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Operations on single valued neutrosophic graphs with application

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Abstract. The concepts of graph theory are applied in many areas of computer science including image segmentation, data mining, clustering, image capturing and networking. Fuzzy graph theory is successfully used in many problems, to handle the uncertainty that occurs in graph theory. A single valued neutrosophic graph (SVNG) is an instance of a neutrosophic graph and a generalization of the fuzzy graph, intuitionistic fuzzy graph, and interval-valued intuitionistic fuzzy graph. In this paper, the basic operations on SVNGs such as direct product, Cartesian product, semi-strong product, strong product, lexicographic product, union, ring sum and join are defined. Moreover, the degree of a vertex in SVNGs formed by these operations in terms of the degree of vertices in the given SVNGs in some particular cases are determined. Finally, an application of single valued neutrosophic digraph (SVNDG) in traval time is provided.

Keywords: Neutrosophic graph, direct product, Cartesian product, semi-strong product, strong product, lexicographic product, degree of a vertex

1. Introduction

Graph representations are generally used for dealing with structural information, in different domains such as operations research, networks, systems analysis, pattern recognition, economics and image interpretation. However, in many situations, some aspects of a graph theoretic problem may be vague or uncertain. For instance, the vehicle travel time or vehicle capacity on a road network may not be known exactly. In such situations, it is natural to deal with the uncertainty applying fuzzy set theory. In 1965, Zadeh [27] originally introduced the concept of fuzzy set. Its characteristic is that a membership degree in [0, 1] is assigned to each element in the set. After the inception of fuzzy set theory, it has become a vigorous area of research in different disciplines including management sciences, medical and life sciences, social sciences, artificial intelligence, signal processing, robotics, expert systems, computer networks, pattern recognition, decision-making, graph theory and automata theory.

Smarandache [21] firstly proposed neutrosophy, a branch of philosophy which discusses the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. The characteristics of neutrosophic set (NS), a generalization of [3, 4, 29], are described by truth-membership, indeterminacy membership and falsity membership degrees independently. NS as a powerful general formal framework expresses and handles imprecise, indeterminate and inconsistent information, existing in real situations. Intuitionistic fuzzy set (IFS) is a generalization of fuzzy set. Its characteristic is that a membership degree and a non-membership degree are assigned to each element in the set. However IFSs and interval-valued intuitionistic fuzzy sets (IVIFSs) cannot deal with all types of uncertainty, such as indeterminate and inconsistent information, therefore, the concept of NS is more extensive and overcomes

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the above-mentioned issues. But NSs are difficult to apply in the real applications. To easily apply it to scientific and engineering fields, Wang et al. [23] initiated the concept of a single valued neutrosophic set (SVNS) and provide its various properties. NS, particularly SVNS has attracted significant interest from researchers in recent years. It has been widely applied in various fields, such as information fusion in which data are combined from different sensors [6], control theory [1], image processing [12], medical diagnosis [26], decision making [25], and graph theory [7, 22], etc.

The concept of fuzzy graphs was initiated by Kafmann [13], based on Zadeh's fuzzy relations. Later, another elaborated definition of fuzzy graph with fuzzy vertex and fuzzy edges was introduced by Rosenfeld [15]. Mordeson and Peng [15] defined some operations on fuzzy graphs and investigated their properties. Later, the degrees of the vertices of the resultant graphs, obtained from two given fuzzy graphs using these operations were determined in [16, 17]. Ghorai and Pal [10, 11] defined the concept of m-polar fuzzy planar graphs. Intuitionistic fuzzy graphs were first introduced by Atanassov [5] and further discussed by Akram [2]. Mishra and Pal [14] initiated the notion of interval-valued intuitionistic fuzzy graphs. Rashmanlou et al. [18, 19] introduced many new concepts, including product of bipolar fuzzy graphs and interval-valued intuitionistic (S, T)-fuzzy graphs. When description of the objects or their relations or both is indeterminate and inconsistent, it cannot be handled by fuzzy, intuitionistic fuzzy and interval valued intuitionistic fuzzy graphs. Therefore some new theories are required. That is why, Smarandache [22] put forward the concept of neutrosophic graphs. Furthermore, Broumi et al. [7–9] discussed several properties of SVNGs and their extensions.

The paper is structured as follows: Section 2 contains a brief background about SVNSs and SVNGs. Section 3 establishes the definitions and properties of direct product, Cartesian product, semi-strong product, strong product, lexicographic product, union, ring sum and join on SVNGs. Section 4 is devoted to the application of SVNDGs in travel time and finally we draw conclusions in Section 5.

2. Preliminaries

In the following, some basic concepts on SVNSs and SVNGs are reviewed to facilitate next sections.

A graph is a pair of sets $G = (V, E)$, satisfying $E(G) \subseteq V \times V$. The elements of $V(G)$ and $E(G)$ are the vertices and edges of the graph G, respectively. The standard products of graphs: direct product (tensor product), Cartesian product, semi-strong product, strong product (symmetric composition) and lexicographic product (composition) of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ will be denoted by $G_1 \times G_2$, $G_1 \square G_2$, $G_1 \bullet G_2$, $G_1 \boxtimes G_2$ and $G_1[G_2]$, respectively. Let $(x_1, x_2), (y_1, y_2) \in V_1 \times V_2$, then

$$\begin{aligned}
&E(G_1 \times G_2)\\
&\quad = \{(x_1, x_2)(y_1, y_2) \mid x_1y_1 \in E_1 \text{ and } x_2y_2 \in E_2\},\\
&E(G_1 \square G_2) = \{(x_1, x_2)(y_1, y_2) \mid x_1\\
&\quad = y_1 \text{ and } x_2y_2 \in E_2, \text{ or } x_1y_1 \in E_1 \text{ and } x_2 = y_2\},\\
&E(G_1 \bullet G_2)\\
&\quad = \{(x_1, x_2)(y_1, y_2) \mid x_1 = y_1 \text{ and } x_2y_2 \in E_2,\\
&\qquad \text{or } x_1y_1 \in E_1 \text{ and } x_2y_2 \in E_2\},\\
&E(G_1 \boxtimes G_2)\\
&\quad = E(G_1 \square G_2) \cup E(G_1 \times G_2),\\
&E(G_1[G_2])\\
&\quad = \{(x_1, x_2)(y_1, y_2) \mid x_1y_1 \in E_1, \text{ or } x_1 = y_1\\
&\qquad \text{and } x_2y_2 \in E_2\}.
\end{aligned}$$

A directed graph (or digraph) is nothing but a graph with directed edges.

Definition 2.1. [20, 27, 28] A fuzzy subset η of a set V is a function $\eta : V \to [0, 1]$. A fuzzy (binary) relation on a set V is a mapping $\mu : V \times V \to [0, 1]$ such that $\mu(x, y) \leq \min\{\eta(x), \eta(y)\}$ for all $x, y \in V$. A fuzzy relation μ is symmetric if $\mu(x, y) = \mu(y, x)$ for all $x, y \in V$. A fuzzy graph is a pair $\mathcal{G} = (\eta, \mu)$, where η is a fuzzy subset of a set V and μ is a (symmetric) fuzzy relation on η.

Definition 2.2. [3] An IFS X in V is an object having the form

$$X = \{\langle x, \mu_X(x), \nu_X(x)\rangle \mid x \in V\},$$

where the functions $\mu_X : V \to [0, 1]$ and $\nu_X : V \to [0, 1]$ define the degree of membership and degree of non-membership of the element $x \in V$, respectively, such that $0 \leq \mu_X(x) + \nu_X(x) \leq 1$ for all $x \in V$.

For each IFS X in V, $\pi_X(x) = 1 - \mu_X(x) - \nu_X(x)$ is called a hesitancy degree of x in X. If $\pi_X(x) = 0$ for all $x \in V$, then IFS reduces to Zadeh's fuzzy set.

Definition 2.3. [21] Let V be a universal set. A NS X in V is an object having the following form

$$X = \{\langle x, T_X(x), I_X(x), F_X(x)\rangle \mid x \in V\},$$

which is characterized by a truth-membership function T_X, an indeterminacy-membership function I_X and a falsity-membership function F_X, where

$$T_X : V \to]0^-, 1^+[, x \in V \to T_X(x) \in]0^-, 1^+[$$
$$I_X : V \to]0^-, 1^+[, x \in V \to I_X(x) \in]0^-, 1^+[$$
$$F_X : V \to]0^-, 1^+[, x \in V \to F_X(x) \in]0^-, 1^+[$$

There is no restriction on the sum of $T_X(x)$, $I_X(x)$ and $F_X(x)$, therefore $0^- \leq \sup T_X(x) + \sup I_X(x) + \sup F_X(x) \leq 3^+$. The functions $T_X(x)$, $I_X(x)$ and $F_X(x)$ are real standard or nonstandard subsets of $]0^-, 1^+[$.

Definition 2.4. [23] Let V be a universal set. A SVNS X in V is an object having the form

$$X = \{\langle x, T_X(x), I_X(x), F_X(x)\rangle \mid x \in V\},$$

which is characterized by a truth-membership function T_X, an indeterminacy-membership function I_X and a falsity-membership function F_X, where

$$T_X : V \to [0, 1], x \in V \to T_X(x) \in [0, 1]$$
$$I_X : V \to [0, 1], x \in V \to I_X(x) \in [0, 1]$$
$$F_X : V \to [0, 1], x \in V \to F_X(x) \in [0, 1]$$

For convenience, $\gamma = \langle T, I, F\rangle$ is called a single valued neutrosophic number, where $T, I, F \in [0, 1]$, $0 \leq T + I + F \leq 3$. To measure the degree of suitability, Zhang et al. [30] presented a score function s of γ as $s(\gamma) = T + 1 - I + 1 - F$.

Definition 2.5. [24] A single valued neutrosophic relation in V, denoted by

$$R = \{\langle xy, T_R(xy), I_R(xy), F_R(xy)\rangle \mid xy \in V \times V\}$$

where $T_R : V \times V \to [0, 1]$, $I_R : V \times V \to [0, 1]$ and $F_R : V \times V \to [0, 1]$ represent the truth-membership function, indeterminacy membership function and falsity-membership function of R, respectively.

Definition 2.6. [7] A SVNG of a (crisp) graph $G = (V, E)$ is defined to be a pair $\mathcal{G} = (X, Y)$, where

(i) the functions $T_X : V \to [0, 1]$, $I_X : V \to [0, 1]$ and $F_X : V \to [0, 1]$ represent the degree of truth-membership, indeterminacy-membership and falsity membership of the element $x \in V$, respectively. There is no restriction on the sum of $T_X(x)$, $I_X(x)$ and $F_X(x)$, therefore $0 \leq T_X(x) + I_X(x) + F_X(x) \leq 3$ for all $x \in V$,

(ii) the functions $T_Y : E \subseteq V \times V \to [0, 1]$, $I_Y : E \subseteq V \times V \to [0, 1]$ and $F_Y : E \subseteq V \times V \to [0, 1]$ are defined by

$$T_Y(xy) \leq \min\{T_X(x), T_X(y)\}, I_Y(xy) \geq \max\{I_X(x), I_X(y)\} \text{ and } F_Y(xy) \geq \max\{F_X(x), F_X(y)\}.$$

There is no restriction on the sum of $T_Y(xy)$, $I_Y(xy)$ and $F_Y(xy)$, therefore $0 \leq T_Y(xy) + I_Y(xy) + F_Y(xy) \leq 3$ for all $xy \in E$. Here X is the single valued neutrosophic vertex set of $\mathcal{G}$ and Y is the single valued neutrosophic edge set of $\mathcal{G}$.

