PERFECT POWERS IN SMARANDACHE N- EXPRESSIONS

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Abstract The main purpose of this paper is to study the concept of Smarandache n-expressions (but with a slight modification), its perfect powers, give conjectures, and proposed future studies.

Keywords: Smarandache n-expressions, Smarandache 5-expressions, Smarandache 2-expressions, perfect powers in Smarandache type expressions.

§1. Introduction

In [1] M. Perez & E. Burton, documented that J. Castillo [2], asked how many primes are there in the Smarandache n-expressions:

$$x_1^{x_2} + x_2^{x_3} + \cdots + x_n^{x_1} \qquad (1)$$

where $n > 1, x_1, x_2, \cdots, x_n > 1$, and $gcd(x_1, x_2, \cdots, x_n) = 1$
In this paper, with only slight modification of (1), we got (2) namely;

$$a^{x_1} + a^{x_2} + \cdots + a^{x_n} \qquad (2)$$

where $a > 1, x_1, x_2, \cdots, x_n \geq 0$, and $gcd(a, x_1, x_2, \cdots, x_n) = 1$
I will study the following cases of equation (2).

§2. Case1 of $5-$Expressions

$$3^{x_1} + 3^{x_2} + 3^{x_3} + 3^{x_4} + 3^{x_5} = k^2, \qquad (3)$$

The solution of (3) is:
$x_1 = 2m, x_2 = 2m+1, x_3 = 2m+2, x_4 = 2m+3, x_5 = 2m+4$, and $k = (11)3^m$.

Proof.

$$\begin{aligned} 3^{x_1}+3^{x_2}+3^{x_3}+3^{x_4}+3^{x_5} &= 3^{2m}+3^{2m+1}+3^{2m+2}+3^{2m+3}+3^{2m+4} \\ &= 3^{2m}(1+3^1+3^2+3^3+3^4) \\ &= 3^{2m}(121) \\ &= k^2. \end{aligned}$$

Examples:

$3^{x_1}+3^{x_2}+3^{x_3}+3^{x_4}+3^{x_5}$	k^2
$3^0+3^1+3^2+3^3+3^4$	11^2
$3^2+3^3+3^4+3^5+3^6$	33^2
$3^4+3^5+3^6+3^7+3^8$	99^2
$3^6+3^7+3^8+3^9+3^{10}$	297^2
$3^8+3^9+3^{10}+3^{11}+3^{12}$	891^2
$3^{10}+3^{11}+3^{12}+3^{13}+3^{14}$	2673^2

The first terms and the nth terms of the sequence are:

$$121, 1089, 9801, 88209, 793881, \cdots, (11)^2(9)^{n-1}, \cdots \tag{4}$$

Where the square roots are:

$$11, 33, 99, 297, 891, \cdots, (11)(3)^{(n-1)}, \cdots \tag{5}$$

Notice that there is no prime numbers in (5), (excluding 11).
The sum of (5) is $\frac{11(3^n)-1}{2}$, and there is no limit, since $\frac{11(3^n)-1}{2}$ becomes large as n approach infinity, the sequence has no limit, therefore it is divergent, but the summation of reciprocal convergent.

Conjecture: if p, q, r, s, t are primes numbers, then the equation $3^p+3^q+3^r+3^s+3^t=k^2$ has no solution.

§3. Case2 of 5-Expressions

$$3^{x_1}+3^{x_2}+3^{x_3}+3^{x_4}+3^{x_5}=k^2+k^2+k^2 \tag{6}$$

The solution of (3) is: $x_1=2m+1, x_2=2m+2, x_3=2m+3, x_4=2m+4, x_5=2m+5$, and $k=11(3)^{\frac{2m+1}{2}}$.

