(t_r)}\hat{T}_{A_{NVB}}(t_r); iff^{-1}(t_r)\neq\Phi \\ 0 \quad ; \quad otherwise \end{cases}$$

$$f_{inf}\left(\hat{I}_{A_{NVB}(s_i)}\right)=\begin{cases} inf_{x_j\in f^{-1}(s_i)}\hat{I}_{A_{NVB}}\left(x_j\right); iff^{-1}\left(s_i\right)\neq\Phi \\ 0 \quad ; \quad otherwise \end{cases}$$

$$f_{inf}\left(\hat{I}_{A_{NVB}(t_r)}\right)=\begin{cases} inf_{y_k\in f^{-1}(t_r)}\hat{I}_{A_{NVB}}\left(y_k\right); iff^{-1}\left(t_r\right)\neq\Phi \\ 0 \quad ; \quad otherwise \end{cases}$$

$$f_{inf}\left(\hat{F}_{A_{NVB}(s_i)}\right)=\begin{cases} inf_{x_j\in f^{-1}(s_i)}\hat{F}_{A_{NVB}}\left(x_j\right); iff^{-1}\left(s_i\right)\neq\Phi \\ 0 \quad ; \quad otherwise \end{cases}$$

$$f_{inf}\left(\hat{F}_{A_{NVB}(t_r)}\right)=\begin{cases} inf_{y_k\in f^{-1}(t_r)}\hat{F}_{A_{NVB}}\left(y_k\right); iff^{-1}\left(t_r\right)\neq\Phi \\ 0 \quad ; \quad otherwise \end{cases}$$

for each $s_i \in V_1$ *and for each* $t_r \in V_2$

Definition 4.3.2. ***(Neutrosophic Vague Binary Continuity):***
Let $\left(U_1, U_2, \tau_{\Delta}^{NVB}\right)$ *and* $\left(V_1, V_2, \sigma_{\Delta}^{NVB}\right)$ *is said to be neutrosophic vague binary continuous (NVB continuous) if for every NVBOS (or NVBCS)* M_{NVB} *of* $\left(V_1, V_2, \sigma_{\Delta}^{NVB}\right)$, $f^{-1}\left(M_{NVB}\right)$ *is a NVBOS (or NVBCS) in* $\left(U_1, U_2, \tau_{\Delta}^{NVB}\right)$

Example 4.3.3.
Let $f=(g,h): M_{NVB}\to P_{NVB}$ *be a function defined as,* $f(\Phi^1_{NVB})=\Phi^2_{NVB}$, $f(M^1_{NVB})=P^1_{NVB}$, $f(M^2_{NVB})=P^1_{NVB}$, $f(U^1_{NVB})=U^2_{NVB}$ *where* $g: U_1\to V_1$ *and* $h: U_2\to V_2$ *be two functions with* $g(u_1)=s_2$, $g(u_2)=s_1$ *and* $h(v_1)=t_1$ *where* $U_1=\{u_1,u_2\}, U_2=\{v_1\}$ *and* $V_1=\{s_1,s_2\}, V_2=\{t_1\}$.

$$Let\ \tau_{\Delta}^{NVB}=\begin{Bmatrix} \Phi^1_{NVB}, M^1_{NVB}, M^2_{NVB}, M^3_{NVB}, M^4_{NVB}, M^5_{NVB}, M^6_{NVB}, \\ M^7_{NVB}, M^8_{NVB}, M^9_{NVB}, M^{10}_{NVB}, M^{11}_{NVB}, U^1_{NVB} \end{Bmatrix}$$

$\sigma_{\Delta}^{NVB} = \{\Phi_{NVB}^{2}, P_{NVB}^{1}, U_{NVB}^{2}\}$ *be their respective NVBT's. Here,*

$$\Phi_{NVB}^{1} = \left\{ \left\langle \frac{[0,0],[1,1],[1,1]}{u_1}, \frac{[0,0],[1,1],[1,1]}{u_2} \right\rangle, \left\langle \frac{[0,0],[1,1],[1,1]}{v_1} \right\rangle \right\}$$

$$M_{NVB}^{1} = \left\{ \begin{array}{l} \left\langle \frac{[0.3,0.4],[0.7,0.8],[0.6,0.7]}{u_1}, \frac{[0.2,0.7],[0.1,0.5],[0.3,0.8]}{u_2} \right\rangle, \\ \left\langle \frac{[0.4,0.9],[0.2,0.6],[0.1,0.6]}{v_1} \right\rangle \end{array} \right\}$$

$$M_{NVB}^{2} = \left\{ \begin{array}{l} \left\langle \frac{[0.1,0.6],[0.6,0.9],[0.4,0.9]}{u_1}, \frac{[0.6,0.8],[0.3,0.7],[0.2,0.4]}{u_2} \right\rangle, \\ \left\langle \frac{[0.2,0.7],[0.2,0.9],[0.3,0.8]}{v_1} \right\rangle \end{array} \right\}$$

$$M_{NVB}^{3} = \left\{ \begin{array}{l} \left\langle \frac{[0.6,0.8],[0.1,0.5],[0.2,0.4]}{u_1}, \frac{[0.3,0.6],[0.6,0.8],[0.4,0.7]}{u_2} \right\rangle, \\ \left\langle \frac{[0.2,0.7],[0.2,0.9],[0.3,0.8]}{v_1} \right\rangle \end{array} \right\}$$

$$M_{NVB}^{4} = \left\{ \begin{array}{l} \left\langle \frac{[0.1,0.4],[0.7,0.9],[0.6,0.9]}{u_1}, \frac{[0.2,0.6],[0.6,0.8],[0.4,0.8]}{u_2} \right\rangle, \\ \left\langle \frac{[0.2,0.7],[0.2,0.9],[0.3,0.8]}{v_1} \right\rangle \end{array} \right\}$$

$$M_{NVB}^{5} = \left\{ \begin{array}{l} \left\langle \frac{[0.6,0.8],[0.1,0.5],[0.2,0.4]}{u_1}, \frac{[0.6,0.8],[0.1,0.5],[0.2,0.4]}{u_2} \right\rangle, \\ \left\langle \frac{[0.4,0.9],[0.2,0.6],[0.1,0.6]}{v_1} \right\rangle \end{array} \right\}$$

$$M_{NVB}^{6} = \left\{ \begin{array}{l} \left\langle \frac{[0.1,0.4],[0.7,0.9],[0.6,0.9]}{u_1}, \frac{[0.2,0.7],[0.3,0.7],[0.3,0.8]}{u_2} \right\rangle, \\ \left\langle \frac{[0.2,0.7],[0.2,0.9],[0.3,0.8]}{v_1} \right\rangle \end{array} \right\}$$

$$M_{NVB}^{7} = \left\{ \begin{array}{l} \left\langle \frac{[0.3,0.6],[0.6,0.8],[0.4,0.7]}{u_1}, \frac{[0.6,0.8],[0.1,0.5],[0.2,0.4]}{u_2} \right\rangle, \\ \left\langle \frac{[0.4,0.9],[0.2,0.6],[0.1,0.6]}{v_1} \right\rangle \end{array} \right\}$$

$$M_{NVB}^{8} = \left\{ \begin{array}{l} \left\langle \frac{[0.1,0.6],[0.6,0.9],[0.4,0.9]}{u_1}, \frac{[0.3,0.6],[0.6,0.8],[0.4,0.7]}{u_2} \right\rangle, \\ \left\langle \frac{[0.2,0.7],[0.2,0.9],[0.3,0.8]}{v_1} \right\rangle \end{array} \right\}$$

$$M_{NVB}^{9} = \left\{ \begin{array}{l} \left\langle \frac{[0.6,0.8],[0.1,0.5],[0.2,0.4]}{u_1}, \frac{[0.6,0.8],[0.3,0.7],[0.2,0.4]}{u_2} \right\rangle, \\ \left\langle \frac{[0.2,0.7],[0.2,0.9],[0.3,0.8]}{v_1} \right\rangle \end{array} \right\}$$

$$M^{10}_{NVB} = \left\{ \begin{array}{l} \left\langle \dfrac{[0.3,0.4],[0.7,0.8],[0.6,0.7]}{u_1}, \dfrac{[0.2,0.6],[0.6,0.8],[0.4,0.8]}{u_2} \right\rangle, \\ \left\langle \dfrac{[0.2,0.7],[0.2,0.9],[0.3,0.8]}{v_1} \right\rangle \end{array} \right\}$$

$$M^{11}_{NVB} = \left\{ \begin{array}{l} \left\langle \dfrac{[0.6,0.8],[0.1,0.5],[0.2,0.4]}{u_1}, \dfrac{[0.3,0.7],[0.1,0.5],[0.3,0.7]}{u_2} \right\rangle, \\ \left\langle \dfrac{[0.4,0.9],[0.2,0.6],[0.1,0.6]}{v_1} \right\rangle \end{array} \right\}$$

$$U^{1}_{NVB} = \left\{ \left\langle \dfrac{[1,1],[0,0],[0,0]}{u_1}, \dfrac{[1,1],[0,0],[0,0]}{u_2} \right\rangle, \left\langle \dfrac{[1,1],[0,0],[0,0]}{v_1} \right\rangle \right\}$$

and $V_1 = \{s_1, s_2\}, V_2 = \{t_1\}$ *be a common universe with*

$$P^{1}_{NVB} = \left\{ \begin{array}{l} \left\langle \dfrac{[0.2,0.3],[0.5,0.6],[0.7,0.8]}{s_1}, \dfrac{[0.3,0.7],[0.4,0.5],[0.3,0.7]}{s_2} \right\rangle, \\ \left\langle \dfrac{[0.2,0.4],[0.5,0.6],[0.6,0.8]}{t_1} \right\rangle \end{array} \right\}$$

$$U^{2}_{NVB} = \left\{ \left\langle \dfrac{[1,1],[0,0],[0,0]}{s_1}, \dfrac{[1,1],[0,0],[0,0]}{s_2} \right\rangle, \left\langle \dfrac{[1,1],[0,0],[0,0]}{t_1} \right\rangle \right\}$$

It is got that, $f^{-1}(\Phi^2_{NVB}) = \Phi^1_{NVB}$, $\quad f^{-1}(P^1_{NVB}) = M^3_{NVB}$, $\quad f^{-1}(U^2_{NVB}) = U^1_{NVB}$. *Then clearly f is a neutrosophic vague binary continuous mapping.*

4.4 Neutrosophic Vague Binary Distance Measure with Application

Let (U_1, U_2) be a binary universe with $U_1 = \{u_1, u_2, --, u_r, ---, u_i\}$ and $U_2 = \{v_1, v_2, --, v_s, ---, v_j\}$. Let the cardinality of $U_1 = \#(U_1) = i$ and cardinality of $U_2 = \#(U_2) = j$. Let $d : VBSS(U_1, U_2) \times VBSS(U_1, U_2) \to [0,1]$ is a mapping. Also, let $M_{NVB}, P_{NVB} \in VBSS(U_1, U_2)$. Four distance measures of these NVBS's are given by the following formulae

(i) Hamming Distance

$d^{H}_{NVB}(M_{NVB}, P_{NVB}) =$

$$\frac{1}{6}\sum_{r=1}^{i}\left[\begin{array}{l} \left|T^{-}_{M_{NVB}}(u_r) - T^{-}_{P_{NVB}}(u_r)\right| + \left|I^{-}_{M_{NVB}}(u_r) - I^{-}_{P_{NVB}}(u_r)\right| + \left|F^{-}_{M_{NVB}}(u_r) - F^{-}_{P_{NVB}}(u_r)\right| + \\ \left|T^{+}_{M_{NVB}}(u_r) - T^{+}_{P_{NVB}}(u_r)\right| + \left|I^{+}_{M_{NVB}}(u_r) - I^{+}_{P_{NVB}}(u_r)\right| + \left|F^{+}_{M_{NVB}}(u_r) - F^{+}_{P_{NVB}}(u_r)\right| \end{array}\right] +$$

$$\frac{1}{6}\sum_{s=1}^{j}\left[\begin{array}{l} \left|T^{-}_{M_{NVB}}(v_s) - T^{-}_{P_{NVB}}(v_s)\right| + \left|I^{-}_{M_{NVB}}(v_s) - I^{-}_{P_{NVB}}(v_s)\right| + \left|F^{-}_{M_{NVB}}(v_s) - F^{-}_{P_{NVB}}(v_s)\right| + \\ \left|T^{+}_{M_{NVB}}(v_s) - T^{+}_{P_{NVB}}(v_s)\right| + \left|I^{+}_{M_{NVB}}(v_s) - I^{+}_{P_{NVB}}(v_s)\right| + \left|F^{+}_{M_{NVB}}(v_s) - F^{+}_{P_{NVB}}(v_s)\right| \end{array}\right]$$

(ii) Normalised Hamming Distance

$d^{nH}_{NVB}(M_{NVB}, P_{NVB}) =$

$$\frac{1}{6i}\sum_{r=1}^{i}\left[\begin{array}{l} \left|T^{-}_{M_{NVB}}(u_r) - T^{-}_{P_{NVB}}(u_r)\right| + \left|I^{-}_{M_{NVB}}(u_r) - I^{-}_{P_{NVB}}(u_r)\right| + \left|F^{-}_{M_{NVB}}(u_r) - F^{-}_{P_{NVB}}(u_r)\right| + \\ \left|T^{+}_{M_{NVB}}(u_r) - T^{+}_{P_{NVB}}(u_r)\right| + \left|I^{+}_{M_{NVB}}(u_r) - I^{+}_{P_{NVB}}(u_r)\right| + \left|F^{+}_{M_{NVB}}(u_r) - F^{+}_{P_{NVB}}(u_r)\right| \end{array}\right] +$$

$$\frac{1}{6j}\sum_{s=1}^{j}\left[\begin{array}{l}\left|T^{-}_{M_{NVB}}(v_s)-T^{-}_{P_{NVB}}(v_s)\right|+\left|I^{-}_{M_{NVB}}(v_s)-I^{-}_{P_{NVB}}(v_s)\right|+\left|F^{-}_{M_{NVB}}(v_s)-F^{-}_{P_{NVB}}(v_s)\right|+\\ \left|T^{+}_{M_{NVB}}(v_s)-T^{+}_{P_{NVB}}(v_s)\right|+\left|I^{+}_{M_{NVB}}(v_s)-I^{+}_{P_{NVB}}(v_s)\right|+\left|F^{+}_{M_{NVB}}(v_s)-F^{+}_{P_{NVB}}(v_s)\right|\end{array}\right]$$

(iii) Euclidean Distance

$d^{E}_{NVB}(M_{NVB}, P_{NVB}) =$

$$\sqrt{\frac{1}{6}\sum_{r=1}^{i}\left[\begin{array}{l}\left|T^{-}_{M_{NVB}}(u_r)-T^{-}_{P_{NVB}}(u_r)\right|^2+\left|I^{-}_{M_{NVB}}(u_r)-I^{-}_{P_{NVB}}(u_r)\right|^2+\left|F^{-}_{M_{NVB}}(u_r)-F^{-}_{P_{NVB}}(u_r)\right|^2+\\ \left|T^{+}_{M_{NVB}}(u_r)-T^{+}_{P_{NVB}}(u_r)\right|^2+\left|I^{+}_{M_{NVB}}(u_r)-I^{+}_{P_{NVB}}(u_r)\right|^2+\left|F^{+}_{M_{NVB}}(u_r)-F^{+}_{P_{NVB}}(u_r)\right|^2\end{array}\right]}+$$

$$\sqrt{\frac{1}{6}\sum_{s=1}^{j}\left[\begin{array}{l}\left|T^{-}_{M_{NVB}}(v_s)-T^{-}_{P_{NVB}}(v_s)\right|^2+\left|I^{-}_{M_{NVB}}(v_s)-I^{-}_{P_{NVB}}(v_s)\right|^2+\left|F^{-}_{M_{NVB}}(v_s)-F^{-}_{P_{NVB}}(v_s)\right|^2+\\ \left|T^{+}_{M_{NVB}}(v_s)-T^{+}_{P_{NVB}}(v_s)\right|^2+\left|I^{+}_{M_{NVB}}(v_s)-I^{+}_{P_{NVB}}(v_s)\right|^2+\left|F^{+}_{M_{NVB}}(v_s)-F^{+}_{P_{NVB}}(v_s)\right|^2\end{array}\right]}$$

(iv) Normalised Euclidean distance

$d^{nE}_{NVB}(M_{NVB}, P_{NVB}) =$

$$\sqrt{\frac{1}{6i}\sum_{r=1}^{i}\left[\begin{array}{l}\left|T^{-}_{M_{NVB}}(u_r)-T^{-}_{P_{NVB}}(u_r)\right|^2+\left|I^{-}_{M_{NVB}}(u_r)-I^{-}_{P_{NVB}}(u_r)\right|^2+\left|F^{-}_{M_{NVB}}(u_r)-F^{-}_{P_{NVB}}(u_r)\right|^2+\\ \left|T^{+}_{M_{NVB}}(u_r)-T^{+}_{P_{NVB}}(u_r)\right|^2+\left|I^{+}_{M_{NVB}}(u_r)-I^{+}_{P_{NVB}}(u_r)\right|^2+\left|F^{+}_{M_{NVB}}(u_r)-F^{+}_{P_{NVB}}(u_r)\right|^2\end{array}\right]}+$$

$$\sqrt{\frac{1}{6j}\sum_{s=1}^{j}\left[\begin{array}{l}\left|T^{-}_{M_{NVB}}(v_s)-T^{-}_{P_{NVB}}(v_s)\right|^2+\left|I^{-}_{M_{NVB}}(v_s)-I^{-}_{P_{NVB}}(v_s)\right|^2+\left|F^{-}_{M_{NVB}}(v_s)-F^{-}_{P_{NVB}}(v_s)\right|^2+\\ \left|T^{+}_{M_{NVB}}(v_s)-T^{+}_{P_{NVB}}(v_s)\right|^2+\left|I^{+}_{M_{NVB}}(v_s)-I^{+}_{P_{NVB}}(v_s)\right|^2+\left|F^{+}_{M_{NVB}}(v_s)-F^{+}_{P_{NVB}}(v_s)\right|^2\end{array}\right]}$$

NVBS's in Medical Diagonosis

This section deals with an application of NVBS's in medical diagnosis.

Table 4.1: Before Treatment

Before Treatment (BT)	P_1	P_2	P_3
Albumin	[0.042, 0.052]	[0.025, 0.052]	[0.052, 0.064]
Globulin Serum	[0.035, 0.045]	[0.033, 0.035]	[0.011, 0.035]
Bilirubin Total	[0.045, 0.100]	[0.070, 0.100]	[0.093, 0.100]

Table 4.1 describes data's collected from three patients after conducting liver function test before treatment which describes first universe.
Table 4.2 describes data's collected after treatment which describes second universe.

Data collected are converted to NVBS's $P^1_{NVB}, P^2_{NVB}, P^3_{NVB}$ as given below:
$P^1_{NVB} =$

$$\left\{\left(\begin{array}{l}\left\langle\frac{[0.042,0.052],[0.948,0.958],[0.948,0.958]}{P_{BT}^{Albumin}},\frac{[0.035,0.045],[0.955,0.965],[0.955,0.965]}{P_{BT}^{GlobulinSerum}},\frac{[0.045,0.100],[0.900,0.955],[0.900,0.955]}{P_{BT}^{BilirubinTotal}},\right\rangle\\ \left\langle\frac{[0.031,0.052],[0.948,0.969],[0.948,0.969]}{P_{AT}^{Albumin}},\frac{[0.021,0.035],[0.965,0.979],[0.965,0.979]}{P_{AT}^{GlobulinSerum}},\frac{[0.025,0.100],[0.900,0.975],[0.900,0.975]}{P_{AT}^{BilirubinTotal}}\right\rangle\end{array}\right)\right\}$$

Table 4.2: After Treatment

After Treatment (AT)	P_1	P_2	P_3
Albumin	[0.031, 0.052]	[0.036, 0.052]	[0.052, 0.064]
Globulin Serum	[0.021, 0.035]	[0.035, 0.042]	[0.019, 0.035]
Bilirubin Total	[0.025, 0.100]	[0.017, 0.100]	[0.099, 0.100]

$P^2_{NVB} =$

$$\left\{\left(\begin{array}{c}\left\langle \frac{[0.025, 0.052], [0.948, 0.975], [0.948, 0.975]}{p_{BT}^{Albumin}}, \frac{[0.033, 0.035], [0.965, 0.967], [0.965, 0.967]}{p_{BT}^{GlobulinSerum}}, \frac{[0.070, 0.100], [0.900, 0.930], [0.900, 0.930]}{p_{BT}^{BilirubinTotal}} \right\rangle \\ \left\langle \frac{[0.036, 0.052], [0.948, 0.964], [0.948, 0.964]}{p_{AT}^{Albumin}}, \frac{[0.035, 0.042], [0.958, 0.965], [0.958, 0.965]}{p_{AT}^{GlobulinSerum}}, \frac{[0.017, 0.100], [0.900, 0.983], [0.900, 0.983]}{p_{AT}^{BilirubinTotal}} \right\rangle\end{array}\right)\right\}$$

$P^3_{NVB} =$

$$\left\{\left(\begin{array}{c}\frac{[0.052, 0.064], [0.936, 0.948], [0.936, 0.948]}{p_{BT}^{Albumin}}, \frac{[0.011, 0.035], [0.965, 0.989], [0.965, 0.989]}{p_{BT}^{GlobulinSerum}}, \frac{[0.011, 0.035], [0.965, 0.989], [0.965, 0.989]}{p_{BT}^{BilirubinTotal}} \\ \frac{[0.052, 0.064], [0.936, 0.948], [0.936, 0.948]}{p_{AT}^{Albumin}}, \frac{[0.019, 0.035], [0.965, 0.981], [0.965, 0.981]}{p_{AT}^{GlobulinSerum}}, \frac{[0.099, 0.100], [0.900, 0.901], [0.900, 0.901]}{p_{AT}^{BilirubinTotal}}\end{array}\right)\right\}$$

D^{LFT}_{NVB} is a NVBS formed, based on the actual range fixed for a liver function test. Ranges for D^{LFT}_{NVB} under a liver function test for albumin, Globulin Serum and Bilirubin Total is given in Table 4.3 and in Table 4.4:

Table 4.3: Before Treatment

Before Treatment (BT)	D^{LFT}_{NVB}
Albumin	$[0.034, 0.052], [0.948, 0.966], [0.948, 0.966]$
Globulin Serum	$[0.015, 0.035], [0.965, 0.985], [0.965, 0.985]$
Bilirubin Total	$[0.000, 0.100], [0.900, 0.100], [0.900, 0.100]$

Table 4.4: After Treatment

After Treatment(AT)	D^{LFT}_{NVB}
Albumin	$[0.034, 0.052], [0.948, 0.966], [0.948, 0.966]$
Globulin Serum	$[0.015, 0.035], [0.965, 0.985], [0.965, 0.985]$
Bilirubin Total	$[0.000, 0.100], [0.900, 0.100], [0.900, 0.100]$

Datas in Table 4.3 and Table 4.4 are converted to NVBS as below.
$D_{NVB}^{LFT} =$

$$\left\{\begin{pmatrix} \frac{[0.034, 0.052], [0.948, 0.966], [0.948, 0.966]}{p_{BT}^{Albumin}}, \frac{[0.015, 0.035], [0.965, 0.985], [0.965, 0.985]}{p_{BT}^{GlobulinSerum}}, \frac{[0.000, 0.100], [0.900, 0.100], [0.900, 0.100]}{p_{BT}^{BilirubinTotal}} \\ \frac{[0.034, 0.052], [0.948, 0.966], [0.948, 0.966]}{p_{BT}^{Albumin}}, \frac{[0.015, 0.035], [0.965, 0.985], [0.965, 0.985]}{p_{BT}^{GlobulinSerum}}, \frac{[0.000, 0.100], [0.900, 0.100], [0.900, 0.100]}{p_{BT}^{BilirubinTotal}} \end{pmatrix}\right\}$$

Neutrosophic Vague Binary Euclidean Distance Measure can be used to diagonise

Table 4.5: After Treatment

$d_{NVBS}^{ED}(P_{NVB}^1, D_{NVB}^{LFT})$	$d_{NVBS}^{ED}(P_{NVB}^2, D_{NVB}^{LFT})$	$d_{NVBS}^{ED}(P_{NVB}^3, D_{NVB}^{LFT})$
0.014856	0.277330	0.745502

which patient is more suffering with liver problems even after treatment. Table 4.5 gives the neutrosophic vague binary euclidean difference between each of the patients from D_{NVB}^{LFT}. Lowest neutrosophic vague binary euclidean difference is for patient I. So patient I suffers more with liver problems even after treatment.

Conclusion

Neutrosophic vague binary sets are developed in this chapter with some basic concepts and examples. Real-life situations demand binary and higher dimensional universes than a unique one. Being the vital concept to homeomorphism - 'which is the underlying principle to any topology'- continuity has an important role in topology. It is also developed for this new concept. Practical applications are tremendous for binary concept in day today life. One real life example in medical diagonosis is discussed above. Several situations demand combined result than ' a unique separate one' - to compare and deal situations in a more fast manner. NVBS is a good tool for comparison in such cases. It could be made use in surveys, case studies and in some other sort of similar situations. Topology is a special type of subset to a universal set - based on which study of all other subsets of the universal set is possible. New study will produce a combined result or net effect than taking a single result.

Chapter 5

Chapter 5

Various Algebras of NVBS's

Artificial Intelligence is a sharp tool made use by human society to upgrade their existing applicational criteria's and tools. These works are mainly assisted with logic. Inter-connection between algebra and logic is strong. This reason persuaded researchers to work with algebra too with the same attention and level as they are working with logic ! As a result, algebra and logic developed parallely and simultaneously. An algebraic structure on a set under consideration is a collection of operations on that set together with a set of axioms. Being a base stone to algebraic structure, notion of *set* is very important. A walk through development route of crisp set theory will make us to meet with different forms like fuzzy, intuitionistic fuzzy, rough, interval- mathematics, vague, neutrosophic, soft, plithogenic etc. At the same time algebra's developmental route equipped with different configurations as BCK/BCI [95], BCH [35], BH [97], BZ [96], MV [37], BZMV [14], K [24], Q [39], QS [84], G [73] etc. Among these four different algebras BCK/BCI, BZMV^{dM}, K and G are selected in this chapter for a wide study.

Chapter Scheme :

Section 5.1 : Neutrosophic Vague Binary BCK/BCI-subalgebra of BCK/BCI-algebra
Section 5.2 : Neutrosophic Vague Binary BZMV^{dM} - subalgebra of BZMV^{dM} - algebra
Section 5.3 : Neutrosophic Vague Binary K - subalgebra of K - algebra
Section 5.4 : Neutrosophic Vague Binary G - subalgebra of G - algebra

5.1 Neutrosophic Vague Binary BCK/BCI-subalgebra of BCK/BCI-algebra

Yasuyuki Imai and Kiyoshi Iseki [95] introduced BCK/BCI - algebra in the year 1966. After that, plenty of works emerged related to this area. In this section, neutrosophic vague binary BCK/BCI -subalgebra is developed in BCK/BCI -algebraic zone. It's various ideals like p-ideal, q-ideal, a-ideal, H -ideal also got framed out. Various cuts and mappings are also verified.

Definition 5.1.1. ***(Neutrosophic Vague Binary BCK/BCI - subalgebra)***
A neutrosophic vague binary BCK/BCI -subalgebra is a structure,
$\mathcal{B}_{M_{NVB}} = \left(U^{\mathcal{B}_{M_{NVB}}}, \star, 0\right)$ *which satisfies the following condition,*
$NVB_{M_{NVB}}\left(u_x \star u_y\right) \succcurlyeq rmin\left\{NVB_{M_{NVB}}\left(u_x\right), NVB_{M_{NVB}}\left(u_y\right)\right\}; \forall u_x, u_y \in U.$

$$\text{That is,} \begin{cases} T_{MNVB}\left(u_x \star u_y\right) \geqslant min\left\{T_{M_{NVB}}(u_x), T_{M_{NVB}}\left(u_y\right)\right\} \\ I_{MNVB}\left(u_x \star u_y\right) \leqslant max\left\{I_{M_{NVB}}\left(u_x\right), I_{M_{NVB}}\left(y\right)\right\} \\ F_{MNVB}\left(u_x \star u_y\right) \leqslant max\left\{F_{M_{NVB}}\left(u_x\right), F_{M_{NVB}}\left(u_y\right)\right\} \end{cases}$$

where

1. $U^{\mathcal{B}_{M_{NVB}}} = (U = \{U_1 \cup U_2\}, \quad \star, \quad 0)$ *is the underlying BCK/BCI - algebraic structure to the neutrosophic vague binary set* M_{NVB}, *with a combined universe* $U = \{U_1 \cup U_2\}$ *[where* (U_1, U_2) *is the binary universe to* M_{NVB} & $\cup$ *is the crisp set union] & with a binary operation* $\star$ *and a constant 0 satisfies the following axioms:* $\forall u_x, u_y, u_z \in U$
   (i) $((u_x \star u_y) \star (u_x \star u_z)) \star (u_z \star u_y) = 0$ *(ii)* $(u_x \star (u_x \star u_y)) \star u_y = 0$
   (iii) $(u_x \star u_x) = 0$ *(iv)* $(u_x \star u_y) = 0$ *and* $(u_y \star u_x) = 0$ *imply* $u_x = u_y$
   (v) $(0 \star u_x) = 0$
2. $\star$ *and 0 are taken as defined in 1.*

Remark 5.1.2.

1. *Every NVB BCK -algebra is NVB BCI - algebra too.*
   Generally, converse not true! (Proof given in Theorem 5.1.14.)
   So distinguishing between structures of these two are important!

To denote NVB BCK - algebra, following structures can be used:
$\mathcal{B}^{BCK}_{M_{NVB}} = \left(U^{\mathcal{B}^{BCK}_{M_{NVB}}}, \star, 0\right)$ *or simply as* $\mathcal{B}^{K}_{M_{NVB}} = \left(U^{\mathcal{B}^{K}_{M_{NVB}}}, \star, 0\right)$.

Similarly, to denote NVB BCI - algebra, following structures can be used:
$\mathcal{B}^{BCI}_{M_{NVB}} = \left(U^{\mathcal{B}^{BCI}}_{M_{NVB}}, \star, 0\right)$ *or simply as* $\mathcal{B}^{I}_{M_{NVB}} = \left(U^{\mathcal{B}^{I}_{M_{NVB}}}, \star, 0\right)$

2. *For NVB BCK - algebra,*
'notation for NVB BCK/BCI- algebra', that is, $\mathcal{B}_{M_{NVB}} = \left(U^{\mathcal{B}_{M_{NVB}}}, \star, 0\right)$ *is used in this section, instead of using those given in remark 5.1.2. (1)*

3. *Neutrosophic Vague Binary membership grade of common elements of* U_1 *and* U_2 *is got by taking their neutrosophic vague binary union. For example, let* $U_1 = \{0, u_1\}$ *and* $U_2 = \{0, u_1, u_2\}$ *be two universes.* $NVB^{U}_{M_{NVB}}(0)$ *is neutrosophic vague binary membership grade of 0 in universe U.*
$\therefore\ U = \{U_1 \cup U_2\} = \{0, u_1, u_2\}; U_1 \cap U_2 = \{0, u_1\}$. $\therefore\ NVB^{U}_{M_{NVB}}(0) = NVB^{U_1}_{M_{NVB}}(0) \cup NVB^{U_2}_{M_{NVB}}(0)$. *Similarly to other common elements.*

4. *It is to be noted that, generally, neutrosophic vague binary membership grade of 0 is not same in universe* U_1 *and in universe* U_2. *That is,* $NVB^{U_1}(0) \neq NVB^{U_2}(0)$. *Similarly, to other common elements!*

Example 5.1.3.
Let $U_1 = \{0, u_a\}$ *and let* $U_2 = \{0, u_1, u_2\}$ *be the universes under consideration. Combined universe* $U = \{U_1 \cup U_2\} = \{0, u_a, u_1, u_2\}$ *with* $(U_1 \cap U_2) = \{0\}$. *Let a non - empty neutrosophic vague binary set* M_{NVB} *with underlying set U is given as :*

$$M_{NVB} = \left\{ \begin{array}{l} \left\langle \frac{[0.3, 0.8], [0.1, 0.3], [0.2, 0.7]}{0}, \frac{[0.2, 0.3], [0.2, 0.5], [0.7, 0.8]}{u_a} \right\rangle, \\ \left\langle \frac{[0.1, 0.7], [0.7, 0.8], [0.3, 0.9]}{0}, \frac{[0.2, 0.6], [0.5, 0.7], [0.4, 0.8]}{u_1}, \frac{[0.2, 0.6], [0.5, 0.7], [0.4, 0.8]}{u_2} \right\rangle \end{array} \right\}$$

Cayley table 5.1 indicates the binary operation $\star$ *for U :*

Table 5.1:

$\star$	0	u_a	u_1	u_2
0	0	0	0	0
u_a	u_a	0	0	u_1
u_1	u_1	u_a	0	u_1
u_2	u_2	u_2	u_2	0

Clearly, $U^{\mathcal{B}_{M_{NVB}}} = (U = \{U_1 \cup U_2\}, \star, 0)$ *is a BCK/BCI - algebra.*
Since 0 is a common element,
$\therefore NVB_{M_{NVB}}(0)$
$= ([0.3, 0.8], [0.1, 0.3], [0.2, 0.7]) \cup ([0.1, 0.7], [0.7, 0.8], [0.3, 0.9])$
$= ([0.3, 0.8], [0.1, 0.3, [0.2, 0.7])$
Since u_a, u_1 *and* u_2 *are not common elements*
$\therefore NVB_{M_{NVB}}(u_a) = ([0.2, 0.3], [0.2, 0.5], [0.7, 0.8])$
$NVB_{M_{NVB}}(u_1) = NVB_{M_{NVB}}(u_2) = ([0.2, 0.6], [0.5, 0.7], [0.4, 0.8])$
$\Rightarrow NVB_{M_{NVB}}(u_r^t) =$

$$\left\{\begin{array}{ll} [0.3, 0.8], [0.1, 0.3], [0.2, 0.7] & ; u_r^t = 0 \\ [0.2, 0.3], [0.2, 0.5], [0.7, 0.8] & ; u_r^t = \{u_a\} \quad and \quad u_r^t \neq 0 \\ [0.2, 0.6], [0.5, 0.7], [0.4, 0.8] & ; u_r^t = \{u_1, u_2\} \quad and \quad u_r^t \neq 0 \end{array}\right\} (for \quad any \quad u_r^t \in U)$$

It is clear after verification that,
$\mathcal{B}_{M_{NVB}} = \left(U^{\mathcal{B}_{M_{NVB}}}, \star, 0\right)$ *is a NVB BCK/BCI - algebra.*

Remark 5.1.4.

1. *If* $U_1 \subseteq U_2$ *then* $U = U_2$
2. *The symbols* $\succeq$ *and* $\not\succeq$ *does not imply our usual* $\geqslant$ *or* $\ngeqslant$
3. *In a Cayley table,*
   (a) *principal diagonal elements of a BCK/BCI - algebra U is always zero, since* $(u_x \star u_x) = 0, \forall u_x \in U$
   (b) *Using the property* $(u_x \star 0) = u_x; \forall u_x \in U$ *of BCI- algebra, it is clear that* $(0 \star 0) = 0$

Neutrosophic Vague Binary BCK/BCI - Ideal

Various ideal for neutrosophic vague binary BCK/BCI -subalgebra is developed in this section. Same idea follows to neutrosophic and neutrosophic vague concepts.