Definition 2.7. [7] The degree of a vertex $x \in V$ in a SVNG $\mathcal{G}$ is defined as $d_{\mathcal{G}}(x) = \langle d_T(x), d_I(x), d_F(x)\rangle$, where $d_T(x) = \sum_{x,y \neq x \in V} T_Y(xy)$, $d_I(x) = \sum_{x,y \neq x \in V} I_Y(xy)$ and $d_F(x) = \sum_{x,y \neq x \in V} F_Y(xy)$.

3. Operations on SVNGs and their degree

In this section, the basic operations on graphs such as direct product, Cartesian product, semi-strong product, strong product, lexicographic product, union, ring sum and join are defined under single valued neutrosophic environment and their properties are investigated.

Definition 3.1. The direct product $\mathcal{G}_1 \times \mathcal{G}_2 = (X_1 \times X_2, Y_1 \times Y_2)$ of two SVNGs $\mathcal{G}_1 = (X_1, Y_1)$ and $\mathcal{G}_2 = (X_2, Y_2)$ of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, is defined as follows:

$$\text{(i)} \begin{cases} (T_{X_1} \times T_{X_2})(x_1, x_2) = \min\{T_{X_1}(x_1), T_{X_2}(x_2)\} \\ (I_{X_1} \times I_{X_2})(x_1, x_2) = \max\{I_{X_1}(x_1), I_{X_2}(x_2)\} \\ (F_{X_1} \times F_{X_2})(x_1, x_2) = \max\{F_{X_1}(x_1), F_{X_2}(x_2)\} \end{cases}$$
for all $(x_1, x_2) \in V_1 \times V_2$,

$$\text{(ii)} \begin{cases} (T_{Y_1} \times T_{Y_2})((x_1, x_2)(y_1, y_2)) \\ \quad = \min\{T_{Y_1}(x_1y_1), T_{Y_2}(x_2y_2)\} \\ (I_{Y_1} \times I_{Y_2})((x_1, x_2)(y_1, y_2)) \\ \quad = \max\{I_{Y_1}(x_1y_1), I_{Y_2}(x_2y_2)\} \\ (F_{Y_1} \times F_{Y_2})((x_1, x_2)(y_1, y_2)) \\ \quad = \max\{F_{Y_1}(x_1y_1), F_{Y_2}(x_2y_2)\} \end{cases}$$
for all $x_1y_1 \in E_1$, for all $x_2y_2 \in E_2$.

Proposition 3.2. *If* $\mathcal{G}_1 = (X_1, Y_1)$ *and* $\mathcal{G}_2 = (X_2, Y_2)$ *are the SVNGs, then* $\mathcal{G}_1 \times \mathcal{G}_2$ *is the SVNG.*

Proof. Consider $x_1y_1 \in E_1, x_2y_2 \in E_2$. Then

$$\begin{aligned}
&(T_{Y_1} \times T_{Y_2})((x_1, x_2)(y_1, y_2))\\
&= \min\{T_{Y_1}(x_1y_1), T_{Y_2}(x_2y_2)\}\\
&\leq \min\{\min\{T_{X_1}(x_1), T_{X_1}(y_1)\},\\
&\quad \min\{T_{X_2}(x_2), T_{X_2}(y_2)\}\}\\
&= \min\{\min\{T_{X_1}(x_1), T_{X_2}(x_2)\},\\
&\quad \min\{T_{X_1}(y_1), T_{X_2}(y_2)\}\}\\
&= \min\{(T_{X_1} \times T_{X_2})(x_1, x_2),\\
&\quad (T_{X_1} \times T_{X_2})(y_1, y_2)\},\\
&(I_{Y_1} \times I_{Y_2})((x_1, x_2)(y_1, y_2))\\
&= \max\{I_{Y_1}(x_1y_1), I_{Y_2}(x_2y_2)\}\\
&\geq \max\{\max\{I_{X_1}(x_1), I_{X_1}(y_1)\},\\
&\quad \max\{I_{X_2}(x_2), I_{X_2}(y_2)\}\}\\
&= \max\{\max\{I_{X_1}(x_1), I_{X_2}(x_2)\},\\
&\quad \max\{I_{X_1}(y_1), I_{X_2}(y_2)\}\}\\
&= \max\{(I_{X_1} \times I_{X_2})(x_1, x_2),\\
&\quad (I_{X_1} \times I_{X_2})(y_1, y_2)\},\\
&(F_{Y_1} \times F_{Y_2})((x_1, x_2)(y_1, y_2))\\
&= \max\{F_{Y_1}(x_1y_1), F_{Y_2}(x_2y_2)\}\\
&\geq \max\{\max\{F_{X_1}(x_1), F_{X_1}(y_1)\},\\
&\quad \max\{F_{X_2}(x_2), F_{X_2}(y_2)\}\}\\
&= \max\{\max\{F_{X_1}(x_1), F_{X_2}(x_2)\},\\
&\quad \max\{F_{X_1}(y_1), F_{X_2}(y_2)\}\}\\
&= \max\{(F_{X_1} \times F_{X_2})(x_1, x_2),\\
&\quad (F_{X_1} \times F_{X_2})(y_1, y_2)\}.
\end{aligned}$$
□

Definition 3.3. Let $\mathcal{G}_1 = (X_1, Y_1)$ and $\mathcal{G}_2 = (X_2, Y_2)$ be two SVNGs. For any vertex $(x_1, x_2) \in V_1 \times V_2$,

$$\begin{aligned}
&(d_T)_{\mathcal{G}_1\times\mathcal{G}_2}(x_1, x_2)\\
&= \sum_{(x_1,x_2)(y_1,y_2)\in E_1\times E_2} (T_{Y_1} \times T_{Y_2})((x_1, x_2)(y_1, y_2))\\
&= \sum_{x_1y_1\in E_1, x_2y_2\in E_2} \min\{T_{Y_1}(x_1y_1), T_{Y_2}(x_2y_2)\},\\
&(d_I)_{\mathcal{G}_1\times\mathcal{G}_2}(x_1, x_2)\\
&= \sum_{(x_1,x_2)(y_1,y_2)\in E_1\times E_2} (I_{Y_1} \times I_{Y_2})((x_1, x_2)(y_1, y_2))\\
&= \sum_{x_1y_1\in E_1, x_2y_2\in E_2} \max\{I_{Y_1}(x_1y_1), I_{Y_2}(x_2y_2)\},\\
&(d_F)_{\mathcal{G}_1\times\mathcal{G}_2}(x_1, x_2)\\
&= \sum_{(x_1,x_2)(y_1,y_2)\in E_1\times E_2} (F_{Y_1} \times F_{Y_2})((x_1, x_2)(y_1, y_2))\\
&= \sum_{x_1y_1\in E_1, x_2y_2\in E_2} \max\{F_{Y_1}(x_1y_1), F_{Y_2}(x_2y_2)\}.
\end{aligned}$$

Theorem 3.4. *Let* $\mathcal{G}_1 = (X_1, Y_1)$ *and* $\mathcal{G}_2 = (X_2, Y_2)$ *be two SVNGs. If* $T_{Y_2} \geq T_{Y_1}, I_{Y_2} \leq I_{Y_1}, F_{Y_2} \leq F_{Y_1}$*, then* $d_{\mathcal{G}_1\times\mathcal{G}_2}(x_1, x_2) = d_{\mathcal{G}_1}(x_1)$ *and if* $T_{Y_1} \geq T_{Y_2}, I_{Y_1} \leq I_{Y_2}, F_{Y_1} \leq F_{Y_2}$*, then* $d_{\mathcal{G}_1\times\mathcal{G}_2}(x_1, x_2) = d_{\mathcal{G}_2}(x_2)$ *for all* $(x_1, x_2) \in V_1 \times V_2$.

Proof. By definition of vertex degree of $\mathcal{G}_1 \times \mathcal{G}_2$, we must have

$$\begin{aligned}
&(d_T)_{\mathcal{G}_1\times\mathcal{G}_2}(x_1, x_2)\\
&= \sum_{x_1y_1\in E_1, x_2y_2\in E_2} \min\{T_{Y_1}(x_1y_1), T_{Y_2}(x_2y_2)\}\\
&= \sum_{x_1y_1\in E_1} T_{Y_1}(x_1y_1) \quad (\text{since } T_{Y_2} \geq T_{Y_1})\\
&= (d_T)_{\mathcal{G}_1}(x_1),\\
&(d_I)_{\mathcal{G}_1\times\mathcal{G}_2}(x_1, x_2)\\
&= \sum_{x_1y_1\in E_1, x_2y_2\in E_2} \max\{I_{Y_1}(x_1y_1), I_{Y_2}(x_2y_2)\}\\
&= \sum_{x_1y_1\in E_1} I_{Y_1}(x_1y_1) \quad (\text{since } I_{Y_2} \leq I_{Y_1})\\
&= (d_I)_{\mathcal{G}_1}(x_1),\\
&(d_F)_{\mathcal{G}_1\times\mathcal{G}_2}(x_1, x_2)\\
&= \sum_{x_1y_1\in E_1, x_2y_2\in E_2} \max\{F_{Y_1}(x_1y_1), F_{Y_2}(x_2y_2)\}\\
&= \sum_{x_1y_1\in E_1} F_{Y_1}(x_1y_1) \quad (\text{since } F_{Y_2} \leq F_{Y_1})\\
&= (d_F)_{\mathcal{G}_1}(x_1).
\end{aligned}$$

Hence $d_{\mathcal{G}_1\times\mathcal{G}_2}(x_1, x_2) = d_{\mathcal{G}_1}(x_1)$. Similarly, it is easy to show that, if $T_{Y_1} \geq T_{Y_2}, I_{Y_1} \leq I_{Y_2}, F_{Y_1} \leq F_{Y_2}$, then $d_{\mathcal{G}_1\times\mathcal{G}_2}(x_1, x_2) = d_{\mathcal{G}_2}(x_2)$. □

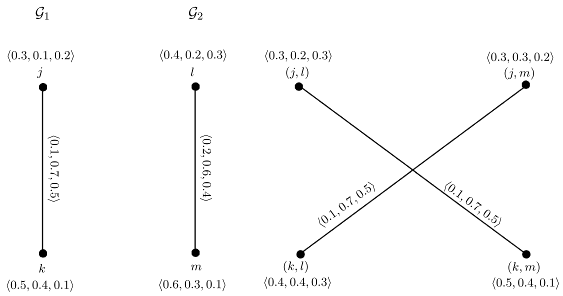

Fig. 1. Direct product of two SVNGs.