Proof.

$$\begin{aligned} 3^{x_1}+3^{x_2}+3^{x_3}+3^{x_4}+3^{x_5} &= 3^{2m+1}+3^{2m+2}+3^{2m+3}+3^{2m+4}+3^{2m+5} \\ &= 3^{2m+1}(1+3^1+3^2+3^3+3^4) \\ &= 3^{2m+1}(121) \\ &= 3k^2. \end{aligned}$$

Examples:

$3^{x_1}+3^{x_2}+3^{x_3}+3^{x_4}+3^{x_5}$	$3k^2$
$3^1+3^2+3^3+3^4+3^5$	$11^2+11^2+11^2$
$3^3+3^4+3^5+3^6+3^7$	$33^2+33^2+33^2$
$3^5+3^6+3^7+3^8+3^9$	$99^2+99^2+99^2$
$3^7+3^8+3^9+3^{10}+3^{11}$	$297^2+297^2+297^2$
$3^9+3^{10}+3^{11}+3^{12}+3^{13}$	$891^2+891^2+891^2$
$3^{11}+3^{12}+3^{13}+3^{14}+3^{15}$	$2673^2+2673^2+2673^2$

The first terms and nth terms of the sequence are:

$$(3)121,(3)1089,(3)9801,(3)88209,(3)793881\cdots(11)^2(3)(9)^{n-1},\cdots \quad (7)$$

The sum of (7) is $\frac{11^2(3)(9^n-1)}{8}$, and there is no limit, since $\frac{11^2(3)(9^n-1)}{8}$ becomes large as n approach infinity, the sequence has no limit, therefore it is divergent, but the summation of reciprocal convergent.

Conjecture: if p,q,r,s,t are primes numbers, then the equation $3^p+3^q+3^r+3^s+3^t=3k^2$ has no solution.

§4. Case3 of 5-Expressions

$$3^{x_1}+3^{x_2}+3^{x_3}+3^{x_4}+3^{x_5}=(11)^2(61)3^{x_1}, \quad (8)$$

The solution of (8) is:
$x_1=2m+1, x_2=2m+3, x_3=2m+5, x_4=2m+7, x_5=2m+9$, and $k=(61^{\frac{1}{2}})(11)(3^{\frac{2m+1}{2}})$.

Proof.

$$\begin{aligned}3^{x_1}+3^{x_2}+3^{x_3}+3^{x_4}+3^{x_5} &= 3^{2m+1}+3^{2m+3}+3^{2m+5}+3^{2m+7}+3^{2m+9}\\ &= (3^{2m+1})(11)^2(61).\end{aligned}$$

Examples:

$3^{x_1}+3^{x_2}+3^{x_3}+3^{x_4}+3^{x_5}$	$(3^{2m+1})(11)^2(61)$
$3^1+3^3+3^5+3^7+3^9$	$(3)(11)^2(61)$
$3^3+3^5+3^7+3^9+3^{11}$	$(3)^3(11)^2(61)$
$3^5+3^7+3^9+3^{11}+3^{13}$	$(3)^5(11)^2(61)$
$3^7+3^9+3^{11}+3^{13}+3^{15}$	$(3)^7(11)^2(61)$
$3^9+3^{11}+3^{13}+3^{15}+3^{17}$	$(3)^9(11)^2(61)$
$3^{11}+3^{13}+3^{15}+3^{17}+3^{19}$	$(3)^{11}(11)^2(61)$

The first terms and nth terms of the sequence are:

$$(3)(11)^2(61),(3)^3(11)^2(61),(3)^5(11)^2(61),(3)^7(11)^2(61),(3)^9(11)^2(61),$$
$$(3)^{11}(11)^2(61),\cdots,(3)(61)(11)^2(9)^{n-1},\cdots \quad (9)$$

The sum of (9) is $\frac{11^2(3)(61)(9^n-1)}{8}$, and there is no limit, since $\frac{11^2(3)(61)(9^n-1)}{8}$ becomes large as n approach infinity, the sequence has no limit, therefore it is divergent, but the summation of reciprocal convergent.

Conjecture: if p, q, r, s, t are primes numbers, then the equation $3^p+3^q+3^r+3^s+3^t=3^{2m+1}(11)^2(61)$ has no solution.

§5. Case4 of 5-Expressions

$$3^{x_1}+3^{x_2}+3^{x_3}+3^{x_4}+3^{x_5}=(11)^2(61)3^{x_1}, \tag{10}$$

The solution of (10) is:
$x_1=2m, x_2=2m+2, x_3=2m+4, x_4=2m+6, x_5=2m+8$, and $k=(61^{\frac{1}{2}})(11)(3^m)$.