Definition 5.1.5. ***(Neutrosophic Vague Binary BCK/BCI - Ideal)***
A non-empty Neutrosophic Vague Binary SubSet (in short, NVBSS) P_{NVB} *of a NVB BCK/BCI- subalgebra* $\mathcal{B}_{M_{NVB}} = \left(U^{\mathcal{B}_{M_{NVB}}}, *, 0\right)$, *is called a NVB BCK/BCI- Ideal of* $\mathcal{B}_{M_{NVB}}$ *if*

1. $NVB_{P_{NVB}}(0) \succcurlyeq NVB_{P_{NVB}}(u_k)$

$$\left.\begin{aligned}\hat{T}_{P_{NVB}}(0) \geqslant \hat{T}_{P_{NVB}}(u_k)\\ \hat{I}_{P_{NVB}}(0) \leqslant \hat{I}_{P_{NVB}}(u_k)\\ \hat{F}_{P_{NVB}}(0) \leqslant \hat{F}_{P_{NVB}}(u_k)\end{aligned}\right\} for\ any\ u_k \in U$$

2. $NVB_{P_{NVB}}(u_a) \succcurlyeq rmin\left\{NVB_{P_{NVB}}\left(u_a \star u_b\right), NVB_{P_{NVB}}(u_b)\right\}$

$$\left.\begin{aligned}\hat{T}_{P_{NVB}}(u_a) \geqslant min\left\{\hat{T}_{P_{NVB}}\left(u_a \star u_b\right), \hat{T}_{P_{NVB}}(u_b)\right\}\\ \hat{I}_{P_{NVB}}(u_a) \leqslant max\left\{\hat{I}_{P_{NVB}}\left(u_a \star u_b\right), \hat{I}_{P_{NVB}}(u_b)\right\}\\ \hat{F}_{P_{NVB}}(u_a) \leqslant max\left\{\hat{F}_{P_{NVB}}\left(u_a \star u_b\right), \hat{F}_{P_{NVB}}(u_b)\right\}\end{aligned}\right\} for\ any\ u_a, u_b \in U$$

Remark 5.1.6.

For NVB BCK-Ideal underlying structure will confine to BCK-algebra and for NVB BCI-Ideal it will confine to BCI- algebra. For different ideals mentioned in definition 5.1.8. the same principle follows

Various Neutrosophic Vague Binary BCK/BCI - Ideals

In this section vague H -ideal is developed first. Then p -ideal, q -ideal, a -ideal and H -ideal are developed for NVB BCK/BCI- algebra $\mathcal{B}_{P_{NVB}} = \left(U^{\mathcal{B}_{P_{NVB}}}, \star, 0\right)$

Definition 5.1.7. ***(Vague H -ideal)***

Let U be a universal set. A vague set A of U is called a vague H -ideal of a BCI -algebra U if it satisfies

1. $V_A(0) \succcurlyeq V_A(u_x); \quad (\forall u_x \in U)$

$$t_A(0) \geqslant t_A(u_x) \quad ; \quad 1 - f_A(0) \geqslant 1 - f_A(0) \qquad or$$
$$t_A(0) \geqslant t_A(u_x) \quad ; \quad f_A(0) \leqslant f_A(u_x)$$

2. $V_A\left(u_x \star u_z\right) \succcurlyeq rmin\left\{V_A\left(u_x \star \left(u_y \star u_z\right)\right), V_A(u_y)\right\}; (\forall u_x, u_y, u_z \in U)$

$$t_A\left(u_x \star u_z\right) \geqslant min\left\{t_A\left(u_x \star \left(u_y \star u_z\right)\right), t_A(u_y)\right\}$$
$$1 - f_A\left(u_x \star u_z\right) \geqslant min\left\{1 - f_A\left(u_x \star \left(u_y \star u_z\right)\right), 1 - f_A\left(u_y\right)\right\}$$

Definition 5.1.8. ***(Comparison of different NVB BCK/BCI -Ideals)***
Let $\mathcal{B}_{M_{NVB}} = \left(U^{\mathcal{B}_{M_{NVB}}}, \star, 0\right)$ *be a NVB BCK/BCI - algebra.*
Conditions for a non - empty NVBSS P_{NVB} *of* $\mathcal{B}_{M_{NVB}}$ *to become*

$$\left.\begin{array}{r} Neutrosophic \quad VagueBinary \quad BCK/BCI - pideal \\ Neutrosophic \quad VagueBinary \quad BCK/BCI - qideal \\ Neutrosophic \quad VagueBinary \quad BCK/BCI - aideal \\ Neutrosophic \quad VagueBinary \quad BCK/BCI - Hideal \end{array}\right\} \textit{are given in the Table 5.2}$$

Table 5.2: Cayley Table

Various Ideals	Condition (1) ; ($\forall u_k \in U$)	Condition(2) ; (for any $u_a, u_b, u_c \in U$)
NVB BCK/BCI p -ideal	$NVB_{P_{NVB}}(0) \succcurlyeq NVB_{P_{NVB}}(u_k)$	$NVB_{P_{NVB}}(u_a) \succcurlyeq rmin \{NVB_{P_{NVB}}\left((u_a \star u_c) \star (u_b \star u_c)\right), NVB_{P_{NVB}}(u_b)\}$
NVB BCK/BCI q -ideal	$NVB_{P_{NVB}}(0) \succcurlyeq NVB_{P_{NVB}}(u_k)$	$NVB_{P_{NVB}}\left(u_a \star u_c\right) \succcurlyeq rmin \{NVB_{P_{NVB}}\left(u_a \star (u_b \star u_c)\right), NVB_{P_{NVB}}(u_b)\}$
NVB BCK/BCI a -ideal	$NVB_{P_{NVB}}(0) \succcurlyeq NVB_{P_{NVB}}(u_k)$	$NVB_{P_{NVB}}\left(u_b \star u_a\right) \succcurlyeq rmin \{NVB_{P_{NVB}}\left((u_a \star u_c) \star (0 \star u_b)\right), NVB_{P_{NVB}}(u_c)\}$
NVB BCK/BCI H -ideal	$NVB_{P_{NVB}}(0) \succcurlyeq NVB_{P_{NVB}}(u_k)$	$NVB_{P_{NVB}}\left(u_a \star u_c\right) \succcurlyeq rmin \{NVB_{P_{NVB}}\left((u_a \star (u_b \star u_c))\right), NVB_{P_{NVB}}(u_b)\}$

Neutrosophic Vague Binary BCK/BCI - Cuts

In this section neutrosophic vague binary BCK/BCI - cuts (in short, NVB BCK/BCI-Cut) is developed

Definition 5.1.9.
(Neutrosophic Vague Binary BCK/BCI
$([\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2]), ([\delta_1, \delta_2], [\rho_1, \rho_2], [\vartheta_1, \vartheta_2])$ ***-Cut) or***
(Neutrosophic Vague Binary BCK/BCI - Cut)
Let the NVBS M_{NVB} *is a NVB BCK/BCI - subalgebra with algebraic structure,* $\mathcal{B}_{M_{NVB}} = \left(U^{\mathcal{B}_{M_{NVB}}}, *, 0\right)$. *Truth membership function, indeterminacy membership function and false membership function of* M_{NVB} *are* $\hat{T}_{M_{NVB}}, \hat{I}_{M_{NVB}}, \hat{F}_{M_{NVB}}$

respectively. A neutrosophic vague binary BCK/BCI $([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]),([\delta_1,\delta_2],[\rho_1,\rho_2],[\vartheta_1,\vartheta_2])$*- Cut of* $\mathcal{B}_{M_{NVB}}$ *is a crisp subset* $M_{NVB_{([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]),([\delta_1,\delta_2],[\rho_1,\rho_2],[\vartheta_1,\vartheta_2])}}$ *of the NVBS* M_{NVB} *given by :*

$$M_{NVB_{([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]),([\delta_1,\delta_2],[\rho_1,\rho_2],[\vartheta_1,\vartheta_2])}}$$

$$=\Big\{ u_k \in U \diagup NVB_{M_{NVB}}(u_k) \succeq$$

$$=\begin{cases} [\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2] & ; \quad if \quad u_k \in U_1 \\ [\delta_1,\delta_2],[\rho_1,\rho_2],[\vartheta_1,\vartheta_2] & ; \quad if \quad u_k \in U_2 \\ [\chi_1,\chi_2],[\Phi_1,\Phi_2],[\Pi_1,\Pi_2] & ; \quad if \quad u_k \in U_1 \cap U_2 \end{cases}$$

Let, *max* $\{[\alpha_1,\alpha_2],[\delta_1,\delta_2] = [\chi_1,\chi_2]\}$ *(say); min* $\{[\beta_1,\beta_2],[\rho_1,\rho_2] = [\Phi_1,\Phi_2]\}$ *(say);* *min* $\{[\gamma_1,\gamma_2],[\vartheta_1,\vartheta_2] = [\Pi_1,\Pi_2]\}$ *(say) with* $\alpha_1,\alpha_2, \beta_1,\beta_2, \gamma_1,\gamma_2,\delta_1,\delta_2,\rho_1,\rho_2,$ $\vartheta_1,\vartheta_2,\chi_1,\chi_2,\Phi_1,\Phi_2,\Pi_1,\Pi_2 \in [0,1]$ *and* $\alpha_1 \le \alpha_2, \beta_1 \le \beta_2, \gamma_1 \le \gamma_2, \delta_1 \le \delta_2,$ $\rho_1 \le \rho_2, \vartheta_1 \le \vartheta_2, \chi_1 \le \chi_2, \Phi_1 \le \Phi_2, \Pi_1 \le \Pi_2$

$\hat{T}_{M_{NVB}}(u_k) \ge [\alpha_1,\alpha_2] \quad ; \quad \hat{I}_{M_{NVB}}(u_k) \le [\beta_1,\beta_2] \quad ; \quad \hat{F}_{M_{NVB}}(u_k) \le [\gamma_1,\gamma_2]$

$\hat{T}_{M_{NVB}}(u_k) \ge [\delta_1,\delta_2] \quad ; \quad \hat{I}_{M_{NVB}}(u_k) \le [\rho_1,\rho_2] \quad ; \quad \hat{F}_{M_{NVB}}(u_k) \le [\vartheta_1,\vartheta_2]$

$\hat{T}_{M_{NVB}}(u_k) \ge [\chi_1,\chi_2] \quad ; \quad \hat{I}_{M_{NVB}}(u_k) \le [\Phi_1,\Phi_2] \quad ; \quad \hat{F}_{M_{NVB}}(u_k) \le [\Pi_1,\Pi_2]$

$T^-(u_k) \ge \alpha_1 \;\&\; T^+(u_k) \ge \alpha_2 \;;\; I^-(u_k) \le \beta_1 \;;\&\; I^+(u_k) \le \beta_2 \;;F^-(u_k) \le \gamma_1 \;\&\; F^+(u_k) \le \gamma_2$

$T^-(u_k) \ge \delta_1 \;\&\; T^+(u_k) \ge \delta_2 \;;\; I^-(u_k) \le \rho_1 \;\&\; I^+(u_k) \le \rho_2 \;;\; F^-(u_k) \le \vartheta_1 \;\&\; F^+(u_k) \le \vartheta_2$

$T^-(u_k) \ge \chi_1 \;\&\; T^+(u_k) \ge \chi_2 \;;\; I^-(u_k) \le \Phi_1 \& I^+(u_k) \le \Phi_2 \;;\; F^-(u_k) \le \Pi_1 \;\&\; F^+(u_k) \le \Pi_2$

Remark 5.1.10.

1. (a) $M_{NVB}\,([0,0],[1,1],[1,1]) = U$

   (b) $M_{NVB}\,\langle([0,0],[1,1],[1,1]),([0,0],[1,1],[1,1])\rangle = U = \{U_1 \cup U_2\}$

2. *If* $[\alpha_1,\alpha_2]$ *and* $[\delta_1,\delta_2]$ *coincides ;* $[\beta_1,\beta_2]$ *and* $[\rho_1,\rho_2]$ *coincides ;* $[\gamma_1,\gamma_2]$ *and* $[\vartheta_1,\vartheta_2]$ *coincides, then* $([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]),([\delta_1,\delta_2],[\rho_1,\rho_2],[\vartheta_1,\vartheta_2])$ *- cuts are called* $([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2])$ *- cuts and is denoted by* $M_{NVB_{([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2])}}$ *instead of* $M_{NVB_{([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]),([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2])}}$

3. *If* $\langle([\alpha_1^*,\alpha_2^*],[\beta_1^*,\beta_2^*],[\gamma_1^*,\gamma_2^*]),([\delta_1^*,\delta_2^*],[\rho_1^*,\rho_2^*],[\vartheta_1^*,\vartheta_2^*])\rangle \ge$ $\langle([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]),([\delta_1,\delta_2],[\rho_1,\rho_2],[\vartheta_1,\vartheta_2])\rangle$ *then* $M_{NVB_{([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]),([\delta_1,\delta_2],[\rho_1,\rho_2],[\vartheta_1,\vartheta_2])}} \subseteq M_{NVB_{([\alpha_1^*,\alpha_2^*],[\beta_1^*,\beta_2^*],[\gamma_1^*,\gamma_2^*]),([\delta_1^*,\delta_2^*],[\rho_1^*,\rho_2^*],[\vartheta_1^*,\vartheta_2^*])}}$

Application

In this section theoretical application of NVB BCK/BCI - subalgebra is developed. Various theorems and propositions are found good to this concept.

Theorem 5.1.11.

Every NVB BCI - subalgebra $\mathcal{B}^{BCI}_{M_{NVB}}$ of a BCI - algebra $U^{\mathcal{B}^{BCI}_{M_{NVB}}}$ satisfies :
$NVB_{M_{NVB}}(0) \succeq NVB_{M_{NVB}}(u_k); \quad \forall \quad u_k \in U = \{U_1 \cap U_2\}$

Proof.

For a $\mathcal{B}^{BCI}_{M_{NVB}}$, underlying BCI - algebraic structure satisfies, $(u_k \star u_k) = 0 \quad ; \forall u_k \in U$

[By property 1(iii) of definition 5.1.1]

$\Rightarrow NVB_{M_{NVB}}(0)$
$= NVB_{M_{NVB}}(u_k \star u_k) \succeq rmin\{NVB_{M_{NVB}}(u_k), NVB_{M_{NVB}}(u_k)\}; \forall u_k \in U$
$= NVB_{M_{NVB}}(u_k)$[By definition 5.1.1]

Theorem 5.1.12.

Every $\mathcal{B}^{BCK}_{M_{NVB}}$ satisfies $NVB_{M_{NVB}}(0) \succeq NVB_{M_{NVB}}(u_k); \quad \forall \quad u_k \in U$

Proof.

For a $\mathcal{B}^{BCK}_{M_{NVB}}$, underlying BCK - algebraic structure satisfies, an additional condition, $(0 \star u_k) = 0, \forall u_k \in U$ besides $(u_k \star u_k) = 0; \forall u_k \in U$. By theorem 5.1.11, we get,
$NVB_{M_{NVB}}(0) \succeq NVB_{M_{NVB}}(u_k); \forall u_k \in U$ &
$NVB_{M_{NVB}}(0) = NVB_{M_{NVB}}(0 \star u_k) \succeq rmin\{NVB_{M_{NVB}}(0), NVB_{M_{NVB}}(u_k)\}$
$\Rightarrow NVB_{M_{NVB}}(0) \succeq rmin\{NVB_{M_{NVB}}(0), NVB_{M_{NVB}}(u_k)\}; \forall u_k \in U$, for $\mathcal{B}^{BCK}_{M_{NVB}}$&
$rmin\{NVB_{M_{NVB}}(0), NVB_{M_{NVB}}(u_k)\}$ will depend upon the given NVBS M_{NVB}
$\Rightarrow NVB_{M_{NVB}}(0) \succeq NVB_{M_{NVB}}(u_k) \qquad$ &
$NVB_{M_{NVB}}(0) \succeq rmin\{NVB_{M_{NVB}}(0), NVB_{M_{NVB}}(u_k)\}$
Even if $rmin\{NVB_{M_{NVB}}(0), NVB_{M_{NVB}}(u_k)\}$ will depend upon the given NVBS, using theorem 5.1.11 $NVB_{M_{NVB}}(0) \succeq rmin\{NVB_{M_{NVB}}(0), NVB_{M_{NVB}}(u_k)\}$ will become $NVB_{M_{NVB}}(0) \succeq NVB_{M_{NVB}}(u_k) \Rightarrow NVB_{M_{NVB}}(0) \succeq NVB_{M_{NVB}}(u_k)$ and $NVB_{M_{NVB}}(0) \succeq NVB_{M_{NVB}}(u_k); \forall u_k \in U$. So, combining both, for a $\mathcal{B}^{BCK}_{M_{NVB}}$ too, $NVB_{M_{NVB}}(0) \succeq NVB_{M_{NVB}}(u_k); \forall u_k \in U$

Remark 5.1.13.

Every $\mathcal{B}^{BCK}_{M_{NVB}}/\mathcal{B}^{BCI}_{M_{NVB}}$ satisfies : $NVB_{M_{NVB}}(0) \succeq NVB_{M_{NVB}}(u_k); \forall u_k \in U$. i.e., Every NVB BCK/BCI - algebra satisfies : $NVB_{M_{NVB}}(0) \succeq NVB_{M_{NVB}}(u_k); \forall u_k \in U$

Theorem 5.1.14.

Every $\mathcal{B}^{BCK}_{M_{NVB}}$ is a $\mathcal{B}^{BCI}_{M_{NVB}}$. But converse not true, generally. i.e., every $\mathcal{B}^{BCI}_{M_{NVB}}$ is not a $\mathcal{B}^{BCK}_{M_{NVB}}$ generally.

Proof.

For a fixed universal set U, underlying BCK - algebraic structure of $\mathcal{B}^{BCK}_{M_{NVB}}$ consists the underlying BCI - structure of $\mathcal{B}^{BCI}_{M_{NVB}} \Rightarrow$ Every $\mathcal{B}^{BCK}_{M_{NVB}}$ is $\mathcal{B}^{BCI}_{M_{NVB}}$. But converse does not hold.

Remark 5.1.15.

Following example illustrates both the cases:

Let $U_1 = \{0\}$ and let $U_2 = \{0, u_1\}$ be the universes under consideration. Combined universe $U = \{U_1 \cup U_2\} = \{0, u_1\}$ with $(U_1 \cap U_2) = \{0\}$

$\therefore$ Cayley Table 5.3 indicates binary operation $\star$ for U

Clearly, $U^{\mathcal{B}^{BCI}_{M_{NVB}}} = (U = \{U_1 \cup U_2\}, \star, 0)$ is a BCI - algebra [Table 5.3]

Table 5.3: BCI-algebra

$*$	0	u_1
0	0	u_1
u_1	u_1	0

Table 5.4: BCK/BCI-algebra

$*$	0	u_1
0	0	0
u_1	u_1	0

$U^{\mathcal{B}^{BCK}_{M_{NVB}}} = (U = \{U_1 \cup U_2\}, \star, 0)$ is a BCK - algebra [Table 5.4]

Case(i): Example for a $\mathcal{B}^{BCI}_{M_{NVB}}$ which is a $\mathcal{B}^{BCK}_{M_{NVB}}$

Let M_{NVB} be a non-empty NVBS with $U^{\mathcal{B}^{BCI}_{M_{NVB}}}$ as underlying algebraic structure:

$M_{NVB} =$

$$\left\{ \left\langle \frac{[0.1, 0.8], [0.1, 0.5], [0.2, 0.9]}{0} \right\rangle, \left\langle \frac{[0.3, 0.7], [0.2, 0.4], [0.3, 0.7]}{0}, \frac{[0.1, 0.4], [0.3, 0.5], [0.6, 0.9]}{u_1} \right\rangle \right\}$$

Here, $(U_1 \cap U_2) = \{0\}$

$NVB_{M_{NVB}}(0) = ([0.1, 0.8], [0.1, 0.5], [0.2, 0.9]) \cap ([0.3, 0.7], [0.2, 0.4], [0.3, 0.7])$

$= [0.3, 0.8], [0.1, 0.4], [0.2, 0.7]$

After verification, clearly M_{NVB} *is a* $\mathcal{B}^{BCI}_{M_{NVB}}$. *Next question is that,- "whether* $\mathcal{B}^{BCI}_{M_{NVB}}$ *is a* $\mathcal{B}^{BCK}_{M_{NVB}}$ *or "not ?" . Additional condition to be satisfied, for a BCK- algebra is,* $(0 \star u_1) = 0$ *from Cayley Table 5.3. Correspondingly,*

$NVB_{M_{NVB}}(0 \star u_1) \succcurlyeq rmin\{NVB_{M_{NVB}}(0), NVB_{M_{NVB}}(u_1)\}$

$\Rightarrow NVB_{M_{NVB}}(0) \succcurlyeq rmin\{NVB_{M_{NVB}}(0), NVB_{M_{NVB}}(u_1)\}$

$\Rightarrow [0.3, 0.8], [0.1, 0.4], [0.2, 0.7]$

$\succcurlyeq rmin\{([0.3, 0.8], [0.1, 0.4], [0.2, 0.7]), ([0.1, 0.4], [0.3, 0.5], [0.6, 0.9])\}$

$\Rightarrow [0.3, 0.8], [0.1, 0.4], [0.2, 0.7] \succcurlyeq [0.1, 0.4], [0.3, 0.5], [0.6, 0.9]$

Since additional condition got satisfied, $\mathcal{B}^{BCI}_{M_{NVB}}$ *is clearly a* $\mathcal{B}^{BCK}_{M_{NVB}}$

Case (ii) : Example for a $\mathcal{B}^{BCI}_{M_{NVB}}$ ***which is not a*** $\mathcal{B}^{BCK}_{M_{NVB}}$

Take binary operation and Cayley Table as taken in Case (i)

Consider another NVBS P_{NVB} *with same conditions as in case (i)*

$P_{NVB} =$

$$\left\{\left\langle\frac{[0.1, 0.5], [0.2, 0.5], [0.5, 0.9]}{0}\right\rangle, \left\langle\frac{[0.1, 0.6], [0.3, 0.3], [0.4, 0.9]}{0}, \frac{[0.1, 0.7], [0.3, 0.4], [0.3, 0.9]}{u_1}\right\rangle\right\}$$

$NVB_{P_{NVB}}(0) = ([0.1, 0.5], [0.2, 0.5], [0.5, 0.9]) \cup ([0.1, 0.6], [0.3, 0.3], [0.4, 0.9])$

$= [0.1, 0.6], [0.2, 0.3], [0.4, 0.9]$

By verification P_{NVB} *is a* $\mathcal{B}^{BCI}_{P_{NVB}}$. *But in this case, additional condition not got satisfied.* $NVB_{P_{NVB}}(0*1) \not\succcurlyeq rmin\{([0.1, 0.8], [0.2, 0.3], [0.4, 0.9]), ([0.1, 0.7], [0.3, 0.4], [0.3, 0.9])\}$. *Since* $[0.1, 0.6], [0.2, 0.3], [0.4, 0.9] \not\succcurlyeq rmin\{([0.1, 0.8], [0.2, 0.3], [0.2, 0.9]), ([0.1, 0.7], [0.3, 0.4], [0.3, 0.9])\}$ *Since* $[0.1, 0.6], [0.2, 0.3], [0.4, 0.9] \not\succcurlyeq [0.1, 0.7], [0.3, 0.4], [0.3, 0.9]$. *In this case clearly,* $\mathcal{B}^{BCI}_{P_{NVB}}$ *is not a* $\mathcal{B}^{BCK}_{P_{NVB}}$

Theorem 5.1.16.

Intersection of two NVB BCK/BCI - algebra remains as a NVB BCK/BCI- algebra.

Proof.

Let M_{NVB} and P_{NVB} be two NVB BCK/BCI - algebras with structures $\mathcal{B}_{M_{NVB}} = \left(U^{\mathcal{B}_{M_{NVB}}}, \star, 0\right)$ and $\mathcal{B}_{P_{NVB}} = \left(U^{\mathcal{B}_{P_{NVB}}}, \star, 0\right)$ respectively, with same universal sets U_1 and U_2. So, $\forall u_1, u_2 \in U$,

$NVB_{(M_{NVB} \cap P_{NVB})}(u_1 \star u_2) = rmin\{NVB_{M_{NVB}}(u_1 \star u_2), NVB_{P_{NVB}}(u_1 \star u_2)\}$

$\succeq rmin\{rmin\{NVB_{M_{NVB}}(u_1), NVB_{M_{NVB}}(u_2)\}, rmin\{NVB_{P_{NVB}}(u_1), NVB_{P_{NVB}}(u_2)\}\}$

$= rmin \left\{NVB_{(M_{NVB} \cap P_{NVB}}(u_1), NVB_{M_{NVB} \cap P_{NVB}}(u_2)\right\} .Therefore,$
$NVB_{(M_{NVB} \cap P_{NVB})}(u_1 \star u_2) \succeq rmin\{NVB_{M_{NVB} \cap P_{NVB}}(u_1), NVB_{M_{NVB} \cap P_{NVB}}(u_2)\}$
$\Rightarrow (M_{NVB} \cap P_{NVB})$ is also a NVB BCK/BCI - algebra.

Theorem 5.1.17.

Every NVB BCI - ideal P_{NVB} of a $\mathcal{B}^{BCI}_{M_{NVB}}$ satisfies: $\forall u_a, u_b, u_c \in U$

1. $u_a \leq u_b \Rightarrow NVB_{P_{NVB}}(u_a) \succeq NVB_{P_{NVB}}(u_b); (\forall u_a, u_b \in U)$
2. $NVB_{P_{NVB}}(u_a \star u_c) \succeq rmin\{NVB_{P_{NVB}}((u_a \star u_b) \star u_c), NVB_{P_{NVB}}(u_b)\}$

Proof.

1 Let $u_a, u_b \in U$ be such that $u_a \leq u_b$. Since P_{NVB} is a NVB BCI - ideal of $\mathcal{B}^{BCI}_{M_{NVB}} \Rightarrow NVB_{P_{NVB}}(u_a) \succeq rmin\{NVB_{P_{NVB}}(u_a \star u_b), NVB_{P_{NVB}}(u_b)\}$, [By condition (2) of definition 5.1.5] $= rmin\{NVB_{P_{NVB}}(0), NVB_{P_{NVB}}(u_b)\}$, [By taking $(u_a \star u_b) = 0$] $= NVB_{P_{NVB}}(u_b)$[By theorem 5.1.11] $\Rightarrow NVB_{P_{NVB}}(u_a) \succeq NVB_{P_{NVB}}(u_b)$

2. Let P_{NVB} be a NVB BCI - ideal of $\mathcal{B}^{BCI}_{M_{NVB}}$
$\Rightarrow NVB_{P_{NVB}}(u_a) \succeq rmin\{NVB_{P_{NVB}}(u_a \star u_b), NVB_{P_{NVB}}(u_b)\}; \forall u_a, u_b \in U$
$\Rightarrow NVB_{P_{NVB}}(u_a \star u_c) \succeq rmin\{NVB_{P_{NVB}}((u_a \star u_c) \star u_b), NVB_{P_{NVB}}(u_b)\};$
$[by \quad putting \quad u_a = (u_a \star u_c); \forall u_a, u_b, u_c \in U]$
$\Rightarrow NVB_{P_{NVB}}(u_a \star u_c) \succeq rmin\{NVB_{P_{NVB}}((u_a \star u_b) \star u_c), NVB_{P_{NVB}}(u_b)\}.$

Theorem 5.1.18.

Let P_{NVB} be a NVB BCI-ideal of $\mathcal{B}_{M^{BCI}_{NVB}}$.
Then, $NVB_{P_{NVB}}(0 \star (0 \star u_k)) \succeq NVB_{P_{NVB}}(u_k); \forall u_k \in U$

Proof.

$NVB_{P_{NVB}}(u_a) \succeq rmin\{NVB_{P_{NVB}}(u_a \star u_b), NVB_{P_{NVB}}(u_b)\}$; for any $u_a, u_b \in U$ [By definition 5.1.8] Let $u_a = (0 \star (0 \star u_k))$ and $u_b = u_k$
$\therefore$For any $u_k \in U$,
$NVB_{P_{NVB}}(0 \star (0 \star u_k)) \succeq rmin\{NVB_{P_{NVB}}((0 \star (0 \star u_k)) \star u_k), NVB_{P_{NVB}}(u_k)\}$
$= rmin\{NVB_{P_{NVB}}((0 \star u_k) \star (0 \star u_k)), NVB_{P_{NVB}}(u_k)\}$
$= rmin\{NVB_{P_{NVB}}(0 \star (u_k \star u_k)), NVB_{P_{NVB}}(u_k)\}$
$= rmin\{NVB_{P_{NVB}}(0 \star 0), NVB_{P_{NVB}}(u_k)\}$
$= rmin\{NVB_{P_{NVB}}(0), NVB_{P_{NVB}}(u_k)\} = NVB_{P_{NVB}}(u_k)$ [By theorem 5.1.11]
$\therefore$ It is concluded that, $NVB_{M_{NVB}}(0 \star (0 \star u_k)) \succeq NVB_{M_{NVB}}(u_k); \forall u_k \in U$

Theorem 5.1.19.

If the NVBS R_{NVB} *of* $\mathcal{B}^{BCI}_{M_{NVB}}$ *is a NVB BCI - subalgebra then it satisfies*
$(u_a \star u_b) \leq u_c \Rightarrow NVB_{R_{NVB}}(u_a) \succeq rmin\{NVB_{R_{NVB}}(u_b), NVB_{R_{NVB}}(u_c)\}$
for any $u_x, u_y, u_z \in U$.

Proof.
Let R_{NVB} be a NVBS of $\mathcal{B}^{BCI}_{M_{NVB}}$ with $(u_a \star u_b) \leq u_c$
$\Rightarrow NVB_{R_{NVB}}(u_c) \succeq NVB_{R_{NVB}}(u_a \star u_b)$ R_{NVB} is a $\mathcal{B}^{BCI}_{R_{NVB}}$
$\Rightarrow NVB_{R_{NVB}}(u_a \star u_b) \succeq rmin\{NVB_{R_{NVB}}(u_a), NVB_{R_{NVB}}(u_b)\}$
$\Rightarrow NVB_{R_{NVB}}(u_c) \succeq NVB_{R_{NVB}}(u_a \star u_b) \succeq rmin\{NVB_{R_{NVB}}(u_a), NVB_{R_{NVB}}(u_b)\}$
$\Rightarrow NVB_{R_{NVB}}(u_c) \succeq rmin\{NVB_{R_{NVB}}(u_a), NVB_{M_{NVB}}(u_b)\}$
$\Rightarrow NVB_{R_{NVB}}(u_a) \succeq rmin\{NVB_{R_{NVB}}(u_c), NVB_{R_{NVB}}(u_b)\}$;
[By putting $u_c = u_a$ & $u_a = u_c$]
$\Rightarrow NVB_{R_{NVB}}(u_a) \succeq rmin\{NVB_{R_{NVB}}(u_b), NVB_{R_{NVB}}(u_c)\}$;

Theorem 5.1.20.

Let S_{NVB} *be both a NVB BCI sub algebra* $\mathcal{B}^{BCI}_{S_{NVB}}$ *and a NVB BCI-ideal of a NVB BCI - sub algebra* $\mathcal{B}^{BCI}_{S_{NVB}}$. *Then* $NVB_{S_{NVB}}(0 \star u_k) \succeq NVB_{S_{NVB}}(u_k)$ *for all* $u_k \in U$

Proof.
Let S_{NVB} be a NVB BCI- subalgebra $\mathcal{B}^{BCI}_{S_{NVB}}$
$\Rightarrow NVB_{S_{NVB}}(u_a \star u_b) \succeq rmin\{NVB_{S_{NVB}}(u_a), NVB_{S_{NVB}}(u_b)\}$; for all $u_a, u_b \in U$
$\Rightarrow NVB_{S_{NVB}}(0 \star u_b) \succeq rmin\{NVB_{S_{NVB}}(0), NVB_{S_{NVB}}(u_b)\}$; [By putting $u_a = 0$]
$\Rightarrow NVB_{S_{NVB}}(0 \star u_b) \succeq NVB_{S_{NVB}}(u_b)$
$\Rightarrow NVB_{S_{NVB}}(0 \star u_k) \succeq NVB_{S_{NVB}}(u_k)$ [By putting $u_b = u_k$]
$\therefore$ For any $u_k \in U, NVB_{S_{NVB}}(0 \star u_k) \succeq NVB_{S_{NVB}}(u_k)$

Theorem 5.1.21.

Let T_{NVB} *be a NVB BCI - ideal of a NVB BCI -sub algebra* $\mathcal{B}^{BCI}_{M_{NVB}}$.$\forall\ u_a, u_b, u_c \in U$, *if* T_{NVB} *satisfies,* $NVB_{T_{NVB}}(u_a \star u_b) \succeq NVB_{T_{NVB}}((u_a \star u_c) \star (u_b \star u_c))$ *then* T_{NVB} *is a NVB BCI p - ideal of* $\mathcal{B}^{BCI}_{M_{NVB}}$

Proof.
T_{NVB} be a NVB BCI - ideal of a NVB BCI - sub algebra $\mathcal{B}^{BCI}_{M_{NVB}}$. for all $u_a, u_b, u_c \in U$
$\Rightarrow NVB_{T_{NVB}}(u_a) \succeq rmin\{NVB_{T_{NVB}}(u_a \star u_b), NVB_{T_{NVB}}(u_b)\}$
$\Rightarrow NVB_{T_{NVB}}(u_a) \succeq rmin\{NVB_{T_{NVB}}((u_a \star u_c) \star (u_b \star u_c)), NVB_{M_{NVB}}(u_b)\}$ for all $u_a, u_b, u_c \in U$ [From given condition]
$\Rightarrow T_{NVB}$ is a NVB BCI p ideal of $\mathcal{B}^{BCI}_{M_{NVB}}$ [By definition 5.1.8]

Theorem 5.1.22.

Any NVB BCI - ideal D_{NVB} of a NVB BCI -subalgebra $\mathcal{B}^{BCI}_{M_{NVB}}$ is a NVB BCI -p ideal $\Leftrightarrow NVB_{D_{NVB}}(u_a) \succeq NVB_{D_{NVB}}(0 \star (0 \star u_a))$; for all $u_a \in U$

Proof.

Let D_{NVB} be a NVB BCI - ideal of a NVB BCI - subalgebra $\mathcal{B}^{BCI}_{M_{NVB}}$.

Also let D_{NVB} is a NVB BCI -p ideal.

$\therefore NVB_{D_{NVB}}(u_a) \succeq rmin\{NVB_{D_{NVB}}((u_a \star u_c) \star (u_b \star u_c)), NVB_{D_{NVB}}(u_b)\}$

for all $u_a, u_b, u_c \in U$ [By definition 5.1.8 of NVB BCI- p ideal]

Put $u_c = u_a$ and $u_b = 0$ in the above,

$\therefore NVB_{D_{NVB}}(u_a) \succeq rmin\{NVB_{D_{NVB}}((u_a \star u_a) \star (0 \star u_a)), NVB_{D_{NVB}}(0)\}$

for all $u_a, u_b, u_c \in U$

$\Rightarrow NVB_{D_{NVB}}(u_a) \succeq rmin\{NVB_{D_{NVB}}(0 \star (0 \star u_a)), NVB_{D_{NVB}}(0)\}$

for all $u_a, u_b \in U$

$= NVB_{D_{NVB}}(0 \star (0 \star u_a))$ for all $u_a \in U$ [By theorem 5.1.18]

$\Rightarrow NVB_{D_{NVB}}(u_a) \succeq NVB_{D_{NVB}}(0 \star (0 \star u_a))$; for all $u_a \in U$

Conversely, let a NVB BCI - ideal D_{NVB} of a NVB BCI - subalgebra $\mathcal{B}^{BCI}_{M_{NVB}}$ satisfies the given condition,

$NVB_{D_{NVB}}(u_a) \succeq NVB_{M_{NVB}}(0 \star (0 \star u_a))$; for all $u_a \in U$.