Example 3.5. Consider two SVNGs $\mathcal{G}_1 = (X_1, Y_1)$ and $\mathcal{G}_2 = (X_2, Y_2)$, where $X_1 = \left\langle \left(\frac{j}{0.3}, \frac{k}{0.5}\right), \left(\frac{j}{0.1}, \frac{k}{0.4}\right), \left(\frac{j}{0.2}, \frac{k}{0.1}\right) \right\rangle$, $Y_1 = \left\langle \frac{jk}{0.1}, \frac{jk}{0.7}, \frac{jk}{0.5} \right\rangle$, $X_2 = \left\langle \left(\frac{l}{0.4}, \frac{m}{0.6}\right), \left(\frac{l}{0.2}, \frac{m}{0.3}\right), \left(\frac{l}{0.3}, \frac{m}{0.1}\right) \right\rangle$ and $Y_2 = \left\langle \frac{lm}{0.2}, \frac{lm}{0.6}, \frac{lm}{0.4} \right\rangle$. SVNGs and their direct product are shown in Fig. 1.

Since $T_{Y_2} \geq T_{Y_1}, I_{Y_2} \leq I_{Y_1}, F_{Y_2} \leq F_{Y_1}$, so, by Theorem 3.4, we have $(d_T)_{\mathcal{G}_1 \times \mathcal{G}_2}(j, l) = (d_T)_{\mathcal{G}_1}(j) = 0.1$, $(d_I)_{\mathcal{G}_1 \times \mathcal{G}_2}(j, l) = (d_I)_{\mathcal{G}_1}(j) = 0.7$ and $(d_F)_{\mathcal{G}_1 \times \mathcal{G}_2}(j, l) = (d_F)_{\mathcal{G}_1}(j) = 0.5$. Therefore, $d_{\mathcal{G}_1 \times \mathcal{G}_2}(j, l) = \langle 0.1, 0.7, 0.5 \rangle$. Similarly, it is easy to find the degree of all vertices in $\mathcal{G}_1 \times \mathcal{G}_2$.

Definition 3.6. The Cartesian product $\mathcal{G}_1 \,\Box\, \mathcal{G}_2 = (X_1 \,\Box\, X_2, Y_1 \,\Box\, Y_2)$ of two SVNGs $\mathcal{G}_1 = (X_1, Y_1)$ and $\mathcal{G}_2 = (X_2, Y_2)$ of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, is defined as follows:

$$\text{(i)} \begin{cases} (T_{X_1} \,\Box\, T_{X_2})(x_1, x_2) = \min\{T_{X_1}(x_1), T_{X_2}(x_2)\} \\ (I_{X_1} \,\Box\, I_{X_2})(x_1, x_2) = \max\{I_{X_1}(x_1), I_{X_2}(x_2)\} \\ (F_{X_1} \,\Box\, F_{X_2})(x_1, x_2) = \max\{F_{X_1}(x_1), F_{X_2}(x_2)\} \end{cases}$$
for all $(x_1, x_2) \in V_1 \times V_2$,

$$\text{(ii)} \begin{cases} (T_{Y_1} \,\Box\, T_{Y_2})((x, x_2)(x, y_2)) = \min\{T_{X_1}(x), T_{Y_2}(x_2y_2)\} \\ (I_{Y_1} \,\Box\, I_{Y_2})((x, x_2)(x, y_2)) = \max\{I_{X_1}(x), I_{Y_2}(x_2y_2)\} \\ (F_{Y_1} \,\Box\, F_{Y_2})((x, x_2)(x, y_2)) = \max\{F_{X_1}(x), F_{Y_2}(x_2y_2)\} \end{cases}$$
for all $x \in V_1$, for all $x_2y_2 \in E_2$,

$$\text{(iii)} \begin{cases} (T_{Y_1} \,\Box\, T_{Y_2})((x_1, z)(y_1, z)) = \min\{T_{Y_1}(x_1y_1), T_{X_2}(z)\} \\ (I_{Y_1} \,\Box\, I_{Y_2})((x_1, z)(y_1, z)) = \max\{I_{Y_1}(x_1y_1), I_{X_2}(z)\} \\ (F_{Y_1} \,\Box\, F_{Y_2})((x_1, z)(y_1, z)) = \max\{F_{Y_1}(x_1y_1), F_{X_2}(z)\} \end{cases}$$
for all $z \in V_2$, for all $x_1y_1 \in E_1$.

Proposition 3.7. *If $\mathcal{G}_1 = (X_1, Y_1)$ and $\mathcal{G}_2 = (X_2, Y_2)$ are the SVNGs, then $\mathcal{G}_1 \,\Box\, \mathcal{G}_2$ is the SVNG.*

Proof. Consider $x \in V_1, x_2y_2 \in E_2$. Then

$$\begin{aligned} &(T_{Y_1} \,\Box\, T_{Y_2})((x, x_2)(x, y_2)) \\ &\quad = \min\{T_{X_1}(x), T_{Y_2}(x_2y_2)\} \\ &\quad \leq \min\{T_{X_1}(x), \min\{T_{X_2}(x_2), T_{X_2}(y_2)\}\} \\ &\quad = \min\{\min\{T_{X_1}(x), T_{X_2}(x_2)\}, \\ &\qquad \min\{T_{X_1}(x), T_{X_2}(y_2)\}\} \\ &\quad = \min\{(T_{X_1} \,\Box\, T_{X_2})(x, x_2), (T_{X_1} \,\Box\, T_{X_2})(x, y_2)\}, \end{aligned}$$

$$\begin{aligned} &(I_{Y_1} \,\Box\, I_{Y_2})((x, x_2)(x, y_2)) \\ &\quad = \max\{I_{X_1}(x), I_{Y_2}(x_2y_2)\} \\ &\quad \geq \max\{I_{X_1}(x), \max\{I_{X_2}(x_2), I_{X_2}(y_2)\}\} \\ &\quad = \max\{\max\{I_{X_1}(x), I_{X_2}(x_2)\}, \\ &\qquad \max\{I_{X_1}(x), I_{X_2}(y_2)\}\} \\ &\quad = \max\{(I_{X_1} \,\Box\, I_{X_2})(x, x_2), (I_{X_1} \,\Box\, I_{X_2})(x, y_2)\}, \end{aligned}$$

$$\begin{aligned} &(F_{Y_1} \,\Box\, F_{Y_2})((x, x_2)(x, y_2)) \\ &\quad = \max\{F_{X_1}(x), F_{Y_2}(x_2y_2)\} \\ &\quad \geq \max\{F_{X_1}(x), \max\{F_{X_2}(x_2), F_{X_2}(y_2)\}\} \\ &\quad = \max\{\max\{F_{X_1}(x), F_{X_2}(x_2)\}, \\ &\qquad \max\{F_{X_1}(x), F_{X_2}(y_2)\}\} \\ &\quad = \max\{(F_{X_1} \,\Box\, F_{X_2})(x, x_2), (F_{X_1} \,\Box\, F_{X_2})(x, y_2)\}. \end{aligned}$$

Similarly, for $z \in V_2, x_1y_1 \in E_1$, we have

$$\begin{aligned} &(T_{Y_1} \,\Box\, T_{Y_2})((x_1, z)(y_1, z)) \\ &\quad = \min\{(T_{X_1} \,\Box\, T_{X_2})(x_1, z), (T_{X_1} \,\Box\, T_{X_2})(y_1, z)\}, \\ &(I_{Y_1} \,\Box\, I_{Y_2})((x_1, z)(y_1, z)) \\ &\quad = \max\{(I_{X_1} \,\Box\, I_{X_2})(x_1, z), (I_{X_1} \,\Box\, I_{X_2})(y_1, z)\}, \\ &(F_{Y_1} \,\Box\, F_{Y_2})((x_1, z)(y_1, z)) \\ &\quad = \max\{(F_{X_1} \,\Box\, F_{X_2})(x_1, z), (F_{X_1} \,\Box\, F_{X_2})(y_1, z)\}. \end{aligned}$$

□

Definition 3.8. Let $\mathcal{G}_1 = (X_1, Y_1)$ and $\mathcal{G}_2 = (X_2, Y_2)$ be two SVNGs. For any vertex $(x_1, x_2) \in V_1 \times V_2$,

$$\begin{aligned} &(d_T)_{\mathcal{G}_1 \,\Box\, \mathcal{G}_2}(x_1, x_2) \\ &= \sum_{(x_1,x_2)(y_1,y_2) \in E_1 \,\Box\, E_2} (T_{Y_1} \,\Box\, T_{Y_2})((x_1, x_2)(y_1, y_2)) \\ &= \sum_{x_1 = y_1, x_2y_2 \in E_2} \min\{T_{X_1}(x_1), T_{Y_2}(x_2y_2)\} \\ &\quad + \sum_{x_2 = y_2, x_1y_1 \in E_1} \min\{T_{X_2}(x_2), T_{Y_1}(x_1y_1)\}, \end{aligned}$$

$$\begin{aligned} &(d_I)_{\mathcal{G}_1 \,\Box\, \mathcal{G}_2}(x_1, x_2) \\ &= \sum_{(x_1,x_2)(y_1,y_2) \in E_1 \,\Box\, E_2} (I_{Y_1} \,\Box\, I_{Y_2})((x_1, x_2)(y_1, y_2)) \\ &= \sum_{x_1 = y_1, x_2y_2 \in E_2} \max\{I_{X_1}(x_1), I_{Y_2}(x_2y_2)\} \end{aligned}$$

$$+ \sum_{x_2=y_2, x_1y_1 \in E_1} \max\{I_{X_2}(x_2), I_{Y_1}(x_1y_1)\},$$

$$(d_F)_{\mathcal{G}_1 \,\square\, \mathcal{G}_2}(x_1, x_2)$$
$$= \sum_{(x_1,x_2)(y_1,y_2)\in E_1 \,\square\, E_2} (F_{Y_1} \,\square\, F_{Y_2})((x_1, x_2)(y_1, y_2))$$
$$= \sum_{x_1=y_1, x_2y_2 \in E_2} \max\{F_{X_1}(x_1), F_{Y_2}(x_2y_2)\}$$
$$+ \sum_{x_2=y_2, x_1y_1 \in E_1} \max\{F_{X_2}(x_2), F_{Y_1}(x_1y_1)\}.$$

Theorem 3.9. *Let $\mathcal{G}_1 = (X_1, Y_1)$ and $\mathcal{G}_2 = (X_2, Y_2)$ be two SVNGs. If $T_{X_1} \geq T_{Y_2}, I_{X_1} \leq I_{Y_2}, F_{X_1} \leq F_{Y_2}$ and $T_{X_2} \geq T_{Y_1}, I_{X_2} \leq I_{Y_1}, F_{X_2} \leq F_{Y_1}$. Then $d_{\mathcal{G}_1 \,\square\, \mathcal{G}_2}(x_1, x_2) = d_{\mathcal{G}_1}(x_1) + d_{\mathcal{G}_2}(x_2)$ for all $(x_1, x_2) \in V_1 \times V_2$.*

Proof. By definition of vertex degree of $\mathcal{G}_1 \,\square\, \mathcal{G}_2$, we must have

$$(d_T)_{\mathcal{G}_1 \,\square\, \mathcal{G}_2}(x_1, x_2)$$
$$= \sum_{x_1=y_1, x_2y_2 \in E_2} \min\{T_{X_1}(x_1), T_{Y_2}(x_2y_2)\}$$
$$+ \sum_{x_2=y_2, x_1y_1 \in E_1} \min\{T_{X_2}(x_2), T_{Y_1}(x_1y_1)\}$$
$$= \sum_{x_2y_2 \in E_2} T_{Y_2}(x_2y_2) + \sum_{x_1y_1 \in E_1} T_{Y_1}(x_1y_1)$$
$$\text{(by using } T_{X_1} \geq T_{Y_2} \text{ and } T_{X_2} \geq T_{Y_1})$$
$$= (d_T)_{\mathcal{G}_1}(x_1) + (d_T)_{\mathcal{G}_2}(x_2),$$

Similarly, it is easy to show that $(d_I)_{\mathcal{G}_1 \,\square\, \mathcal{G}_2}(x_1, x_2) = (d_I)_{\mathcal{G}_1}(x_1) + (d_I)_{\mathcal{G}_2}(x_2)$ and $(d_F)_{\mathcal{G}_1 \,\square\, \mathcal{G}_2}(x_1, x_2) = (d_F)_{\mathcal{G}_1}(x_1) + (d_F)_{\mathcal{G}_2}(x_2)$. Hence $d_{\mathcal{G}_1 \,\square\, \mathcal{G}_2}(x_1, x_2) = d_{\mathcal{G}_1}(x_1) + d_{\mathcal{G}_2}(x_2)$. □

Example 3.10. Consider two SVNGs $\mathcal{G}_1$ and $\mathcal{G}_2$ as in Example 3.5, where $T_{X_1} \geq T_{Y_2}, I_{X_1} \leq I_{Y_2}, F_{X_1} \leq F_{Y_2}$ and $T_{X_2} \geq T_{Y_1}, I_{X_2} \leq I_{Y_1}, F_{X_2} \leq F_{Y_1}$. Their Cartesian product $\mathcal{G}_1 \,\square\, \mathcal{G}_2$ is shown in Fig. 2.