Proof.

$$\begin{aligned} 3^{x_1}+3^{x_2}+3^{x_3}+3^{x_4}+3^{x_5} &= 3^{2m}+3^{2m+2}+3^{2m+4}+3^{2m+6}+3^{2m+8} \\ &= 3^{2m}(11)^2(61). \end{aligned}$$

Examples:

$3^{x_1}+3^{x_2}+3^{x_3}+3^{x_4}+3^{x_5}$	$3^{2m}(11)^2(61)$
$3^2+3^4+3^6+3^8+3^{10}$	$(3)^2(11)^2(61)$
$3^4+3^6+3^8+3^{10}+3^{12}$	$(3)^4(11)^2(61)$
$3^6+3^8+3^{10}+3^{12}+3^{14}$	$(3)^6(11)^2(61)$
$3^8+3^{10}+3^{12}+3^{14}+3^{16}$	$(3)^8(11)^2(61)$
$3^{10}+3^{12}+3^{14}+3^{16}+3^{18}$	$(3)^{10}(11)^2(61)$
$3^{12}+3^{14}+3^{16}+3^{18}+3^{20}$	$(3)^{12}(11)^2(61)$

The first terms and nth terms of the sequence are:

$$(3)^2(11)^2(61), (3)^4(11)^2(61), (3)^6(11)^2(61), (3)^8(11)^2(61),$$
$$\cdots(61)(11)^2(3)^2(9)^{n-1}, \cdots \tag{11}$$

The sum of (11) is $\frac{(3^2)11^2(61)(9^n-1)}{8}$, and there is no limit, since $\frac{(3^2)11^2(61)(9^n-1)}{8}$ becomes large as n approach infinity, the sequence has no limit, therefore it is divergent, but the summation of reciprocal convergent.

Conjecture: if p, q, r, s, t are primes numbers, then the equation $3^p+3^q+3^r+3^s+3^t=3^{2m}(11)^2(61)$ has no solution.

§6. Case5 of 2-Expressions

$$3^x+3^y=z^2, \tag{12}$$

The solution of (12) is $x = 2m, y = 2m + 1$, and $z = 2(3)^m$
Proof. $3^x + 3^y = 3^{2m} + 3^{2m+1} = 3^{2m}(1 + 3) = z^2$

Examples:

$3^x + 3^y$	z^2
$3^2 + 3^3$	6^2
$3^4 + 3^5$	18^2
$3^6 + 3^7$	54^2
$3^8 + 3^9$	162^2
$3^{10} + 3^{11}$	486^2
$3^{12} + 3^{13}$	1458^2

The first terms and nth terms of the sequence are:
$36, 324, 2916, 26244, 236196, \cdots, (6)^2(9)^(n-1), \cdots$ (13)
Where the square roots are:

$$6, 18, 54, 162, 486, 1458, \cdots, (6)(3)(n-1), \cdots \quad (14)$$

The sum of the first n terms of the sequence (14) is given by the following formula.

$$\frac{6 - 6(3)^n}{1 - 3} = 3(3^n - 1).$$

and there is no limit, since $3(3^n - 1)$ becomes large as n approach infinity, the sequence has no limit, therefore it is divergent, but the summation of reciprocal convergent.

Conjecture:
1) The equation $3^x + 3^y = z^2$ has one solution in prime numbers, if x, and y are prime numbers, namely $(x, y) = (2, 3)$.
2) The equation $3^{x^2} + 3^{y^2} = z^2$ has unique solution, if x, and y are prime numbers, namely $(x, y, z) = (3, 2, 162)$.

§7. Case6 of 2-Expressions

$$3^x + 3^y = 3z^2, \quad (15)$$

The solution of (15) is $x = 2m + 1, y = 2m + 2$, and $z = 2(3)^{\frac{2m+1}{2}}$
Proof. $3^x + 3^y = 3^{2m+1} + 3^{2m+2} = 3^{2m}(3 + 9) = 3^{2m}(12) = 3z^2$

Examples:

$3^x + 3^y$	$3z^2$
$3^1 + 3^2$	$12 = 2^2 + 2^2 + 2^2$
$3^3 + 3^4$	$108 = 6^2 + 6^2 + 6^2$
$3^5 + 3^6$	$972 = 18^2 + 18^2 + 18^2$
$3^7 + 3^8$	$8748 = 54^2 + 54^2 + 54^2$
$3^9 + 3^{10}$	$78732 = 162^2 + 162^2 + 162^2$
$3^{11} + 3^{12}$	$708588 = 486^2 + 486^2 + 486^2$
$3^{13} + 3^{14}$	$6377292 = 1458^2 + 1458^2 + 1458^2$