By theorem 5.1.18, " Let P_{NVB} be a NVB BCI-ideal of $\mathcal{B}^{BCI}_{M_{NVB}}$.

Then, $NVB_{P_{NVB}}(0 \star (0 \star u_k)) \succeq NVB_{P_{NVB}}(u_k); \forall u_k \in U$"

$\Rightarrow NVB_{D_{NVB}}((u_a \star u_c) \star (u_b \star u_c)) \preceq NVB_{M_{NVB}}(0 \star (0 \star ((u_a \star u_c) \star (u_b \star u_c))))$

[By putting $u_k = (u_a \star u_c) \star (u_b \star u_c)$ in theorem 5.1.18]

$= NVB_{D_{NVB}}((0 \star u_b) \star (0 \star u_a))$

$= NVB_{D_{NVB}}(0 \star (0 \star (u_a \star u_b)))$;

$= NVB_{D_{NVB}}(0 \star (u_a \star u_b))$;

$= NVB_{D_{NVB}}(u_a \star u_b)$;

$\Rightarrow NVB_{D_{NVB}}(u_a \star u_b) \succeq NVB_{D_{NVB}}((u_a \star u_c) \star (u_b \star u_c))$

$\Rightarrow NVB_{D_{NVB}}$ is a NVB BCI - p ideal[By theorem 5.1.21]

Theorem 5.1.23.

Every NVB BCI - p ideal of a NVB BCI - subalgebra $\mathcal{B}^{BCI}_{M_{NVB}}$ is a NVB BCI - ideal of $\mathcal{B}^{BCI}_{M_{NVB}}$.

Proof.

Let M_{NVB} be a NVB BCI - p ideal of a NVB BCI - subalgebra $\mathcal{B}^{BCI}_{M_{NVB}}$. By definition,

$NVB_{M_{NVB}}(u_x) \succeq rmin\left\{NVB_{M_{NVB}}\left((u_x \star u_z) \star (u_y \star u_z)\right), NVB_{M_{NVB}}(u_y)\right\}$
for all $u_x, u_y, u_z \in U$. Put $u_z = 0$ then, $\forall u_x, u_y \in U$, the above becomes,
$NVB_{M_{NVB}}(u_x) \succeq rmin\left\{NVB_{M_{NVB}}\left((u_x \star 0) \star (u_y \star 0)\right), NVB_{M_{NVB}}(u_y)\right\}$
$= rmin\left\{NVB_{M_{NVB}}\left(u_x \star u_y\right), NVB_{M_{NVB}}(u_y)\right\}$ for all $u_x, u_y \in U$
$\Rightarrow NVB_{M_{NVB}}(u_x) \succeq rmin\left\{NVB_{M_{NVB}}\left(u_x \star u_y\right), NVB_{M_{NVB}}(u_y)\right\}$ for all $u_x, u_y \in U$
Obviously, M_{NVB} is a NVB BCI - ideal.

Theorem 5.1.24.

Every NVB BCK/BCI H - ideal of a NVB BCK/BCI - subalgebra $\mathcal{B}_{M_{NVB}}$ acts both as

1. *NVB BCK/BCI - ideal of $\mathcal{B}_{M_{NVB}}$*
2. *NVB BCK/BCI - subalgebra $\mathcal{B}_{M_{NVB}}$*

Proof.

Let I_{NVB} be a NVB BCK/BCI- H ideal of a NVB BCK/BCI subalgebra $\mathcal{B}_{M_{NVB}}$

1. From definition of NVB BCK/BCI- H ideal,
   $NVB_{I_{NVB}}\left(u_a \star u_c\right) \succeq min\left\{NVB_{I_{NVB}}\left(u_a \star (u_b \star u_c)\right), NVB_{I_{NVB}}(u_b)\right\}$
   for all $u_a, u_b, u_c \in U$
   Put $u_c = 0$. Then for all $u_a, u_b, u_c \in U$
   $NVB_{I_{NVB}}\left(u_a \star 0\right) \succeq min\left\{NVB_{I_{NVB}}\left(u_a \star (u_b \star 0)\right), NVB_{M_{NVB}}(u_b)\right\}$
   $\Rightarrow NVB_{I_{NVB}}(u_a) \succeq min\left\{NVB_{I_{NVB}}\left(u_a \star u_b\right), NVB_{M_{NVB}}(u_b)\right\}$
   Since I_{NVB} is a NVB BCK/BCI- H ideal
   $\Rightarrow NVB_{I_{NVB}}(0) \succeq NVB_{I_{NVB}}(u_k)$; for any $u_k \in U$
   $\therefore I_{NVB}$ is a NVB BCK/BCI - ideal of $\mathcal{B}_{M_{NVB}}$
2. Let I_{NVB} be a NVB BCK/BCI- H ideal of $\mathcal{B}_{M_{NVB}}$. for all $u_a, u_b, u_c \in U$
   $\therefore NVB_{I_{NVB}}\left(u_a \star u_c\right) \succeq rmin\left\{NVB_{I_{NVB}}\left(u_a \star (u_b \star u_c)\right), NVB_{I_{NVB}}(u_b)\right\}$
   $\Rightarrow \ NVB_{I_{NVB}}\left(u_a \star u_b\right) \ \succeq \ rmin\left\{NVB_{I_{NVB}}\left(u_a \star (u_b \star u_b)\right), NVB_{I_{NVB}}(u_b)\right\}$;
   [By putting $u_c = u_b$]
   $\Rightarrow NVB_{I_{NVB}}\left(u_a \star u_b\right) \succeq rmin\left\{NVB_{I_{NVB}}\left(u_a \star 0\right), NVB_{I_{NVB}}(u_b)\right\}$;
   $\Rightarrow NVB_{I_{NVB}}\left(u_a \star u_b\right) \succeq rmin\left\{NVB_(I_NVB)(u_a), NVB_(I_NVB)(u_b)\right\}$;
   $\Rightarrow I_{NVB}$ be a NVB BCK/BCI -subalgebra of $\mathcal{B}_{M_{NVB}}$

Theorem 5.1.25.

P_{NVB} be a NVBS of a NVB BCK/BCI - subalgebra $\mathcal{B}_{M_{NVB}}$. Then P_{NVB} is a NVB BCK/BCI -ideal of $\mathcal{B}_{M_{NVB}} \Leftrightarrow$ it satisfies the following conditions: $\forall u_a, u_m, u_n \in U$

1. $NVB_{P_{NVB}}(u_a \star u_b) \succeq NVB_{P_{NVB}}(u_b)$
2. $NVB_{P_{NVB}}(u_a \star ((u_a \star u_m) \star u_n)) \succeq rmin\{NVB_{P_{NVB}}(u_m), NVB_{P_{NVB}}(u_n)\}$

Proof.

Let P_{NVB} be NVB BCK/BCI - ideal of $\mathcal{B}_{M_{NVB}}$. By definition,
$NVB_{P_{NVB}}(u_a) \succeq rmin\{NVB_{P_{NVB}}(u_a \star u_b), NVB_{P_{NVB}}(u_b)\}; \forall u_a, u_b \in U$

1. Put $u_a = (u_a \star u_b)$ and $u_b = u_a$ in the above,
   $NVB_{P_{NVB}}(u_a \star u_b) \succeq rmin\{NVB_{P_{NVB}}((u_a \star u_b) \star u_a), NVB_{P_{NVB}}(u_a)\}$
   $\Rightarrow NVB_{P_{NVB}}(u_a \star u_b) \succeq rmin\{NVB_{P_{NVB}}((u_a \star u_a) \star u_b), NVB_{P_{NVB}}(u_a)\}$
   $\Rightarrow NVB_{P_{NVB}}(u_a \star u_b) \succeq rmin\{NVB_{P_{NVB}}(0 \star u_b), NVB_{P_{NVB}}(u_a)\}$
   $\Rightarrow NVB_{P_{NVB}}(u_a \star u_b) \succeq rmin\{NVB_{P_{NVB}}(0 \star u_b), NVB_{P_{NVB}}(u_b)\}$
   [By assumption $u_a = u_b$]
   $\Rightarrow NVB_{P_{NVB}}(u_a \star u_b) \succeq rmin\{NVB_{P_{NVB}}(0), NVB_{P_{NVB}}(u_b)\}$
   $\Rightarrow NVB_{P_{NVB}}(u_a \star u_b) \succeq NVB_{P_{NVB}}(u_b)$

2. Consider, $(u_a \star ((u_a \star u_m) \star u_n)) \star u_m = (u_a \star u_m) \star ((u_a \star u_m) \star u_n) \leq u_n$
   We have, $(u_x \star (u_x \star u_y)) \star u_y = 0 \Rightarrow u_x \star (u_x \star y) \leq u_y$.
   Here, $(u_a \star ((u_a \star u_m) \star u_n)) \star u_m = (u_a \star u_m) \star ((u_a \star u_m) \star u_n)$
   $= ((u_a \star u_m) \star (u_a \star u_m)) \star u_n = (0 \star u_n) = 0.$
   [Since $(u_a \star u_m) \star ((u_a \star u_m) \star u_n) = 0$, $(u_a \star u_m) \star ((u_a \star u_m) \star u_n) \leq u_n$]
   $\therefore$ Above can be written as, $(u_a \star ((u_a \star u_m) \star u_n)) \star u_m \leq u_n$
   $\Rightarrow NVB_{P_{NVB}}((u_a \star ((u_a \star u_m) \star u_n)) \star u_m) \succeq NVB_{P_{NVB}}(u_n)$ [By theorem 5.1.17(1)]
   P_{NVB} is a NVB BCK/BCI -ideal of $\mathcal{B}_{M_{NVB}}$
   $\Rightarrow NVB_{P_{NVB}}(u_a) \succeq rmin\{NVB_{P_{NVB}}(u_a \star u_b), NVB_{P_{NVB}}(u_b)\}$
   Put $u_a = (u_a \star ((u_a \star u_m) \star u_n))$ & $u_b = u_m$ in above,
   $NVB_{P_{NVB}}(u_a \star ((u_a \star u_m) \star u_n))$
   $\succeq rmin\{NVB_{P_{NVB}}((u_a \star ((u_a \star u_m) \star u_n)) \star u_m), NVB_{P_{NVB}}(u_m)\}$
   $= rmin\{NVB_{P_{NVB}}(u_n), NVB_{P_{NVB}}(u_m)\}$ [proved above]
   $\succeq rmin\{NVB_{P_{NVB}}(u_m), NVB_{P_{NVB}}(u_n)\}$
   $\Rightarrow NVB_{M_{NVB}}(u_a \star ((u_a \star u_m) \star u_n)) \succeq rmin\{NVB_{P_{NVB}}(u_m), NVB_{P_{NVB}}(u_n)\}$
   ; $[\forall u_a, u_m, u_n \in U]$

Conversely, Let P_{NVB} be a NVBS of a NVB BCK/BCI subalgebra $\mathcal{B}_{M_{NVB}}$ satisfying, the given conditions,

$NVB_{P_{NVB}}(u_a \star u_b) \succeq NVB_{P_{NVB}}(u_a); [\forall u_a, u_b \in U\ ]$

$NVB_{P_{NVB}}(u_a \star ((u_a \star u_m) \star u_n)) \succeq rmin\{NVB_{P_{NVB}}(u_m), NVB_{P_{NVB}}(u_n)\};$ $[\forall u_a, u_m, u_n \in U]$

To prove condition (1) of a NVB BCK/BCI - ideal,

Take $u_b = u_a$ in (i) and (ii) respectively,

(i) $\Rightarrow NVB_{P_{NVB}}(u_a \star u_a) \succeq NVB_{P_{NVB}}(u_a)$

$\Rightarrow NVB_{P_{NVB}}(0) \succeq NVB_{P_{NVB}}(u_a)$; [By property 1(iii)of definition 5.1.1]

To prove condition (2) of a NVB BCK/BCI - ideal,

Take, $NVB_{P_{NVB}}(u_a) = NVB_{P_{NVB}}(u_a \star 0)$

$= NVB_{P_{NVB}}(u_a \star ((u_a \star u_b) \star (u_a \star u_b)))$

$= NVB_{P_{NVB}}(u_a \star ((u_a \star (u_a \star u_b)) \star u_b))$

$= NVB_{P_{NVB}}(u_a \star ((u_a \star u_m) \star u_n))$; [By putting $(u_a \star u_b) = u_m$ and $u_b = u_n$]

$\succeq rmin\{NVB_{P_{NVB}}(u_m), NVB_{P_{NVB}}(u_n)\}$;[By condition(ii)in the assumption]

$= rmin\{NVB_{P_{NVB}}(u_a \star u_b), NVB_{P_{NVB}}(u_b)\}$;

[By putting $(u_a \star u_b) = u_m$ and $u_b = u_n$]

$\Rightarrow NVB_{P_{NVB}}(u_a) \succeq rmin\{NVB_{P_{NVB}}(u_a \star u_b), NVB_{P_{NVB}}(u_b)\}$

$\therefore P_{NVB}$ is a NVB BCK/BCI - ideal of $\mathcal{B}_{M_{NVB}}$

Theorem 5.1.26.

Let M_{NVB} be a NVB BCK/BCI - subalgebra $\mathcal{B}_{M_{NVB}}$. Then any NVB BCK/BCI - cut of M_{NVB} is a crisp NVB BCK/BCI - subalgebra of $\mathcal{B}_{M_{NVB}}$

Proof.

Let for any $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \rho_1, \rho_2, \vartheta_1, \vartheta_2 \in [0, 1]$,

$M_{NVB_{[\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2],[\delta_1,\delta_2],[\rho_1,\rho_2],[\vartheta_1,\vartheta_2]}}$ be a NVB BCK/BCI -cut of M_{NVB}.

Assume $u_x, u_y \in M_{NVB_{[\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2],[\delta_1,\delta_2],[\rho_1,\rho_2],[\vartheta_1,\vartheta_2]}}$

$$\Rightarrow NVB_{M_{NVB}}(u_x) \geq [\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2] \quad \& \quad NVB_{M_{NVB}}(u_x) \geq ([\delta_1, \delta_2], [\rho_1, \rho_2], [\vartheta_1, \vartheta_2])$$

$$NVB_{M_{NVB}}(u_y) \geq [\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2] \quad \& \quad NVB_{M_{NVB}}(u_y) \geq ([\delta_1, \delta_2], [\rho_1, \rho_2], [\vartheta_1, \vartheta_2])$$

$$\Rightarrow \hat{T}_{M_{NVB}}(u_x) \geq [\alpha_1, \alpha_2]; \hat{I}_{M_{NVB}}(u_x) \leq [\beta_1, \beta_2]; \hat{F}_{M_{NVB}}(u_x) \leq [\gamma_1, \gamma_2] \quad \&$$
$$\hat{T}_{M_{NVB}}(u_x) \geq [\delta_1, \delta_2]; \hat{I}_{M_{NVB}}(u_x) \leq [\rho_1, \rho_2]; \hat{F}_{M_{NVB}}(u_x) \leq [\vartheta_1, \vartheta_2] \quad \&$$
$$\hat{T}_{M_{NVB}}(u_y) \geq [\alpha_1, \alpha_2]; \hat{I}_{M_{NVB}}(u_y) \leq [\beta_1, \beta_2]; \hat{F}_{M_{NVB}}(u_y) \leq [\gamma_1, \gamma_2] \quad \&$$
$$\hat{T}_{M_{NVB}}(u_y) \geq [\delta_1, \delta_2]; \hat{I}_{M_{NVB}}(u_y) \leq [\rho_1, \rho_2]; \hat{F}_{M_{NVB}}(u_y) \leq [\vartheta_1, \vartheta_2]$$

M_{NVB} is a NVB BCK/BCI -subalgebra $\mathcal{B}_{M_{NVB}}$

$\Rightarrow NVB_{M_{NVB}}(u_x \star u_y) \succeq rmin\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\}$

$$\Rightarrow \begin{cases} \hat{T}_{M_{NVB}}(u_x \star u_y) \geq min\left\{\hat{T}_{M_{NVB}}(u_x), \hat{T}_{M_{NVB}}(u_y)\right\} \\ \hat{I}_{M_{NVB}}(u_x \star u_y) \leq max\left\{\hat{I}_{M_{NVB}}(u_x), \hat{I}_{M_{NVB}}(u_y)\right\} \\ \hat{F}_{M_{NVB}}(u_x \star u_y) \leq max\left\{\hat{F}_{M_{NVB}}(u_x), \hat{F}_{M_{NVB}}(u_y)\right\} \end{cases}$$

$$\Rightarrow \left\{ (u_x \star u_y) \in M_{NVB_{(([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]),([\delta_1,\delta_2],[\rho_1,\rho_2],[\vartheta_1,\vartheta_2]))}} \right.$$

$\Rightarrow$ NVB BCK/BCI - cut $M_{NVB_{([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]),([\delta_1,\delta_2],[\rho_1,\rho_2],[\vartheta_1,\vartheta_2])}}$ of M_{NVB} is a crisp NVB BCK/BCI - subalgebra of $\mathcal{B}_{M_{NVB}}$

5.2 Neutrosophic Vague Binary BZMV^{dM} sub-algebra of BZMV^{dM} algebra

$BZMV^{dM}$ -algebra is a combination of BZ and MV -algebra in demorgan concept zone. It is a logical algebra. Z -algebra and BZ-algebra are different, first one developed around 2017 only. But BZ- algebra is a concept developed long before that. In this section neutrosophic vague BZMV ^{dM} sub - algebra is developed for neutrosophic vague binary set theory.

Definition 5.2.1. ***(Neutrosophic Vague Binary BZMV^{dM} subalgebra)***
A neutrosophic vague binary BZMV$^{dM}-$ subalgebra (NVB BZMV^{dM} sub-algebra, in short) is a structure $\mathcal{M}_{M_{NVB}} = \left(U^{\mathcal{M}_{M_{NVB}}}, \oplus, \neg, \sim, 0\right)$ *with,* $\hat{T} = [T^-, T^+]; \hat{I} = [I^-, I^+]; \hat{F} = [F^-, F^+]$ *which satisfies, the following two* $\mathcal{M}_{M_{NVB}}$ *inequalities :*

$\mathcal{M}_{M_{NVB}}$ ***inequality(1):*** $\forall u_v, \quad u_x, u_y \quad \in U$

$$inf_{u_v \in (u_x \oplus u_y)} NVB_{M_{NVB}}(u_v)) \succeq rmin\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\}$$

$$inf_{u_v \in (u_x \oplus u_y)} \hat{T}_{M_{NVB}}(u_v) \geq min\left\{\hat{T}_{M_{NVB}}(u_x), \hat{T}_{M_{NVB}}(u_y)\right\}$$
$$inf_{u_v \in (u_x \oplus u_y)} \hat{I}_{M_{NVB}}(u_v) \leq max\left\{I_{M_{NVB}}(u_x), \hat{I}_{M_{NVB}}(u_y)\right\}$$
$$inf_{u_v \in (u_x \oplus u_y)} \hat{F}_{M_{NVB}}(u_v)) \leq max\left\{\hat{F}_{M_{NVB}}(u_x), \hat{F}_{M_{NVB}}(u_y)\right\}$$

$M_{M_{NVB}}$ ***inequality (2) :*** $\forall \quad u_x \in U$

1. $NVB_{M_{NVB}}(u_x) \succeq NVB_{M_{NVB}}(\neg u_x)$

$$\hat{T}_{M_{NVB}}(u_x) \geq \hat{T}_{M_{NVB}}(\neg u_x)$$
$$\hat{I}_{M_{NVB}}(u_x) \leq \hat{I}_{M_{NVB}}(\neg u_x)$$
$$\hat{F}_{M_{NVB}}(u_x) \leq \hat{F}_{M_{NVB}}(\neg u_x)$$

2. $NVB_{M_{NVB}}(u_x) \succeq NVB_{M_{NVB}}(\sim u_x)$

$$\hat{T}_{M_{NVB}}(u_x) \geq \hat{T}_{M_{NVB}}(\sim u_x)$$
$$\hat{I}_{M_{NVB}}(u_x) \leq \hat{I}_{M_{NVB}}(\sim u_x)$$
$$\hat{F}_{M_{NVB}}(u_x) \leq \hat{F}_{M_{NVB}}(\sim u_x)$$

Here

• $U^{\mathcal{M}_{M_{NVB}}} = (U = \{U_1 \cap U_2\}, \oplus, \neg, \sim, 0)$ *is a* $BZMV^{dM}$ *algebraic structure of the underlying universal set U [which is got by combining the two universes of the given neutrosophic vague binary set* M_{NVB}*], with a binary operation* $\oplus$ *, two unary operations* $\neg, \sim$ *and with a constant 0 which satisfies the following* $BZMV^{dM}-$ *axioms:* $\forall u_x, u_y, u_z \in U$

(1) $(u_x \oplus u_y) \oplus u_z = (u_y \oplus u_z) \oplus u_x$

(2) $(u_x \oplus 0) = u_x$

(3) $\neg(\neg u_x) = u_x$

(4) $\neg(\neg u_x \oplus u_y) \oplus u_y = \neg(u_x \oplus \neg u_y) \oplus u_x$

(5) $\sim u_x \oplus \sim\sim u_x = \neg 0$

(6) $u_x \oplus \sim\sim u_x = \sim\sim u_x$

(7) $\sim \neg[(\neg(u_x \oplus \neg u_y)) \oplus \neg u_y)] = \neg(\sim\sim u_x \oplus \neg \sim\sim u_y) \oplus \neg \sim\sim u_y$

Remark 5.2.2.

In a NVB $BZMV^{dM}$ *- subalgebra, possible operations that can also be further derived are:*

1. $(u_x \odot u_y) = \neg(\neg u_x \odot \neg u_y)$

2. $(u_x \vee u_y) = \neg(\neg u_x \oplus u_y) \oplus u_y$

3. $(u_x \wedge u_y) = \neg(\neg(u_x \oplus \neg u_y) \oplus \neg u_y)$

Example 5.2.3.

Let $U_1 = \{0, u_p, u_q, 1\}$ *and* $U_2 = \{0, u_r, u_s, 1\}$ *be two universes with neutrosophic vague binary membership grades as given below:*

$$NVB_{M_{NVB}}(u_p) = \begin{cases} [0.8, 0.9][0.1, 0.6][0.1, 0.2] & ; u_p = 0 \quad and \quad 1 \\ [0.9, 0.9]\,[0.1, 0.2]\,[0.1, 0.1] & ; 0 < u_p < 1 \end{cases} \quad ; \quad \forall u_p \in U_1$$

$$NVB_{M_{NVB}}(u_q) = \begin{cases} [0.7, 0.9][0.2, 0.5][0.1, 0.3] & ; u_q = 0 \quad and \quad 1 \\ [0.8, 0.9][0.1, 0.4][0.1, 0.2] & ; 0 < u_q < 1 \end{cases} \quad ; \quad \forall u_q \in U_2$$

Combined universal set is $U = \{U_1 \cup U_2\} = \{0, u_p, u_q, u_r, u_s, 1\}$ & $\{U_1 \cap U_2\} = \{0, 1\}$
Neutrosophic vague binary union of common elements are given by,

$$\begin{aligned} NVB_{M_{NVB}}(0) &= [0.8, 0.9][0.1, 0.6][0.1, 0.2] \;\cup\; [0.7, 0.9][0.2, 0.5][0.1, 0.3] \\ &= [0.8, 0.9][0.1, 0.5][0.1, 0.2] = NVB_{M_{NVB}}(1) \end{aligned}$$

Combined neutrosophic vague binary membership grades are given as follows:

$$NVB_{M_{NVB}}(u_t) \quad = \quad \left\{ \begin{array}{l} [0.8, 0.9][0.1, 0.5][0.1, 0.2]; u_t = 0 \\ [0.9, 0.9][0.1, 0.2][0.1, 0.1]; u_t = u_p \\ [0.9, 0.9][0.1, 0.2][0.1, 0.1]; u_t = u_q \\ [0.8, 0.9][0.1, 0.4][0.1, 0.2]; u_t = u_r \\ [0.8, 0.9][0.1, 0.4][0.1, 0.2]; u_t = u_s \\ [0.8, 0.9][0.1, 0.5][0.1, 0.2]; u_t = 1 \end{array} \right\}$$

Algebraic structure $U^{\mathcal{M}_{M_{NVB}}} = (U = \{U_1 \cap U_2\}, \star, \neg \sim, 0)$ with binary and unary operations defined as in Cayley Table 5.6 and 5.7 respectively, given below clearly indicates a BZMV$^{dM}-$ subalgebra.

Now have to verify, neutrosophic vague binary concept !

For that check the inequalities given in definition 5.2.1.

Table 5.5: Cayley Table for unary operations $\neg$ and $\sim$ are given below :

	0	u_p	u_q	u_r	u_s	1
$\neg$	1	u_r	u_s	u_s	u_r	0
$\sim$	1	0	0	0	0	0

Table 5.6: Cayley table for binary operation $\star$ is given below

$\star$	0	u_p	u_q	u_r	u_s	1
0	0	u_p	u_q	u_r	u_s	1
u_p	u_p	u_p	u_r	u_r	1	1
u_q	u_q	u_r	u_s	1	u_s	1
u_r	u_r	u_r	1	1	1	1
u_s	u_s	1	u_s	1	u_s	1
1	1	1	1	1	1	1

$\mathcal{M}_{M_{NVB}}$ **inequality (1): (Binary Operation)** $\forall u_x, u_y \in U$

$$\left(inf_{u_v \in (u_x \star u_y)}\, NVB_{M_{NVB}}(u_v)\right) \succeq rmin\left\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\right\}$$

$$\Rightarrow \left(inf_{u_v \in (0,u_p,u_q,u_r,u_s,1)}\, NVB_{M_{NVB}}(u_v))\right) \succeq rmin\left\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\right\}$$

$$\Rightarrow \left(glb_{u_v \in (0,u_p,u_q,u_r,u_s,1)}\quad NVB_{M_{NVB}}(u_v))\right) \succeq rmin\left\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\right\}$$

$$\Rightarrow \left(min_{u_v \in (0,u_p,u_q,u_r,u_s,1)}\quad NVB_{M_{NVB}}(u_v))\right) \succeq rmin\left\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\right\}$$

$$= \min\left[\begin{Bmatrix} [0.8,0.9][0.1,0.5][0.1,0.2] & ; u_v = 0 \\ [0.9,0.9][0.1,0.2][0.1,0.1] & ; u_v = u_p \\ [0.9,0.9][0.1,0.2][0.1,0.1] & ; u_v = u_q \\ [0.8,0.9][0.1,0.4][0.1,0.2] & ; u_v = u_r \\ [0.8,0.9][0.1,0.4][0.1,0.2] & ; u_v = u_s \\ [0.8,0.9][0.1,0.5][0.1,0.2] & ; u_v = 1 \end{Bmatrix} \cap \begin{Bmatrix} [0.8,0.9][0.1,0.5][0.1,0.2] & ; u_v = 0 \\ [0.9,0.9][0.1,0.2][0.1,0.1] & ; u_v = u_p \\ [0.9,0.9][0.1,0.2][0.1,0.1] & ; u_v = u_q \\ [0.8,0.9][0.1,0.4][0.1,0.2] & ; u_v = u_r \\ [0.8,0.9][0.1,0.4][0.1,0.2] & ; u_v = u_s \\ [0.8,0.9][0.1,0.5][0.1,0.2] & ; u_v = 1 \end{Bmatrix}\right]$$

= [0.8,0.9][0.1,0.5][0.1,0.2]. [In neutrosophic concept,minimum concept has been taken as intersection. ie., in this case,(Min,Max,Max)]

$$\therefore \left(inf_{u_v \in (u_x \star u_y)}\ NVB_{M_{NVB}}(u_v)\right) = [0.8, 0.9][0.1, 0.5][0.1, 0.2]$$

Here, NVB BZMV^{dM}-inequality-(1) got satisfied, for any pair of elements from U.

$$\left(inf_{u_v \in (u_x \oplus u_y)}\quad NVB_{M_{NVB}}(u_v))\right) \succeq rmin\left\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\right\}$$

$\mathcal{M}_{M_{NVB}}$ inequality(2):(Unary operations)

Next to check, two inequalities of $\mathcal{M}_{M_{NVB}}$ for all elements of U. [using Cayley Table 5.6 & by Combining given data]

(i) For unary operation $\neg$ (Kleene or Zadeh or fuzzy orthocomplementation)

$NVB_{M_{NVB}}(u_x) \succeq NVB_{M_{NVB}}(\neg u_x)\,; \forall u_x \in U$ as showed in table.

u_x	$NVB_{M_{NVB}}(u_x)$	$NVB_{M_{NVB}}(\neg u_x)$
0	$[0.8, 0.9][0.1, 0.5][0.1, 0.2]$	$[0.8, 0.9][0.1, 0.5][0.1, 0.2]$
u_p	$[0.9, 0.9][0.1, 0.2][0.1, 0.1]$	$[0.8, 0.9][0.1, 0.4][0.1, 0.2]$
u_q	$[0.9, 0.9][0.1, 0.2][0.1, 0.1]$	$[0.8, 0.9][0.1, 0.4][0.1, 0.2]$
u_r	$[0.8, 0.9][0.1, 0.4][0.1, 0.2]$	$[0.8, 0.9][0.1, 0.4][0.1, 0.2]$
u_s	$[0.8, 0.9][0.1, 0.4][0.1, 0.2]$	$[0.8, 0.9][0.1, 0.4][0.1, 0.2]$
1	$[0.8, 0.9][0.1, 0.5][0.1, 0.2]$	$[0.8, 0.9][0.1, 0.5][0.1, 0.2]$

(ii) For unary operation $\sim$ (Brower orthocomplemention)

$NVB_{M_{NVB}}(u_x) \succeq NVB_{M_{NVB}}(\sim u_x)\,; \forall u_x \in U$ as showed in table.

u_x	$NVB_{M_{NVB}}(u_x)$	$NVB_{M_{NVB}}(\sim u_x)$
0	$[0.8, 0.9][0.1, 0.5][0.1, 0.2]$	$[0.8, 0.9][0.1, 0.5][0.1, 0.2]$
u_p	$[0.9, 0.9][0.1, 0.2][0.1, 0.1]$	$[0.8, 0.9][0.1, 0.5][0.1, 0.2]$
u_q	$[0.9, 0.9][0.1, 0.2][0.1, 0.1]$	$[0.8, 0.9][0.1, 0.5][0.1, 0.2]$
u_r	$[0.8, 0.9][0.1, 0.5][0.1, 0.2]$	$[0.8, 0.9][0.1, 0.5][0.1, 0.2]$
u_s	$[0.8, 0.9][0.1, 0.5][0.1, 0.2]$	$[0.8, 0.9][0.1, 0.5][0.1, 0.2]$
1	$[0.8, 0.9][0.1, 0.5][0.1, 0.2]$	$[0.8, 0.9][0.1, 0.5][0.1, 0.2]$

So given example is a $\mathcal{M}_{M_{NVB}}$ with structure $\left(U^{M_{M_{NVB}}}, \star, \neg, \sim, 0\right)$

Remark 5.2.4.

It is to be noted that,

(i) first column of the Cayley table for binary operation will be a copy of column of operands, using definition 5.2.1- $BZMV^{dM}$ property (2)

(ii) last row and column of the Cayley table for binary operation will be always 1 for a $BZMV^{\,dM}-$ algebra.

Theorem 5.2.5.

If M_{NVB} is a $\mathcal{M}_{M_{NVB}}$ then the following results are true:

1. $NVB_{M_{NVB}}\,(u_x \oplus u_y) = NVB_{M_{NVB}}\,(u_y \oplus u_x)$
   [i.e.,commutative law holds for Binary Operation]

2. $NVB_{M_{NVB}}\,((u_x \oplus u_y) \oplus u_z) = NVB_{M_{NVB}}\,(u_x \oplus (u_y \oplus u_z))$
   [i.e.,associative law holds for Binary Operation]

3. $NVB_{M_{NVB}}\,(u_x \oplus 1) = NVB_{M_{NVB}}\,(1)$

4. $NVB_{M_{NVB}}\,(u_x \oplus \neg u_x) = NVB_{M_{NVB}}\,(1)$
   [Neutrosophic vague binary membership grade of an element binary operated with it's kleene complement ¬ always produce the neutrosophic vague binary membership grade of the maximum element 1]

5. $NVB_{M_{NVB}}\,(\neg\,(u_x \oplus \sim\sim u_x) \oplus \sim\sim u_x) = NVB_{M_{NVB}}(1)$

6. $NVB_{M_{NVB}}(\neg u_x \oplus \sim\sim u_x) = NVB_{M_{NVB}}(1)$

7. $NVB_{M_{NVB}}\,(u_x \wedge \sim\sim u_x) = NVB_{M_{NVB}}(u_x)$

8. $NVB_{M_{NVB}}\,(\neg \sim u_x) = NVB_{M_{NVB}}\,(\sim\sim u_x)$

9. $NVB_{M_{NVB}}\,(\sim (u_x \wedge u_y)) = NVB_{M_{NVB}}\,(\sim u_x \vee \sim u_y)$

10. $NVB_{M_{NVB}}\,(\sim (u_x \vee u_y)) = NVB_{M_{NVB}}\,(\sim u_x \wedge \sim u_y)$
    $(Equivalently, u_x \leq u_y \quad \Rightarrow \quad \sim u_y \leq \sim u_x)$

11. $NVB_{M_{NVB}}\,(u_x \wedge \sim u_x) = NVB_{M_{NVB}}(0)$

12. $NVB_{M_{NVB}}\,(\sim u_x) = NVB_{M_{NVB}}\,(\sim\sim\sim u_x)$

13. $NVB_{M_{NVB}}\,(\sim u_x \oplus \sim u_x) = NVB_{M_{NVB}}\,(\sim u_x)$

14. $NVB_{M_{NVB}}(\neg 0) = NVB_{M_{NVB}}(\sim 0)$

Proof.