Then by Theorem 3.9, we have $(d_T)_{\mathcal{G}_1 \,\square\, \mathcal{G}_2}(j, l) = 0.3 = 0.1 + 0.2 = (d_T)_{\mathcal{G}_1}(j) + (d_T)_{\mathcal{G}_2}(l)$, $(d_I)_{\mathcal{G}_1 \,\square\, \mathcal{G}_2}(j, l) = 1.3 = 0.7 + 0.6 = (d_I)_{\mathcal{G}_1}(j) + (d_I)_{\mathcal{G}_2}(l)$ and $(d_F)_{\mathcal{G}_1 \,\square\, \mathcal{G}_2}(j, l) = 0.9 = 0.5 + 0.4 = (d_F)_{\mathcal{G}_1}(j) + (d_F)_{\mathcal{G}_2}(l)$. Therefore, $d_{\mathcal{G}_1 \,\square\, \mathcal{G}_2}(j, l) = \langle 0.3, 1.3, 0.9\rangle$. Similarly, we can find the degree of all the vertices in $\mathcal{G}_1 \,\square\, \mathcal{G}_2$.

Definition 3.11. The semi-strong product $\mathcal{G}_1 \bullet \mathcal{G}_2 = (X_1 \bullet X_2, Y_1 \bullet Y_2)$ of two SVNGs $\mathcal{G}_1 = (X_1, Y_1)$ and $\mathcal{G}_2 = (X_2, Y_2)$ of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, is defined as follows:

(i) $\begin{cases} (T_{X_1} \bullet T_{X_2})(x_1, x_2) = \min\{T_{X_1}(x_1), T_{X_2}(x_2)\} \\ (I_{X_1} \bullet I_{X_2})(x_1, x_2) = \max\{I_{X_1}(x_1), I_{X_2}(x_2)\} \\ (F_{X_1} \bullet F_{X_2})(x_1, x_2) = \max\{F_{X_1}(x_1), F_{X_2}(x_2)\} \end{cases}$
for all $(x_1, x_2) \in V_1 \times V_2$,

(ii) $\begin{cases} (T_{Y_1} \bullet T_{Y_2})((x, x_2)(x, y_2)) = \min\{T_{X_1}(x), T_{Y_2}(x_2y_2)\} \\ (I_{Y_1} \bullet I_{Y_2})((x, x_2)(x, y_2)) = \max\{I_{X_1}(x), I_{Y_2}(x_2y_2)\} \\ (F_{Y_1} \bullet F_{Y_2})((x, x_2)(x, y_2)) = \max\{F_{X_1}(x), F_{Y_2}(x_2y_2)\} \end{cases}$
for all $x \in V_1$, for all $x_2y_2 \in E_2$,

(iii) $\begin{cases} (T_{Y_1} \bullet T_{Y_2})((x_1, x_2)(y_1, y_2)) = \min\{T_{Y_1}(x_1y_1), T_{Y_2}(x_2y_2)\} \\ (I_{Y_1} \bullet I_{Y_2})((x_1, x_2)(y_1, y_2)) = \max\{I_{Y_1}(x_1y_1), I_{Y_2}(x_2y_2)\} \\ (F_{Y_1} \bullet F_{Y_2})((x_1, x_2)(y_1, y_2)) = \max\{F_{Y_1}(x_1y_1), F_{Y_2}(x_2y_2)\} \end{cases}$
for all $x_1y_1 \in E_1$, for all $x_2y_2 \in E_2$.

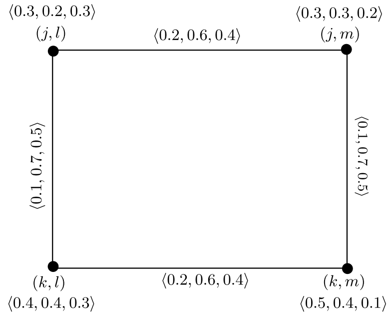

Fig. 2. Cartesian product of two SVNGs.

Proposition 3.12. *If $\mathcal{G}_1 = (X_1, Y_1)$ and $\mathcal{G}_2 = (X_2, Y_2)$ are the SVNGs, then $\mathcal{G}_1 \bullet \mathcal{G}_2$ is the SVNG.*

Proof. The proof follows from proof of Propositions 3.2 and 3.7. □

Definition 3.13. Let $\mathcal{G}_1 = (X_1, Y_1)$ and $\mathcal{G}_2 = (X_2, Y_2)$ be two SVNGs. For any vertex $(x_1, x_2) \in V_1 \times V_2$,

$$(d_T)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(x_1, x_2)$$
$$= \sum_{(x_1,x_2)(y_1,y_2)\in E_1 \bullet E_2} (T_{Y_1} \bullet T_{Y_2})((x_1, x_2)(y_1, y_2))$$
$$= \sum_{x_1=y_1, x_2y_2 \in E_2} \min\{T_{X_1}(x_1), T_{Y_2}(x_2y_2)\}$$
$$+ \sum_{x_1y_1 \in E_1, x_2y_2 \in E_2} \min\{T_{Y_1}(x_1y_1), T_{Y_2}(x_2y_2)\},$$

$$(d_I)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(x_1, x_2)$$
$$= \sum_{(x_1,x_2)(y_1,y_2)\in E_1 \bullet E_2} (I_{Y_1} \bullet I_{Y_2})((x_1, x_2)(y_1, y_2))$$
$$= \sum_{x_1=y_1, x_2y_2 \in E_2} \max\{I_{X_1}(x_1), I_{Y_2}(x_2y_2)\}$$

$$+ \sum_{x_1y_1 \in E_1, x_2y_2 \in E_2} \max\{I_{Y_1}(x_1y_1), I_{Y_2}(x_2y_2)\},$$

$$(d_F)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(x_1, x_2) = \sum_{(x_1,x_2)(y_1,y_2) \in E_1 \bullet E_2} (F_{Y_1} \bullet F_{Y_2})((x_1, x_2)(y_1, y_2))$$

$$= \sum_{x_1=y_1, x_2y_2 \in E_2} \max\{F_{X_1}(x_1), F_{Y_2}(x_2y_2)\} + \sum_{x_1y_1 \in E_1, x_2y_2 \in E_2} \max\{F_{Y_1}(x_1y_1), F_{Y_2}(x_2y_2)\}.$$

Theorem 3.14. *Let $\mathcal{G}_1 = (X_1, Y_1)$ and $\mathcal{G}_2 = (X_2, Y_2)$ be two SVNGs. If $T_{X_1} \geq T_{Y_2}, I_{X_1} \leq I_{Y_2}, F_{X_1} \leq F_{Y_2}$, $T_{Y_1} \leq T_{Y_2}, I_{Y_1} \geq I_{Y_2}, F_{Y_1} \geq F_{Y_2}$. Then $d_{\mathcal{G}_1 \bullet \mathcal{G}_2}(x_1, x_2) = d_{\mathcal{G}_1}(x_1) + d_{\mathcal{G}_2}(x_2)$ for all $(x_1, x_2) \in V_1 \times V_2$.*

Proof. By definition of vertex degree of $\mathcal{G}_1 \times \mathcal{G}_2$, we must have

$$(d_T)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(x_1, x_2) = \sum_{x_1=y_1, x_2y_2 \in E_2} \min\{T_{X_1}(x_1), T_{Y_2}(x_2y_2)\} + \sum_{x_1y_1 \in E_1, x_2y_2 \in E_2} \min\{T_{Y_1}(x_1y_1), T_{Y_2}(x_2y_2)\}$$

$$= \sum_{x_2y_2 \in E_2} T_{Y_2}(x_2y_2) + \sum_{x_1y_1 \in E_1} T_{Y_1}(x_1y_1)$$

(Since $T_{X_1} \geq T_{Y_2}$ and $T_{Y_1} \leq T_{Y_2}$)

$$= (d_T)_{\mathcal{G}_2}(x_2) + (d_T)_{\mathcal{G}_1}(x_1).$$

Analogously, it is easy to show that $(d_I)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(x_1, x_2) = (d_I)_{\mathcal{G}_1}(x_1) + (d_I)_{\mathcal{G}_2}(x_2)$ and $(d_F)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(x_1, x_2) = (d_F)_{\mathcal{G}_1}(x_1) + (d_F)_{\mathcal{G}_2}(x_2)$. Hence $d_{\mathcal{G}_1 \bullet \mathcal{G}_2}(x_1, x_2) = d_{\mathcal{G}_1}(x_1) + d_{\mathcal{G}_2}(x_2)$. □

Example 3.15. Consider two SVNGs $\mathcal{G}_1$ and $\mathcal{G}_2$ as given in Example 3.5, where $T_{X_1} \geq T_{Y_2}, I_{X_1} \leq I_{Y_2}, F_{X_1} \leq F_{Y_2}$, $T_{Y_1} \leq T_{Y_2}, I_{Y_1} \geq I_{Y_2}, F_{Y_1} \geq F_{Y_2}$, and their semi-strong product $\mathcal{G}_1 \bullet \mathcal{G}_2$ is shown in Fig. 3.

So, by Theorem 3.14, we have $(d_T)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(j, m) = 0.3 = 0.1 + 0.2 = (d_T)_{\mathcal{G}_1}(j) + (d_T)_{\mathcal{G}_2}(m)$, $(d_I)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(j, m) = 1.3 = 0.7 + 0.6 = (d_I)_{\mathcal{G}_1}(j) + (d_I)_{\mathcal{G}_2}(m)$ and $(d_F)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(j, m) = 0.9 = 0.5 + 0.4 = (d_F)_{\mathcal{G}_1}(j) + (d_F)_{\mathcal{G}_2}(m)$. Therefore, $d_{\mathcal{G}_1 \bullet \mathcal{G}_2}(j, m) = \langle 0.3, 1.3, 0.9\rangle$. Similarly, we can find the degree of all the vertices in $\mathcal{G}_1 \bullet \mathcal{G}_2$.