The first terms and nth terms of the sequence are:
$12, 108, 972, 8748, 78732, \cdots, 12(9)n-1, \cdots$ (16)
The sum of the first n terms of the sequence (16) is given by the following formula.

$$\frac{12-12(9)^n}{1-9} = \frac{3(9^n-1)}{2}.$$

and there is no limit, since $\frac{3(9^n-1)}{2}$ becomes large as n approach infinity, the sequence has no limit, therefore it is divergent, but the summation of reciprocal convergent.
Conjecture: The equation $3^x + 3^y = 12(3)^{2m}$ has no solution, if x, and y are prime numbers.

§8. Case7 of 2-Expressions

$$3^x + 3^y = (10)3^{2m+1}, \quad (17)$$

The solution of (17) is $x = 2m+1, y = 2m+3$.
Proof: $3^x + 3^y = 3^{2m+1} + 3^{2m+3} = 3^{2m+1}(1+9) = 3^{2m+1}(10)$

Examples:

$3^x + 3^y$	$10(3)^{2m+1}$
$3^1 + 3^3$	30
$3^3 + 3^5$	270
$3^5 + 3^7$	2430
$3^7 + 3^9$	21870
$3^9 + 3^{11}$	196830
$3^{11} + 3^{13}$	1771470
$3^{13} + 3^{15}$	15943230

The first terms and the nth terms of the sequence are:

$$30, 270, 2430, 21870, 196830, \cdots, 30(9)n-1, \cdots \quad (18)$$

The sum of the first n terms of the sequence (18) is given by the following formula.

$$\frac{30-30(9)^n}{1-9} = \frac{15(9^n-1)}{4}.$$

and there is no limit, since $\frac{15(9^n-1)}{4}$ becomes large as n approach infinity, the sequence has no limit, therefore it is divergent, but the summation of reciprocal convergent.
Conjecture: The equation $3^x+3^y = 10(3)^{2m+1}$ has infinitely many solutions, if x, and y are prime numbers.

§9. Case8 of 2-Expressions

$$3^x + 3^y = 3^{2m}(10), \tag{19}$$

The solution of (19) is $x = 2m, y = 2m + 2$.
Proof. $3^x + 3^y = 3^{2m} + 3^{2m+2} = 3^{2m}(1 + 9) = 3^{2m}(10)$

Examples:

$3^x + 3^y$	$10(3)^{2m}$
$3^2 + 3^4$	90
$3^4 + 3^6$	810
$3^6 + 3^8$	7290
$3^8 + 3^{10}$	65610
$3^{10} + 3^{12}$	590490
$3^{12} + 3^{14}$	5314410
$3^{14} + 3^{16}$	47829690

The first terms and the nth terms of the sequence are:

$$90, 810, 7290, 65610, 590490, \cdots, 90(9)^{n-1}, \cdots \tag{20}$$

The sum of the first n terms of the sequence (20) is given by the following formula.

$$\frac{90 - 90(9)^n}{1 - 9} = \frac{45(9^n - 1)}{4}.$$

and there is no limit, since $\frac{45(9^n-1)}{4}$ becomes large as n approach infinity, the sequence has no limit, therefore it is divergent, but the summation of reciprocal convergent.
Conjecture: The equation $3^x + 3^y = 10(3)^{2m}$ has infinitely many solutions, if x, and y are prime numbers.