By using definition and axioms of $BZMV^{dM}$ given in 5.2.1, will get the following results :

1. $NVB_{M_{NVB}}(u_x \oplus u_y) = NVB_{M_{NVB}}((u_x \oplus u_y) \oplus 0)$, by putting $u_x = (u_x \oplus u_y)$
   $= NVB_{M_{NVB}}((u_y \oplus 0) \oplus u_x) = NVB_{M_{NVB}}(u_y \oplus u_x)$

2. $NVB_{M_{NVB}}((u_x \oplus u_y) \oplus u_z) = NVB_{M_{NVB}}((u_y \oplus u_z) \oplus u_x)$
   $= NVB_{M_{NVB}}(u_x \oplus (u_y \oplus u_z))$

3. $NVB_{M_{NVB}}(u_x \oplus 1) = NVB_{M_{NVB}}(u_x \oplus \neg 0)$, since $1 = \neg 0$
   $= NVB_{M_{NVB}}(u_x \oplus (\sim u_x \oplus \sim\sim u_x)) = NVB_{M_{NVB}}((\sim u_x \oplus \sim\sim u_x) \oplus u_x)$
   $= NVB_{M_{NVB}}((\sim\sim u_x \oplus u_x) \oplus \sim u_x) = NVB_{M_{NVB}}((u_x \oplus \sim\sim u_x) \oplus \sim u_x)$
   $= NVB_{M_{NVB}}(\sim\sim u_x \oplus \sim u_x) = NVB_{M_{NVB}}(\sim u_x \oplus \sim\sim u_x)$
   $= NVB_{M_{NVB}}(\neg 0) = NVB_{M_{NVB}}(1)$, since $\neg 0 = 1$

4. $NVB_{M_{NVB}}(u_x \oplus \neg u_x) = NVB_{M_{NVB}}(\neg\neg u_x \oplus \neg u_x)$
   $= NVB_{M_{NVB}}(\neg(\neg u_x \oplus 0) \oplus \neg u_x) = NVB_{M_{NVB}}(\neg(0 \oplus \neg u_x) \oplus \neg u_x)$
   $= NVB_{M_{NVB}}(\neg(u_x \oplus \neg 0) \oplus \neg 0) = NVB_{M_{NVB}}(\neg(u_x \oplus 1) \oplus 1)$, since $\neg 0 = 1$
   $= NVB_{M_{NVB}}(\neg 0) = NVB_{M_{NVB}}(1)$, since $\neg 0 = 1$

5. $NVB_{M_{NVB}}(\neg(u_x \oplus \sim\sim u_x) \oplus \sim\sim u_x) = NVB_{M_{NVB}}(\neg(\sim\sim u_x) \oplus \sim\sim u_x)$
   $= NVB_{M_{NVB}}(\sim\sim u_x \oplus \neg(\sim\sim u_x)) = NVB_{M_{NVB}}(1)$

6. $NVB_{M_{NVB}}(\neg u_x \oplus \sim\sim u_x) = NVB_{M_{NVB}}(\neg u_x \oplus (u_x \oplus \sim\sim u_x))$
   $= NVB_{M_{NVB}}((\neg u_x \oplus u_x) \oplus \sim\sim u_x) = NVB_{M_{NVB}}((u_x \oplus \neg u_x) \oplus \sim\sim u_x)$
   $= NVB_{M_{NVB}}(1 \oplus \sim\sim u_x) = NVB_{M_{NVB}}(\sim\sim u_x \oplus 1)$
   $= NVB_{M_{NVB}}(1)$, by putting $u_x = \sim\sim u_x$

7. $NVB_{M_{NVB}}(u_x \wedge \sim\sim u_x)$
   $= NVB_{M_{NVB}}(\neg(\neg(u_x \oplus \neg \sim\sim u_x) \oplus \neg \sim\sim u_x))$
   [$(u_x \wedge u_y) = \neg(\neg(u_x \oplus \neg u_y) \oplus \neg u_y)$, by putting $u_y = \sim\sim u_x$]
   $= NVB_{M_{NVB}}(\neg(\neg(\neg \sim\sim u_x \oplus u_x) \oplus \neg \sim\sim u_x))$
   $= NVB_{M_{NVB}}(\neg(\neg(u_x \oplus \neg \sim\sim u_x) \oplus \neg \sim\sim u_x))$
   $= NVB_{M_{NVB}}(\neg(\neg(u_x \oplus \neg \sim\sim u_x) \oplus \sim\sim\sim u_x))$
   $= NVB_{M_{NVB}}(\neg(\neg(u_x \oplus \neg \sim\sim u_x) \oplus u_x))$
   $= NVB_{M_{NVB}}(\neg(\neg(u_x \oplus \neg \sim\sim u_x) \oplus u_x))$

$= NVB_{M_{NVB}}\left(\neg\left(\neg\left(\neg u_x \oplus \sim\sim u_x\right) \oplus \sim\sim u_x\right)\right)$

$\left[\neg\left(\neg u_x \oplus u_y\right) \oplus u_y = \neg\left(u_x \oplus \neg u_y\right) \oplus u_x\right]$

$= NVB_{M_{NVB}} NVB_{M_{NVB}}\left(\neg\left(\neg 1 \oplus \sim\sim u_x\right)\right)$

$= NVB_{M_{NVB}}\left(\neg\left(0 \oplus \sim\sim u_x\right)\right)$, since $\neg 1 = 0$

$= NVB_{M_{NVB}}\left(\neg\left(\sim\sim u_x \oplus 0\right)\right)$

$= NVB_{M_{NVB}}\left(\neg\left(\sim\sim u_x\right)\right)$

$= NVB_{M_{NVB}}\left(\sim\sim\sim u_x\right)$, since $\neg\left(\sim\sim u_x\right) = \sim\sim\sim u_x$

$= NVB_{M_{NVB}}\left(u_x\right)$, since $\sim\sim\sim u_x = u_x$

8. $NVB_{M_{NVB}}\left(\sim\neg\left[\neg\left(u_x \oplus \neg u_y\right) \oplus \neg u_y\right]\right)$

$= NVB_{M_{NVB}}\left(\neg\left(\sim\sim u_x \oplus \neg \sim\sim u_y\right) \oplus \neg \sim\sim u_y\right)$

By putting $u_y = u_x$ in the above,

$\Rightarrow NVB_{M_{NVB}}\left(\sim\neg\left[\neg\left(u_x \oplus \neg u_x\right) \oplus \neg u_x\right]\right)$

$= NVB_{M_{NVB}}\left(\neg\left(\sim\sim u_x \oplus \neg \sim\sim u_x\right) \oplus \neg \sim\sim u_x\right)$

$\Rightarrow NVB_{M_{NVB}}\left(\sim\neg\left[\neg 1 \oplus \neg u_x\right]\right) = NVB_{M_{NVB}}\left(\neg 1 \oplus \neg \sim\sim u_x\right)$

$\Rightarrow NVB_{M_{NVB}}\left(\sim\neg\left[0 \oplus \neg u_x\right]\right) = NVB_{M_{NVB}}\left(0 \oplus \neg \sim\sim u_x\right)$

$\Rightarrow NVB_{M_{NVB}}\left(\sim\neg\left[\neg u_x \oplus 0\right]\right) = NVB_{M_{NVB}}\left(\neg \sim\sim u_x \oplus 0\right)$

$\Rightarrow NVB_{M_{NVB}}\left(\sim\neg\left[\neg u_x\right]\right) = NVB_{M_{NVB}}\left(\neg \sim\sim u_x\right)$

$\Rightarrow NVB_{M_{NVB}}\left(\sim u_x\right) = NVB_{M_{NVB}}\left(\neg \sim\sim u_x\right)$

$\Rightarrow NVB_{M_{NVB}}\left(\neg \sim u_x\right) = NVB_{M_{NVB}}\left(\neg\neg \sim\sim u_x\right)$,

by applying $\neg$ on both sides.

$\Rightarrow NVB_{M_{NVB}}\left(\neg \sim u_x\right) = NVB_{M_{NVB}}\left(\sim\sim u_x\right)$

9. $\Rightarrow NVB_{M_{NVB}}\left(\sim\neg\left[\neg\left(u_x \oplus \neg u_y\right) \oplus \neg u_y\right]\right)$

$= NVB_{M_{NVB}}\left(\neg\left(\sim\sim u_x \oplus \neg \sim\sim u_y\right) \oplus \neg \sim\sim u_y\right)$

$\Rightarrow NVB_{M_{NVB}}\left(\sim\left(u_x \wedge u_y\right)\right) = NVB_{M_{NVB}}\left(\neg\left(\neg \sim u_x \oplus \neg\neg \sim u_y\right) \oplus \neg\neg \sim u_y\right)$

[By using auxiliary operation,

$\left(u_x \wedge u_y\right) = \neg\left[\neg\left(u_x \oplus \neg u_y\right) \oplus \neg u_y\right]$ & $\left[\neg \sim u_x = \sim\sim u_x\right]$

$\Rightarrow NVB_{M_{NVB}}\left(\sim\left(u_x \wedge u_y\right)\right) = NVB_{M_{NVB}}\left(\neg\left(\sim\sim u_x \oplus \sim u_y\right) \oplus \sim u_y\right)$,

$\Rightarrow NVB_{M_{NVB}}\left(\sim\left(u_x \wedge u_y\right)\right) = NVB_{M_{NVB}}\left(\neg\left(\neg \sim u_x \oplus \sim u_y\right) \oplus \sim u_y\right)$,

by using $\sim\sim u_x = \neg \sim u_x$

$\Rightarrow NVB_{M_{NVB}}(\sim\left(u_x \wedge u_y\right) = NVB_{M_{NVB}}\left(\sim u_x \vee \sim u_y\right)$

[By using auxiliary operation,

$NVB_{M_{NVB}}\left(u_x \vee u_y\right) = NVB_{M_{NVB}}\left(\neg\left[\neg\left(u_x \oplus u_y\right) \oplus u_y\right]\right)$]

10. Let $NVB_{M_{NVB}}(u_x) \preceq NVB_{M_{NVB}}(u_y)$

$\Rightarrow NVB_{M_{NVB}}(u_x) = NVB_{M_{NVB}}(u_x \wedge u_y)$

$\Rightarrow NVB_{M_{NVB}}(\sim u_x) = NVB_{M_{NVB}}(\sim (u_x \wedge u_y))$, by applying $\sim$ on both sides and by using (9)

$\Rightarrow NVB_{M_{NVB}}(\sim u_x) = NVB_{M_{NVB}}(\sim u_x \vee \sim u_y)$,

$\Rightarrow NVB_{M_{NVB}}(\sim u_y) \preceq NVB_{M_{NVB}}(\sim u_x)$

$\forall u_x, \quad NVB_{M_{NVB}}(u_x) \preceq NVB_{M_{NVB}}(\sim\sim u_x)$.

So now the contraposition law is equivalent to the de Morgan law:

$NVB_{M_{NVB}}(\sim (u_x \vee u_y)) = NVB_{M_{NVB}}(\sim u_x \wedge \sim u_y)$

11. $NVB_{M_{NVB}}(u_x \wedge \sim u_x)$

$= NVB_{M_{NVB}}(\neg(\neg(u_x \oplus \neg \sim u_x) \oplus \neg \sim u_x))$

$= NVB_{M_{NVB}}(\neg(\neg(u_x \oplus \sim\sim u_x) \oplus \sim\sim u_x))$,

$= NVB_{M_{NVB}}(\neg 1), = NVB_{M_{NVB}}(0)$

12. $NVB_{M_{NVB}}(\neg \sim u_x) = NVB_{M_{NVB}}(\sim\sim u_x)$

Put $u_x = \sim u_x$, in the above then,

$NVB_{M_{NVB}}(\neg \sim\sim u_x) = NVB_{M_{NVB}}(\sim\sim\sim u_x)$

$\Rightarrow NVB_{M_{NVB}}(\neg\neg \sim u_x) = NVB_{M_{NVB}}(\sim\sim\sim u_x)$, since $\sim\sim u_x = \neg \sim u_x$

$\Rightarrow NVB_{M_{NVB}}(\sim u_x) = NVB_{M_{NVB}}(\sim\sim\sim u_x)$, since $\neg\neg \sim u_x = \sim u_x$

13. $NVB_{M_{NVB}}(u_x \wedge \sim\sim u_x) = NVB_{M_{NVB}}(\sim\sim u_x)$,

Taking Brouwerian orthocomplementation to both sides,

$NVB_{M_{NVB}}(\sim u_x \wedge \sim\sim\sim u_x) = NVB_{M_{NVB}}(\sim\sim\sim u_x)$

$\Rightarrow NVB_{M_{NVB}}(\sim u_x \wedge \sim u_x) = NVB_{M_{NVB}}(\sim u_x)$

$[\sim\sim\sim u_x = \sim u_x]$

14. To prove that, $NVB_{M_{NVB}}(\neg 0) = NVB_{M_{NVB}}(\sim 0)$. It is enough to prove that, $NVB_{M_{NVB}}(\neg 0) \preceq NVB_{M_{NVB}}(\sim 0)$ and $NVB_{M_{NVB}}(\sim 0) \preceq NVB_{M_{NVB}}(\neg 0)$ We know that, $NVB_{M_{NVB}}(1) = NVB_{M_{NVB}}(\neg 0) \, \forall u_x \in M_{NVB}$, where M_{NVB} is a $\mathcal{M}_{M_{NVB}}$, $\quad NVB_{M_{NVB}}(u_x) \preceq NVB_{M_{NVB}}(1)$, since 1 is the maximum element. In particular, $NVB_{M_{NVB}}(\sim 0) \preceq NVB_{M_{NVB}}(1) \Rightarrow NVB_{M_{NVB}}(\sim 0) \preceq NVB_{M_{NVB}}(\neg 0)$. Similarly, being the least element, $\forall u_x \in M_{NVB}$, where M_{NVB} is a $\mathcal{M}_{M_{NVB}}$, $\quad NVB_{M_{NVB}}(u_x) \preceq NVB_{M_{NVB}}(\sim\sim u_x) \leq NVB_{M_{NVB}}(\sim 0)$ In particular, $NVB_{M_{NVB}}(\neg 0) \preceq NVB_{M_{NVB}}(\sim 0)$

Theorem 5.2.6.

Let M_{NVB} is a $\mathcal{M}_{M_{NVB}}$. Then

1. $\forall u_x, u_y \in M_{NVB}$,
   $NVB_{M_{NVB}}(u_x \wedge u_y) = NVB_{M_{NVB}}(0) \Leftrightarrow NVB_{M_{NVB}}(u_y) \preceq NVB_{M_{NVB}}(\sim u_x)$
   Equivalently,
   $NVB_{M_{NVB}}(u_x \wedge u_y) = NVB_{M_{NVB}}(0) \Leftrightarrow NVB_{M_{NVB}}(u_x) \preceq NVB_{M_{NVB}}(\sim u_y)$

2. *Let $u_x \in M_{NVB}$ be such that $NVB_{M_{NVB}}(u_x \oplus u_x) = NVB_{M_{NVB}}(u_x)$*
   Then $\forall u_y \in M_{NVB}$,
   $NVB_{M_{NVB}}(u_x \wedge u_y) = NVB_{M_{NVB}}(0) \Leftrightarrow NVB_{M_{NVB}}(u_x) \preceq NVB_{M_{NVB}}(\neg u_y)$

Proof.

1. Assume $(u_x \wedge u_y) = 0$. Now $NVB_{M_{NVB}}(u_y \wedge \sim u_x) = NVB_{M_{NVB}}((u_y \wedge \sim u_x) \vee 0)$,
   since in any lattice $(u_x \vee 0) = u_x$
   $\Rightarrow NVB_{M_{NVB}}(u_y \wedge \sim u_x) = NVB_{M_{NVB}}((u_y \wedge \sim u_x) \vee (u_y \wedge \sim u_y))$,
   by a result $(u_y \wedge \sim u_y) = 0$, of BZMV$^{dM}-$ algebra
   $= NVB_{M_{NVB}}(u_y \wedge (\sim u_x \vee \sim u_y))$, by theorem 5.2.5 (9)
   $= NVB_{M_{NVB}}(u_y \wedge \sim (u_x \wedge u_y))$, by theorem 5.2.5 (9)
   $= NVB_{M_{NVB}}(u_y \wedge \sim 0)$, by assumption
   $= NVB_{M_{NVB}}(u_y) = NVB_{M_{NVB}}(u_y \wedge 1)$, [since $\sim 0 = \neg 0 = 1$]
   $= NVB_{M_{NVB}}(u_y)$,
   [since in any lattice $(u_y \wedge 1) = u_y \Rightarrow NVB_{M_{NVB}}(u_y \wedge 1) = NVB_{M_{NVB}}(u_y)$]
   $\Rightarrow NVB_{M_{NVB}}(u_y) \preceq NVB_{M_{NVB}}(\sim u_x)$
   Conversely, suppose $NVB_{M_{NVB}}(u_y) \preceq NVB_{M_{NVB}}(\sim u_x)$ then
   $\Rightarrow NVB_{M_{NVB}}(u_x \wedge u_y) = NVB_{M_{NVB}}(u_x \wedge (u_y \wedge \sim u_x))$
   $= NVB_{M_{NVB}}(u_y \wedge (u_x \wedge \sim u_x))$, by associativity
   $= NVB_{M_{NVB}}(u_y \wedge 0)$, by theorem 5.2.5 (11)
   $= NVB_{M_{NVB}}(0)$, since in any lattice $(u_y \wedge 0) = u_y$
   Equivalently, $NVB_{M_{NVB}}(u_x \wedge u_y) = NVB_{M_{NVB}}(0)$
   $\Leftrightarrow NVB_{M_{NVB}}(u_x) \preceq NVB_{M_{NVB}}(\sim u_y)$, can be proved

2. Suppose $NVB_{M_{NVB}}(u_x) = NVB_{M_{NVB}}(u_x \oplus u_x)$.
   Then, $NVB_{M_{NVB}}(u_x \oplus u_x) = NVB_(M_N VB)(u_x \wedge u_y)$.
   Thus we got, $NVB_{M_{NVB}}(u_x) \preceq NVB_{M_{NVB}}(\neg u_y)$
   $\Leftrightarrow NVB_{M_{NVB}}(u_x \wedge u_y) = NVB_{M_{NVB}}(0)$

Theorem 5.2.7.

In a $\mathcal{M}_{M_{NVB}}$ the following holds :

$NVB_{M_{NVB}}(\sim\sim u_x) = NVB_{M_{NVB}}(u_x) \Leftrightarrow NVB_{M_{NVB}}(\sim u_x \oplus u_x) = NVB_{M_{NVB}}(1)$

$\Leftrightarrow NVB_{M_{NVB}}(u_x \oplus u_x) = NVB_{M_{NVB}}(u_x)$

Proof.

Assume $\sim\sim u_x = u_x$.. Then, $NVB_{M_{NVB}}(\sim\sim u_x) = NVB_{M_{NVB}}(u_x)$

$\Rightarrow NVB_{M_{NVB}}(u_x \oplus \sim\sim u_x) = NVB_{M_{NVB}}(\sim\sim u_x)$

[Using BZMV^{dM}-property (6) from Definition 5.2.1]

$\Rightarrow NVB_{M_{NVB}}(u_x \oplus u_x) = NVB_{M_{NVB}}(u_x)$

Again, from definition 5.2.1, using BZMV^{dM}-property-(5),

$NVB_{M_{NVB}}(\sim u_x \oplus \sim\sim u_x) = NVB_{M_{NVB}}(\neg 0)$

$\Rightarrow NVB_{M_{NVB}}(\sim u_x \oplus u_x) = NVB_{M_{NVB}}(1)$.

Assume, $NVB_{M_{NVB}}(u_x \oplus u_x) = NVB_{M_{NVB}}(u_x)$.

Since $\neg u_x \in M_{M_{NVB}}$ and $(u_x \wedge \neg u_x) = 0$

$\Rightarrow NVB_{M_{NVB}}(u_x \wedge \neg u_x) = NVB_{M_{NVB}}(0)$

$\Rightarrow NVB_{M_{NVB}}(\neg u_x) \preceq NVB_{M_{NVB}}(\sim u_x)$ [By theorem 5.2.6(i)]

$\Rightarrow NVB_{M_{NVB}}(\neg \sim u_x) \preceq NVB_{M_{NVB}}(\sim\sim u_x)$, by putting $u_x = \sim u_x$

$\Rightarrow NVB_{M_{NVB}}(\sim\sim u_x) = NVB_{M_{NVB}}(u_x)$

Under condition $NVB_{M_{NVB}}(\sim u_x \oplus u_x) = NVB_{M_{NVB}}(1)$, it is clear that,

$NVB_{M_{NVB}}(u_x \wedge \sim\sim u_x) = NVB_{M_{NVB}}(\sim\sim u_x)$

$\Rightarrow NVB_{M_{NVB}}(\sim\sim u_x) \preceq NVB_{M_{NVB}}(u_x)$

In fact,

$NVB_{M_{NVB}}(u_x \wedge \sim\sim u_x) = NVB_{M_{NVB}}(\neg[\neg(u_x \oplus \neg \sim\sim u_x) \oplus \neg \sim\sim u_x])$

[Using definition of $\wedge$]

$= NVB_{M_{NVB}}(\neg[\neg(u_x \oplus \neg\neg \sim u_x) \oplus \neg\neg \sim u_x])$, [since $\neg \sim u_x = \sim\sim u_x$]

$= NVB_{M_{NVB}}(\neg[\neg(u_x \oplus \sim u_x) \oplus \sim u_x])$, [since $\neg\neg u_x = u_x$]

$= NVB_{M_{NVB}}(\neg(0 \oplus \sim u_x))$

$= NVB_{M_{NVB}}(\neg(\sim u_x \oplus 0))$, by theorem 5.2.5(1)

$= NVB_{M_{NVB}}(\neg \sim u_x)$, from definition 5.2.1. BZMV^{dM} property-(2)

$= NVB_{M_{NVB}}(\sim\sim u_x)$. [since $\neg \sim u_x = \sim\sim u_x$]

Similarly, we get, $NVB_{M_{NVB}}(u_x) = NVB_{M_{NVB}}(\sim\sim u_x)$.

Theorem 5.2.8.

Let M_{NVB} be a $\mathcal{M}_{M_{NVB}}$; then for any $u_x, u_y, u_z \in M_{NVB}$,

$NVB_{M_{NVB}}((u_x \oplus u_y) \wedge u_z) = NVB_{M_{NVB}}(0)$
$\Leftrightarrow NVB_{M_{NVB}}(u_x \wedge u_z) = NVB_{M_{NVB}}(0)$ *and* $NVB_{M_{NVB}}(u_y \wedge u_z) = NVB_{M_{NVB}}(0)$

Proof.
Suppose, $NVB_{M_{NVB}}((u_x \oplus u_y) \wedge u_z) = NVB_{M_{NVB}}(0)$
$\Rightarrow NVB_{M_{NVB}}(u_x \wedge u_z) \preceq NVB_{M_{NVB}}((u_x \oplus u_y) \wedge u_z) = NVB_{M_{NVB}}(0)$ and
$NVB_{M_{NVB}}(u_y \wedge u_z) \preceq NVB_{M_{NVB}}((u_x \oplus u_y) \wedge u_z) = NVB_{M_{NVB}}(0)$
$\Rightarrow NVB_{M_{NVB}}(u_x \wedge u_z) = NVB_{M_{NVB}}(0) = NVB_{M_{NVB}}(u_y \wedge u_z)$ by theorem 5.2.6.
Conversely, let $NVB_{M_{NVB}}(u_x \wedge u_z) = NVB_{M_{NVB}}(0)$
and $NVB_{M_{NVB}}(u_y \wedge u_z) = NVB_{M_{NVB}}(0)$.
$NVB_{M_{NVB}}(u_x \wedge \neg u_z) = NVB_{M_{NVB}}(\neg u_z) = NVB_{M_{NVB}}(u_y \wedge \neg u_z)$. So,
$NVB_{M_{NVB}}((u_x \oplus u_y) \oplus \neg u_z) = NVB_{M_{NVB}}(u_x \oplus (u_y \oplus \neg u_z))$, by Definition 5.2.5(2)
$= NVB_{M_{NVB}}(u_x \oplus \neg u_z) = NVB_{M_{NVB}}(\neg u_z)$, from assumption
$\therefore NVB_{M_{NVB}}((u_x \oplus u_y) \wedge u_z)$
$= NVB_{M_{NVB}}(\neg[\neg(u_x \oplus \neg \sim\sim u_x) \oplus \neg\neg \sim u_x])$
$= NVB_{M_{NVB}}(((u_x \oplus u_y) \oplus \neg u_z) \oplus \neg u_z)$
$= NVB_{M_{NVB}}(0 \oplus \neg u_z)$
$= NVB_{M_{NVB}}(u_x \oplus 0) = NVB_{M_{NVB}}(0)$

Ideals in Neutrosophic vague binary BZMV^{dM} subalgebra

Concept of ideal with three different kinds are developed in this section

Definition 5.2.9. ***(NVB BZMV^{dM-}ideal)***
Let M_{NVB} be a $\mathcal{M}_{M_{NVB}}$ and I_{NVB} be a nonempty subset of M_{NVB}.
Then I_{NVB} is a neutrosophic vague binary BZMV$^{dM}-$ ideal (NVB-BZMV$^{dM}-$ ideal) if the following inequalities got satisfied:

1. $NVB_{I_{NVB}}(0) \succeq NVB_{I_{NVB}}(u_x) \quad ; \quad \forall \quad u_x \in I_{NVB}$
2. $NVB_{I_{NVB}}(u_x \oplus u_y) \succeq rmin\{NVB_{I_{NVB}}(u_x), NVB_{I_{NVB}}(u_y)\}; \forall u_x, u_y \in I_{NVB}$
3. $NVB_{I_{NVB}}(u_y) \succeq rmin\{NVB_{I_{NVB}}(u_x), NVB_{I_{NVB}}(u_y \leq u_x)\}; \forall u_x, u_y \in I_{NVB}$

Definition 5.2.10.
(prime ideal, $\sim$ ideal, normal ideal of a NVB BZMV$^{dM}-$subalgebra)
Let M_{NVB} be a $\mathcal{M}_{M_{NVB}}$ and I_{NVB} be a NVB BZMV^{dM-} ideal of M_{NVB}.
$\forall u_x, u_y \in M_{NVB}$. I_{NVB} is called,

1. *A neutrosophic vague binary* $BZMV^{dM-}$ *prime ideal* $\left(NVB\ BZMV^{dM} - prime\ ideal\right)$ *of* M_{NVB}
$\Leftrightarrow NVB_{I_{NVB}}\left(u_x \oplus \neg u_y\right) \in M_{NVB} or NVB_{I_{NVB}}\left(\neg u_x \oplus u_y\right) \in M_{NVB}$

2. *A neutrosophic vague binary* $BZMV^{dM} \sim$ *ideal* $\left(NVB \quad BZMV^{dM} \sim ideal\right)$ *of* M_{NVB} *if it satisfies:*
$NVB_{I_{NVB}}\left(\sim\sim u_x \oplus \sim \neg u_y \in I_{NVB}\right) \succeq NVB_{I_{NVB}}\left(u_x \oplus u_y\right)$

3. *A neutrosophic vague binary* $BZMV^{dM}$ *normal ideal* $\left(NVB \quad BZMV^{dM} \quad normal \quad ideal\right)$ *of* M_{NVB} *whenever*
$NVB_{I_{NVB}}\left(\neg u_x \oplus u_y\right) \succeq NVB_{I_{NVB}}\left(\sim u_x \oplus u_y\right)$ *and*
$NVB_{I_{NVB}}\left(\neg u_x \oplus u_y\right) \preceq NVB_{I_{NVB}}\left(\sim u_x \oplus u_y\right)$
i.e., $NVB_{I_{NVB}}\left(\neg u_x \oplus u_y\right) \Leftrightarrow NVB_{I_{NVB}}\left(\sim u_x \oplus u_y\right)$

Theorem 5.2.11.
I_{NVB} *is a* $NVB \quad BZMV^{dM}$ *p ideal of* $M_{M_{NVB}} \Leftrightarrow$

$$\left\{\begin{array}{l} NVB_{I_{NVB}}\left(u_x\right) \succeq NVB_{I_{NVB}}\left(u_x \wedge u_y\right) \quad or \\ NVB_{I_{NVB}}\left(u_y\right) \succeq NVB_{I_{NVB}}\left(u_x \wedge u_y\right) \end{array}\right\}$$

Proof.
Assume I_{NVB} is a $NVB \quad BZMV^{dM}$ p ideal of $\mathcal{M}_{M_{NVB}} \Rightarrow$

$$\begin{cases} NVB_{I_{NVB}}\left(u_x \oplus \neg u_y\right) \succeq rmin\left\{NVB_{M_{NVB}}\left(u_x\right), NVB_{M_{NVB}}\left(u_y\right)\right\}\ ;\ \forall u_x, u_y \in M_{NVB} \quad or \\ NVB_{I_{NVB}}\left(\neg u_x \oplus u_y\right) \succeq rmin\left\{NVB_{M_{NVB}}\left(u_x\right), NVB_{M_{NVB}}\left(u_y\right)\right\}\ ;\ \forall u_x, u_y \in M_{NVB} \end{cases}$$

Without loss of generality, consider
$NVB_{I_{NVB}}\left(\neg u_x \oplus u_y\right) \succeq rmin\left\{NVB_{M_{NVB}}\left(u_x\right), NVB_{M_{NVB}}(u_y)\right\}$
$NVB_{I_{NVB}}\left(u_x \wedge u_y\right) = NVB_{I_{NVB}}\left(u_y \wedge u_x\right)$ [using commutativity of $\wedge$]
$= NVB_{I_{NVB}}\left(u_y \oplus \left(u_x \oplus \neg u_y\right)\right)$ [by using $\left(u_a \wedge u_b\right) = u_a \oplus \left(u_b \oplus \neg u_a\right)$]
$= NVB_{I_{NVB}}\left(\neg\left(\neg u_y \oplus \neg\left(u_x \oplus \neg u_y\right)\right)\right)$ [by using $\left(u_a \oplus u_b\right)$
$= \neg\left(\neg u_a \oplus \neg u_b\right)$] $= NVB_{I_{NVB}}\left(\neg\left(\neg u_y \oplus \neg\left(\neg u_y \oplus \oplus u_x\right)\right)\right)$
[by commutativity of $\oplus$]
$= NVB_{I_{NVB}}\left(\neg\left(\neg u_y \oplus \neg\left(\neg u_y \oplus \neg\neg u_x\right)\right)\right)$ $[since \neg\neg u_x = u_x]$
$= NVB_{I_{NVB}}\left(\neg\left(\neg u_y \oplus \left(u_y \oplus \neg u_x\right)\right)\right)$ $[since \neg\left(\neg u_y \oplus \neg\neg u_x\right)$
$= \left(u_y \oplus \neg u_x\right)] \in I_{NVB}$
$NVB_{I_{NVB}}\left(\neg\left(\neg u_y \oplus \left(u_y \oplus \neg u_x\right)\right)\right) \succeq NVB_{I_{NVB}}\left(u_x \wedge u_y\right) and$
$NVB_{I_{NVB}}\left(\neg u_x \oplus u_y\right) \succeq rmin\left\{NVB_{M_{NVB}}\left(u_x\right), NVB_{M_{NVB}}\left(u_y\right)\right\}$

$\Rightarrow NVB_{I_{NVB}}(\neg(\neg u_y \oplus (u_y \oplus \neg u_x)) \oplus (\neg u_x \oplus u_y)) \in I_{NVB}$
$\Rightarrow NVB_{I_{NVB}}(\neg(\neg u_y \oplus (u_y \oplus \neg u_x)) \oplus (u_y \oplus \neg u_x)) \in I_{NVB}$
Hence, $NVB_{I_{NVB}}(u_y \vee (u_y \oplus \neg u_x)) \in I_{NVB}$
$\Rightarrow NVB_{I_{NVB}}(u_y) \in I_{NVB}, since\,(u_y \oplus \neg u_x) \leq u_y$
$\therefore NVB_{I_{NVB}}(u_y) \succeq NVB_{I_{NVB}}(\neg u_x \oplus u_y)$
$Similarly, if NVB_{I_{NVB}}(\neg u_y \oplus u_x) \in I_{NVB}$
$\Rightarrow NVB_{I_{NVB}}(u_x) \in I_{NVB}$
$\Rightarrow NVB_{I_{NVB}}(u_x) \succeq NVB_{I_{NVB}}(\neg u_y \oplus u_x)$
Conversely,
$NVB_{I_{NVB}}((\neg u_x \oplus u_y) \wedge (u_x \oplus \neg u_y)) = NVB_{I_{NVB}}(0) \in I_{NVB}$
[from definition of $NVB \quad BZMV^{dM}$ ideal]
$\Rightarrow NVB_{I_{NVB}}(\neg u_x \oplus u_y) \in I_{NVB} or NVB_{I_{NVB}}(u_x \oplus \neg u_y) \in I_{NVB}$
$\Rightarrow I_{NVB}$ is a $NVB \quad BZMV^{dM}$ prime ideal of $\mathcal{M}_{M_{NVB}}$

Theorem 5.2.12.

Let I_{NVB} be an NVB ideal of a neutrosophic vague binary $BZMV^{dM}$ subalgebra $\mathcal{M}_{M_{NVB}}$ and $\sim\sim u_x = u_x$ for all $u_x \in M_{NVB}$. Then the following conditions are equivalent :

1. *I_{NVB} is a $NVB \quad BZMV^{dM}$ normal ideal*
2. *I_{NVB} is a $NVB \quad BZMV^{dM} \sim$ ideal*
3. *$NVB_{I_{NVB}}(\sim u_x) \in I_{NVB} \Leftrightarrow NVB_{I_{NVB}}(\neg u_x) \in I_{NVB}$*

Proof.
(1) $\Rightarrow$(2)
Let I_{NVB} is a $NVB \quad BZMV^{dM-}$ normal ideal of $\mathcal{M}_{M_{NVB}}$ Then, $\forall u_x, u_y \in M_{NVB}$,
$NVB_{M_{NVB}}(u_x \oplus u_y) \in I_{NVB} \Rightarrow NVB_{M_{NVB}}(\neg\neg u_x \oplus u_y) \in I_{NVB}$
[by using $u_x = \neg\neg u_x$]
$\Rightarrow NVB_{M_{NVB}}(\sim \neg u_x \oplus u_y) \in I_{NVB}$
[by property of $BZMV^{dM}$ subalgebra $\neg\neg u_x = \sim \neg u_x$]
$\Rightarrow NVB_{M_{NVB}}(\sim \neg u_x \oplus \sim\sim u_y) \in I_{NVB}$ [since given $\sim\sim u_x = u_x$]
$\Rightarrow NVB_{M_{NVB}}(\sim\sim u_y \oplus \sim \neg u_x) \in I_{NVB}$ [by commutativity]
$\therefore NVB_{M_{NVB}}(\sim\sim u_y \oplus \sim \neg u_x) \succeq NVB_{M_{NVB}}(u_x \oplus u_y), \forall u_x, u_y \in M_{NVB}$
$\Rightarrow I_{NVB}$ is a neutrosophic vague binary $BZMV^{dM} \sim$ ideal of M_{NVB}
[by definition 5.2.10 (ii)]

(2)⇒(1)

Let I_{NVB} be a neutrosophic vague binary$BZMV^{dM}\sim$ ideal.

Then, $\forall u_x, u_y \in M_{NVB}$,

$NVB_{M_{NVB}}(\neg u_x \oplus u_y) \in I_{NVB}$

$\Rightarrow NVB_{M_{NVB}}(\sim\neg(\neg u_x) \oplus \sim\sim u_y) \in I_{NVB}$

$\Rightarrow NVB_{M_{NVB}}(\sim(\neg\neg u_x) \oplus \sim\sim u_y) \in I_{NVB}$

$\Rightarrow NVB_{M_{NVB}}(\sim u_x \oplus u_y) \in I_{NVB}$

$\Rightarrow NVB_{M_{NVB}}(\sim\neg(\sim u_x) \oplus \sim\sim u_y) \in I_{NVB}$

$\Rightarrow NVB_{M_{NVB}}(\sim(\neg\sim u_x) \oplus \sim\sim u_y) \in I_{NVB}$

$\Rightarrow NVB_{M_{NVB}}(\sim\sim\sim u_x \oplus \sim\sim u_y) \in I_{NVB}$, since $[\neg\sim u_x = \sim\sim u_x]$

$\Rightarrow NVB_{M_{NVB}}(\neg(\sim\sim u_x) \oplus \sim\sim u_y) \in I_{NVB}$

$\Rightarrow NVB_{M_{NVB}}(\neg u_x \oplus u_y) \in I_{NVB}$,

so I_{NVB} is a $NVB \quad BZMV^{dM}$ normal ideal of M_{NVB}.