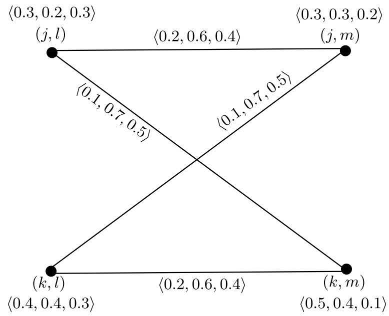

Fig. 3. Semi-strong product of two SVNGs.

Definition 3.16. The strong product $\mathcal{G}_1 \boxtimes \mathcal{G}_2 = (X_1 \boxtimes X_2, Y_1 \boxtimes Y_2)$ of two SVNGs $\mathcal{G}_1 = (X_1, Y_1)$ and $\mathcal{G}_2 = (X_2, Y_2)$ of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, is defined as follows:

(i) $\begin{cases} (T_{X_1} \boxtimes T_{X_2})(x_1, x_2) = \min\{T_{X_1}(x_1), T_{X_2}(x_2)\} \\ (I_{X_1} \boxtimes I_{X_2})(x_1, x_2) = \max\{I_{X_1}(x_1), I_{X_2}(x_2)\} \\ (F_{X_1} \boxtimes F_{X_2})(x_1, x_2) = \max\{F_{X_1}(x_1), F_{X_2}(x_2)\} \end{cases}$
for all $(x_1, x_2) \in V_1 \times V_2$,

(ii) $\begin{cases} (T_{Y_1} \boxtimes T_{Y_2})((x, x_2)(x, y_2)) = \min\{T_{X_1}(x), T_{Y_2}(x_2y_2)\} \\ (I_{Y_1} \boxtimes I_{Y_2})((x, x_2)(x, y_2)) = \max\{I_{X_1}(x), I_{Y_2}(x_2y_2)\} \\ (F_{Y_1} \boxtimes F_{Y_2})((x, x_2)(x, y_2)) = \max\{F_{X_1}(x), F_{Y_2}(x_2y_2)\} \end{cases}$
for all $x \in V_1$, for all $x_2y_2 \in E_2$,

(iii) $\begin{cases} (T_{Y_1} \boxtimes T_{Y_2})((x_1, z)(y_1, z)) = \min\{T_{Y_1}(x_1y_1), T_{X_2}(z)\} \\ (I_{Y_1} \boxtimes I_{Y_2})((x_1, z)(y_1, z)) = \max\{I_{Y_1}(x_1y_1), I_{X_2}(z)\} \\ (F_{Y_1} \boxtimes F_{Y_2})((x_1, z)(y_1, z)) = \max\{F_{Y_1}(x_1y_1), F_{X_2}(z)\} \end{cases}$
for all $z \in V_2$, for all $x_1y_1 \in E_1$.

(iv) $\begin{cases} (T_{Y_1} \boxtimes T_{Y_2})((x_1, x_2)(y_1, y_2)) = \min\{T_{Y_1}(x_1y_1), T_{Y_2}(x_2y_2)\} \\ (I_{Y_1} \boxtimes I_{Y_2})((x_1, x_2)(y_1, y_2)) = \max\{I_{Y_1}(x_1y_1), I_{Y_2}(x_2y_2)\} \\ (F_{Y_1} \boxtimes F_{Y_2})((x_1, x_2)(y_1, y_2)) = \max\{F_{Y_1}(x_1y_1), F_{Y_2}(x_2y_2)\} \end{cases}$
for all $x_1y_1 \in E_1$, for all $x_2y_2 \in E_2$.

Proposition 3.17. *If $\mathcal{G}_1 = (X_1, Y_1)$ and $\mathcal{G}_2 = (X_2, Y_2)$ are the SVNGs, then $\mathcal{G}_1 \boxtimes \mathcal{G}_2$ is the SVNG.*

Proof. The proof follows from proof of Propositions 3.2 and 3.7. □

Definition 3.18. Let $\mathcal{G}_1 = (X_1, Y_1)$ and $\mathcal{G}_2 = (X_2, Y_2)$ be two SVNGs. For any vertex $(x_1, x_2) \in V_1 \times V_2$,

$$(d_T)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(x_1, x_2) = \sum_{(x_1,x_2)(y_1,y_2) \in E_1 \boxtimes E_2} (T_{Y_1} \boxtimes T_{Y_2})((x_1, x_2)(y_1, y_2))$$

$$= \sum_{x_1=y_1, x_2y_2 \in E_2} \min\{T_{X_1}(x_1), T_{Y_2}(x_2y_2)\} + \sum_{x_2=y_2, x_1y_1 \in E_1} \min\{T_{X_2}(x_2), T_{Y_1}(x_1y_1)\}$$

$$+ \sum_{x_1y_1 \in E_1, x_2y_2 \in E_2} \min\{T_{Y_1}(x_1y_1), T_{Y_2}(x_2y_2)\},$$

$$(d_I)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(x_1, x_2)$$
$$= \sum_{(x_1,x_2)(y_1,y_2) \in E_1 \boxtimes E_2} (I_{Y_1} \boxtimes I_{Y_2})((x_1, x_2)(y_1, y_2))$$
$$= \sum_{x_1=y_1, x_2y_2 \in E_2} \max\{I_{X_1}(x_1), I_{Y_2}(x_2y_2)\}$$
$$+ \sum_{x_2=y_2, x_1y_1 \in E_1} \max\{I_{X_2}(x_2), I_{Y_1}(x_1y_1)\}$$
$$+ \sum_{x_1y_1 \in E_1, x_2y_2 \in E_2} \max\{I_{Y_1}(x_1y_1), I_{Y_2}(x_2y_2)\},$$

$$(d_F)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(x_1, x_2)$$
$$= \sum_{(x_1,x_2)(y_1,y_2) \in E_1 \boxtimes E_2} (F_{Y_1} \boxtimes F_{Y_2})((x_1, x_2)(y_1, y_2))$$
$$= \sum_{x_1=y_1, x_2y_2 \in E_2} \max\{F_{X_1}(x_1), F_{Y_2}(x_2y_2)\}$$
$$+ \sum_{x_2=y_2, x_1y_1 \in E_1} \max\{F_{X_2}(x_2), F_{Y_1}(x_1y_1)\}$$
$$+ \sum_{x_1y_1 \in E_1, x_2y_2 \in E_2} \max\{F_{Y_1}(x_1y_1), F_{Y_2}(x_2y_2)\}.$$

Theorem 3.19. *Let $\mathcal{G}_1 = (X_1, Y_1)$ and $\mathcal{G}_2 = (X_2, Y_2)$ be two SVNGs. If $T_{X_1} \geq T_{Y_2}$, $I_{X_1} \leq I_{Y_2}$, $F_{X_1} \leq F_{Y_2}$, $T_{X_2} \geq T_{Y_1}$, $I_{X_2} \leq I_{Y_1}$, $F_{X_2} \leq F_{Y_1}$, $T_{Y_1} \leq T_{Y_2}$, $I_{Y_1} \geq I_{Y_2}$, $F_{Y_1} \geq F_{Y_2}$. Then $d_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(x_1, x_2) = |V_2| d_{\mathcal{G}_1}(x_1) + d_{\mathcal{G}_2}(x_2)$ for all $(x_1, x_2) \in V_1 \times V_2$.*

Proof. By definition of vertex degree of $\mathcal{G}_1 \times \mathcal{G}_2$, we must have

$$(d_T)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(x_1, x_2)$$
$$= \sum_{x_1=y_1, x_2y_2 \in E_2} \min\{T_{X_1}(x_1), T_{Y_2}(x_2y_2)\}$$
$$+ \sum_{x_2=y_2, x_1y_1 \in E_1} \min\{T_{X_2}(x_2), T_{Y_1}(x_1y_1)\}$$
$$+ \sum_{x_1y_1 \in E_1, x_2y_2 \in E_2} \min\{T_{Y_1}(x_1y_1), T_{Y_2}(x_2y_2)\}$$
$$= \sum_{x_2y_2 \in E_2} T_{Y_2}(x_2y_2) + \sum_{x_1y_1 \in E_1} T_{Y_1}(x_1y_1)$$
$$+ \sum_{x_1y_1 \in E_1} T_{Y_1}(x_1y_1)$$

(Since $T_{X_1} \geq T_{Y_2}$, $T_{X_2} \geq T_{Y_1}$ and $T_{Y_1} \leq T_{Y_2}$)

$$= |V_2|(d_T)_{\mathcal{G}_1}(x_1) + (d_T)_{\mathcal{G}_2}(x_2).$$

Analogously, it is easy to show that $(d_I)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(x_1, x_2) = |V_2|(d_I)_{\mathcal{G}_1}(x_1) + (d_I)_{\mathcal{G}_2}(x_2)$ and $(d_F)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(x_1, x_2) = |V_2|(d_F)_{\mathcal{G}_1}(x_1) + (d_F)_{\mathcal{G}_2}(x_2)$. Hence $d_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(x_1, x_2) = |V_2| d_{\mathcal{G}_1}(x_1) + d_{\mathcal{G}_2}(x_2)$. □

Example 3.20. Consider two SVNGs $\mathcal{G}_1$ and $\mathcal{G}_2$ as in Example 3.5, where $T_{X_1} \geq T_{Y_2}$, $I_{X_1} \leq I_{Y_2}$, $F_{X_1} \leq F_{Y_2}$, $T_{X_2} \geq T_{Y_1}$, $I_{X_2} \leq I_{Y_1}$, $F_{X_2} \leq F_{Y_1}$, $T_{Y_1} \leq T_{Y_2}$, $I_{Y_1} \geq I_{Y_2}$, $F_{Y_1} \geq F_{Y_2}$ and their strong product $\mathcal{G}_1 \boxtimes \mathcal{G}_2$ is shown in Fig. 4.

Then by Theorem 3.19, we have $(d_T)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(j, l) = 0.4 = 2(0.1) + 0.2 = |V_2|(d_T)_{\mathcal{G}_1}(j) + (d_T)_{\mathcal{G}_2}(l)$, $(d_I)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(j, l) = 2 = 2(0.7) + 0.6 = |V_2|(d_I)_{\mathcal{G}_1}(j) + (d_I)_{\mathcal{G}_2}(l)$, $(d_F)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(j, l) = 1.4 = 2(0.5) + 0.4 = |V_2|(d_F)_{\mathcal{G}_1}(j) + (d_F)_{\mathcal{G}_2}(l)$. Therefore, $d_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(j, l) = \langle 0.4, 2, 1.4 \rangle$. Similarly, we can find the degree of all the vertices in $\mathcal{G}_1 \boxtimes \mathcal{G}_2$.