§10. Case9 of 2-Expressions

$$3^x - 3^y = 2(3)^y, \tag{21}$$

The solution of (21) is $x = 6m - 2, y = 6m - 3$.
Proof. $3^x - 3^y = 3^{6m-2} - 3^{6m-3} = 2(3)^{6m-3}$

Examples:

$3^x - 3^y$	$2(3)^{6m-3}$
$3^4 - 3^3$	$2(3)^3$
$3^{10} - 3^9$	$2(3)^9$
$3^{16} - 3^{15}$	$2(3)^{15}$
$3^{22} - 3^{21}$	$2(3)^{21}$
$3^{28} - 3^{27}$	$2(3)^{27}$
$3^{34} - 3^{33}$	$2(3)^{33}$
$3^{40} - 3^{39}$	$2(3)^{39}$

The first terms and the nth terms of the sequence are:

$$2(3)^3, 2(3)^9, 2(3)^{15}, 2(3)^{21}, 2(3)^{27}, \cdots, 2(3)^3(729)^{n-1}, \cdots \quad (22)$$

The sum of the first n terms of the sequence (20) is given by the following formula.

$$\frac{54 - 54(729)^n}{1 - 729} = \frac{27(729^n - 1)}{364}.$$

and there is no limit, since $\frac{27(729^n-1)}{364}$ becomes large as n approach infinity, the sequence has no limit, therefore it is divergent, but the summation of reciprocal convergent.
Conjecture: The equation $3^x - 3^y = 2(3)^y$ has no solutions, if x, and y are prime numbers.

§11. Case10 of 2-Expressions

$$3^x + 3^y = 4(3)^y, \quad (23)$$

The solution of (23) is $x = 6m - 2, y = 6m - 3$.
Proof. $3^x + 3^y = 3^{6m-2} + 3^{6m-3} = 4(3)^{6m-3}$
Examples:

$3^x + 3^y$	$4(3)^{6m-3}$
$3^4 + 3^3$	$4(3)^3$
$3^{10} + 3^9$	$4(3)^9$
$3^{16} + 3^{15}$	$4(3)^{15}$
$3^{22} + 3^{21}$	$4(3)^{21}$
$3^{28} + 3^{27}$	$4(3)^{27}$
$3^{34} + 3^{33}$	$4(3)^{33}$
$3^{40} + 3^{39}$	$4(3)^{39}$

The first terms and the nth terms of the sequence are:

$$4(3)3, 4(3)9, 4(3)15, 4(3)21, 4(3)27, \cdots, 4(3)3(729)n - 1, \cdots \quad (24)$$

The sum of the first n terms of the sequence (24) is given by the following formula.

$$\frac{108 - 108(729)^n}{1 - 729} = \frac{27(729^n - 1)}{182}.$$

and there is no limit, since $\frac{27(729^n-1)}{182}$ becomes large as n approach infinity, the sequence has no limit, therefore it is divergent, but the summation of reciprocal convergent.
Conjecture: The equation $3^x + 3^y = 4(3)^y$ has no solutions, if x, and y are prime numbers.

§12. Case11 of 2-Expressions

$$2^x + 2^y = z^2, \quad (25)$$

The solution of (25) is $x = 2m - 2, y = 2m + 1$, and $z = 3(2)^{m-1}$.
Proof. $2^x + 2^y = 2^{2m-2} + 2^{2m+1} = 2^{2m}(2^{-2} + 2^1) = 9(2)^{2m-2} = z^2$
Examples:

$2^x + 2^y$	Z^2
$2^0 + 2^3$	3^2
$2^2 + 2^5$	6^2
$2^4 + 2^7$	12^2
$2^6 + 2^9$	24^2
$2^8 + 2^{11}$	48^2
$2^{10} + 2^{13}$	96^2
$2^{12} + 2^{15}$	192^2

The first terms and the nth terms of the sequence are:

$$9, 36, 144, 576, 2304 \cdots, (9)(4)n - 1, \cdots \quad (26)$$

Where the square roots are:

$$3, 6, 12, 24, 48, 96 \cdots, (3)(2)(n - 1), \cdots \quad (27)$$

The sum of the first n terms of the sequence (27) is given by the following formula.

$$\frac{3 - 3(2)^n}{1 - 2} = 3(2^n - 1).$$

and there is no limit, since $3(2^n - 1)$ becomes large as n approach infinity, the sequence has no limit, therefore it is divergent, but the summation of reciprocal convergent.
Conjecture: The equation $2^x + 2^y = z^2$ has one solution if x, and y are prime numbers, i.e. $(x, y) = (2, 5)$.

Future Studies

The Smarandache n-expressions suggest that there may be future interesting n-expressions yet to be revealed.

Reference

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[2] J. Castillo, The Smarandache n-expressions, Mathematical Spectrum **29** (1997/8), 21.