(1) ⇒ (3)

Let I_{NVB} be a $NVB \quad BZMV^{dM}$ normal ideal of $M_{NVB} \Rightarrow NVB_{I_{NVB}}(\neg u_x \oplus u_y)$

$\Leftrightarrow NVB_{I_{NVB}}(\sim u_x \oplus u_y), \forall u_x, u_y \in M_{NVB}$

$\Rightarrow NVB_{I_{NVB}}(\neg u_x \oplus 1)$

$\Leftrightarrow NVB_{I_{NVB}}(\sim u_x \oplus 1), \forall u_x, u_y \in M_{NVB} \quad [by \quad putting \quad y = 1]$

$\Rightarrow NVB_{I_{NVB}}(\neg(\neg\neg u_x \oplus \neg 1))$

$\Leftrightarrow NVB_{I_{NVB}}(\neg(\neg\sim u_x \oplus \neg 1), \forall u_x, u_y \in M_{NVB} \quad [by \quad definition \quad of \quad \oplus]$

$\Rightarrow NVB_{I_{NVB}}(\neg(\neg\neg u_x \oplus 0))$

$\Leftrightarrow NVB_{I_{NVB}}(\neg(\neg\sim u_x \oplus 0)), \forall u_x, u_y \in M_{NVB}$

$\Rightarrow NVB_{I_{NVB}}(\neg(\neg\neg u_x)) \Leftrightarrow NVB_{I_{NVB}}(\neg(\neg\sim u_x)), \forall u_x, u_y \in M_{NVB}$

$[since\,(u_x \oplus 0) = u_x]$

$\Rightarrow NVB_{I_{NVB}}(\neg\neg(\neg u_x))$

$\Leftrightarrow NVB_{I_{NVB}}(\neg\neg(\sim u_x)), \forall u_x, u_y \in M_{NVB}$

$\Rightarrow NVB_{I_{NVB}}(\neg u_x)$

$\Leftrightarrow NVB_{I_{NVB}}(\sim u_x), \forall u_x, u_y \in M_{NVB} \quad [since \neg\neg(u_x) = u_x]$

(3)⇒(1)

Suppose,

$NVB_{I_{NVB}}(\sim u_x) \in I_{NVB} \Leftrightarrow NVB_{I_{NVB}}(\neg u_x) \in I_{NVB}$

$NVB_{I_{NVB}}(\sim u_x \oplus u_y) \in I_{NVB} \Leftrightarrow NVB_{I_{NVB}}(\sim u_x \oplus \sim\sim u_y) \in I_{NVB}$

$[since \quad u_y = \sim\sim u_y, by \quad definition]$

$\Rightarrow NVB_{I_{NVB}}(\neg(\neg\sim u_x \oplus \neg u_y)) \in I_{NVB}$ [by definition of $\oplus$]

$\Rightarrow NVB_{I_{NVB}}(\neg(\sim\sim u_x \oplus \neg u_y)) \in I_{NVB}[\neg \sim u_x = \sim\sim u_x]$
$\Rightarrow NVB_{I_{NVB}}(\neg(u_x \oplus \sim u_y)) \in I_{NVB}\,[given \sim\sim u_x = u_x]$
$\Rightarrow NVB_{I_{NVB}}(\neg(u_x \oplus \neg \sim u_y)) \in I_{NVB}$
$\Rightarrow NVB_{I_{NVB}}(\neg u_x \oplus \neg \sim u_y) \in I_{NVB}$ [by property of $\oplus$]
$\Rightarrow NVB_{I_{NVB}}(\neg u_x \oplus \sim\sim u_y) \in I_{NVB}\,[\neg \sim u_y = \sim\sim u_y]$
$\Rightarrow NVB_{I_{NVB}}(\neg u_x \oplus u_y) \in I_{NVB}$ [by using the given property $\sim\sim u_x = u_x$]
$\therefore I_{NVB}$ is a NVB BZMV^{dM} ideal

Direct sum of NVB BZMV^{dM} subalgebra

In this section, we obtain a NVB BZMV^{dM} subalgebra by combining two NVB BZMV$^{dM}-$ subalgebras having $\{0,1\}$ as common elements.

Theorem 5.2.13.
Let $\mathcal{M}_{M_{NVB}} = \langle U^{\mathcal{M}_{M_{NVB}}}, \oplus_1, \neg_1, \sim_1, 0, 1\rangle$ and $\mathcal{M}_{P_{NVB}} = \langle U^{\mathcal{M}_{P_{NVB}}}, \oplus_2, \neg_2, \sim_2, 0, 1\rangle$ be two NVB BZMV ^{dM} subalgebras such that $\left(U^{\mathcal{M}_{M_{NVB}}} \cap U^{\mathcal{M}_{P_{NVB}}}\right) = \{0,1\}$. Let $U^{\mathcal{M}_{W_{NVB}}} = \left(U^{\mathcal{M}_{M_{NVB}}} \cup U^{\mathcal{M}_{P_{NVB}}}\right)$ and let a binary operation $\ominus$ be defined on Z as follows:

$$(u_a \ominus u_b) = \begin{cases} (u_a \oplus_1 u_b) & if\, u_a, u_b \in U^{\mathcal{M}_{M_{NVB}}} \\ (u_a \oplus_2 u_b) & if\, u_a, u_b \in U^{\mathcal{M}_{P_{NVB}}} \\ u_a & otherwise \end{cases}$$

$$\neg^{\circledast} u_a = \begin{cases} \neg_1 u_a & if \quad u_a \in U^{\mathcal{M}_{M_{NVB}}} \\ \neg_2 u_a & if \quad u_a \in U^{\mathcal{M}_{P_{NVB}}} \end{cases}$$

$$\sim^{\circledast} u_a = \begin{cases} \sim_1 u_a & if \quad u_a \in U^{\mathcal{M}_{M_{NVB}}} \\ \sim_2 u_a & if \quad u_a \in U^{\mathcal{M}_{P_{NVB}}} \end{cases}$$

Then, $\langle U^{M_{W_{NVB}}}, \ominus, \neg^{\circledast}, \sim^{\circledast}, 0, 1\rangle$ is a NVB BZMV^{dM} subalgebra $M_{Z_{NVB}}$.
Here, $\ominus$ denotes direct sum.

Proof.

1. Let $u_a, u_b \in U^{\mathcal{M}_{M_{NVB}}}$ and $u_c \in U^{\mathcal{M}_{P_{NVB}}}$
   $NVB_{M_{NVB}}((u_a \oplus u_b) \oplus u_c) = NVB_{M_{NVB}}((u_a \oplus_1 u_b) \oplus u_c) = NVB_{M_{NVB}}(u_a \oplus_1 u_b)$
   $NVB_{M_{NVB}}((u_b \oplus u_c) \oplus u_a) = NVB_{M_{NVB}}(u_b \oplus u_a) = NVB_{M_{NVB}}(u_a \oplus u_b)$
   $= NVB_{M_{NVB}}(u_a \oplus_1 u_b)$

2. Let $u_a \in U^{\mathcal{M}_{M_{NVB}}}$ and $0 \in U^{\mathcal{M}_{P_{NVB}}}$
   $NVB_{M_{NVB}}(u_a \oplus 0) = NVB_{M_{NVB}}(u_a)$
   Similarly, all the axioms for a BZMV^{dM} subalgebra can be verified

Case (i) : $u_a, u_b \in U^{\mathcal{M}_{M_{NVB}}}, \left(\forall u_a, u_b \in U^{\mathcal{M}_{M_{NVB}}}\right)$
$(1)\left(inf_{u_v \in (u_a \ominus u_b)} NVB_{M_{NVB}}(u_v)\right)$
$= \left(inf_{u_v \in (u_a \oplus_1 u_b)} NVB_{M_{NVB}}(u_v)\right) \succeq rmin\left\{NVB_{M_{NVB}}(u_a), NVB_{M_{NVB}}(u_b)\right\}$
(2) $\left(\forall u_a \in U^{\mathcal{M}_{M_{NVB}}}\right)$
(i) $NVB_{M_{NVB}}(u_a) \succeq NVB_{M_{NVB}}(\neg^{\circledast} u_a) \Rightarrow NVB_{M_{NVB}}(u_a) \succeq NVB_{M_{NVB}}(\neg_1 \quad u_a)$
(ii)$NVB_{M_{NVB}}(u_a) \succeq NVB_{M_{NVB}}(\sim^{\circledast} u_a) \Rightarrow NVB_{M_{NVB}}(u_a) \succeq NVB_{M_{NVB}}(\sim_1 u_a)$
[Since, $\left\langle U^{\mathcal{M}_{M_{NVB}}}, \oplus_1, \neg_1, \sim_1, 0, 1\right\rangle$ is a NVB BZMV^{dM} subalgebra]

Case (ii) : $u_a, u_b \in U^{\mathcal{M}_{P_{NVB}}}, \left(\forall u_a, u_b \in U^{\mathcal{M}_{P_{NVB}}}\right)$
$(1)\left(inf_{u_v \in (u_a \ominus u_b)} NVB_{P_{NVB}}(u_v)\right)$
$= \left(inf_{u_v \in (u_a \oplus_2 u_b)} NVB_{P_{NVB}}(u_v)\right) \succeq rmin\left\{NVB_{P_{NVB}}(u_a), NVB_{P_{NVB}}(u_b)\right\}$
(2) $\left(\forall u_a \in U^{\mathcal{M}_{P_{NVB}}}\right)$
(i) $NVB_{M_{NVB}}(u_a) \succeq NVB_{P_{NVB}}(\neg^{\circledast} u_a) \Rightarrow NVB_{P_{NVB}}(u_a) \succeq NVB_{P_{NVB}}(\neg_2 \quad u_a)$
(ii)$NVB_{P_{NVB}}(u_a) \succeq NVB_{P_{NVB}}(\sim^{\circledast} u_a) \Rightarrow NVB_{P_{NVB}}(u_a) \succeq NVB_{P_{NVB}}(\sim_2 u_a)$
[Since, $\left\langle U^{\mathcal{M}_{P_{NVB}}}, \oplus_2, \neg_2, \sim_2, 0, 1\right\rangle$ is a NVB BZMV^{dM} subalgebra]

Case (iii) : $\forall u_a \in U^{\mathcal{M}_{M_{NVB}}}, \quad u_b \in U^{\mathcal{M}_{P_{NVB}}} \quad$ or $\quad u_a \in U^{\mathcal{M}_{P_{NVB}}}, u_b \in U^{\mathcal{M}_{M_{NVB}}}$
(1) $\left(\forall u_x \in U^{\mathcal{M}_{M_{NVB}}}, u_y \in U^{\mathcal{M}_{P_{NVB}}} \quad or \quad u_x \in U^{\mathcal{M}_{P_{NVB}}}, u_y \in U^{\mathcal{M}_{M_{NVB}}}\right)$
$\left(inf_{u_v \in (u_a \ominus u_b)} NVB_{M_{NVB}}(u_v)\right)$
$= \left(inf_{u_v \in (u_a)} NVB_{M_{NVB}}(u_v)\right) \succeq rmin\left\{NVB_{M_{NVB}}(u_a), NVB_{M_{NVB}}(u_b)\right\}$
Being a unary operation, 2^{nd} axiom does not exists.
Clearly all the conditions for a NVB BZMV^{dM} subalgebra is verified. It is clear that combining of two NVB BZMV^{dM} subalgebras, will produce the same.

5.3 Neutrosophic Vague Binary K - subalgebra of K - algebra

Pioneer work to K - algebra has done by Akram and Dar and can be found in the papers [3, 23, 24]. Being strongly related to groups, K - algebras are renamed as K(G)-algebras where G indicates the underlying group structure. This algebraic structure has a two faced behaviour one is its non - commutative face and the other one is non-associative. This algebra possess right identity u_e , but left identity law violates for this algebra. Underlying structure $(G, \cdot, u_e)$ of K(G) has two different choices, one as a non-abelian group and the other one to be as an abelian group.

K - algebra in Neutrosophic Vague Binary Sets

In this section, an attempt has done to apply K - algebra into NVBS. Ideals, homomorphism, direct product and some related theorems are also discussed.

Definition 5.3.1.

(Neutrosophic Vague Binary K -subalgebra of K - algebra)

A neutrosophic vague binary K-subalgebra (NVB K- subalgebra, in short) is an algebraic structure $\mathcal{K}_{M_{NVB}} = \left(U^{\mathcal{K}_{M_{NVB}}}, \cdot, \odot, u_e\right)$ *which satisfies the following two* $\mathcal{K}_{M_{NVB}}$ *inequalities*

$\mathcal{K}_{M_{NVB}}$ ***inequality (1):*** $NVB_{M_{NVB}}(u_e) \succeq NVB_{M_{NVB}}(u_s); \quad \forall \quad u_s \quad \in \quad U$

That is, $\hat{T}_{M_{NVB}}(u_e) \geqslant \hat{T}_{M_{NVB}}(u_s); \hat{I}_{M_{NVB}}(u_e) \leqslant \hat{I}_{M_{NVB}}(u_s); \hat{F}_{M_{NVB}}(u_e) \leqslant \hat{F}_{M_{NVB}}(u_s)$

$\mathcal{K}_{M_{NVB}}$ ***inequality (2):*** $NVB_{M_{NVB}}(u_a \odot u_b) \succeq rmin\left\{NVB_{M_{NVB}}(u_a), NVB_{M_{NVB}}(u_b)\right\}$

$$i.e., \quad \left.\begin{aligned} \hat{T}_{M_{NVB}}(u_a \odot u_b) &\geqslant min\left\{\hat{T}_{M_{NVB}}(u_a), \hat{T}_{M_{NVB}}(u_b)\right\} \\ \hat{I}_{M_{NVB}}(u_a \odot u_b) &\leqslant max\left\{\hat{I}_{M_{NVB}}(u_a), \hat{I}_{M_{NVB}}(u_b)\right\} \\ \hat{F}_{M_{NVB}}(u_a \odot u_b) &\leqslant max\left\{\hat{F}_{M_{NVB}}(u_a), \hat{F}_{M_{NVB}}(u_b)\right\} \end{aligned}\right\} \forall u_a, u_b \in U$$

[$\cdot$, $\odot$ *and* u_e *are as in* $U^{\mathcal{K}_{M_{NVB}}}$ & $\hat{T} = [T^-, T^+], \hat{I} = [I^-, I^+], \hat{F} = [F^-, F^+]$]

Here,

- $U^{\mathcal{K}_{M_{NVB}}} = (U = \{U_1 \cup U_2\}, \cdot, \odot, u_e)$ *is a K- algebraic structure with a binary operation* $\cdot$*, an induced binary operation* $\odot : U \times U \to U$ *is defined by* $\odot : (u_x, u_y) = (u_x \odot u_y) = u_x.(u_y)^{-1}$ *and an identity element* u_e *defined on the group* $(U = \{U_1 \cup U_2\}, \cdot, u_e)$ *in which each non-identity element is not of order 2 and satisfying the following* $\odot$ *axioms :*

$\forall u_x, u_y, u_z \in U$

(i) $(u_x \odot u_y) \odot (u_x \odot u_z) = (u_x \odot ((u_e \odot u_z) \odot (u_e \odot u_y))) \odot u_x$

(ii) $u_x \odot (u_x \odot u_y) = (u_x \odot (u_y)^{-1}) \odot u_x = (u_x \odot (u_e \odot u_y)) \odot u_x$

(iii) $(u_x \odot u_x) = u_e$

(iv) $(u_x \odot u_e) = u_x$

(v) $(u_e \odot u_x) = (u_x)^{-1}$

Remark 5.3.2.

$u_x = u_y \Rightarrow NVB_{M_{NVB}}(u_x) = NVB_{M_{NVB}}(u_y) \quad ; \quad \forall u_x, u_y \in U = \{U_1 \cup U_2\}$

[Above is true only for Cantorian Sets. For multisets this won't be true in general]

Converse need not be true.

That is, $NVB_{M_{NVB}}(u_x) = NVB_{M_{NVB}}(u_y) \nRightarrow u_x = u_y$ *in general !*

i.e., same neutrosophic vague binary membership grades do not always imply equality of those elements.

For example, consider,

$$NVB_{M_{NVB}}(u_s) = \begin{cases} [0.7, 0.9], [0.4, 0.4], [0.1, 0.3]; u_s = u_e \\ [0.7, 0.8], [0.4, 0.7], [0.2, 0.3]; u_s = u_a \\ [0.7, 0.9], [0.4, 0.7], [0.1, 0.3]; u_s = u_b \\ [0.7, 0.8], [0.4, 0.7], [0.2, 0.3]; u_s = u_c \end{cases}$$

$NVB_{M_{NVB}}(u_a) = NVB_{M_{NVB}}(u_c) \nRightarrow u_a = u_c$.

That is, here, $NVB_{M_{NVB}}(u_a) = NVB_{M_{NVB}}(u_c) \Rightarrow u_a \neq u_c$

Theorem 5.3.3.

Let $\left(U^{\mathcal{K}_{M_{NVB}}}, \cdot, \odot, u_e\right)$ *be a NVB K - subalgebra* $\mathcal{K}_{M_{NVB}}$ *on abelian group* $(U, \cdot)$ *which is not elementary abelian 2 - group. Then the following results hold within* $\left(U^{\mathcal{K}_{M_{NVB}}}, \cdot, \odot, u_e\right)$

1. $NVB_{M_{NVB}}\left((u_x \odot u_y) \odot u_z\right) = NVB_{M_{NVB}}\left((u_x \odot u_z) \odot u_y\right)$

2. $NVB_{M_{NVB}}\left(u_x \odot u_y\right) = NVB_{M_{NVB}}(u_e) \Leftrightarrow NVB_{M_{NVB}}(u_x) = NVB_{M_{NVB}}(u_y)$

3. $NVB_{M_{NVB}}\left(u_e \odot u_x\right) = NVB_{M_{NVB}}(u_e) \Leftrightarrow NVB_{M_{NVB}}(u_x) = NVB_{M_{NVB}}(u_e)$

4. $NVB_{M_{NVB}}\left(u_e \odot u_x\right) = NVB_{M_{NVB}}(u_x) \Leftrightarrow u_x$ *is of order 2 in* $(U, \cdot)$

Proof.

1. $NVB_{M_{NVB}}\left((u_x \odot u_y) \odot u_z\right) = NVB_{M_{NVB}}\left((u_x \cdot (u_y)^{-1}) \odot u_z\right)$
   $= NVB_{M_{NVB}}\left((u_x \cdot (u_y)^{-1}) \cdot (u_z)^{-1}\right)$
   $= NVB_{M_{NVB}}\left((u_x \cdot (u_z)^{-1}) \cdot (u_y)^{-1}\right)$ [because $(U, \cdot)$ is abelian]
   $NVB_{M_{NVB}}\left((u_x \odot u_z) \cdot (u_y)^{-1}\right) = NVB_{M_{NVB}}\left((u_x \odot u_z) \odot u_y\right),$
   $\forall u_x, u_y, u_z \in \text{U}$

2. $NVB_{M_{NVB}}(u_x \odot u_y) = NVB_{M_{NVB}}(u_e)$
   $\Leftrightarrow NVB_{M_{NVB}}(u_x \cdot (u_y)^{-1}) = NVB_{M_{NVB}}(u_e)$
   $\Leftrightarrow (u_x \cdot (u_y)^{-1}) = u_e$, for all $u_x, u_y \in U$
   $\Leftrightarrow u_x = u_y$, for all $u_x, u_y \in U$
   $\Leftrightarrow NVB_{M_{NVB}}(u_x) = NVB_{M_{NVB}}(u_y)$, for all $u_x, u_y \in U$

3. $NVB_{M_{NVB}}(u_e \odot u_x) = NVB_{M_{NVB}}(u_e)$
   $\Leftrightarrow NVB_{M_{NVB}}(u_e \cdot (u_x)^{-1}) = NVB_{M_{NVB}}(u_e)$
   $\Leftrightarrow (u_x)^{-1} = u_e \Leftrightarrow u_x = u_e$, since $(U, \cdot)$ is abelian $(u_x)^{-1} = u_x$
   $\Leftrightarrow NVB_{M_{NVB}}(u_x) = NVB_{M_{NVB}}(u_e)$

4. If $NVB_{M_{NVB}}(u_e \odot u_x) = NVB_{M_{NVB}}(u_x)$
   $\Leftrightarrow NVB_{M_{NVB}}(u_e \cdot (u_x)^{-1}) = NVB_{M_{NVB}}(u_x)$
   $\Leftrightarrow NVB_{M_{NVB}}(u_x) = NVB_{M_{NVB}}(u_x)^{-1}$
   $\Leftrightarrow u_x = (u_x)^{-1}$
   $\Leftrightarrow u_x$ in U is of order 2 in $(U, \cdot)$

Theorem 5.3.4.
$NVB_{M_{NVB}}((u_x \odot u_y) \odot u_z) = NVB_{M_{NVB}}(u_x \odot (u_y \odot u_y (u_e \odot u_z)))$,
$\forall \quad u_x, u_y, u_z \in U$

Proof.
$NVB_{M_{NVB}}((u_x \odot u_y) \odot u_z) = NVB_{M_{NVB}}((u_x \cdot (u_y)^{-1}) \odot u_z)$
$= NVB_{M_{NVB}}((u_x \cdot (u_y)^{-1}) \cdot (u_z)^{-1}) = NVB_{M_{NVB}}\left((u_x \cdot (u_y \cdot u_z)^{-1})\right)$
$= NVB_{M_{NVB}}(u_x \odot (u_y \cdot u_z)) = NVB_{M_{NVB}}((u_x \odot (u_y \odot (u_z)^{-1})))$
$= NVB_{M_{NVB}}((u_x \odot (u_y \odot (u_e \odot u_z)))) \ \forall u_x, u_y, u_z \in U$

Theorem 5.3.5.
$NVB_{M_{NVB}}(u_x \odot (u_y \odot u_z)) = NVB_{M_{NVB}}((u_x \odot u_y) \odot (u_e \odot u_z))$

Proof.
$\forall u_x, u_y, u_z \in U$,
$NVB_{M_{NVB}}(u_x \odot (u_y \odot u_z))$
$= NVB_{M_{NVB}}(u_x \odot (u_y \cdot u_z^{-1})) = NVB_{M_{NVB}}\left(u_x \cdot (u_y \cdot u_z^{-1})^{-1}\right)$
$= NVB_{M_{NVB}}\left(u_x \cdot (u_y^{-1} \cdot u_z)\right) = NVB_{M_{NVB}}\left((u_x \cdot u_y^{-1}) \cdot u_z\right)$
$= NVB_{M_{NVB}}((u_x \odot u_y) \cdot u_z^{-1}) = NVB_{M_{NVB}}((u_x \odot u_y) \odot (u_e \odot u_z))$

Theorem 5.3.6.

Let $\mathcal{K}_{M_{NVB}}$ be a NVB K - subalgebra on a non-abelian group U.
Then the following identities hold in $\mathcal{K}_{M_{NVB}}$, for all $u_x, u_y, u_z \in U$

1. $NVB_{M_{NVB}}[u_x \odot (u_y \odot u_z)] = NVB_{M_{NVB}}[u_x \odot (u_z \odot (u_e \odot u_y))]$
2. $NVB_{M_{NVB}}[(u_x \odot u_y) \odot u_z] = NVB_{M_{NVB}}[u_x \odot (u_z \odot (u_e \odot u_y))]$
3. $NVB_{M_{NVB}}[u_e \odot (u_x \odot u_y)] = NVB_{M_{NVB}}(u_y \odot u_x)$

Proof.
(1) $\Rightarrow$ (2)
Assume, $NVB_{M_{NVB}}[u_x \odot (u_y \odot u_z)] = NVB_{M_{NVB}}[u_x \odot (u_z \odot (u_e \odot u_y))]$
Consider, $NVB_{M_{NVB}}[(u_x \odot u_y) \odot u_z] = NVB_{M_{NVB}}\left[\left(u_x \cdot u_y^{-1}\right) \odot u_z\right]$
$= NVB_{M_{NVB}}\left[\left(u_x \cdot u_y^{-1}\right) \cdot u_z^{-1}\right] = NVB_{M_{NVB}}\left[u_x \cdot \left(u_y^{-1} \cdot u_z^{-1}\right)\right]$
$= NVB_{M_{NVB}}\left[u_x \cdot (u_z \cdot u_y)^{-1}\right] = NVB_{M_{NVB}}[u_x \odot (u_z \cdot u_y)]$
$= NVB_{M_{NVB}}\left[u_x \odot \left(u_z \odot u_y^{-1}\right)\right] = NVB_{M_{NVB}}[u_x \odot (u_z \odot (u_e \odot u_y))]$
(2) $\Rightarrow$ (3)
Assume (2). Consider,
$NVB_{M_{NVB}}[u_e \odot (u_x \odot u_y)]$
$= NVB_{M_{NVB}}\left[u_e \odot \left(u_z \odot \left(u_x \cdot u_y^{-1}\right)\right)\right]$
$= NVB_{M_{NVB}}\left[u_e \cdot \left(u_x \cdot u_y^{-1}\right)^{-1}\right]$
$= NVB_{M_{NVB}}\left[u_e \cdot \left(u_y \cdot u_x^{-1}\right)\right]$
$= NVB_{M_{NVB}}\left[(u_e \cdot u_y) \cdot u_x^{-1}\right]$
$= NVB_{M_{NVB}}(u_e \cdot u_y) \odot u_x$
$= NVB_{M_{NVB}}\left[\left(u_e \odot \left(u_y^{-1}\right) \odot u_x\right)\right]$
$= NVB_{M_{NVB}}\left[u_e \odot \left(u_x \odot \left(u_e \odot u_y^{-1}\right)\right)\right]$, by using assumption
$= NVB_{M_{NVB}}[u_e \odot (u_x \odot u_y)]$, [since $(U, \cdot)$ is a non-abelian group]
$= NVB_{M_{NVB}}\left[(u_x \odot u_y)^{-1}\right]$, [since $(U, \cdot)$ is a non-abelian group]
$= NVB_{M_{NVB}}\left(u_x \cdot u_y^{-1}\right)^{-1}$
$= NVB_{M_{NVB}}\left(u_y \cdot u_x^{-1}\right)$, [since $(U, \cdot)$ is a non-abelian group]
$= NVB_{M_{NVB}}(u_y \odot u_x)$
(3)$\Rightarrow$ (1)
Assume (3). $NVB_{M_{NVB}}[u_e \odot (u_x \odot u_y)] = NVB_{M_{NVB}}(u_y \odot u_x)$
Consider, $NVB_{M_{NVB}}(u_x \odot (u_y \odot u_z)) = NVB_{M_{NVB}}([u_e \odot (u_z \odot u_y)])$, by (3)
$= NVB_{M_{NVB}}(u_x \odot (u_y \odot u_z))$, since for non - abelian groups
$NVB_{M_{NVB}}[u_e \odot (u_z \odot u_y)] = NVB_{M_{NVB}}(u_y \odot u_z)$

$= NVB_{M_{NVB}}\left(u_x \odot (u_y \cdot u_z^{-1})\right) = NVB_{M_{NVB}}\left(u_x \cdot (u_y \cdot u_z^{-1})^{-1}\right)$
$= NVB_{M_{NVB}}\left(u_x \cdot (u_z \cdot u_y^{-1})\right) = NVB_{M_{NVB}}\left(u_x \cdot (u_z \odot u_y)\right)$
$= NVB_{M_{NVB}}\left(u_x \odot (u_z \odot u_y)^{-1}\right)$

Abelian Neutrosophic Vague Binary K - Subalgebra of K - algebra

This section explores abelian neutrosophic vague binary K - subalgebra of K - algebra. If in a NVB K - subalgebra $\mathcal{K}_{M_{NVB}}$, the underlying group structure is abelian then it is called an abelian neutrosophic vague binary K - subalgebra. It's structure is given by $\mathcal{K}_{M_{NVB}}^{Abelian} = \left(U^{\mathcal{K}_{M_{NVB}}^{Abelian}}, \cdot, \odot, u_e\right)$ where $\mathcal{K}_{M_{NVB}}^{Abelian} = (U = \{U_1 \cup U_2\}, \cdot, u_e)$ is an abelian group. i.e., the binary operation "$\cdot$" will be commutative.

Definition 5.3.7.

(Abelian Neutrosophic Vague Binary K - subalgebra of K - algebra)

A NVB K - subalgebra $\mathcal{K}_{M_{NVB}}$ is called abelian

$\Leftrightarrow NVB_{M_{NVB}}\left(u_x \odot (u_e \odot u_y)\right) = NVB_{M_{NVB}}\left(u_y \odot (u_e \odot u_x)\right)$ *; for all $u_x, u_y \in U$*

Remark 5.3.8.

Above can be written as, $\left(u_x \odot (u_e \odot u_y)\right) = \left(u_x \odot u_y^{-1}\right) = \left(u_x \odot \left(u_y^{-1}\right)^{-1}\right) = (u_x \cdot u_y)$

Similarly, $\left(u_y \odot (u_e \odot u_x)\right) = \left(u_y \odot u_x^{-1}\right) = \left(u_y \cdot (u_x^{-1})^{-1}\right) = (u_x \cdot u_y)$

That is, $NVB_{M_{NVB}}(u_x \cdot u_y) = NVB_{M_{NVB}}(u_y \cdot u_x)$, *commutativity is satisfied for the binary operation "$\cdot$"*

Remark 5.3.9.

(Abelian Neutrosophic Vague Binary K - subalgebra of K - algebra)

An abelian neutrosphic vague binary K - subalgebra(abelian NVB K - subalgebra, in short) is a structure $\mathcal{K}_{M_{NVB}}^{Abelian} = \left(U^{\mathcal{K}_{M_{NVB}}^{Abelian}}, \cdot, \odot, u_e\right)$ *which satisfies, the following two abelian* $\mathcal{K}_{M_{NVB}}^{Abelian}$ *inequalities:*

Abelian $\mathcal{K}_{M_{NVB}}^{Abelian}$ inequality (1) :

$NVB_{M_{NVB}}(u_e) \succcurlyeq NVB_{M_{NVB}}(u_s); \forall u_s \in U$

i.e., $\hat{T}_{M_{NVB}}(u_e) \geqslant \hat{T}_{M_{NVB}}(u_s); \hat{I}_{M_{NVB}}(u_e) \leqslant \hat{I}_{M_{NVB}}(u_s); \hat{F}_{M_{NVB}}(u_e) \leqslant \hat{F}_{M_{NVB}}(u_s)$

Abelian $\mathcal{K}_{M_{NVB}}^{Abelian}$ inequality (2) :

$NVB_{M_{NVB}}(u_a \odot u_b) \succcurlyeq rmin\left\{NVB_{M_{NVB}}(u_a), NVB_{M_{NVB}}(u_b)\right\}; \forall u_a, u_b \in U$

That is,

$$\hat{T}_{M_{NVB}}(u_a \odot u_b) \leqslant max\left\{\hat{T}_{M_{NVB}}(u_a), \hat{T}_{M_{NVB}}(u_b)\right\}$$
$$\hat{I}_{M_{NVB}}(u_a \odot u_b) \geqslant min\left\{\hat{I}_{M_{NVB}}(u_a), \hat{I}_{M_{NVB}}(u_b)\right\}$$
$$\hat{F}_{M_{NVB}}(u_a \odot u_b) \leqslant max\left\{\hat{F}_{M_{NVB}}(u_a), \hat{F}_{M_{NVB}}(u_b)\right\}$$

[$\cdot$, $\odot$ *and* u_e *are as in* $U^{\mathcal{K}^{Abelian}_{M_{NVB}}}$ $\&\hat{T} = [T^-, T^+]; \hat{I} = [I^-, I^+]; \hat{F} = [F^-, F^+]$]

Here,

- $U^{\mathcal{K}^{Abelian}_{M_{NVB}}} = (U = \{U_1 \cup U_2\}, \cdot, \odot, u_e)$ *is an abelian K - algebraic structure, with a binary operation* $\cdot$, *an induced operation* $\odot : U \times U \rightarrow U$ *is defined by* $\odot(u_x, u_y) = (u_x \odot u_y) = u_x.(u_y)^{-1}$ *and an identity element* u_e *defined on an abelian group* $(U = \{U_1 \cup U_2\}, \cdot, u_e)$ *in which each non-identity element is not of order 2, satisfying the following* $\odot$ *- axioms:* $\forall u_x, u_y, u_z \in U$

(i) $(u_x \odot u_y) \odot (u_x \odot u_z) = (u_z \odot u_y)$ *(ii)* $u_x \odot (u_x \odot u_y) = u_y$

(iii) $(u_x \odot u_x) = u_e$ *(iv)* $(u_x \odot u_e) = u_x$ *(v)* $(u_e \odot u_x) = (u_x)^{-1}$

Example 5.3.10.

Let $U_1 = \{u_e, u_a, u_b\}$ *and* $U_2 = \{u_e, u_b, u_c\}$ *be two universes under consideration with identity element* u_e. *Based on this, consider a NVBS,*

$M_{NVB} =$

$$\left\{\begin{array}{l}\left\langle \frac{[0.6,0.9],[0.4,0.4],[0.1,0.4]}{u_e}, \frac{[0.7,0.8],[0.4,0.7],[0.2,0.3]}{u_a}, \frac{[0.7,0.8],[0.5,0.7],[0.2,0.3]}{u_b}\right\rangle, \\ \left\langle \frac{[0.7,0.8],[0.4,0.6],[0.2,0.3]}{u_e}, \frac{[0.4,0.9],[0.4,0.9],[0.1,0.6]}{u_b}, \frac{[0.7,0.8],[0.4,0.7],[0.2,0.3]}{u_c}\right\rangle\end{array}\right\}$$

Combined universe is given by, $U = \{u_e, u_a, u_b, u_c\}$

Let $U^{\mathcal{K}_{M_{NVB}}} = (U, ., \odot, u_e)$ *be a K - algebra (formed based on this combined universe) and the induced binary operation is defined by Cayley Table 5.10.*

Combined neutrosophic vague binary membership grades are given as follows:

$$NVB_{M_{NVB}}(u_s) = \begin{cases} [0.7,0.9],[0.4,0.4],[0.1,0.3]; u_s = u_e \\ [0.7,0.8],[0.4,0.7],[0.2,0.3]; u_s = u_a \\ [0.7,0.9],[0.4,0.7],[0.1,0.3]; u_s = u_b \\ [0.7,0.8],[0.4,0.7],[0.2,0.3]; u_s = u_c \end{cases}$$

Table 5.7: Cayley Table

$\odot$	u_e	u_a	u_b	u_c
u_e	u_e	u_a	u_b	u_c
u_a	u_a	u_e	u_c	u_b
u_b	u_b	u_c	u_e	u_a
u_c	u_c	u_b	u_a	u_e

[*Combined neutrosophic vague binary membership grades are calculated for common elements by taking their neutrosophic vague binary union.*
In this example, $\{U_1 \cup U_2\} = \{u_e, u_b\}$

$$NVB_{M_{NVB}}(u_e) = ([0.6, 0.9], [0.4, 0.4], [0.1, 0.4]) \cup ([0.7, 0.8], [0.4, 0.6], [0.2, 0.3])$$
$$= [0.7, 0.9], [0.4, 0.4], [0.1, 0.3]$$
$$NVB_{M_{NVB}}(u_b) = ([0.7, 0.8], [0.5, 0.7], [0.2, 0.3]) \cup ([0.4, 0.9], [0.4, 0.9], [0.1, 0.6])$$
$$= [0.7, 0.9], [0.4, 0.7], [0.1, 0.3]]$$

$U^{\mathcal{K}_{M_{NVB}}}$ *is called a* $\mathcal{K}_{M_{NVB}}$ *if* $\mathcal{K}_{M_{NVB}}$ *- inequalities got satisfied. After verification it is clear that,* M_{NVB} *is a neutrosophic vague binary K - subalgebra. That is, considered* M_{NVB} *is* $\mathcal{K}^{Abelian}_{M_{NVB}}$

Theorem 5.3.11.