Definition 3.21. The lexicographic product $\mathcal{G}_1 \circ \mathcal{G}_2 = (X_1 \circ X_2, Y_1 \circ Y_2)$ of SVNGs $\mathcal{G}_1 = (X_1, Y_1)$ of $G_1 = (V_1, E_1)$ and $\mathcal{G}_2 = (X_2, Y_2)$ of $G_2 = (V_2, E_2)$ is defined as follows:

(i) $\begin{cases} (T_{X_1} \circ T_{X_2})(x_1, x_2) = \min\{T_{X_1}(x_1), T_{X_2}(x_2)\} \\ (I_{X_1} \circ I_{X_2})(x_1, x_2) = \max\{I_{X_1}(x_1), I_{X_2}(x_2)\} \\ (F_{X_1} \circ F_{X_2})(x_1, x_2) = \max\{F_{X_1}(x_1), F_{X_2}(x_2)\} \end{cases}$
for all $(x_1, x_2) \in V_1 \times V_2$,

(ii) $\begin{cases} (T_{Y_1} \circ T_{Y_2})((x, x_2)(x, y_2)) = \min\{T_{X_1}(x), T_{Y_2}(x_2y_2)\} \\ (I_{Y_1} \circ I_{Y_2})((x, x_2)(x, y_2)) = \max\{I_{X_1}(x), I_{Y_2}(x_2y_2)\} \\ (F_{Y_1} \circ F_{Y_2})((x, x_2)(x, y_2)) = \max\{F_{X_1}(x), F_{Y_2}(x_2y_2)\} \end{cases}$
for all $x \in V_1$, for all $x_2y_2 \in E_2$,

(iii) $\begin{cases} (T_{Y_1} \circ T_{Y_2})((x_1, z)(y_1, z)) = \min\{T_{Y_1}(x_1y_1), T_{X_2}(z)\} \\ (I_{Y_1} \circ I_{Y_2})((x_1, z)(y_1, z)) = \max\{I_{Y_1}(x_1y_1), I_{X_2}(z)\} \\ (F_{Y_1} \circ F_{Y_2})((x_1, z)(y_1, z)) = \max\{F_{Y_1}(x_1y_1), F_{X_2}(z)\} \end{cases}$
for all $z \in V_2$, for all $x_1y_1 \in E_1$,

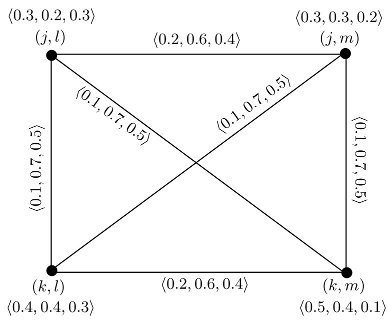

Fig. 4. Strong product of two SVNGs.

$$\text{(iv)}\quad\begin{cases}(T_{Y_1}\circ T_{Y_2})((x_1,x_2)(y_1,y_2))\\ \quad=\min\{T_{X_2}(x_2),T_{X_2}(y_2),T_{Y_1}(x_1y_1)\}\\ (I_{Y_1}\circ I_{Y_2})((x_1,x_2)(y_1,y_2))\\ \quad=\max\{I_{X_2}(x_2),I_{X_2}(y_2),I_{Y_1}(x_1y_1)\}\\ (F_{Y_1}\circ F_{Y_2})((x_1,x_2)(y_1,y_2))\\ \quad=\max\{F_{X_2}(x_2),F_{X_2}(y_2),F_{Y_1}(x_1y_1)\}\\ \quad\text{for all } x_1y_1\in E_1,\ x_2\neq y_2.\end{cases}$$

Proposition 3.22. *If $\mathcal{G}_1=(X_1,Y_1)$ and $\mathcal{G}_2=(X_2,Y_2)$ are the SVNGs, then $\mathcal{G}_1[\mathcal{G}_2]$ is the SVNG.*

Proof. From the proof of Proposition 3.7, it follows that

$$\begin{aligned}&(T_{Y_1}\circ T_{Y_2})((x,x_2)(x,y_2))\\&\quad=\min\{(T_{X_1}\circ T_{X_2})(x,x_2),(T_{X_1}\circ T_{X_2})(x,y_2)\},\\&(I_{Y_1}\circ I_{Y_2})((x,x_2)(x,y_2))\\&\quad=\max\{(I_{X_1}\circ I_{X_2})(x,x_2),(I_{X_1}\circ I_{X_2})(x,y_2)\},\\&(F_{Y_1}\circ F_{Y_2})((x,x_2)(x,y_2))\\&\quad=\max\{(F_{X_1}\circ F_{X_2})(x,x_2),(F_{X_1}\circ F_{X_2})(x,y_2)\}\\&\qquad\text{for all } x\in V_1,\ x_2y_2\in E_2,\\&(T_{Y_1}\circ T_{Y_2})((x_1,z)(y_1,z))\\&\quad=\min\{(T_{X_1}\circ T_{X_2})(x_1,z),(T_{X_1}\circ T_{X_2})(y_1,z)\},\\&(I_{Y_1}\circ I_{Y_2})((x_1,z)(y_1,z))\\&\quad=\max\{(I_{X_1}\circ I_{X_2})(x_1,z),(I_{X_1}\circ I_{X_2})(y_1,z)\},\\&(F_{Y_1}\circ F_{Y_2})((x_1,z)(y_1,z))\\&\quad=\max\{(F_{X_1}\circ F_{X_2})(x_1,z),(F_{X_1}\circ F_{X_2})(y_1,z)\}\\&\qquad\text{for all } z\in V_2,\ x_1y_1\in E_1.\end{aligned}$$

Now consider $x_1y_1\in E_1$, $x_2\neq y_2$. Then

$$\begin{aligned}&(T_{Y_1}\circ T_{Y_2})((x_1,x_2)(y_1,y_2))\\&\quad=\min\{T_{X_2}(x_2),T_{X_2}(y_2),T_{Y_1}(x_1y_1)\}\\&\quad\leq\min\{T_{X_2}(x_2),T_{X_2}(y_2),\\&\qquad\min\{T_{X_1}(x_1),T_{X_1}(y_1)\}\}\\&\quad=\min\{\min\{T_{X_1}(x_1),T_{X_2}(x_2)\},\\&\qquad\min\{T_{X_1}(y_1),T_{X_2}(y_2)\}\}\\&\quad=\min\{(T_{X_1}\circ T_{X_2})(x_1,x_2),(T_{X_1}\circ T_{X_2})(y_1,y_2)\}.\end{aligned}$$

Similarly, $(I_{Y_1}\circ I_{Y_2})((x_1,x_2)(y_1,y_2))=\max\{(I_{X_1}\circ I_{X_2})(x_1,x_2),(I_{X_1}\circ I_{X_2})(y_1,y_2)\}$ and $(F_{Y_1}\circ F_{Y_2})((x_1,x_2)(y_1,y_2))=\max\{(F_{X_1}\circ F_{X_2})(x_1,x_2),(F_{X_1}\circ F_{X_2})(y_1,y_2)\}$. □

Definition 3.23. Let $\mathcal{G}_1=(X_1,Y_1)$ and $\mathcal{G}_2=(X_2,Y_2)$ be two SVNGs. For any vertex $(x_1,x_2)\in V_1\times V_2$,

$$\begin{aligned}&(d_T)_{\mathcal{G}_1[\mathcal{G}_2]}(x_1,x_2)\\&\quad=\sum_{(x_1,x_2)(y_1,y_2)\in E_1\circ E_2}(T_{Y_1}\circ T_{Y_2})((x_1,x_2)(y_1,y_2))\\&\quad=\sum_{x_1=y_1,x_2y_2\in E_2}\min\{T_{X_1}(x_1),T_{Y_2}(x_2y_2)\}\\&\qquad+\sum_{x_2=y_2,x_1y_1\in E_1}\min\{T_{X_2}(x_2),T_{Y_1}(x_1y_1)\}\\&\qquad+\sum_{x_2\neq y_2,x_1y_1\in E_1}\min\{T_{X_2}(y_2),T_{X_2}(x_2),T_{Y_1}(x_1y_1)\},\\&(d_I)_{\mathcal{G}_1[\mathcal{G}_2]}(x_1,x_2)\\&\quad=\sum_{(x_1,x_2)(y_1,y_2)\in E_1\circ E_2}(I_{Y_1}\circ I_{Y_2})((x_1,x_2)(y_1,y_2))\\&\quad=\sum_{x_1=y_1,x_2y_2\in E_2}\max\{I_{X_1}(x_1),I_{Y_2}(x_2y_2)\}\\&\qquad+\sum_{x_2=y_2,x_1y_1\in E_1}\max\{I_{X_2}(x_2),I_{Y_1}(x_1y_1)\}\\&\qquad+\sum_{x_2\neq y_2,x_1y_1\in E_1}\max\{I_{X_2}(y_2),I_{X_2}(x_2),I_{Y_1}(x_1y_1)\},\\&(d_F)_{\mathcal{G}_1[\mathcal{G}_2]}(x_1,x_2)\\&\quad=\sum_{(x_1,x_2)(y_1,y_2)\in E_1\circ E_2}(F_{Y_1}\circ F_{Y_2})((x_1,x_2)(y_1,y_2))\\&\quad=\sum_{x_1=y_1,x_2y_2\in E_2}\max\{F_{X_1}(x_1),F_{Y_2}(x_2y_2)\}\\&\qquad+\sum_{x_2=y_2,x_1y_1\in E_1}\max\{F_{X_2}(x_2),F_{Y_1}(x_1y_1)\}\\&\qquad+\sum_{x_2\neq y_2,x_1y_1\in E_1}\max\{F_{X_2}(y_2),F_{X_2}(x_2),F_{Y_1}(x_1y_1)\}.\end{aligned}$$

Theorem 3.24. *Let $\mathcal{G}_1=(X_1,Y_1)$ and $\mathcal{G}_2=(X_2,Y_2)$ be two SVNGs. If $T_{X_1}\geq T_{Y_2}$, $I_{X_1}\leq I_{Y_2}$, $F_{X_1}\leq F_{Y_2}$ and $T_{X_2}\geq T_{Y_1}$, $I_{X_2}\leq I_{Y_1}$, $F_{X_2}\leq F_{Y_1}$. Then $d_{\mathcal{G}_1[\mathcal{G}_2]}(x_1,x_2)=|V_2|d_{\mathcal{G}_1}(x_1)+d_{\mathcal{G}_2}(x_2)$ for all $(x_1,x_2)\in V_1\times V_2$.*

Proof. For any vertex $(x_1,x_2)\in V_1\times V_2$,

$$\begin{aligned}&(d_T)_{\mathcal{G}_1[\mathcal{G}_2]}(x_1,x_2)\\&\quad=\sum_{x_1=y_1,x_2y_2\in E_2}\min\{T_{X_1}(x_1),T_{Y_2}(x_2y_2)\}\end{aligned}$$

$$+\sum_{x_2=y_2,x_1y_1\in E_1}\min\{T_{X_2}(x_2),T_{Y_1}(x_1y_1)\}$$
$$+\sum_{x_2\neq y_2,x_1y_1\in E_1}\min\{T_{X_2}(y_2),T_{X_2}(x_2),T_{Y_1}(x_1y_1)\}$$
$$=\sum_{x_2y_2\in E_2}T_{Y_2}(x_2y_2)+\sum_{x_1y_1\in E_1}T_{Y_1}(x_1y_1)$$
$$+\sum_{x_1y_1\in E_1}T_{Y_1}(x_1y_1)$$
(Since $T_{X_1}\geq T_{Y_2}$ and $T_{X_2}\geq T_{Y_1}$)
$$=(d_T)_{\mathcal{G}_2}(x_2)+|V_2|(d_T)_{\mathcal{G}_1}(x_1).$$

Analogously, we can show that $(d_I)_{\mathcal{G}_1[\mathcal{G}_2]}(x_1,x_2)=(d_I)_{\mathcal{G}_2}(x_2)+|V_2|(d_I)_{\mathcal{G}_1}(x_1)$ and $(d_F)_{\mathcal{G}_1[\mathcal{G}_2]}(x_1,x_2)=(d_F)_{\mathcal{G}_2}(x_2)+|V_2|(d_F)_{\mathcal{G}_1}(x_1)$. Hence $d_{\mathcal{G}_1[\mathcal{G}_2]}(x_1,x_2)=d_{\mathcal{G}_2}(x_2)+|V_2|d_{\mathcal{G}_1}(x_1)$. □

Definition 3.25. The union $\mathcal{G}_1\cup\mathcal{G}_2=(X_1\cup X_2, Y_1\cup Y_2)$ of two SVNGs $\mathcal{G}_1=(X_1,Y_1)$ and $\mathcal{G}_2=(X_2,Y_2)$ of the graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$, respectively, is defined as follows:

$$(T_{X_1}\cup T_{X_2})(x)=\begin{cases}T_{X_1}(x) & \text{if } x\in V_1\setminus V_2,\\ T_{X_2}(x) & \text{if } x\in V_2\setminus V_1,\\ \max\{T_{X_1}(x),T_{X_2}(x)\} & \text{if } x\in V_1\cap V_2.\end{cases}$$
$$(I_{X_1}\cup I_{X_2})(x)=\begin{cases}I_{X_1}(x) & \text{if } x\in V_1\setminus V_2,\\ I_{X_2}(x) & \text{if } x\in V_2\setminus V_1,\\ \min\{I_{X_1}(x),I_{X_2}(x)\} & \text{if } x\in V_1\cap V_2.\end{cases}$$
$$(F_{X_1}\cup F_{X_2})(x)=\begin{cases}F_{X_1}(x) & \text{if } x\in V_1\setminus V_2,\\ F_{X_2}(x) & \text{if } x\in V_2\setminus V_1,\\ \min\{F_{X_1}(x),F_{X_2}(x)\} & \text{if } x\in V_1\cap V_2.\end{cases}$$
$$(T_{Y_1}\cup T_{Y_2})(xy)=\begin{cases}T_{Y_1}(xy) & \text{if } xy\in E_1\setminus E_2,\\ T_{Y_2}(xy) & \text{if } xy\in E_2\setminus E_1,\\ \max\{T_{Y_1}(xy),T_{Y_2}(xy)\} & \text{if } xy\in E_1\cap E_2.\end{cases}$$
$$(I_{Y_1}\cup I_{Y_2})(xy)=\begin{cases}I_{Y_1}(xy) & \text{if } xy\in E_1\setminus E_2,\\ I_{Y_2}(xy) & \text{if } xy\in E_2\setminus E_1,\\ \min\{I_{Y_1}(xy),I_{Y_2}(xy)\} & \text{if } xy\in E_1\cap E_2.\end{cases}$$
$$(F_{Y_1}\cup F_{Y_2})(xy)=\begin{cases}F_{Y_1}(xy) & \text{if } xy\in E_1\setminus E_2,\\ F_{Y_2}(xy) & \text{if } xy\in E_2\setminus E_1,\\ \min\{F_{Y_1}(xy),F_{Y_2}(xy)\} & \text{if } xy\in E_1\cap E_2.\end{cases}$$

Proposition 3.26. *If $\mathcal{G}_1=(X_1,Y_1)$ and $\mathcal{G}_2=(X_2,Y_2)$ are the SVNGs, then $\mathcal{G}_1\cup\mathcal{G}_2$ is the SVNG.*

Proof. Consider $xy\in E_1\setminus E_2$. Then there are three different cases, (i) $x,y\in V_1\setminus V_2$, (ii) $x\in V_1\setminus V_2,\ y\in V_1\cap V_2$ and (iii) $x,y\in V_1\cap V_2$.

(i) Suppose $x,y\in V_1\setminus V_2$
$$(T_{Y_1}\cup T_{Y_2})(xy)=T_{Y_1}(xy)\leq\min\{T_{X_1}(x),T_{X_1}(y)\}=\min\{(T_{X_1}\cup T_{X_2})(x),(T_{X_1}\cup T_{X_2})(y)\}.$$

(ii) Suppose $x\in V_1\setminus V_2,\ y\in V_1\cap V_2$. Then
$$(T_{Y_1}\cup T_{Y_2})(xy)=T_{Y_1}(xy)\leq\min\{T_{X_1}(x),T_{X_1}(y)\}\leq\min\{(T_{X_1}\cup T_{X_2})(x),\max\{T_{X_1}(y),T_{X_2}(y)\}\}=\min\{(T_{X_1}\cup T_{X_2})(x),(T_{X_1}\cup T_{X_2})(y)\}.$$

(iii) Suppose $x\in V_1\cap V_2,\ y\in V_1\cap V_2$. Then
$$(T_{Y_1}\cup T_{Y_2})(xy)=T_{Y_1}(xy)\leq\min\{T_{X_1}(x),T_{X_1}(y)\}\leq\min\{\max\{T_{X_1}(x),T_{X_2}(x)\},\max\{T_{X_1}(y),T_{X_2}(y)\}\}=\min\{(T_{X_1}\cup T_{X_2})(x),(T_{X_1}\cup T_{X_2})(y)\}$$

Also, it is easy to find $(I_{Y_1}\cup I_{Y_2})(xy)\geq\max\{(I_{X_1}\cup I_{X_2})(x),(I_{X_1}\cup I_{X_2})(y)\}$ and $(F_{Y_1}\cup F_{Y_2})(xy)\geq\max\{(F_{X_1}\cup F_{X_2})(x),(F_{X_1}\cup F_{X_2})(y)\}$ in the three possible cases. Similarly, if $xy\in E_2\setminus E_1$, then $(T_{Y_1}\cup T_{Y_2})(xy)\leq\min\{(T_{X_1}\cup T_{X_2})(x),(T_{X_1}\cup T_{X_2})(y)\}$, $(I_{Y_1}\cup I_{Y_2})(xy)\geq\max\{(I_{X_1}\cup I_{X_2})(x),(I_{X_1}\cup I_{X_2})(y)\}$, $(F_{Y_1}\cup F_{Y_2})(xy)\geq\max\{(F_{X_1}\cup F_{X_2})(x),(F_{X_1}\cup F_{X_2})(y)\}$.

Let $xy\in E_1\cap E_2$. Then
$$(T_{Y_1}\cup T_{Y_2})(xy)=\max\{T_{Y_1}(xy),T_{Y_2}(xy)\}\leq\max\{\min\{T_{X_1}(x),T_{X_1}(y)\},\min\{T_{X_2}(x),T_{X_2}(y)\}\}\leq\min\{\max\{T_{X_1}(x),T_{X_2}(x)\},\max\{T_{X_1}(y),T_{X_2}(y)\}\}$$

$$= \min\{(T_{X_1} \cup T_{X_2})(x), (T_{X_1} \cup T_{X_2})(y)\},$$

$$(I_{Y_1} \cup I_{Y_2})(xy) = \min\{I_{Y_1}(xy), I_{Y_2}(xy)\}$$
$$\geq \min\{\max\{I_{X_1}(x), I_{X_1}(y)\}, \max\{I_{X_2}(x), I_{X_2}(y)\}\}$$
$$\geq \max\{\min\{I_{X_1}(x), I_{X_2}(x)\}, \min\{I_{X_1}(y), I_{X_2}(y)\}\}$$
$$= \max\{(I_{X_1} \cup I_{X_2})(x), (I_{X_1} \cup I_{X_2})(y)\},$$

$$(F_{Y_1} \cup F_{Y_2})(xy) = \min\{F_{Y_1}(xy), F_{Y_2}(xy)\}$$
$$\geq \min\{\max\{F_{X_1}(x), F_{X_1}(y)\}, \max\{F_{X_2}(x), F_{X_2}(y)\}\}$$
$$\geq \max\{\min\{F_{X_1}(x), F_{X_2}(x)\}, \min\{F_{X_1}(y), F_{X_2}(y)\}\}$$
$$= \max\{(F_{X_1} \cup F_{X_2})(x), (F_{X_1} \cup F_{X_2})(y)\}.$$

The converse of above result holds if $V_1 \cap V_2 = \emptyset$. □

Theorem 3.27. *The union $\mathcal{G}_1 \cup \mathcal{G}_2$ of $\mathcal{G}_1$ and $\mathcal{G}_2$ is a SVNG of $G_1 \cup G_2$ if and only if $\mathcal{G}_1$ and $\mathcal{G}_2$ are SVNGs of G_1 and G_2, respectively, where $V_1 \cap V_2 = \emptyset$.*

Proof. Assume that $\mathcal{G}_1 \cup \mathcal{G}_2$ is a SVNG. Let $xy \in E_1$, then $xy \notin E_2$ and $x, y \in V_1 \setminus V_2$. Thus

$$T_{Y_1}(xy) = (T_{Y_1} \cup T_{Y_2})(xy)$$
$$\leq \min\{(T_{X_1} \cup T_{X_2})(x), (T_{X_1} \cup T_{X_2})(y)\}$$
$$= \min\{T_{X_1}(x), T_{X_1}(y)\},$$

$$I_{Y_1}(xy) = (I_{Y_1} \cup I_{Y_2})(xy)$$
$$\geq \max\{(I_{X_1} \cup I_{X_2})(x), (I_{X_1} \cup I_{X_2})(y)\}$$
$$= \max\{I_{X_1}(x), I_{X_1}(y)\},$$

$$F_{Y_1}(xy) = (F_{Y_1} \cup F_{Y_2})(xy)$$
$$\geq \max\{(F_{X_1} \cup F_{X_2})(x), (F_{X_1} \cup F_{X_2})(y)\}$$
$$= \max\{F_{X_1}(x), F_{X_1}(y)\}.$$

Thus $\mathcal{G}_1$ is a SVNG of G_1. Similarly, it is easy to show that $\mathcal{G}_2$ is a SVNG of G_2. □

Definition 3.28. Let $\mathcal{G}_1 = (X_1, Y_1)$ and $\mathcal{G}_2 = (X_2, Y_2)$ be two SVNGs. For any vertex $x \in V_1 \cup V_2$, there are three cases to consider.