$\mathcal{K}_{M_{NVB}}$ *is* $\mathcal{K}^{Abelian}_{M_{NVB}}$ *then the following results hold good*

1. $NVB_{M_{NVB}}(u_e \odot u_x) = NVB_{M_{NVB}}(u_x) \Leftrightarrow u_x$ *is of order 2 in* U, $\forall u_x \in U$

2. $NVB_{M_{NVB}}(u_e \odot u_x) = NVB_{M_{NVB}}(u_e)$
   $\Leftrightarrow NVB_{M_{NVB}}(u_x) = NVB_{M_{NVB}}(u_e); \quad \forall u_x \in U$

Proof.

1. $\mathcal{K}_{M_{NVB}}$ is $\mathcal{K}^{Abelian}_{M_{NVB}}$
   $\Rightarrow NVB_{M_{NVB}}\left(u_x \odot (u_e \odot u_y)\right) = NVB_{M_{NVB}}\left(u_y \odot (u_e \odot u_x)\right); \forall u_x \in U$
   Let $NVB_{M_{NVB}}(u_e \odot u_x) = NVB_{M_{NVB}}(u_x) \Rightarrow NVB_{M_{NVB}}(u_e.(u_x)^{-1})$
   $= NVB_{M_{NVB}}(u_x) \Leftrightarrow NVB_{M_{NVB}}((u_x)^{-1}) = NVB_{M_{NVB}}(u_x)$
   $\Leftrightarrow$ there exists 2 possibilities

   (a) $(u_x)^{-1} = u_x$

   (b) $(u_x)^{-1} \neq u_x$

$\Leftrightarrow$ since underlying group structure is abelian only (1), is admissible
$\Leftrightarrow$ u_x is of order 2 in U

2. $NVB_{M_{NVB}}\left(u_x \odot (u_e \odot u_y)\right) = NVB_{M_{NVB}}\left(u_y \odot (u_e \odot u_x)\right)$
$\Leftrightarrow NVB_{M_{NVB}}\left(u_x \odot (u_e \odot u_e)\right) = NVB_{M_{NVB}}\left(u_e \odot (u_e \odot u_x)\right)$, [put $u_y = u_e$]
$\Leftrightarrow NVB_{M_{NVB}}(u_x \odot u_e) = NVB_{M_{NVB}}(u_e \odot u_e)$
$\Leftrightarrow NVB_{M_{NVB}}(u_x.(u_e)^{-1}) = NVB_{M_{NVB}}(u_e)$, for all $u_x \in U$
$\Leftrightarrow NVB_{M_{NVB}}(u_x.u_e) = NVB_{M_{NVB}}(u_e)$, for all $u_x \in$ U
$\Leftrightarrow NVB_{M_{NVB}}(u_x) = NVB_{M_{NVB}}(u_e)$, for all $u_x \in U$

Theorem 5.3.12.
In a NVB K - subalgebra $\mathcal{K}_{M_{NVB}}$ the following statements are equivalent:
$\forall u_x, u_y, u_z \in U$

1. $\mathcal{K}_{M_{NVB}}$ *is* $\mathcal{K}^{Abelian}_{M_{NVB}}$
2. $NVB_{M_{NVB}}\left(u_x \odot (u_e \odot u_y)\right) = NVB_{M_{NVB}}\left(u_y \odot (u_e \odot u_x)\right)$
3. $NVB_{M_{NVB}}\left(u_x \odot (u_x \odot u_y)\right) = NVB_{M_{NVB}}\left(u_x\right)$
4. $NVB_{M_{NVB}}\left((u_x \odot u_y) \odot u_z\right) = NVB_{M_{NVB}}\left((u_x \odot u_z) \odot u_y\right)$
5. $NVB_{M_{NVB}}\left((u_e \odot u_x) \odot (u_e \odot u_y)\right) = NVB_{M_{NVB}}\left(u_e \odot (u_x \odot u_y)\right)$
6. $NVB_{M_{NVB}}\left((u_x \odot u_y) \odot (u_x \odot u_z)\right) = NVB_{M_{NVB}}\left(u_z \odot u_y\right)$

Proof.
Let $\mathcal{K}_{M_{NVB}}$ be a NVB K - subalgebra
(1) $\Rightarrow$ (2).
Let $\mathcal{K}_{M_{NVB}}$ is $\mathcal{K}^{Abelian}_{M_{NVB}}$
$\Rightarrow NVB_{M_{NVB}}\left(u_x \odot (u_e \odot u_y)\right) = NVB_{M_{NVB}}\left(u_y \odot (u_e \odot u_x)\right)$,
[From Definition 5.3.7]
(2)$\Rightarrow$ (3).
Let $NVB_{M_{NVB}}\left(u_x \odot (u_e \odot u_y)\right) = NVB_{M_{NVB}}\left(u_y \odot (u_e \odot u_x)\right)$
$\Rightarrow NVB_{M_{NVB}}\left(u_x \odot (u_x \odot u_y)\right) = NVB_{M_{NVB}}\left(u_y \odot (u_x \odot u_x)\right)$, by putting $u_e = u_x$
$\Rightarrow NVB_{M_{NVB}}\left(u_x \odot (u_x \odot u_y)\right) = NVB_{M_{NVB}}\left(u_y \odot u_e\right)$,by definition 5.3.1 (iii)
$\Rightarrow NVB_{M_{NVB}}\left(u_x \odot (u_x \odot u_y)\right) = NVB_{M_{NVB}}\left(u_y\right)$, by definition 5. 3.1 (iv)
(3)$\Rightarrow$ (4).

Let $NVB_{M_{NVB}}(u_x \odot (u_x \odot u_y)) = NVB_{M_{NVB}}(u_y)$.
Consider, $NVB_{M_{NVB}}((u_x \odot u_y) \odot u_z) = NVB_{M_{NVB}}\left(\left(u_x.(u_y)^{-1}\right).(u_z)^{-1}\right)$
$= NVB_{M_{NVB}}\left(\left(u_x.(u_z)^{-1}\right).(u_y)^{-1}\right)$, being a group, $(U, .)$ is associative
$= NVB_{M_{NVB}}((u_x.u_z).u_y)$
(4)⇒ (5).
Assume $NVB_{M_{NVB}}((u_x \odot u_y) \odot u_z) = NVB_{M_{NVB}}((u_x \odot u_z) \odot u_y)$.
Consider $NVB_{M_{NVB}}((u_e \odot u_x) \odot (u_e \odot u_y)) = NVB_{M_{NVB}}(u_y \odot (u_e \odot (u_e \odot u_x)))$
$\Rightarrow NVB_{M_{NVB}}((u_e \odot u_x) \odot (u_e \odot u_y)) = NVB_{M_{NVB}}(u_y \odot u_x)$
$\Rightarrow NVB_{M_{NVB}}((u_e \odot u_x) \odot (u_e \odot u_y)) = NVB_{M_{NVB}}(u_e \odot (u_x \odot u_y))$
[since $(u_x \odot u_x) = u_e \Rightarrow (u_x)^{-1} = u_x$ & $(u_y \odot u_y) = u_e \Rightarrow (u_y)^{-1} = (u_y)$].
$\therefore u_e \odot (u_x \odot u_y) = (u_x \odot u_y)^{-1} = \left((u_y)^{-1} \odot (u_x)^{-1}\right) = (u_y \odot u_x)$
(5) ⇒ (6).
Assume (5). i.e., $NVB_{M_{NVB}}((u_e \odot u_x) \odot (u_e \odot u_y)) = NVB_{M_{NVB}}(u_e \odot (u_x \odot u_y))$.
Consider, $NVB_{M_{NVB}}((u_x \odot u_y) \odot (u_x \odot u_z)) = NVB_{M_{NVB}}((u_e \odot u_y) \odot (u_e \odot u_z))$,
[by putting $u_x = u_e$]
$= NVB_{M_{NVB}}(u_e \odot (y \odot u_z)) = NVB_{M_{NVB}}(u_z \odot u_y)$
(6)⇒(1).
Assume (6). i.e., $NVB_{M_{NVB}}((u_x \odot u_y) \odot (u_x \odot u_z)) = NVB_{M_{NVB}}(u_z \odot u_y)$
Consider, $NVB_{M_{NVB}}((u_x \odot u_y) \odot (u_x \odot u_z)) = NVB_{M_{NVB}}(u_z \odot u_y)$
$\Rightarrow NVB_{M_{NVB}}(u_z \odot (u_e \odot u_y)) = NVB_{M_{NVB}}(u_y \odot (u_e \odot u_z))$,
[by putting $(u_x \odot u_y) = u_z$, $u_x = u_e$, $u_z = u_y$ &
by putting $(u_e \odot u_z) = u_z$ since $u_x = u_e$ & , $u_z = u_y$]
$\Rightarrow \mathcal{K}_{M_{NVB}}$ is a $\mathcal{K}^{Abelian}_{M_{NVB}}$

Theorem 5.3.13.

If the class of NVB K -subalgebra is $\mathcal{K}^{Abelian}_{M_{NVB}}$. Then the following identities hold:

1. $NVB_{M_{NVB}}(u_x \odot (u_e \odot u_y)) = NVB_{M_{NVB}}(u_y \odot (u_e \odot u_x))$
2. $NVB_{M_{NVB}}((u_x \odot u_y) \odot u_z) = NVB_{M_{NVB}}((u_x \odot u_z) \odot u_y)$
3. $NVB_{M_{NVB}}((u_x \odot (u_x \odot u_y)) \odot u_y) = NVB_{M_{NVB}}(u_e)$
4. $NVB_{M_{NVB}}(u_e \odot (u_x \odot u_y)) = NVB_{M_{NVB}}((u_e \odot u_x) \odot (u_e \odot u_y))$
$= NVB_{M_{NVB}}(u_y \odot u_x)$

Proof.

1. Obvious from definition

2. Obvious from definition 5.3.12 (4)

3. $NVB_{M_{NVB}}((u_x \odot (u_x \odot u_y)) \odot u_y) = NVB_{M_{NVB}}(u_y \odot u_y) = NVB_{M_{NVB}}(u_e)$

4. $NVB_{M_{NVB}}(u_e \odot (u_x \odot u_y)) = NVB_{M_{NVB}}((u_e \odot u_x) \odot (u_e \odot u_y))$
   $= NVB_{M_{NVB}}((u_z \odot u_x) \odot (u_z \odot u_y))$, by putting $u_e = u_z, u_x = u_x$ and $u_y = u_y$ in definition $= NVB_{M_{NVB}}(u_y \odot u_x)$

Hence the proof

Theorem 5.3.14.
In an abelian NVB K - subalgebra $\mathcal{K}^{Abelian}_{M_{NVB}}$, the following assertions are equivalent:

1. $NVB_{M_{NVB}}(u_x \odot (u_y \odot u_z))$
2. $NVB_{M_{NVB}}((u_x \odot u_y) \odot (u_e \odot u_z))$
3. $NVB_{M_{NVB}}(u_z \odot (u_y \odot u_x))$

Proof.
(1) ⇒(2)
$NVB_{M_{NVB}}(u_x \odot (u_y \odot u_z)) = NVB_{M_{NVB}}(u_x \odot (u_e \odot (u_z \odot u_y)))$
$= NVB_{M_{NVB}}((u_z \odot u_y) \odot (u_e \odot u_x))$, by definition of $K^{Abelian}_{M_{NVB}}$
$= NVB_{M_{NVB}}((u_z \odot (u_e \odot u_x)) \odot u_y)$,
$= NVB_{M_{NVB}}((u_x \odot (u_e \odot u_z) \odot u_y)$, by definition of $K^{Abelian}_{M_{NVB}}$
$= NVB_{M_{NVB}}((u_x \odot u_y) \odot (u_e \odot u_z))$,
(2) ⇒ (3)
$NVB_{M_{NVB}}((u_x \odot u_y) \odot (u_e \odot u_z))$
$= NVB_{M_{NVB}}(u_z \odot (u_e \odot (u_x \odot u_y)))$, by definition of $K^{Abelian}_{M_{NVB}}$
$= NVB_{M_{NVB}}[u_z \odot (u_y \odot u_x)]$
(3) ⇒(1)
$NVB_{M_{NVB}}(u_z \odot (u_y \odot u_x)) = NVB_{M_{NVB}}(u_z \odot (u_y . (u_x)^{-1}))$
$= NVB_{M_{NVB}}\left(u_z . (u_y . (u_x)^{-1})^{-1}\right) = NVB_{M_{NVB}}(u_z . (u_x . (u_y)^{-1}))$
$= NVB_{M_{NVB}}((u_z . u_x) . (u_y)^{-1}) = NVB_{M_{NVB}}((u_x . u_z) . (u_y)^{-1})$
$= NVB_{M_{NVB}}(u_x . (u_z . (u_y)^{-1})) = NVB_{M_{NVB}}\left(u_x \odot (u_z . (u_y)^{-1})^{-1}\right)$
$= NVB_{M_{NVB}}(u_x \odot (u_y . (u_z)^{-1})) = NVB_{M_{NVB}}(u_x \odot (u_y \odot u_z))$

Homomorphism for NVB K- subalgebra

In this section homomorphism and some related theorems for NVB K subalgebra are discussed.

Definition 5.3.15. ***(Homomorphism between two NVB K-subalgebra)***
Neutrosophic Vague Binary K-homomorphism or Homomorphism between two neutrosophic vague binary K-subalgebras
$K_{M_{NVB}} = \left(U^{K_{M_{NVB}}}, \star, \circledcirc, u_e\right)$ *and* $K_{P_{NVB}} = \left(V^{K_{P_{NVB}}}, \circ, \circledcirc, v_e\right)$ *is a mapping*
$\Psi^{\kappa} : K_{M_{NVB}} \to K_{P_{NVB}}$ *such that,* $\left(NVB_{M_{NVB}\Psi^{\kappa}(u_a)} = NVB_{M_{NVB}}(\Psi^{\kappa}(u_a))\right),$
$\forall \quad u_a \in K_{M_{NVB}}$

Theorem 5.3.16.
Let $K_{M_{NVB}} = \left(U^{(}K_{M_{NVB}}, \star, \circledast, u_e\right)$ *and* $K_{P_{NVB}} = \left(V^{K_{P_{NVB}}}, \circ, \circledcirc, v_e\right)$ *be two NVB K subalgebra and* $\Omega^K \in$ *NVB K Hom* $(K_{M_{NVB}}, K_{P_{NVB}})$. *i.e.,* $\Omega^K : K_{M_{NVB}} \to K_{P_{NVB}}$. *Then for* $u_x, u_y \in K_{M_{NVB}}, \Omega(u_x), \Omega(u_y) \in K_{P_{NVB}}$. *Then we conclude that,*

1. $\Omega^K(u_e) = v_e$
2. $\Omega^K(u_x) = \Omega^K(u_x)^{-1}$
3. $\Omega^K(u_e \circledast u_x) = v_e \circledcirc \Omega^K(u_x)$
4. $\Omega^K(u_x \circledast u_y) = v_e$ *if and only if* $\Omega^K(u_x) = \Omega^K(u_y)$

Proof.

1. Let $K_{M_{NVB}} = \left(U^{K_{M_{NVB}}}, \star, \circledast, u_e\right)$ and $K_{P_{NVB}} = \left(V^{K_{P_{NVB}}}, \circ, \circledcirc, v_e\right)$ be two NVB K subalgebras and let $\Omega^K : K_{M_{NVB}} \to K_{P_{NVB}}$ be a NVB K homomorphism.
   Let $u_x, u_y \in K_{M_{NVB}}$ & also let $\Omega(u_x) = v_x$, $\Omega(u_y) = v_y \in K_{P_{NVB}}$
   $\Omega^K(u_x \circledast u_x) = \Omega^K(u_x) \circledcirc \Omega^K(u_x) = v_x \circledcirc v_x = v_e$
   But, $(u_x \circledast u_x) = u_e \Rightarrow \Omega^K(u_x \circledast u_x) = \Omega^K(u_e) \Rightarrow \Omega^K(u_x) = \Omega^K(u_x)^{-1}$

2. $\Omega^K(u_x \circledast u_e) = \Omega^K(u_x) \Rightarrow \Omega^K(u_x \circledast (u_x \circledast u_x)) = \Omega^K(u_x)$[Since $u_e = (u_x \odot u_x)$]
   $\Rightarrow \Omega^K((u_x \circledast (u_x)^{-1}) \circledast u_x) = \Omega^K(u_x)$ [*Since* $u_x \odot (u_x \odot u_y) = (u_x \odot (u_y)^{-1}) \odot u_x$]
   $\Rightarrow \Omega^K(u_e \circledast u_x) = \Omega^K(u_x) \Rightarrow \Omega^K(u_x)^{-1} = \Omega^K(u_x)$

3. $\Omega^K(u_e \circledast u_x) = \Omega^K(u_e) \circledcirc \Omega^K(u_x) = v_e \circledcirc \Omega^K(u_x)$ [From 5.3.16 (i)]

4. Let $\Omega^K (u_x \circledast u_y) = v_e \Leftrightarrow \Omega^K(u_x) \circledcirc \Omega^K(u_y) = v_e$
$\Rightarrow \Omega^K (u_x) = \Omega^K (u_y)^{-1} \Rightarrow \Omega^K(u_x) = \Omega^K(u_y)$

5.4 Neutrosophic Vague Binary G - subalgebra of G - algebra

G-algebra is an extension work to QS-algebra [84]. In this section neutrosophic vague binary G - subalgebra of G - algebra is developed with its properties and with some theorems. It's axioms are few in number and also very simple to handle. But this is not a reason to discard it. Most of the time, major theories have found to develop from minor elementary facts. So G-algebra also deserves it's importance.

Definition 5.4.1.
Let M_{NVB} be a neutrosophic vague binary set (in short, NVBS) with a binary universe (U_1, U_2). A neutrosophic vague binary G - subalgebra of G - algebra is a structure $\mathcal{G}_{M_{NVB}} = \left(U^{\mathcal{G}_{M_{NVB}}}, \star, 0\right)$ which satisfies the following $\mathcal{G}_{M_{NVB}}$ inequality :
$\mathcal{G}_{M_{NVB}}$ ***inequality :***
$NVB_{M_{NVB}}(u_x \star u_y) \succeq rmin\left\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\right\}; \forall \quad u_x, u_y \in U$
That is,

$$\left.\begin{aligned} \hat{T}_{M_{NVB}}(u_x \star u_y) &\geq min\left\{\hat{T}_{M_{NVB}}(u_x), \hat{T}_{M_{NVB}}(u_x)\right\} \\ \hat{I}_{M_{NVB}}(u_x \star u_y) &\leq max\left\{\hat{I}_{M_{NVB}}(u_x), \hat{I}_{M_{NVB}}(u_x)\right\} \\ \hat{F}_{M_{NVB}}(u_x \star u_y) &\leq max\left\{\hat{F}_{M_{NVB}}(u_x), \hat{F}_{M_{NVB}}(u_x)\right\} \end{aligned}\right. \quad ; \forall\, u_x,\ u_y \in U$$

$[\star$ and $0]$ are as in $U^{\mathcal{G}_{M_{NVB}}}$ & $\hat{T} = [T^-, T^+]; \hat{I} = [I^-, I^+]; \hat{F} = [F^-, F^+]$
where,
• *$U^{\mathcal{G}_{M_{NVB}}} = (U = \{U_1 \cap U_2\}, \star, 0)$ is the underlying G - algebraic structure with a binary operation $\star$ & with a constant 0 which satisfies the following axioms:*
$\forall u_x,\ u_y \in U$
(i) $(u_x \star u_x) = 0$ *(ii)* $(u_x \star (u_x \star u_y)) = u_y$

Example 5.4.2.
Let $U_1 = \{0, u_a, u_b\}, U_2 = \{0, u_b, u_c\}$ be two universes.

Combined universe $U = \{U_1 \cup U_2\} = \{0, u_a, u_b, u_c\}$.

Binary operation $\star$ *is defined as given by the Cayley Table 5.8:*

Table 5.8: Cayley Table

$\star$	0	u_a	u_b	u_c
0	0	u_c	u_b	u_a
u_a	u_a	0	u_c	u_b
u_b	u_b	u_a	0	u_c
u_c	u_c	u_b	u_a	0

Clearly, $(U, \star, 0)$ is a G - algebra. Consider a NVBS formed based on U_1 & U_2

$M_{NVB} =$

$$\left\{\begin{array}{l} \left\langle \frac{[0.9,0.9],[0.2,0.8],[0.1,0.1]}{0}, \frac{[0.8,0.9],[0.3,0.7],[0.1,0.2]}{u_a}, \frac{[0.6,0.9],[0.4,0.6],[0.1,0.4]}{u_b} \right\rangle \\ \left\langle \frac{[0.6,0.9],[0.3,0.6],[0.1,0.4]}{0}, \frac{[0.8,0.8],[0.2,0.7],[0.2,0.2]}{u_b}, \frac{[0.8,0.9],[0.3,0.7],[0.1,0.2]}{u_c} \right\rangle \end{array}\right\}$$

Combined neutrosophic vague binary membership grade is given by,

$$M_{NVB}(u_s) = \begin{cases} [0.9,0.9],[0.2,0.6],[0.1,0.1]; & if \quad u_s = 0; \\ [0.8,0.9],[0.3,0.7],[0.1,0.2]; & if \quad u_s = u_a \\ [0.8,0.9],[0.2,0.6],[0.1,0.2]; & if \quad u_s = u_b \\ [0.8,0.9],[0.3,0.7],[0.1,0.2]; & if \quad u_s = u_c \end{cases}$$

Calculations shows that M_{NVB} is a NVB G - subalgebra

Remark 5.4.3.

In a NVB G - algebra, construction of the underlying G - algebraic structure, using a binary operation $\star$ *deserves prime importance. Instead of* $\star$ *different symbols like* $+, -, \times, +_4$ *etc can be applied. Binary operation can be formed in different ways. Construction of G - algebra using the following points always defines a G - algebra. In the Binary Operation,*

1. *If "first operand = second operand" then the output will be zero.*
   [Using definition of G - algebra, $(u_x \star u_x) = 0 \Rightarrow$ *principal diagonal elements should occupy with constant 0, in the Cayley Table of a G - algebra]*

2. *If "first operand $\neq$ second operand " with "first operand $\neq 0$ & second operand $= 0$", then output will be first operand*

3. *If "first operand $\neq$ second operand " with "first operand $\neq 0$ & "the second operand $\neq 0$ ", then output will be second operand.*
Cayley Table 5.9 will make idea clear. $U = \{0, a_1 \neq 0, a_2 \neq 0, - - - - - -, a_n \neq 0\}$

Table 5.9: Cayley Table

$\star$	0	$u_1 \neq 0$	$u_2 \neq 0$	$- - - \neq 0$	$- - - \neq 0$	$u_n \neq 0$
0	0[1]	u_1[3]	u_2[3]	—	—	u_n[3]
$u_1 \neq 0$	u_1[2]	0[1]	u_2[3]	—	—	u_n[3]
$u_2 \neq 0$	u_2 [2]	u_1[3]	0[1]	—	—	u_n[3]
— $\neq 0$	— [2]	— [3]	—[3]	—	—	[3]
— $\neq 0$	— [2]	— [3]	—[3]	—	—	u_n[3]
$u_n \neq 0$	u_n [2]	u_1[3]	u_2[3]	—	—	0[1]

Numbers in square brackets indicates specific points used from remark 5.4.3. to frame the output.

Different Notions of NVB G - subalgebra

Definition 5.4.4.
Let M_{NVB} be a NVB G -subalgebra with structure $\mathcal{G}_{\mathcal{M}_{NVB}} = \left(U^{\mathcal{G}_{M_{NVB}}}, \star, 0\right)$

1. ***G -part of a Neutrosophic Vague Binary G - subalgebra***
Let S_{NVB} be any NVBSS of M_{NVB}.
Define, $G(S_{NVB})\{u_x \in S_{NVB} / NVB_{S_{NVB}}(0 \star u_x) = NVB_{S_{NVB}}(u_x).\}$.
In particular, if $S_{NVB} = M_{NVB}$ then $G(M_{NVB})$ is called the neutrosophic vague binary G- G part (in short, NVB G - G part) of the NVB G - subalgebra

2. ***p - radical of a Neutrosophic Vague Binary G - subalgebra***
$B(M_{NVB}) = \{u_x \in U / NVB_{M_{NVB}}(0 \star u_x) = NVB_{M_{NVB}}(0)\}$ *is called a neutrosophic vague binary G - p radical(in short, NVB G - p radical) of the NVB G - subalgebra M_{NVB}*

3. ***p - semi simple of a Neutrosophic Vague Binary G - subalgebra***

   M_{NVB} is called neutrosophic vague binary G - p semi simple (in short, NVB G - p semi simple),

   if $B(M_{NVB}) = \{u_x \in U \diagup NVB_{M_{NVB}}(0 \star u_x) = NVB_{M_{NVB}}(0)\} = \{0\}$

4. ***Minimal Element for Neutrosophic Vague Binary G - subalgebra***

   Any element $u_x \in U$ in neutrosophic vague binary G - minimal element (in short, NVB G - minimal element), if

   $NVB_{M_{NVB}}(u_x \star u_y) = NVB_{M_{NVB}}(0) \Rightarrow NVB_{M_{NVB}}(u_y) = NVB_{M_{NVB}}(u_x)$

Theorem 5.4.5.

Let M_{NVB} be a NVB G - subalgebra with structure $\mathcal{G}_{M_{NVB}} = \left(U^{\mathcal{G}_{M_{NVB}}}, \star, 0\right)$. Then, $u_x \in \mathcal{G}_{M_{NVB}}$ if and only if $NVB_{M_{NVB}}(0 \star u_x) \in \mathcal{G}_{M_{NVB}}$

Proof.

$u_x \in \mathcal{G}_{M_{NVB}} \Rightarrow NVB_{M_{NVB}}(0 \star u_x) = NVB_{M_{NVB}}(u_x)$

$\Rightarrow NVB_{M_{NVB}}(0 \star u_x) = NVB_{M_{NVB}}(0 \star (0 \star u_x))$

$\Rightarrow (0 \star u_x) \in \mathcal{G}_{M_{NVB}}$, [By definition 5.4.1]

Conversely,

if $(0 \star u_x) \in \mathcal{G}_{M_{NVB}}$, then $NVB_{M_{NVB}}(0 \star (0 \star u_x)) = NVB_{M_{NVB}}(0 \star u_x)$

$\Rightarrow NVB_{M_{NVB}}(u_x) = NVB_{M_{NVB}}(0 \star u_x)$

$\Rightarrow u_x \in \mathcal{G}_{M_{NVB}}$

Theorem 5.4.6.

Let M_{NVB} be a NVB G - subalgebra with structure $\mathcal{G}_{M_{NVB}} = \left(U^{\mathcal{G}_{M_{NVB}}}, \star, 0\right)$

1. *M_{NVB} is NVB G - p semi simple.*

2. *Every element in U is a NVB G - minimal element*

Proof.

1. From definition 5.4.4 (iii),

   $B(M_{NVB}) = \{u_x \in U \diagup NVB_{M_{NVB}}(0 \star u_x) = NVB_{M_{NVB}}(0)\}$

   $\Rightarrow B(M_{NVB}) = \{u_x \in U \diagup NVB_{M_{NVB}}(0) = NVB_{M_{NVB}}(u_x)\} = \{0\}$

2. Assume (ii).

   Let u_x be an arbitrary element in U such that $u_y \leq u_x$ for some $u_y \in U$

   $\Rightarrow (u_x \star u_y) = 0$

   $\Rightarrow NVB_{M_{NVB}}(u_x \star u_y) = NVB_{M_{NVB}}(0)$

   $\Rightarrow NVB_{M_{NVB}}(u_x) = NVB_{M_{NVB}}(u_y)$

Theorem 5.4.7.

If $G(M_{NVB}) = \mathcal{G}_{M_{NVB}}$, then M_{NVB} is NVB G - p semi simple. That is, if a NVB G - subalgebra coincides with its G - part then it is NVB G - p semi simple

Proof.

Let M_{NVB} be a NVB G - subalgebra with structure $\mathcal{G}_{M_{NVB}} = (U^{\mathcal{G}}_{M_{NVB}}, \star, 0)$.

$G(M_{NVB}) = \{u_x \in M_{NVB} / NVB_{M_{NVB}}(0 \star u_x) = NVB_{M_{NVB}}(u_x)\}$,

[From definition 5.4.1 (i).]

If $G(M_{NVB}) = \mathcal{G}_{M_{NVB}}$ then $B(M_{NVB}) = 0 \Rightarrow M_{NVB}$ is NVB G - p semi simple

Remark 5.4.8.

1. *In any NVB G - subalgebra:*
   $NVB_{M_{NVB}}(u_x) \preceq NVB_{M_{NVB}}(u_y) \Leftrightarrow NVB_{M_{NVB}}(u_y \star u_x) = NVB_{M_{NVB}}(0)$

2. *Denote $NVB_{M_{NVB}}[u_y \star (u_y \star u_x)]$ by $NVB_{M_{NVB}}(u_x \wedge u_y)$ for all $u_x, u_y \in U$.*
   $NVB_{M_{NVB}}(u_x) = NVB_{M_{NVB}}(u_x \wedge u_y)$,
   [From axiom (2) given in definition 5.4.1]

Theorem 5.4.9.

Let M_{NVB} be a NVB G - subalgebra with structure $\mathcal{G}_{M_{NVB}} = \left(U^{\mathcal{G}_{M_{NVB}}}, \star, 0\right)$. Then for any $u_x, u_y, u_z \in U$,

1. *For $u_x \neq u_y, NVB_{M_{NVB}}(u_x \wedge u_y) \neq NVB_{M_{NVB}}(u_y \wedge u_x)$*

2. $NVB_{M_{NVB}}[u_x \wedge (u_y \wedge u_z)] = NVB_{M_{NVB}}[(u_x \wedge u_y) \wedge u_z)]$

3. $NVB_{M_{NVB}}(u_x \wedge 0) = NVB_{M_{NVB}}(u_x)$ *and* $NVB_{M_{NVB}}(0 \wedge u_x) = NVB_{M_{NVB}}(0)$

4. *For $u_x \neq 0, NVB_{M_{NVB}}[u_x \wedge (u_y \wedge u_z)] \neq NVB_{M_{NVB}}[(u_x \wedge u_y) \star (u_x \wedge u_z)]$*

Proof.

1. For a NVB G - subalgebra,
   $NVB_{M_{NVB}}(u_x \wedge u_y) = NVB_{M_{NVB}}(u_x) \& NVB_{M_{NVB}}(u_y \wedge u_x) = NVB_{M_{NVB}}(u_y)$

2. $NVB_{M_{NVB}}[u_x \wedge (u_y \wedge u_z)] = NVB_{M_{NVB}}(u_x \wedge u_y) = NVB_{M_{NVB}}(u_x)$.
   $\therefore NVB_{M_{NVB}}[u_x \wedge (u_y \wedge u_z)] = NVB_{M_{NVB}}[(u_x \wedge u_y) \wedge u_z]$

3. For a NVB G - subalgebra,
   $NVB_{M_{NVB}}(u_x \wedge 0) = NVB_{M_{NVB}}[0 \star (0 \star u_x)] = NVB_{M_{NVB}}(u_x) \&$
   $NVB_{M_{NVB}}(0 \wedge u_x) = NVB_{M_{NVB}}[u_x \star (u_x \star 0)] = NVB_{M_{NVB}}(0)$

4. For a NVB G subalgebra,
   $NVB_{M_{NVB}}[u_x \wedge (u_y \wedge u_z)] = NVB_{M_{NVB}}(u_x \wedge u_y) = NVB_{M_{NVB}}(u_x)$,
   $NVB_{M_{NVB}}[(u_x \wedge u_y) \star (u_x \wedge u_z)] = NVB_{M_{NVB}}(u_x \star u_x) = NVB_{M_{NVB}}(0)$,
   $\therefore$ It is clear that, for a NVB G - subalgebra, $NVB_{M_{NVB}}(u_x) \neq NVB_{M_{NVB}}(0)$
   $\Rightarrow NVB_{M_{NVB}}[u_x \wedge (u_y \wedge u_z)] \neq NVB_{M_{NVB}}[(u_x \wedge u_y) \star (u_x \wedge u_z)]$

Theorem 5.4.10.
Every NVB G - subalgebra satisfies the inequality,
$NVB_{M_{NVB}}(0) \succeq NVB_{M_{NVB}}(u_x);\ \forall u_x \in U$

Proof.
$$NVB_{M_{NVB}}(0) = NVB_{M_{NVB}}(u_x \star u_x) \succeq rmin\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_x)\} = NVB_{M_{NVB}}(u_x)$$
$\therefore NVB_{M_{NVB}}(0) \succeq NVB_{M_{NVB}}(u_x) \quad ; \quad \forall u_x \in U$

Theorem 5.4.11.
Let $\mathcal{G}_{M_{NVB}} = \left(U^{M_{NVB}}, \star, 0\right)$ *be a NVB G - subalgebra.*
Then the following conditions hold:

1. $NVB_{M_{NVB}}(u_x \star 0) = NVB_{M_{NVB}}(u_x), \quad \forall u_x \in U$

2. $NVB_{M_{NVB}}(0 \star (0 \star u_x)) = NVB_{M_{NVB}}(u_x); \quad \forall u_x \in U$

Proof.
Let $\mathcal{G}_{M_{NVB}} = (U^{\mathcal{G}_{M_{NVB}}}, \star, 0)$ be a NVB G - subalgebra and $x, y \in \mathcal{G}_{M_{NVB}}$. Then,

1. $NVB_{M_{NVB}}(u_x \star 0) = NVB_{M_{NVB}}(u_x \star (u_x \star u_x))$
   [Using first condition in the G - algebraic structure of NVB G - subalgebra]
   $= NVB_{M_{NVB}}(u_x)$
   [Using 2^{nd} condition in the G - algebraic structure of NVB G - subalgebra]

2. Since $\mathcal{G}_{M_{NVB}}$ is a NVB G - subalgebra, $NVB_{M_{NVB}}(u_x \star (u_x \star u_y)) = NVB_{M_{NVB}}(u_y)$
   [Using 2^{nd} condition in the G - algebraic structure of NVB G - subalgebra]
   Put $u_x = 0$ and $u_y = u_x$ in the above then (2) follows

Theorem 5.4.12.

Let $\mathcal{G}_{M_{NVB}} = (U^{\mathcal{G}_{M_{NVB}}}, \star, 0)$ *be a NVB G - subalgebra.*
Then following conditions hold: $\forall u_x, u_y \in U$

1. $NVB_{M_{NVB}}((u_x \star (u_x \star u_y) \star u_y)) = NVB_{M_{NVB}}(0)$
2. $NVB_{M_{NVB}}(u_x \star u_y) = NVB_{M_{NVB}}(0) \Rightarrow NVB_{M_{NVB}}(u_x) = NVB_{M_{NVB}}(u_y)$
3. $NVB_{M_{NVB}}(0 \star u_x) = NVB_{M_{NVB}}(0 \star u_y) \Rightarrow NVB_{M_{NVB}}(u_x) = NVB_{M_{NVB}}(u_y)$

Proof.

1. $NVB_{M_{NVB}}((u_x \star (u_x \star u_y)) \star u_y) = NVB_{M_{NVB}}((u_y \star (u_y \star u_y)) \star u_y)$
   by putting $u_x = u_y$
   $= NVB_{M_{NVB}}((u_y \star 0) \star u_y) = NVB_{M_{NVB}}(u_y \star u_y) = NVB_{M_{NVB}}(0)$
2. Assume $NVB_{M_{NVB}}(u_x \star u_y) = NVB_{M_{NVB}}(0)$
   $\therefore NVB_{M_{NVB}}(u_x) = NVB_{M_{NVB}}(u_x \star 0) = NVB_{M_{NVB}}(u_x \star (u_x \star u_y)),$
   [by assumption]
   $= NVB_{M_{NVB}}(u_y)$
3. Assume $NVB_{M_{NVB}}(0 \star u_x) = NVB_{M_{NVB}}(0 \star u_y)$
   $\therefore NVB_{M_{NVB}}(u_x) = NVB_{M_{NVB}}(0 \star (0 \star u_x)) = NVB_{M_{NVB}}(0 \star (0 \star u_y)),$
   [by assumption]
   $= NVB_{M_{NVB}}(u_y)$

Theorem 5.4.13.

Let $\mathcal{G}_{M_{NVB}} = \left(U^{\mathcal{G}_{M_{NVB}}}, \star, 0\right)$ *be a NVB G - subalgebra. Then,*
$NVB_{M_{NVB}}(u_a \star u_x) = NVB_{M_{NVB}}(u_a \star u_y) \Rightarrow NVB_{M_{NVB}}(u_x) = NVB_{M_{NVB}}(u_y),$
for any $u_a, u_x, u_y \in U$

Proof.
If $\mathcal{G}_{M_{NVB}} = \left(U^{\mathcal{G}_{M_{NVB}}}, \star, 0\right)$ be a NVB G - subalgebra satisfying,
$NVB_{M_{NVB}}(u_a \star u_x) = NVB_{M_{NVB}}(u_a \star u_y)$, for any $u_a, u_x, u_y \in U$.
Then,

$$NVB_{M_{NVB}}(u_x) = NVB_{M_{NVB}}(u_a \star (u_a \star u_x))$$
$$= NVB_{M_{NVB}}(u_a \star (u_a \star u_y)) = NVB_{M_{NVB}}(u_y)$$

Theorem 5.4.14.

Let $\mathcal{G}_{M_{NVB}} = \left(U^{\mathcal{G}_{M_{NVB}}}, \star, 0\right)$ *be a NVB G - subalgebra.*
Then the following are equivalent:

1. $NVB_{M_{NVB}}\left((u_x \star u_y) \star (u_x \star u_z)\right) = NVB_{M_{NVB}}\left(u_z \star u_y\right); \forall u_x, u_y, u_z \in U$

2. $NVB_{M_{NVB}}\left((u_x \star u_z) \star (u_y \star u_z)\right) = NVB_{M_{NVB}}\left(u_x \star u_y\right); \forall u_x, u_y, u_z \in U$

Proof.
(i) $\Rightarrow$ (ii)
Assume (i).
i.e., $NVB_{M_{NVB}}\left((u_x \star u_y) \star (u_x \star u_z)\right) = NVB_{M_{NVB}}\left(u_z \star u_y\right); \forall u_x, u_y, u_z \in U$
$\therefore NVB_{M_{NVB}}\left((u_x \star u_z) \star (u_x \star u_y)\right) = NVB_{M_{NVB}}\left(u_y \star u_z\right)$
Consider
$NVB_{M_{NVB}}\left((u_x \star u_z) \star (u_y \star u_z)\right) = NVB_{M_{NVB}}\left((u_x \star u_z) \star ((u_x \star u_z) \star (u_x \star u_y))\right)$
$= NVB_{M_{NVB}}\left(u_x \star u_y\right)$, since $NVB_{M_{NVB}}\left(u_x \star (u_x \star u_y)\right) = NVB_{M_{NVB}}(u_y)$
(ii) $\Rightarrow$ (i)
Assume (ii). i.e., $NVB_{M_{NVB}}\left((u_x \star u_z) \star (u_y \star u_z)\right) = NVB_{M_{NVB}}\left(u_x \star u_y\right)$
$\therefore NVB_{M_{NVB}}\left((u_x \star u_y) \star (u_z \star u_y)\right) = NVB_{M_{NVB}}\left(u_x \star u_z\right)$
Consider,
$NVB_{M_{NVB}}\left((u_x \star u_y) \star (u_x \star u_z)\right) = NVB_{M_{NVB}}\left((u_x \star u_y) \star ((u_x \star u_y) \star (u_z \star u_y))\right)$
$= NVB_{M_{NVB}}\left(u_z \star u_y\right)$, since $NVB_{M_{NVB}}\left(u_x \star (u_x \star u_y)\right) = NVB_{M_{NVB}}(u_y)$

Neutrosophic Vague Binary G - normal subalgebra

In this section neutrosophic vague binary G - normal subalgebra is introduced

Definition 5.4.15. ***(Neutrosophic Vague Binary G - normal subalgebra)***
Let M_{NVB} *be a neutrosophic vague binary set (in short, NVBS) with a binary universe* (U_1, U_2)*. Neutrosophic Vague Binary G - normal subalgebra is a structure* $\mathcal{G}^{\mathcal{N}_{M_{NVB}}} = \left(U^{\mathcal{N}}_{M_{NVB}}, \star, 0\right)$ *which satisfies, the following 2 conditions known as* $\mathcal{G}^{\mathcal{N}}_{M_{NVB}}$ *inequalities:*
$\mathcal{G}^{\mathcal{N}_{M_{NVB}}}$ ***inequality (1):***
$NVB_{M_{NVB}}\left(u_x \star u_y\right) \succeq \quad r \quad min\left\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\right\}; \forall u_x, u_y \in U$

$$\hat{T}_{M_{NVB}}(u_x \star u_y) \geq min\left\{\hat{T}_{M_{NVB}}(u_x), \hat{T}_{M_{NVB}}(u_y)\right\}$$
$$\hat{I}_{M_{NVB}}(u_x \star u_y) \leq max\left\{\hat{I}_{M_{NVB}}(u_x), \hat{I}_{M_{NVB}}(u_y)\right\}$$
$$\hat{F}_{M_{NVB}}(u_x \star u_y) \leq max\left\{\hat{F}_{M_{NVB}}(u_x), \hat{F}_{M_{NVB}}(u_y);\right\}$$

$\mathcal{G}^{\mathcal{N}}_{M_{NVB}}$ ***inequality (2):***

$NVB_{M_{NVB}}((u_x \star u_a) \star (u_y \star u_b)) \succeq rmin\{NVB_{M_{NVB}}(u_x \star u_y), NVB_{M_{NVB}}(u_a \star u_b)\}$
$\forall \quad u_a, u_b, u_x, u_y \in U$

That is, $\forall \quad u_a, u_b, u_x, u_y \in U$

$$\hat{T}_{M_{NVB}}((u_x \star u_a) \star (u_y \star u_b)) \geq min\left\{\hat{T}_{M_{NVB}}(u_x \star u_y), \hat{T}_{M_{NVB}}(u_a \star u_b)\right\}$$
$$\hat{I}_{M_{NVB}}((u_x \star u_a) \star (u_y \star u_b)) \leq max\left\{\hat{I}_{M_{NVB}}(u_x \star u_y), \hat{I}_{M_{NVB}}(u_a \star u_b)\right\}$$
$$\hat{F}_{M_{NVB}}((u_x \star u_a) \star (u_y \star u_b)) \leq max\left\{\hat{F}_{M_{NVB}}(u_x \star u_y), \hat{F}_{M_{NVB}}(u_a \star u_b)\right\}$$

[$\star$ *and* 0 *are as in* $U^{\mathcal{G}\mathcal{N}}_{M_{NVB}}$ & $\hat{T} = [T^-, T^+], \hat{I} = [I^-, I^+], \hat{F} = [F^-, F^+]$]

Here,

- $U^{\mathcal{G}\mathcal{N}}_{M_{NVB}} = (U = \{U_1 \bigcup U_2\}, \star, 0)$ *is a G - algebraic structure with a binary operation* $\star$*& a constant 0, which satisfies following axioms :*

$\forall u_x, u_y \in U, \quad (i) \quad (u_x \star u_x) = 0; \quad (ii) \quad u_x \star (u_x \star u_y) = u_y$

Definition 5.4.16. ***(Neutrosophic Vague Binary G - Normal Set)***

Let M_{NVB} *be a NVBS with a binary universe* (U_1, U_2)*. Take* $U = \{U_1 \cup U_2\}$*.*

A NVBS M_{NVB} *in U is said to be NVB G - normal set if it satisfies the inequality:*

$\forall u_x, u_y, u_a, u_b \in U,$

$NVB_{M_{NVB}}((u_x \star u_a) \star (u_y \star u_b)) \succeq rmin\{NVB_{M_{NVB}}(u_x \star u_y), NVB_{M_{NVB}}(u_a \star u_b)\}$

That is, $\forall u_x, u_y, u_a, u_b \in U$

$$\hat{T}_{M_{NVB}}((u_x \star u_a) \star (u_y \star u_b)) \geq min\left\{\hat{T}_{M_{NVB}}(u_x \star u_y), \hat{T}_{M_{NVB}}(u_a \star u_b)\right\}$$
$$\hat{I}_{M_{NVB}}((u_x \star u_a) \star (u_y \star u_b)) \geq min\left\{\hat{I}_{M_{NVB}}(u_x \star u_y), \hat{I}_{M_{NVB}}(u_a \star u_b)\right\}$$
$$\hat{F}_{M_{NVB}}((u_x \star u_a) \star (u_y \star u_b)) \geq min\left\{\hat{F}_{M_{NVB}}(u_x \star u_y), \hat{F}_{M_{NVB}}(u_a \star u_b)\right\}$$

Theorem 5.4.17.

Every NVB G - Normal set M_{NVB} *in U is a NVB G - subalgebra of U.*

Proof.

Let M_{NVB} be a NVB G - normal set in U

$\Rightarrow NVB_{M_{NVB}}((u_x \star u_a) \star (u_y \star u_b))$
$\succeq rmin\{NVB_{M_{NVB}}(u_x \star u_y), NVB_{M_{NVB}}(u_a \star u_b)\}$
Consider, $NVB_{M_{NVB}}(u_x \star u_y) = NVB_{M_{NVB}}((u_x \star u_y) \star (0 \star 0))$
$\succeq rmin\{NVB_{M_{NVB}}(u_x \star 0), NVB_{M_{NVB}}(u_y \star 0)\}$
$= rmin\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\}$
$\Rightarrow NVB_{M_{NVB}}(u_x \star u_y) \succeq rmin\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\}; \forall u_x, u_y \in U$
$\Rightarrow M_{NVB}$ is a NVB G - subalgebra

Remark 5.4.18.

Converse of theorem 5.4.17 is not true.
That is, a NVB G - subalgebra M_{NVB} in U is not a NVB G - normal set, generally

Proof.
Consider example 5.4.2, in which M_{NVB} is a NVB G - subalgebra. In this example,
$NVB_{M_{NVB}}((u_a \star u_a) \star (u_b \star u_a)) \not\succeq rmin\{NVB_{M_{NVB}}(u_a \star u_b), NVB_{M_{NVB}}(u_a \star u_a)\}$
$\Rightarrow M_{NVB}$ is not a NVB G - normal set

Theorem 5.4.19.

If a neutrosophic vague binary set M_{NVB} in U is a NVB G - normal subalgebra, then $NVB_{M_{NVB}}(u_x \star u_y) = NVB_{M_{NVB}}(u_y \star u_x); \forall u_x, u_y \in U$

Proof.
Let $u_x, u_y \in$ U.
Then, $NVB_{M_{NVB}}(u_x \star u_y) = NVB_{M_{NVB}}((u_x \star u_y) \star 0)$
$= NVB_{M_{NVB}}((u_x \star u_y) \star (u_x \star u_x)) \succeq rmin\{NVB_{M_{NVB}}(u_x \star u_x), NVB_{M_{NVB}}(u_y \star u_x)\}$
$= rmin\{NVB_{M_{NVB}}(0), NVB_{M_{NVB}}(u_y \star u_x)\}$
$= NVB_{M_{NVB}}(u_y \star u_x)$
$\therefore NVB_{M_{NVB}}(u_x \star u_y) \succeq NVB_{M_{NVB}}(u_y \star u_x)$.
Similarly,
$NVB_{M_{NVB}}(u_y \star u_x) \succeq NVB_{M_{NVB}}(u_x \star u_y)$

$$\Rightarrow NVB_{M_{NVB}}(u_x \star u_y) = NVB_{M_{NVB}}(u_y \star u_x).$$

Derivations of Neutrosophic Vague Binary G - subalgebra

In this section following points are developed
i. neutrosophic vague binary G - derivation
ii. neutrosophic vague binary G - regular derivation

Definition 5.4.20.

(G - derivation of neutrosophic vague binary G - subalgebra)

Let M_{NVB} be a NVBS with a binary universe (U_1, U_2).
Also let, considered M_{NVB} is a NVB G - subalgebra with structure
$\mathcal{G}_{M_{NVB}} = \left(U^{\mathcal{G}_{M_{NVB}}}, \star, 0\right)$ *and with a self-map $d : U \to U$ on M_{NVB}*
with $U = \{U_1 \cup U_2\}$. Then,

1. *d is (l,r) neutrosophic vague binary G - derivation of M_{NVB} if,*
   $NVB_{M_{NVB}}[d(u_x \star u_y)] = NVB_{M_{NVB}}[d(u_x) \star u_y \wedge (u_x \star d(u_y))]$
2. *d is (r, l) neutrosophic vague binary G - derivation of M_{NVB} if,*
   $NVB_{M_{NVB}}[d(u_x \star u_y)] = NVB_{M_{NVB}}[(u_x \star d(u_y)) \wedge (d(u_x) \star u_y)]$

d is a neutrosophic vague binary G - derivation (in short, NVB G - derivation) of M_{NVB} only if d is both (l, r) neutrosophic vague binary G - derivation [in short, (l,r) NVB G - derivation] & (r, l) neutrosophic vague binary G - derivation [in short, (r,l) NVB G - derivation]. In this derivation, (l,r) indicates left-right and (r,l) indicates right-left.

Remark 5.4.21.

For a NVB G - subalgebra, $NVB_{M_{NVB}}(u_x \wedge u_y) = NVB_{M_{NVB}}(u_x)$

Remark 5.4.22.

1. *To check, d is (l,r) NVB G - derivation of M_{NVB}, it is enough to check,*
   $NVB_{M_{NVB}}(d(u_x \star u_y)) = NVB_{M_{NVB}}(d(u_x) \star u_y)$;
   [*By definition 5.4.20 & remark 5.4.21*]
2. *To check, $d_{\mathcal{G}_{M_{NVB}}}$ is (r,l) NVB G - derivation of M_{NVB}, it is enough to check,*
   $NVB_{M_{NVB}}(d(u_x \star u_y)) = NVB_{M_{NVB}}(u_x \star d(u_y))$
   [*Using definition 5.4.20 & by remark 5.4.21*]
   $\therefore$ *5.4.20 can be re-written as, definition 5.4.23*

Definition 5.4.23.

Let $\mathcal{G}_{M_{NVB}}$ be a NVB G - subalgebra and d be a self-map on U.
d is a neutrosophic vague binary G - derivation of U if
(i) d is (l,r)- neutrosophic vague binary G - derivation of U
i.e., $NVB_{M_{NVB}}(d(u_x \star u_y)) = NVB_{M_{NVB}}(d(u_x) \star u_y)$ *; for all $u_x, u_y \in U$*
& it is denoted by $d_{(l,r)}^{\mathcal{G}_{M_{NVB}}}$

(ii) d is (r,l) −neutrosophic vague binary G - derivation of U.
i.e., $NVB_{M_{NVB}}(d(u_x \star u_y)) = NVB_{M_{NVB}}(u_x \star d(u_y))$ *; for all* $u_x, u_y \in U$
& it is denoted by $d_{(r,l)}^{\mathcal{G}_{M_{NVB}}}$

Definition 5.4.24.

(Regular Derivation of a Neutrosophic Vague Binary G - subalgebra)
A derivation $d_{M_{NVB}}^{\mathcal{G}}$ *of a NVB G - subalgebra is said to be regular if,*
$NVB_{M_{NVB}}(d(0)) = NVB_{M_{NVB}}(0)$. *It is denoted by* $d_r^{\mathcal{G}_{M_{NVB}}}$

Example 5.4.25.

From example 5.4.2, M_{NVB} *is a* $\mathcal{G}_{M_{NVB}}$
Case (i) : *Define a self - map,* $d : U = \{0, u_a, u_b\} \to U = \{0, u_a, u_b\}$ *by*

$$d(u_s) = \begin{cases} 0 \quad if \quad u_s = 0 \\ u_a \quad if \quad u_s = u_a \\ u_b \quad if \quad u_s = u_b \end{cases}$$

Here, the given self-map is an identity map. From calculations, d is a $d_{(l,r)}^{\mathcal{G}_{M_{NVB}}}$ *&* $d_{(r,l)}^{\mathcal{G}_{M_{NVB}}} \Rightarrow$ *d is a* $d_{M_{NVB}}^{\mathcal{G}}$

Case (ii) *Define a self-map,* $d : U = \{0, u_a, u_b\} \to U = \{0, u_a, u_b\}$ *by*

$$d(u_s) = \begin{cases} u_a \quad if \quad u_s = 0 \\ 0 \quad if \quad u_s = u_a \\ u_b \quad if \quad u_s = u_b \end{cases}$$

d is not a NVB G - derivation on M_{NVB}.
One violation is attached below:

$$\begin{aligned} NVB_{M_{NVB}}(d(u_b \star u_a)) &= NVB_{M_{NVB}}(d(u_a)) \\ &= NVB_{M_{NVB}}(0) = [0.9, 0.9], [0.1, 0.1], [0.1, 0.1] \end{aligned}$$

$$\begin{aligned} NVB_{M_{NVB}}((d(u_b) \star u_a))) &= NVB_{M_{NVB}}(u_b \star u_a) \\ &= NVB_{M_{NVB}}(u_a) = [0.7, 0.9], [0.3, 0.4], [0.1, 0.3] \end{aligned}$$

$d_{(l,r)}^{\mathcal{G}_{M_{NVB}}}(u_b \star u_a)$ *does not exist, since* $NVB_{M_{NVB}}(d(u_b \star u_a)) \neq NVB_{M_{NVB}}(d(u_b) \star u_a)$

$$\begin{aligned} NVB_{M_{NVB}}[u_b \star d(u_a)] &= NVB_{M_{NVB}}(u_b \star 0) \\ &= NVB_{M_{NVB}}(u_b) = [0.2, 0.6], [0.1, 0.2], [0.4, 0.8] \end{aligned}$$

$d_{(r,l)}^{\mathcal{G}_{M_{NVB}}}(u_b \star u_a)$ *does not exist, since* $NVB_{M_{NVB}}(d(u_b \star u_a)) \neq MVB_{M_{NVB}}(u_b \star d(u_a))$
$\therefore$ *d is not a* $d_{M_{NVB}}^{\mathcal{G}}$

Theorem 5.4.26.

In a $\mathcal{G}_{M_{NVB}}$ the identity map d on U is a $d^{\mathcal{G}_{M_{NVB}}}$. Converse not true in general. But if $d^{\mathcal{G}_{M_{NVB}}}$ is a $d_r^{\mathcal{G}_{M_{NVB}}}$ then converse hold good. That is, if $d^{\mathcal{G}_{M_{NVB}}}$ is a $d_r^{\mathcal{G}_{M_{NVB}}}$ then d is the identity map on U

Proof.

(i) Let $u_x, u_y \in$ U & also let d is an identity map on U.

Case (i):$u_x = u_y; u_y \neq 0$

$NVB_{M_{NVB}}(d(u_x \star u_y)) = NVB_{M_{NVB}}(d(u_x \star u_x)) = NVB_{M_{NVB}}(d(0)) = NVB_{M_{NVB}}(0)$

$NVB_{M_{NVB}}(d(u_x) \star u_y) = NVB_{M_{NVB}}(d(u_x) \star u_x) = NVB_{M_{NVB}}(u_x \star u_x) = NVB_{M_{NVB}}(0)$

$NVB_{M_{NVB}}(u_x \star d(u_y)) = NVB_{M_{NVB}}(u_x \star d(u_x)) = NVB_{M_{NVB}}(u_x \star u_x) = NVB_{M_{NVB}}(0)$

$\therefore NVB_{M_{NVB}}(d(u_x \star u_y)) = NVB_{M_{NVB}}(d(u_x) \star u_y) = NVB_{M_{NVB}}(u_x \star d(u_y))$

Case(ii): $u_x \neq u_y; u_y \neq 0$

Either $NVB_{M_{NVB}}(d(u_x \star u_y)) = NVB_{M_{NVB}}(d(u_x)) = NVB_{M_{NVB}}(u_x)$

Or $NVB_{M_{NVB}}(d(u_x \star u_y)) = NVB_{M_{NVB}}(d(u_y)) = NVB_{M_{NVB}}(u_y)$

$\Rightarrow d(u_x \star u_y) = d(u_x)$ or $d(u_x \star u_y) = d(u_y)$

$\Rightarrow$ either $(u_x \star u_y) = u_x$ or $(u_x \star u_y) = u_y$, since d is identity map

$\Rightarrow$ either $u_y = 0$ or $u_y \neq 0$

Consider $u_y \neq 0$, i.e., $d(u_x \star u_y) = d(u_y)$. i.e., $(u_x \star u_y) = u_y$

$NVB_{M_{NVB}}(d(u_x \star u_y)) = NVB_{M_{NVB}}(d(u_y)) = NVB_{M_{NVB}}(u_y)$

$NVB_{M_{NVB}}(d(u_x) \star u_y) = NVB_{M_{NVB}}(u_x \star u_y) = NVB_{M_{NVB}}(u_y)$

$NVB_{M_{NVB}}(u_x \star d(u_y)) = NVB_{M_{NVB}}(u_x \star u_y) = NVB_{M_{NVB}}(u_y)$

$\therefore NVB_{M_{NVB}}(d(u_x \star u_y)) = NVB_{M_{NVB}}(d(u_x) \star u_y) = NVB_{M_{NVB}}(u_x \star d(u_y))$

Case(iii):$u_x \neq u_y; u_y = 0$

Either $NVB_{M_{NVB}}(d(u_x \star u_y)) = NVB_{M_{NVB}}(d(u_x)) = NVB_{M_{NVB}}(u_x)$

Or $NVB_{M_{NVB}}(d(u_x \star u_y)) = NVB_{M_{NVB}}(d(u_y)) = NVB_{M_{NVB}}(u_y)$

$\Rightarrow d(u_x \star u_y) = d(u_x)$ or $d(u_x \star u_y) = d(u_y)$

$\Rightarrow$ either $(u_x \star u_y) = u_x$ or $(u_x \star u_y) = u_y$, since d is identity map.

$\Rightarrow$ either $u_y = 0$ or $u_y \neq 0$

Consider $u_y = 0$, i.e., $d(u_x \star u_y) = d(u_x)$. i.e., $(u_x \star u_y) = u_x$

$NVB_{M_{NVB}}(d(u_x \star u_y)) = NVB_{M_{NVB}}(d(u_x)) = NVB_{M_{NVB}}(u_x)$

$NVB_{M_{NVB}}(d(u_x) \star u_y) = NVB_{M_{NVB}}(u_x \star u_y) = NVB_{M_{NVB}}(u_x)$

$NVB_{M_{NVB}}(u_x \star d(u_y)) = NVB_{M_{NVB}}(u_x \star u_y) = NVB_{M_{NVB}}(u_x)$

$\therefore NVB_{M_{NVB}}(d(u_x \star u_y)) = NVB_{M_{NVB}}(d(u_x) \star u_y) = NVB_{M_{NVB}}(u_x \star d(u_y))$

$\therefore$ d is both $d_{(l,r)}^{\mathcal{G}_{M_{NVB}}}$ & $d_{(r,l)}^{\mathcal{G}_{M_{NVB}}}$. Hence d is a $d^{\mathcal{G}_{M_{NVB}}}$

Converse

$d^{\mathcal{G}_{M_{NVB}}}$ is a $d_r^{\mathcal{G}_{M_{NVB}}} \Rightarrow NVB_{M_{NVB}}(d(0)) = NVB_{M_{NVB}}(0)$

$\Rightarrow NVB_{M_{NVB}}(d(u_x \star u_x)) = NVB_{M_{NVB}}(0)$

$\Rightarrow d(u_x) = u_x$

$\Rightarrow$ d is the identity map on U

Remark 5.4.27.

Let M_{NVB} be a NVB G - subalgebra with structure $\mathcal{G}_{M_{NVB}} = \left(U^{\mathcal{G}_{M_{NVB}}}, \star, 0\right)$.
A NVB G - derivation on M_{NVB} is a mapping $d : U \to U$ such that, $\forall u_x, u_y \in U$,
$NVB_{M_{NVB}}(d(u_x \star u_y)) = NVB_{M_{NVB}}(d(u_x) \star u_y) = NVB_{M_{NVB}}(u_x \star d(u_y))$.

Set of all neutrosophic vague binary G - derivations on M_{NVB} is denoted as $\Gamma^{d^{M_{NVB}}}$

Neutrosophic Vague Binary G - Coset

General properties that are true for abstract algebra and G - algebra may not be true in the case of Neutrosophic Vague Binary G - subalgebra. In this section, coset for neutrosophic vague binary G - subalgebra is developed. Neutrosophic Vague Binary G -Coset is considered as a shifted (or translated) neutrosophic vague binary G - subalgebra. Existence of identity element and inverse element can't be assured in every neutrosophic vague binary G - subalgebra. In generalization process, this will become a crisis. As a result, generalization is confined to a particular area. It will lead to the formation of different concepts like Lagrange neutrosophic vague binary G - subalgebra etc.

Definition 5.4.28.

***Neutrosophic Vague Binary G - Right Coset** &*
Neutrosophic Vague Binary G - left Coset

Let M_{NVB} be a neutrosophic vague binary set with a binary universe (U_1, U_2) and also let the considered M_{NVB} is a NVB G - subalgebra of a G - algebra with algebraic structure $\mathcal{G}_{M_{NVB}} = \left(U^{\mathcal{G}_{M_{NVB}}}, \star, 0\right)$ where $U^{\mathcal{G}_{M_{NVB}}} = (U, \star, 0_{M_{NVB}})$. Also $\hat{T} = [T^-, T^+]\,;\,\hat{I} = [I^-, I^+]\,;\,\hat{F} = [F^-, F^+]$ and $U = \{U_1 \cup U_2\}$

Case (i) (Neutrosophic Vague Binary G - Right Coset)

Let $u_a \in U_1$ and $u_b \in U_2$ be fixed elements. Then define, for every $u_c \in U_1$ and for *every* $d \in U_2$, *a neutrosophic vague binary G - right coset of* M_{NVB} *is denoted by* $M_{NVB}(u_a, u_b)$ *and defined by,*

$$(M_{NVB}(u_a, u_b))(u_c, u_d)$$

$$= NVB_{M_{NVB}}(u_a, u_b)(u_c, u_d)$$

$$= \left\{ \left\langle NVB_{M_{NVB}}\left(u_c \star (u_a)^{-1} | \forall u_c \in U_1\right)\right\rangle, \left\langle NVB_{M_{NVB}}\left(u_d \star (u_b)^{-1} | \forall u_d \in U_2\right)\right\rangle \right\}$$

$$= \left\{ \begin{array}{c} \left\langle \left(\hat{T}_{M_{NVB(u_a)}}(u_c), \hat{I}_{M_{NVB(u_a)}}(u_c), \hat{F}_{M_{NVB(u_a)}}(u_c)\right) | \forall u_c \in U_1 \right\rangle, \\ \left\langle \left(\hat{T}_{M_{NVB(u_b)}}(u_d), \hat{I}_{M_{NVB(u_b)}}(u_d), \hat{F}_{M_{NVB(u_b)}}(u_d)\right) | \forall u_d \in U_2 \right\rangle \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left\langle \left(\hat{T}_{M_{NVB}}(u_c \star (u_a)^{-1}), \hat{I}_{M_{NVB}}(u_c \star (u_a)^{-1}), \hat{F}_{M_{NVB}}(u_c \star (u_a)^{-1}) | \forall u_c \in U_1\right) \right\rangle, \\ \left\langle \left(\hat{T}_{M_{NVB}}(u_d \star (u_b)^{-1}), \hat{I}_{M_{NVB}}(u_d \star (u_b)^{-1}), \hat{F}_{M_{NVB}}(u_d \star (u_b)^{-1}) | \forall d \in U_2\right) \right\rangle \end{array} \right\}$$

Then $M_{NVB}(u_a, u_b)$ *is called a neutrosophic vague binary G - Right Coset(in short NVB G - Right Coset) determined by* M_{NVB} *and* (u_a, u_b)

Case (ii) (Neutrosophic Vague Binary G - Left Coset)

Let $u_a \in U_1$ *and* $u_b \in U_2$ *be fixed elements. Then define, for every* $u_c \in U_1$ *and for every* $u_d \in U_2$, *a neutrosophic vague binary G - left coset of* M_{NVB} *is denoted by* $(u_a, u_b)M_{NVB}$ *and defined by,*

$$((u_a, u_b) M_{NVB})(u_c, u_d)$$

$$= NVB_{(u_a, u_b)M_{NVB}}(u_c, u_d)$$

$$= \left\{ \left\langle NVB_{M_{NVB}}\left((u_a)^{-1} \star u_c | \forall u_c \in U_1\right)\right\rangle, \left\langle NVB_{M_{NVB}}\left((u_b)^{-1} \star u_d | \forall u_d \in U_2\right)\right\rangle \right\}$$

$$= \left\{ \begin{array}{c} \left\langle \left(\hat{T}_{(a)M_{NVB}}(u_c), \hat{I}_{(u_a)M_{NVB}}(u_c), \hat{F}_{(a)M_{NVB}}(u_c) | \forall u_c \in U_1\right) \right\rangle, \\ \left\langle \left(\hat{T}_{(u_b)M_{NVB}}(u_d), \hat{I}_{(u_b)M_{NVB}}(u_d), \hat{F}_{(u_b)M_{NVB}}(u_d) | \forall u_d \in U_2\right) \right\rangle \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left\langle \left(\hat{T}_{M_{NVB}}((u_a)^{-1} \star u_c), \hat{I}_{M_{NVB}}((u_a)^{-1} \star u_c), \hat{F}_{M_{NVB}}((u_a)^{-1} \star u_c) | \forall u_c \in U_1\right) \right\rangle, \\ \left\langle \left(\hat{T}_{M_{NVB}}((u_b)^{-1} \star u_d), \hat{I}_{M_{NVB}}((u_b)^{-1} \star u_d), \hat{F}_{M_{NVB}}((u_b)^{-1} \star u_d) | \forall u_d \in U_2\right) \right\rangle \end{array} \right\}$$

Then $(u_a, u_b) M_{NVB}$ *is called a neutrosophic vague binary G - Left Coset (in short NVB G - Left Coset) determined by* M_{NVB} *and* (u_a, u_b)

Remark 5.4.29.

NVB G - Right Coset is a NVBS. Similarly, a NVB G -Left Coset is a NVBS

Definition 5.4.30. ***(Neutrosophic Vague Binary G - Coset)***

Let the neutrosophic vague binary set M_{NVB} *be a neutrosophic vague binary*

G - subalgebra of a G - algebra. If M_{NVB} is both neutrosophic vague binary G - right coset and neutrosophic vague binary G - left coset then M_{NVB} is called as a Neutrosophic Vague Binary G - Coset.

Example 5.4.31.
Let $U_1 = \{0, u_1, u_3\}$ and $U_2 = \{0, u_2, u_4, u_5\}$ be two universes.
Let $M_{NVB} =$

$$\left\{ \begin{array}{l} \left\langle \frac{[0.7,0.8],[0.3,0.4],[0.2,0.3]}{0}, \frac{[0.2,0.7],[0.5,0.7],[0.3,0.8]}{u_1}, \frac{[0.6,0.7],[0.1,0.4],[0.3,0.4]}{u_3} \right\rangle, \\ \left\langle \frac{[0.2,0.9],[0.1,0.7],[0.1,0.8]}{0}, \frac{[0.3,0.5],[0.6,0.7],[0.5,0.7]}{u_2}, \frac{[0.2,0.8],[0.4,0.7],[0.2,0.8]}{u_4}, \frac{[0.6,0.9],[0.3,0.7],[0.1,0.4]}{u_5} \right\rangle \end{array} \right\}$$

be a NVBS. Here, combined universe $U = \{0, u_1, u_2, u_3, u_4, u_5\}$ & combined NVB membership grades are,

$$NVB_{M_{NVB}}(u_s) = \begin{cases} [0.7,0.9],[0.1,0.4],[0.1,0.3] & ; \quad u_s = 0 \\ [0.2,0.7],[0.5,0.7],[0.3,0.8] & ; \quad u_s = u_1 \\ [0.3,0.5],[0.6,0.7],[0.5,0.7] & ; \quad u_s = u_2 \\ [0.6,0.7],[0.1,0.4],[0.3,0.4] & ; \quad u_s = u_3 \\ [0.2,0.8],[0.4,0.7],[0.2,0.8] & ; \quad u_s = u_4 \\ [0.6,0.9],[0.3,0.7],[0.1,0.4] & ; \quad u_s = u_5 \end{cases}$$

Corresponding Cayley Table is 5.10:

Table 5.10: Cayley Table

$*$	0	u_1	u_2	u_3	u_4	u_5
0	0	u_1	u_2	u_3	u_4	u_5
u_1	u_1	0	u_2	u_3	u_4	u_5
u_2	u_2	u_1	0	u_3	u_4	u_5
u_3	u_3	u_1	u_2	0	u_4	u_5
u_4	u_4	u_1	u_2	u_3	0	u_5
u_5	u_5	u_1	u_2	u_3	u_4	0

Obviously, M_{NVB} is a NVB G - subalgebra. In every G - algebra 0 may not be the identity element. But in the present case it is clear that 0 acts as an identity element. Hence inverses got as:

$(0)^{-1} = 0 \quad ; \quad (u_1)^{-1} = u_1 \quad ; \quad (u_2)^{-1} = u_2 \quad ;$

$(u_3)^{-1} = u_3 \quad ; \quad (u_4)^{-1} = u_4 \quad ; \quad (u_5)^{-1} = u_5$

To construct a NVB G -Right Coset

Let $u_a = u_1 \in U_1$ and $\forall u_c \in U_1 = \{0, u_1, u_3\}$

$$NVB_{M_{NVB}(u_1)}(0) = NVB_{M_{NVB}}\,(0 \star u_1) = NVB_{M_{NVB}}(u_1)$$
$$= [0.2, 0.7], [0.5, 0.7], [0.3, 0.8]$$
$$NVB_{M_{NVB}(u_1)}(u_1) = NVB_{M_{NVB}}\,(u_1 \star (u_1)^{-1}) = NVB_{M_{NVB}}\,(u_1 \star u_1) = NVB_{M_{NVB}}(0)$$
$$= [0.7, 0.9], [0.1, 0.4], [0.1, 0.3]$$
$$NVB_{M_{NVB}(u_1)}(u_3) = NVB_{M_{NVB}}\,(u_3 \star (u_1)^{-1}) = NVB_{M_{NVB}}\,(u_3 \star u_1) = NVB_{M_{NVB}}(u_1)$$
$$= [0.2, 0.7], [0.5, 0.7], [0.3, 0.8]$$

& Let $u_b = u_2 \in U_2$ and $\forall u_d \in U_2 = \{0, u_2, u_4, u_5\}$

$$NVB_{M_{NVB}(u_2)}(0) = NVB_{M_{NVB}}\,(0 \star (u_2)^{-1}) = NVB_{M_{NVB}}\,(0 \star u_2) = NVB_{M_{NVB}}(u_2)$$
$$= [0.3, 0.5], [0.6, 0.7], [0.5, 0.7]$$
$$NVB_{M_{NVB}(u_2)}(u_2) = NVB_{M_{NVB}}\,(u_2 \star (u_2)^{-1}) = NVB_{M_{NVB}}\,(u_2 \star u_2) = NVB_{M_{NVB}}(0)$$
$$= [0.7, 0.9], [0.1, 0.4], [0.1, 0.3]$$
$$NVB_{M_{NVB}(u_2)}(u_4) = NVB_{M_{NVB}}\,(u_4 \star (u_2)^{-1}) = NVB_{M_{NVB}}\,(u_4 \star u_2) = NVB_{M_{NVB}}(u_2)$$
$$= [0.3, 0.5], [0.6, 0.7], [0.5, 0.7]$$
$$NVB_{M_{NVB}(u_2)}(u_5) = NVB_{M_{NVB}}\,(u_5 \star (u_2)^{-1}) = NVB_{M_{NVB}}\,(u_5 \star u_2) = NVB_{M_{NVB}}(u_2)$$
$$= [0.3, 0.5], [0.6, 0.7], [0.5, 0.7]$$

$M_{NVB}\,\{u_1 : u_2\} =$

$$\left\{ \begin{array}{l} \left\langle \frac{[0.2,0.7],[0.5,0.7],[0.3,0.8]}{0}, \frac{[0.7,0.9],[0.1,0.4],[0.1,0.3]}{u_1}, \frac{[0.2,0.7],[0.5,0.7],[0.3,0.8]}{u_3} \right\rangle, \\ \left\langle \frac{[0.3,0.5],[0.6,0.7],[0.5,0.7]}{0}, \frac{[0.7,0.9],[0.1,0.4],[0.1,0.3]}{u_2}, \frac{[0.3,0.5],[0.6,0.7],[0.5,0.7]}{u_4}, \frac{[0.3,0.5],[0.6,0.7],[0.5,0.7]}{u_5} \right\rangle \end{array} \right\}$$

To construct a NVB G -Left Coset

Let $u_a = u_1 \in U_1$ and $\forall u_c \in U_1 = \{0, u_1, u_3\}$

$$NVB_{(u_1)M_{NVB}}\,(0) = NVB_{M_{NVB}}\,((u_1)^{-1} \star 0)$$
$$= NVB_{M_{NVB}}\,(u_1 \star 0) = NVB_{M_{NVB}}\,(u_1)$$
$$= [0.2, 0.7], [0.5, 0.7], [0.3, 0.8]$$
$$NVB_{(u_1)M_{NVB}}\,(u_1) = NVB_{M_{NVB}}\,(u_1 \star (u_2)^{-1}) = NVB_{M_{NVB}}\,(u_1 \star u_2) = NVB_{M_{NVB}}\,(u_2)$$
$$= [0.3, 0.5], [0.6, 0.7], [0.5, 0.7]$$
$$NVB_{(u_1)M_{NVB}}\,(u_3) = NVB_{M_{NVB}}\,\left((u_1)^{-1} \star u_3\right) = NVB_{M_{NVB}}\,(u_1 \star u_3) = NVB_{M_{NVB}}\,(u_3)$$
$$= [0.6, 0.7], [0.1, 0.4], [0.3, 0.4]$$

& Let $u_b = u_2 \in U_2$ and $\forall u_d \in U_2 = \{0, u_2, u_4, u_5\}$

$$NVB_{(u_2)M_{NVB}}(0) = NVB_{M_{NVB}}\left((u_2)^{-1} \star 0\right) = NVB_{M_{NVB}}(u_2 \star 0) = NVB_{M_{NVB}}(u_2) = [0.3, 0.5], [0.6, 0.7], [0.5, 0.7]$$

$$NVB_{(u_2)M_{NVB}}(u_2) = NVB_{M_{NVB}}\left((u_2)^{-1} \star u_2\right) = NVB_{M_{NVB}}(u_2 \star u_2) = NVB_{M_{NVB}}(0) = [0.7, 0.9], [0.1, 0.4], [0.1, 0.3]$$

$$NVB_{(u_2)M_{NVB}}(u_4) = NVB_{M_{NVB}}\left((u_2)^{-1} \star u_4\right) = NVB_{M_{NVB}}(u_2 \star u_4) = NVB_{M_{NVB}}(u_4) = [0.2, 0.8], [0.4, 0.7], [0.2, 0.8]$$

$$NVB_{(u_2)M_{NVB}}(u_5) = NVB_{M_{NVB}}\left((u_2)^{-1} \star u_5\right) = NVB_{M_{NVB}}(u_2 \star u_5) = NVB_{M_{NVB}}(u_5) = [0.6, 0.9], [0.3, 0.7], [0.1, 0.4]$$

$(u_1 : u_2)M_{NVB} =$

$$\left\{ \begin{array}{l} \left\langle \frac{[0.2, 0.7], [0.5, 0.7], [0.3, 0.8]}{0}, \frac{[0.3, 0.5], [0.6, 0.7], [0.5, 0.7]}{u_1}, \frac{[0.6, 0.7], [0.1, 0.4], [0.3, 0.4]}{u_3} \right\rangle, \\ \left\langle \frac{[0.3, 0.5], [0.6, 0.7], [0.5, 0.7]}{0}, \frac{[0.7, 0.9], [0.1, 0.4], [0.1, 0.3]}{u_2}, \frac{[0.2, 0.8], [0.4, 0.7], [0.2, 0.8]}{u_4}, \frac{[0.6, 0.9], [0.3, 0.7], [0.1, 0.4]}{u_5} \right\rangle \end{array} \right\}$$

Remark 5.4.32.

1. *In example 5.4.31,* $M_{NVB}(u_1 : u_2) \neq (u_1 : u_2)M_{NVB}$

2. *Constant 0 is not an identity element in G - algebra.*
   For example, let $U = \{0, u_1, u_2, u_3, u_4, u_5\}$. $(U, \star, 0)$ *is a G - algebra, with binary operation* $\star$ *is defined by Cayley Table 5.11.*

Table 5.11: Cayley Table

$*$	0	u_1	u_2	u_3	u_4	u_5
0	0	u_2	u_1	u_3	u_4	u_5
u_1	u_1	0	u_3	u_2	u_5	u_4
u_2	u_2	u_4	0	u_5	u_1	u_3
u_3	u_3	u_5	u_4	0	u_2	u_1
u_4	u_4	u_3	u_5	u_1	0	u_2
u_5	u_5	u_1	u_2	u_4	u_3	0

It is clear that U is a G - algebra without an identity element. And hence inverse does not exist. So neutrosophic vague binary G - cosets cannot construct in this case. This construction is possible, only for those cases where identity element exists in the basis G -algebraic structure.

3. *If the basic G - algebraic structure is formed using the following rules, then definitely there exist identity element and hence can construct a NVB G -right coset & NVB G -left coset*

Conclusion

In this chapter, in first section, two logical algebras viz., BCK and BCI are developed for neutrosophic vague binary sets. Its ideal and cut are also got discussed. Different kinds of ideals like p ideal, q ideal, a ideal, H ideal for neutrosophic vague binary BCK/BCI -algebra have been investigated. In second section, neutrosophic vague binary $BZMV^{dM}$ Sub-algebra of $BZMV^{dM}$algebra is developed. This idea will provide a combined effect of the distributive Brouwer Zadeh lattice with Many Valued or Multi - Valued algebra when stipulated into the de-Morgans zone. Third section took a floor for defining neutrosophic vague binary K-subalgebra of the logical K- algebra. Its wide scope in applicational facets made its study more important among logical algebras. In fourth section, NVB G-subalgebraic structure is developed with its properties for NVBS's. Formation of cosets is a basic idea in any algebraic structure. Cosets for neutrosophic vague binary G - subalgebra is also got developed.

As a future scope neutrosophic vague binary models can be tried to use in hazard detection, especially in switching circuits. Binary concept leads us to handle the situations with two universal sets which are found to be common in real-life. Another application can be given in geographical area. Development of a neutrosophic vague binary spatial algebra could be more helpful in this area than the already existing crisp spatial algebraic concepts. Since the already existing pattern got failed to provide an accurate output when collected data becomes vague.

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[1] Vague Binary Sets and their properties, International Journal of Engineering, Science and Mathematics, 7(11), (2018), 56 −73

[2]Vague Binary Soft Topological Spaces, Compliance Engineering Journal, 11 (1), (2020), 20-30

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[1] Remya.P.B and Dr. Francina Shalini. A, Vague Soft Matrix in decision making Problems, Nirmala Annual Research Congress (NARC-2018), Organized by the Internal Quality Assurance Cell, December 4th 2018

[2] Remya.P.B and Dr. Francina Shalini.A, Trigonometric Normalised Hamming Similarity Measure Between Vague Binary Soft Sets, Nirmala Annual Research Congress (NARC - 2019), Organized by the Internal Quality Assurance Cell, November 27th 2019

[3] Remya.P.B and Dr.Francina Shalini.A, Presented a paper entitled on various distance measures of vague binary soft sets, in International Conference on "Recent Trends In Computational Mechanics"(ICRTCM - 2019) organized by the PG and Research Department of Mathematics on 2nd August, 2019

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Neutrosophic Vague Binary G – subalgebra of G - algebra

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Abstract: Nowadays, human society is using artificial intelligence in a large manner so as to upgrade the present existing applicational criteria's and tools. Logic is the underlying principle to these works. Algebra is inevitably inter-connected with logic. Hence its achievements to the scientific research outputs have to be addressed. For these reasons, nowadays, research on various algebraic structures are going on wide. Crisp set has also got developed in a parallel way in the forms as fuzzy, intuitionistic fuzzy, rough, vague, neutrosophic, plithogenic etc. Sets with one or more algebraic operations will form different new algebraic structures for giving assistance to these logics, which in turn acts to as, a support to artificial intelligence. BCH/BCI/BCK- are some algebras developed in the first phase of algebraic development output. After that, so many outputs got flashed out, individually and in combinations in no time. Q- algebra and QS –algebra are some of these and could be showed as such kind of productions. G- algebra is considered as an extension to QS – algebra. In this paper neutrosophic vague binary G – subalgebra of G – algebra is generated with example. Notions like, 0 – commutative G - subalgebra, minimal element, normal subset etc. are investigated. Conditions to define derivation and regular derivation for this novel concept are clearly presented with example. Constant of G – algebra can't be treated as the identity element, generally. In this paper, it is well explained with example. Cosets for neutrosophic vague binary G – subalgebra of G - algebra is developed with proper explanation. Homomorphism for this new concept has been also got commented. Its kernel, monomorphism and isomorphism are also have discussed with proper attention.

Keywords: neutrosophic vague binary G - subalgebra, neutrosophic vague binary G - normal set, neutrosophic vague binary G – normal subalgebra, neutrosophic vague binary G G - part, neutrosophic vague binary G - p radical , neutrosophic vague binary G - p semisimple, neutrosophic vague binary G - minimal element, 0- commutative neutrosophic vague binary G – subalgebra, neutrosophic vague binary G – Derivation, neutrosophic vague binary G – Regular Derivation, neutrosophic vague binary G - Coset, Kernel of neutrosophic vague binary G – Homomorphism.

Notations: NVBS – neutrosophic vague binary set, NVBSS – neutrosophic vague binary subset. In this paper NVB is used to indicate neutrosophic vague binary and NV is used to indicate neutrosophic vague and N is used to indicate neutrosophic.

1.Introduction

Without mathematics mobility in human-life even became an unthinkable process. But when get into the mathematical world, one faces with, versatile facets of maths, which again get take diversions. The thing is that, dry subject is less get commented on or even less get touched with!

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Neutrosophic Vague Binary BCK/BCI-algebra

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Abstract: Ineradicable hindrances of the existing mathematical models widespread from probabilities to soft sets. These difficulties made up way for the opening of "neutrosophic set model'. Set theory of 'vague' values is an already established branch of mathematics. Complex situations which arose in problem solving, demanded more accurate models. As a result, 'neutrosophic vague' came into screen. At present, research works in this area are very few. But it is on the way of its moves. Algebra and topology are well connected, as algebra and geometry. So, anything related to geometric topology is equally important in algebraic topology too. Separate growth of algebra and topology will slow down the development of each branch. And in one sense it is imperfect! In this paper a new algebraic structure, BCK/BCI is developed for 'neutrosophic' and to 'neutrosophic vague' concept with 'single' and 'double' universe. It's sub-algebra, different kinds of ideals and cuts are developed in this paper with suitable examples where necessary. Several theorems connected to this are also got verified.

Keywords: Vague H - ideal**,** neutrosophic vague binary BCK/BCI - algebra, neutrosophic vague binary BCK/BCI – subalgebra, neutrosophic vague binary BCK/BCI - ideal, neutrosophic vague binary BCK/BCI p- ideal, neutrosophic vague binary BCK/BCI q - ideal, neutrosophic vague binary BCK/BCI a-ideal, neutrosophic vague binary BCK/BCI H - ideal, neutrosophic vague binary BCK/BCI - cut

Notations: NVBS : neutrosophic vague binary set, NVBSS : neutrosophic vague binary subset, NVBI : neutrosophic vague binary ideal, N BCK/BCI - algebra : neutrosophic BCK/BCI-algebra, NV BCK/BCI - algebra : neutrosophic vague BCK/BCI-algebra, NVB BCK/BCI - algebra : neutrosophic vague binary BCK/BCI - algebra, N BCK/BCI - subalgebra : neutrosophic BCK/BCI - subalgebra, NV BCK/BCI - subalgebra : neutrosophic vague BCK/BCI - subalgebra, NVB BCK/BCI – subalgebra : neutrosophic vague binary BCK/BCI - subalgebra, N BCK/BCI - ideal : neutrosophic BCK/BCI –ideal, NV BCK/BCI - ideal : neutrosophic vague BCK/BCI - ideal , NVB BCK/BCI- ideal : neutrosophic vague binary BCK/BCI - ideal, NVB BCK/BCI p-ideal : neutrosophic vague binary BCK/BCI p-ideal, NVB BCK/BCI q - ideal : neutrosophic vague binary BCK/BCI q - ideal, NVB BCK/BCI a - ideal : neutrosophic vague binary BCK/BCI a - ideal, NVB BCK/BCI H - ideal : neutrosophic vague binary BCK/BCI H - ideal

1. Introduction

Before 1990's, mathematicians and researchers made use of different mathematical models for problem solving viz. , Probability theory, Hard set theory, Fuzzy set theory, Rough set theory,

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Neutrosophic Vague Binary Sets

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Abstract: Vague sets and neutrosophic sets play an inevitable role in the developing scenario of mathematical world. In this modern era of artificial intelligence most of the real life situations are found to be immersed with unclear data. Even the newly developed concepts are found to fail with such problems. So new sets like Plithogenic and new combinations like neutrosophic vague arose. Classical set theory dealt with single universe and can be studied by taking it's subsets. Situations demand two universes instead of a unique one in certain problems. In this paper two universes are introduced simultaneously and under consideration in a neutrosophic vague environment. It's basic operations, topology and continuity are also discussed with examples. A real life example is also discussed.

Keywords: binary set, fuzzy binary set, vague binary set, neutrosophic vague binary sets, neutrosophic vague binary topology, neutrosophic vague binary continuity

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1. Introduction

Functions are tightly packed but relations are not. They are more general than functions. Decimal system deals with ten digits while binary with two - only with 0 and 1. For detecting electrical signal's on or off state binary system can be used more effectively. It is the prime reason of selecting binary language in computers. Binary operations in algebra will give another idea! After a binary operation, 'operands' produce an element which is also a member of the parent set - means 'domain and co-domain' are in the same set. But binary relations are quite different from the ideas mentioned above. They are subsets of the cartesian product of the sets under consideration, taken in a special way. It is clear that binary stands for two. In point-set topology information from elements of topology will give information about subsets of the universal set under consideration. But real life can't be confined into a single universal set. It may be two or more than two. Being an extension of classical sets [George Cantor, 1874-1897] [27], fuzzy sets (FS's) [Zadeh, 1965] [29] can deal with partial membership. In intuitionistic fuzzy sets (IFS's) [Attanassov, 1986] [12] two membership grades are there - truth and false. As an extension of fuzzy sets Gau and Buehrer [9] introduced vague sets in 1993. Neutro-sophy means knowledge of neutral thought. It is a new branch of philosophy introduced by Florentin Smarandache [6] in 1995 - by giving an additional component - indeterminacy. Movement of paradoxism was set up by him in early 1980's. New concept dealt with the principle of using non-artistic elements to set artistic. Within no time so many hybrid structures developed by using the merits of the newly developed theory. In 2014, Alblowmi. S. A and Mohmed Eisa [1] gave some new concepts of neutrosophic sets. In 1996, Dontchev [5] developed Contra-continous functions and strongly s-closed spaces. In 2014, Salama A.A, Florentin Smarandache and Valeri Kromov [25] developed neutrosophic closed set and neutrosophic continous functions.

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VAGUE BINARY SOFT SETS AND THEIR PROPERTIES

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Abstract: The aim of this paper is to introduce the novel concept vague binary soft sets and to characterize some of its properties.
Keywords: Vague binary set, vague binary soft set, vague binary soft equal set, vague binary soft complement, vague binary soft AND operation, vague binary soft OR operation
Notations: $\ddot{F}$(e) represents the e-approximate element under the mapping $\ddot{F}$. Usual set theoretical operations with double dot on top will be used for vague binary soft set operations. Vague binary soft set is denoted by VBSS. Collection of all vague binary soft sets over the common universe U_1, U_2 under fixed parameter set A is denoted by VBSS$(U_1, U_2)_A$

1. Introduction

In most of the real life situations, humanity is bound to face with loss of data, unclear data, game of chance etc. To overcome such situations, classical probability theory (Gerolamo Cardano, 1501-1575)[5], fuzzy set theory (Zadeh, 1965)[2,5], rough set theory (Pawlak, 1982)[2,5], intuitionistic fuzzy set theory (Attanassov, 1986)[2,5],vague set theory (Gaue & Buehrer, 1993)[5], theory of interval mathematics (Moore, 1996)[5], neutrosophic set theory (Smarandache, 2005)[2] etc have played an important role. Inspired from 'Pawlak's work done in 1993', Molodtsov introduced soft sets in 1999. It has loosened all the existing rigid structure of classical sets by providing plenty of parameterization tools. Hybrid structures like fuzzy soft [3], soft fuzzy [3], neutrosophic soft [2], vague soft [2,4] etc developed later to make things more easier. All of them found rich with parameterization tools.

Later in 2016, Ahu Acikg$\ddot{o}$z [1] introduced binary soft sets with its operations and concluded that soft set can be given on n-dimension initial universal sets with a parameter set like F: A $\longrightarrow \prod_{i=1}^{n} P(U_i)$, where $U_i$are initial universal sets for 1⩽i⩽n and A is the parameter set. Vague binary sets and Vague binary soft sets with two initial universal sets are introduced. Some of the basic operations for vague binary soft sets like union, intersection, complement, AND operation, OR operation, Cartesian product are introduced in this paper. Terms like null vague binary soft set, absolute vague binary soft set are also introduced.

2.Preliminaries

Definition 2.1:[2,5]

A vague set A in the universe of discourse U={u_1, u_2, …….,u_n} is characterized by two membership functions given by
(1) a truth membership function t_A: U ⟶ [0,1]
(2) a false membership function f_A: U⟶ [0,1] where $t_A(u_i)$ is a lower bound of the grade of membership of $u_i$derived from the " evidence for u_i" and $f_A(u_i)$ is a lower bound on the negation

Vague Binary Soft Topological Spaces

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Abstract *– Vague binary soft topological spaces over two initial universal sets with a fixed parameter set is developed in this paper. Notions like interior, closure, exterior, boundary, neighborhood, separation axioms are also discussed with some of its properties*

Keywords – Vague binary soft topology, vague binary soft interior, vague binary soft exterior, vague binary soft boundary, vague binary soft neighborhood, vague binary soft separation axioms

AMS 2010 Mathematics subject classification: 54A05, 54D10, 03E72, 03B52

Notations: VBSS, VBSSS, VBST, VBSTS, VBSOS,VBSOSS, VBSCS,VBSCSS, VBSP, VBSS $(U_1, U_2)_A$ denotes vague binary soft set, vague binary soft subset, vague binary soft topology, vague binary soft topological space, vague binary soft open set,vague binary soft open subset, vague binary soft closed set,vague binary soft closed subset, vague binary soft point, set of all vague binary soft sets over U_1, U_2 under a fixed parameter set A respectively in this paper

1. INTRODUCTION

Modern topology got a strong base on Georg Cantor's [12] classical set theory developed in the 19th century. Numerous theories like fuzzy set theory [14] [Zadeh, 1965], Rough set theory [9] [Pawlak, 1982], Intuitionist fuzzy set theory [7] [Attanassov,1986], Vague set theory[6] [Gau and Buehrer, 1993],Theory of interval mathematics[10][Moore, 1995], Neutrosophic set theory[4] [Smarandache, 2005] developed. Molostsov [8] introduced Soft set theory in 1999. It is rich with its parameterization tools and found more effective and useful in comparison with other branches.

Ahu Acikgöz and Nihal Tas [1] introduced the concept of binary soft set theory in 2016, with two initial universal sets and a fixed parameter set and studied some of its properties. It widened the growth of Soft set theory to a new direction. Concept of binary soft topology and related basics are a continuation work done by Benchalli et al. [11]. They also introduced separation axioms [2] for binary soft sets. In 2014, Chang Wang and Yaya Li [13] introduced topological structure of vague soft sets. In 2010 Wei Xu et.al., [13] developed vague soft sets and used it in decision making problems. In 2014, Chang Wang and Yaya Li [3] introduced topological structure of vague soft sets. Francina Shalini. A and Remya. P. B [5] developed a hybrid structure, vague binary soft sets in 2018 and studied some of its properties. This paper aims its continuation work vague binary soft topological spaces and some of its basic notions. Vague binary soft separation axioms are also developed and some of its basic properties are verified.

2. PRELIMINARIES

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On Various Distance Measures Of Vague Binary Soft Sets

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Abstract—**Classical set theory has gone a long way in its developmental scenario. Among this soft set theory provides a strong tool to measure uncertainty due to its abundance in parameter set. Vague sets play equal importance when handling with uncertainty. In this paper various distance measures are discussed using a hybrid structure vague binary softsets.**

Keywords—**Vague Binary Soft Sets, Hamming distance of vague binary soft sets, Normalized hamming distance of vague binary soft sets, Euclidean distance of vague binary soft sets, Normalized Euclidean distance of vague binary soft sets**

Notations:**VSS(U), vss, vbss denotes set of all vague soft sets over U , vague soft set, vague binary soft set respectively in this paper**

***AMS Classification Code*— 97E60, 03EXX, 03B52, 51Kxx**

1. INTRODUCTION

George Cantor's classical set theory was inadequate in certain real life situationswhich made 'researchers and mathematicians' to think for some other tools. As a resultresearches in 'set theory' burst out in different ways and outlooks. Fuzzy set theory(1965, L.A.Zadeh)[5], Rough Set theory(1982, Pawlak),Intuitionistic Fuzzy Set theory(1986,Attanassov), Vague set theory (1993, Gau & Buehrer)[5, 7], Neutrosophic Set Thery (1995, Smarandache)[5], Interval Mathematics(1996, Moore)[5], Soft set theory(1999, Molodtsov) [5]are some of them. These theories have their own difficulties and negatives when dealing with certain situations. To solve this difficulty 'hybrid structures'of these theories are developed. They handled uncertainties in a more flexible way than the original single set. 'Vague soft set'is such a hybrid structure developed by Xu et al.,[4]in 2010. These hybrid structures extract the beneficial properties of their parent sets, which gave researchers & mathematicians 'strong tools' than the existing ones. Potential of soft sets are high due to their abundancy in parameter set. Vague set provides an interval instead of a single value. Binary soft set theory was introduced by Ahu Acikgöz and Nihal Tas[1, 5] in 2016. It deals with two universes instead of a unique one. Later Dr. Francina Shalini.A and Remya.P.B [5] developed vague

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Measures of Similarity between Vague Binary Soft Sets

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Abstract -Vague sets are one of the extensions of fuzzy sets. Vague binary soft set is one hybrid structure developed for dealing complicated situations with uncertainties with two universes. Similarity measures have a major role in application field of set theory. In this paper a similarity measure and weighted similarity measure is developed for measuring the degree of similarity between vague binary soft sets.

Keywords - vague binary soft set ; similarity measure ; weighted similarity measure
Notations – In this paper, VBSS denotes vague binary soft set. M((F̈, E), (G̈, E)), W((F̈, E), (G̈, E)) is used to denote similarity measures and weighted similarity measures of vague binary soft sets.

1. INTRODUCTION

Georg Cantor's classical set theory was inadequate to handle with several real life situations due to it's hard nature. So it need some reformations in time as a result so many other theories comes out viz., fuzzy set theory, intuitionistic fuzzy set theory, rough set theory, interval mathematics, vague set theory, soft set theory, neutrosophic set theory etc. These theories have their own positives and negatives. To overcome the negatives and extracting the positives some new hybrid structures like fuzzy soft, soft fuzzy, intuitionistic fuzzy, rough neutrosophic, vague soft, neutrosophic soft etc arose. These theories are used in a great extent to practical problems consisting uncertainties and vagueness where hard set theory always found to fail. Also the researches involving them are moving forward in a rapid speed globally. This paper concentrates in a particular area called similarity measures of one hybrid structure vague binary soft sets, developed by Dr. Francina Shalini. A. and Remya.P. B [5] in 2018. Similarity measures can be used to measure how much two sets or patterns or images are alike. In other words they can tell 'How much fuzzy ? '- a fuzzy set is ! or ' How much vague ?'-a vague set is! etc. Similarity measures based on set theoretical approach, distance and matching function which satisfying some axiomatic conditions are well known. Entropy and distance measures can also furnish the same role as that of similarity measures. i.e., they are also some kind of measurement tools used in the above mentioned theories with uncertainties. So these three together can work remarkably and can produce several useful theorems and formulae in this area. Similarity measures and distances are duals since large distance show low degree of similarity and vice versa. So distance measures could be used to define similarity measures. Problem under consideration decides which kind of similarity measure to be chosen. Wide applications of this topic in pattern recognition, decision making medical diagnosis, signal detection, security verification systems etc attracted the attention of researchers, in a commenting manner to this area, nowadays.

Molodtsov [5] introduced soft set theory in 1999 to remove all the existing difficulties of traditional set theory. Free of restrictions in describing the parameter set made soft set theory more convenient and user friendly. In 2010 Athar Kharal [1] introduced distance measures and similarity measures for soft sets. 'Majumdar' and 'Samanta' introduced similarity measure for soft sets based on distances using soft matrices. They [9] also introduced in 2011 similarity measures for fuzzy soft sets based on three different measures set-theoretical approach, matching function and distance. In 1995 Shyi-Ming Chen [12] proposed two similarity measures for measuring the degree of similarity between vague sets. In 2005, Jingli lu et al.,[7] presented a new similarity measure for vague sets. In 2006 , Faxin Zhao et al., [4] gave similarity measures for vague sets based on set theoretical method. Feng Sheng Xu [6] gave a new method on measures of similarity between vague sets in 2009. Qinrong Feng and Weinan Zheng [11] gave new similarity measures for fuzzy soft sets in 2013 based on different distance measures viz., normalized hamming distance, normalized Euclidean, normalized hausdorff, hamming-hausdorff, chebyshev etc. They also gave one application based on distances. In 2014 Zhicai Liu et al., [14] introduced so many similarity measures for fuzzy soft sets and pointed out draw backs of some of them . In 2015 Wenyi Zeng, Yibin Zhao and Yundong Gu [13] proposed similarity measure for vague sets based on implication functions. Chang Wang and Anjing Qu [2] proposed axiomatic definition soft entropy, similarity measure and distance measure for vague soft sets in 2013. They also put forward some formulas to calculate them and some relative theorems. In 2014 Dan Hu, Zhiyong Hong and Yong Wang [3] proposed a new approach to

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Trigonometric Normalised Hamming Similarity Measure BetweenVague Binary Soft sets

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Abstract:

Trigonometric Normalised Hamming Similarity measure between vague binary soft sets aredeveloped in this paper

Keywords: Cosine Normalised Hamming Similarity measure, Sine Normalised Hamming Similaritymeasure, Cotangent Normalised Hamming Similarity measure

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Notations: In this paper VBSS denotes vague binary soft set

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1. Introduction

Distance measure, Similarity measure and Entropy are found to have a great role in data mining problems involving uncertainties. Distances and Similarities are dual to each other. To overcome the deficiencies of classical sets, so manynew types of sets burst out - in which soft sets are novel. Gau and Buehrer [7] introduced vague sets in 1993.In 2005, Jingli Lu et al., [8] gave a new similarity measure for vague sets. In 2006, Faxin Zhao and Z.M. Ma [5] discussed similarity

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Trigonometric Normalized Euclidean Similarity Measure for Vague Binary Soft Sets

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Cantor's set theory got several extensions from nineteenth century to till now. In certain situations, by adding parameters, results could be got in a more clear-cut way. Soft set theory extends in that way. By adding vague concept to soft, researchers make use of the combined effect of these sets. Vague binary soft concept is the underlying idea in vague binary soft sets. It is a clear discussion with two universes. Similarity measures and distance measures are very useful in practical problems. Moreover, they are dual to each other. In this paper, similarity measures for vague binary soft sets by inserting trigonometric and normalized Euclidean ideas are developed

Keywords: Similarity Measure, Distance Measure, Cosine Normalized Euclidean Similarity Measure, Sine Normalized Euclidean Similarity Measure, Cotangent Normalized Euclidean Similarity Measure

Introduction

Strategies of fuzzy set theory allowed partial membership of elements to a set which helped to overcome the negativity of classical sets up to an extent. But uncertain nature of real-life situations demands more powerful tools to deal with certain crisis. In 1993, vague sets are introduced by Gau and Buehrer [7]. Vague sets allow non-memberships of elements which also became useful in the developing scenario of research world. In 1995, Shyi-Ming Chen [10] introduced some similarity measures between vague sets. Later Florentin Smarandache [3] introduced neutrosophic set in which uncertainties are allowed. In 2005, Jingli Lu, Xiaowei Yan, Dingrong Yuan and Zhangyan [8] gave a new similarity approach which serves practical purpose of vague sets. In 2010, Athar Kharal [1] introduced similarity measures for soft sets-i.e., the sets known for serving parameter datas. In 2013, Similarity measures for vague soft sets are introduced by Chang et al., [2]. In 2013, Quinrong Feng and Wenan Zheng [9] introduced new similarity measures of fuzzy soft sets based on distance measures. In 2014, Weibin Deng, Changlin Xu and Feng Hu [11] gave a novel distance measure between vague sets and its application in decision making. In 2018, Francina Shalini. A and Remya.P.B [4] developed vague binary soft sets - a hybrid structure of 'vague and soft'- with two universes and discussed their various laws and other basic properties. In 2019, Francina Shalini. A and Remya. P. B [5] developed measures of similarity between vague binary soft sets. In 2019, Francina Shalini. A and Remya. P.B [6] extended it to trigonometric normalized hamming similarity measure between vague binary soft sets. Similarity measures are very useful to measure similarities between objects. Entropy and distance measures are also found to be useful to the same extent as similarity measures - while measuring uncertainties in day to day real life. Different kinds of distance measures viz., Hamming, Normalized Hamming, Euclidean, Normalized Euclidean are found to be useful in an effective way in day to day life. In this paper, one of the distance measure - 'Normalized Euclidean' is combined with trigonometric functions to measure 'similarities' between vague binary soft sets.

2. Preliminaries

***Definition 2.1* : [4] (vague binary soft set)** Let $U_1 = \{x_1, x_2, ---, x_j\}$, $U_2 = \{y_1, y_2, ---, y_k\}$ be two initial universes which is common to a fixed set $A \subseteq E = \{e_1, e_2, --, e_m\}$ of parameters. Let $V(U_1)$, $V(U_2)$ denote the power set of vague sets on U_1, U_2 respectively. A pair $(\ddot{F}, A)$ is said to be a vague binary

Pythagorean Vague Binary Soft Sets

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Abstract— **Pythagorean vague binary soft sets are developed in this paper. Various distance measures are mentioned with example and application. It's higher dimension stage q-rung orthopair vague binary soft sets are also discussed.**

Keywords— **Pythagorean vague soft set, Pythagorean vague binary soft set, Operations on pythagorean vague binary soft sets, distance measure, q-rung orthopair vague binary soft set**

I. Introduction

Inadequacy of George Cantor's [4] Classical set theory made researchers to seek new tools to handle complex real life situations. Zadeh [4] succeeded in that and introduced fuzzy set theory in 1965, in which partial membership is allowed. Later in 1986, Attanassov [4] introduced classical intuitionistic fuzzy sets. IFS's helped to handle the uncertainty or hesitation region of fuzzy sets. He also introduced intuitionistic fuzzy sets (IFS's) of second type [3] in 1989 in which square sum of the membership grades is less than or equal to one. This sum takes greater than or equal to one in some real situations which demands some other tools to overcome this difficulty. Thus Pythagorean fuzzy sets [8, 9, 10, 14] arose which handled both these situations flexibly. Pythagorean aspects were firstly used in fuzzy set theory. Thus Pythagorean fuzzy sets are introduced in 2016 by Yager [10] to extend intuitionistic fuzzy sets. Later intuitionistic pythagorean fuzzy sets [10] are introduced. q-rung orthopair fuzzy set (q - ROFS with $q \geqslant 3$) is developed by yager [13] in 2018. Comparing with pythagorean fuzzy set it seems to be more stronger when dealing with uncertainty problems. In some situations membership degrees take values inside an interval and not a precise one. Interval valued pythagorean fuzzy set introduced by X.Peng [9] handled these situations more reliably. Vague sets were proposed by Gau and Buehrer [1, 2, 4] as an extension of fuzzy set theory. Pythagorean vague sets are introduced by Nirmala Irudayam [11] and Vinnarasi in 2018. Duojie et al., [3] introduced possibility pythagorean fuzzy soft set and its application in 2019. Pythagorean fuzzy number is introduced by Zhang and Xu [11]. They also introduced comparison laws for Pythagorean fuzzy numbers. Weakness of the existing correlation coefficients in intuitionistic fuzzy set theory made Harish Garg [5] to develop a novel correlation coefficient in pythagorean fuzzy sets in 2016. Later Harish Garg [6] introduced hesitant pythagorean fuzzy set by combining the concepts pythagorean and hesitant fuzzy set in 2018. Vague binary soft sets were introduced by Dr.Francina Shalini.A [4] and Remya.P.B. in 2018. They discussed some of it's properties. Pythagorean membership grade takes more space than intuitionistic membership grades. An intuitionistic membership grade is always Pythagorean but the converse need not be! Pythagorean vague membership grade is a point on a unit circle. This paper aims to develop Pythagorean nature of vague binary soft sets via Pythagorean vague sets and vague binary soft sets. Some distance measures are developed. Using that a decision making problem is also discussed. Besides it's higher stage, q-rung orthopair vague binary soft sets are also developed.