Case 1. Either $x \in V_1 \setminus V_2$ or $x \in V_2 \setminus V_1$. Then no edge incident at x lies in $E_1 \cap E_2$. So, for $x \in V_1 \setminus V_2$

$$(d_T)_{\mathcal{G}_1 \cup \mathcal{G}_2}(x) = \sum_{xy \in E_1} T_{Y_1}(xy) = (d_T)_{\mathcal{G}_1}(x),$$

$$(d_I)_{\mathcal{G}_1 \cup \mathcal{G}_2}(x) = \sum_{xy \in E_1} I_{Y_1}(xy) = (d_I)_{\mathcal{G}_1}(x),$$

$$(d_F)_{\mathcal{G}_1 \cup \mathcal{G}_2}(x) = \sum_{xy \in E_1} F_{Y_1}(xy) = (d_F)_{\mathcal{G}_1}(x).$$

For $x \in V_2 \setminus V_1$,

$$(d_T)_{\mathcal{G}_1 \cup \mathcal{G}_2}(x) = \sum_{xy \in E_2} T_{Y_2}(xy) = (d_T)_{\mathcal{G}_2}(x),$$

$$(d_I)_{\mathcal{G}_1 \cup \mathcal{G}_2}(x) = \sum_{xy \in E_2} I_{Y_2}(xy) = (d_I)_{\mathcal{G}_2}(x),$$

$$(d_F)_{\mathcal{G}_1 \cup \mathcal{G}_2}(x) = \sum_{xy \in E_2} F_{Y_2}(xy) = (d_F)_{\mathcal{G}_2}(x).$$

Case 2. $x \in V_1 \cap V_2$ but no edge incident at x lies in $E_1 \cap E_2$. Then any edge incident at x is either in $E_1 \setminus E_2$ or in $E_2 \setminus E_1$.

$$(d_T)_{\mathcal{G}_1 \cup \mathcal{G}_2}(x) = \sum_{xy \in E_1 \cup E_2} (T_{Y_1} \cup T_{Y_2})(xy)$$
$$= \sum_{xy \in E_1} T_{Y_1}(xy) + \sum_{xy \in E_2} T_{Y_2}(xy)$$
$$= (d_T)_{\mathcal{G}_1}(x) + (d_T)_{\mathcal{G}_2}(x)$$

Similarly, $(d_I)_{\mathcal{G}_1 \cup \mathcal{G}_2}(x) = (d_I)_{\mathcal{G}_1}(x) + (d_I)_{\mathcal{G}_2}(x)$ and $(d_F)_{\mathcal{G}_1 \cup \mathcal{G}_2}(x) = (d_F)_{\mathcal{G}_1}(x) + (d_F)_{\mathcal{G}_2}(x)$.

Case 3. $x \in V_1 \cap V_2$ and some edges incident at x are in $E_1 \cap E_2$.

$$(d_T)_{\mathcal{G}_1 \cup \mathcal{G}_2}(x) = \sum_{xy \in E_1 \cup E_2} (T_{Y_1} \cup T_{Y_2})(xy)$$
$$= \sum_{xy \in E_1 \setminus E_2} T_{Y_1}(xy) + \sum_{xy \in E_2 \setminus E_1} T_{Y_2}(xy) + \sum_{xy \in E_1 \cap E_2} \max\{T_{Y_1}(xy), T_{Y_2}(xy)\}$$
$$= \left(\sum_{xy \in E_1 \setminus E_2} T_{Y_1}(xy) + \sum_{xy \in E_2 \setminus E_1} T_{Y_2}(xy) + \sum_{xy \in E_1 \cap E_2} \max\{T_{Y_1}(xy), T_{Y_2}(xy)\} + \sum_{xy \in E_1 \cap E_2} \min\{T_{Y_1}(xy), T_{Y_2}(xy)\} \right)$$
$$- \sum_{xy \in E_1 \cap E_2} \min\{T_{Y_1}(xy), T_{Y_2}(xy)\}$$

$$= \sum_{xy\in E_1} T_{Y_1}(xy) + \sum_{xy\in E_2} T_{Y_2}(xy) - \sum_{xy\in E_1\cap E_2} \min\{T_{Y_1}(xy), T_{Y_2}(xy)\}$$

$$= (d_T)_{\mathcal{G}_1}(x) + (d_T)_{\mathcal{G}_2}(x) - \sum_{xy\in E_1\cap E_2} \min\{T_{Y_1}(xy), T_{Y_2}(xy)\}.$$

Similarly, $(d_I)_{\mathcal{G}_1\cup\mathcal{G}_2}(x) = (d_I)_{\mathcal{G}_1}(x) + (d_I)_{\mathcal{G}_2}(x) - \sum_{xy\in E_1\cap E_2} \max\{I_{Y_1}(xy), I_{Y_2}(xy)\}$ and $(d_F)_{\mathcal{G}_1\cup\mathcal{G}_2}(x) = (d_F)_{\mathcal{G}_1}(x) + (d_F)_{\mathcal{G}_2}(x) - \sum_{xy\in E_1\cap E_2} \max\{F_{Y_1}(xy), F_{Y_2}(xy)\}$.

Definition 3.29. The ring sum $\mathcal{G}_1 \oplus \mathcal{G}_2 = (X_1 \oplus X_2, Y_1 \oplus Y_2)$ of two SVNGs $\mathcal{G}_1 = (X_1, Y_1)$ and $\mathcal{G}_2 = (X_2, Y_2)$ of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, is defined as follows:

$$(T_{X_1} \oplus T_{X_2})(x) = (T_{X_1} \cup T_{X_2})(x),$$
$$(I_{X_1} \oplus I_{X_2})(x) = (I_{X_1} \cup I_{X_2})(x),$$
$$(F_{X_1} \oplus F_{X_2})(x) = (F_{X_1} \cup F_{X_2})(x) \quad \text{if } x \in V_1 \cup V_2,$$

$$(T_{Y_1} \oplus T_{Y_2})(xy) = \begin{cases} T_{Y_1}(xy) & \text{if } xy \in E_1 \setminus E_2, \\ T_{Y_2}(xy) & \text{if } xy \in E_2 \setminus E_1, \\ 0 & \text{if } xy \in E_1 \cap E_2. \end{cases}$$

$$(I_{Y_1} \oplus I_{Y_2})(xy) = \begin{cases} I_{Y_1}(xy) & \text{if } xy \in E_1 \setminus E_2, \\ I_{Y_2}(xy) & \text{if } xy \in E_2 \setminus E_1, \\ 0 & \text{if } xy \in E_1 \cap E_2. \end{cases}$$

$$(F_{Y_1} \oplus F_{Y_2})(xy) = \begin{cases} F_{Y_1}(xy) & \text{if } xy \in E_1 \setminus E_2, \\ F_{Y_2}(xy) & \text{if } xy \in E_2 \setminus E_1, \\ 0 & \text{if } xy \in E_1 \cap E_2. \end{cases}$$

Proposition 3.30. *If $\mathcal{G}_1 = (X_1, Y_1)$ and $\mathcal{G}_2 = (X_2, Y_2)$ are the SVNGs, then $\mathcal{G}_1 \oplus \mathcal{G}_2$ is the SVNG.*

Definition 3.31. The join $\mathcal{G}_1 + \mathcal{G}_2 = (X_1 + X_2, Y_1 + Y_2)$ of two SVNGs $\mathcal{G}_1 = (X_1, Y_1)$ and $\mathcal{G}_2 = (X_2, Y_2)$ of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, is defined as follows:

$$\text{(i)} \begin{cases} (T_{X_1} + T_{X_2})(x) = (T_{X_1} \cup T_{X_2})(x) \\ (I_{X_1} + I_{X_2})(x) = (I_{X_1} \cup I_{X_2})(x) \\ (F_{X_1} + F_{X_2})(x) = (F_{X_1} \cup F_{X_2})(x) \quad \text{if } x \in V_1 \cup V_2, \end{cases}$$

$$\text{(ii)} \begin{cases} (T_{Y_1} + T_{Y_2})(xy) = (T_{Y_1} \cup T_{Y_2})(xy) \\ (I_{Y_1} + I_{Y_2})(xy) = (I_{Y_1} \cup I_{Y_2})(xy) \\ (F_{Y_1} + F_{Y_2})(xy) = (F_{Y_1} \cup F_{Y_2})(xy) \quad \text{if } xy \in E_1 \cup E_2, \end{cases}$$

$$\text{(iii)} \begin{cases} (T_{Y_1} + T_{Y_2})(xy) = \min\{T_{X_1}(x), T_{X_2}(y)\} \\ (I_{Y_1} + I_{Y_2})(xy) = \max\{I_{X_1}(x), I_{X_2}(y)\} \\ (F_{Y_1} + F_{Y_2})(xy) = \max\{F_{X_1}(x), F_{X_2}(y)\} \quad \text{if } xy \in E', \end{cases}$$

where E' is the set of all edges joining the vertices of V_1 and V_2, $V_1 \cap V_2 = \emptyset$.

Theorem 3.32. *The join $\mathcal{G}_1 + \mathcal{G}_2$ of $\mathcal{G}_1$ and $\mathcal{G}_2$ is a SVNG of $G_1 + G_2$ if and only if $\mathcal{G}_1$ and $\mathcal{G}_2$ are SVNGs of G_1 and G_2, respectively, where $V_1 \cap V_2 = \emptyset$.*

Proof. Suppose that $\mathcal{G}_1 + \mathcal{G}_2$ is a SVNG. Then from Theorem 3.27, $\mathcal{G}_1$ and $\mathcal{G}_2$ are SVNGs.

Conversely, assume that $\mathcal{G}_1$ and $\mathcal{G}_2$ are SVNGs of G_1 and G_2, respectively. Consider $xy \in E_1 \cup E_2$. Then the required result follows from Proposition 3.26. Let $xy \in E'$. Then

$$(T_{Y_1} + T_{Y_2})(xy) = \min\{T_{X_1}(x), T_{X_2}(y)\} = \min\{(T_{X_1} \cup T_{X_2})(x), (T_{X_1} \cup T_{X_2})(y)\} = \min\{(T_{X_1} + T_{X_2})(x), (T_{X_1} + T_{X_2})(y)\},$$

Similarly, we can show $(I_{Y_1} + I_{Y_2})(xy) = \max\{(I_{X_1} + I_{X_2})(x), (I_{X_1} + I_{X_2})(y)\}$ and $(F_{Y_1} + F_{Y_2})(xy) = \max\{(F_{X_1} + F_{X_2})(x), (F_{X_1} + F_{X_2})(y)\}$. □

Definition 3.33. Let $\mathcal{G}_1 = (X_1, Y_1)$ and $\mathcal{G}_2 = (X_2, Y_2)$ be two SVNGs. For any vertex $x \in V_1 + V_2$,

$$(d_T)_{\mathcal{G}_1+\mathcal{G}_2}(x) = \sum_{xy\in E_1\cup E_2} (T_{Y_1} \cup T_{Y_2})(xy) + \sum_{xy\in E'} \min\{T_{X_1}(x), T_{X_2}(y)\},$$

$$(d_I)_{\mathcal{G}_1+\mathcal{G}_2}(x) = \sum_{xy\in E_1\cup E_2} (I_{Y_1} \cup I_{Y_2})(xy) + \sum_{xy\in E'} \max\{I_{X_1}(x), I_{X_2}(y)\},$$

$$(d_F)_{\mathcal{G}_1+\mathcal{G}_2}(x) = \sum_{xy\in E_1\cup E_2} (F_{Y_1} \cup F_{Y_2})(xy) + \sum_{xy\in E'} \max\{F_{X_1}(x), F_{X_2}(y)\}.$$

For any $x \in V_1$,

$$(d_T)_{\mathcal{G}_1+\mathcal{G}_2}(x) = \sum_{xy\in E_1} T_{Y_1}(xy) + \sum_{xy\in E'} \min\{T_{X_1}(x), T_{X_2}(y)\},$$

$$= (d_T)_{\mathcal{G}_1}(x) + \sum_{xy \in E'} \min\{T_{X_1}(x), T_{X_2}(y)\}.$$

Similarly, $(d_I)_{\mathcal{G}_1+\mathcal{G}_2}(x) = (d_I)_{\mathcal{G}_1}(x) + \sum_{xy \in E'} \max \{I_{X_1}(x), I_{X_2}(y)\}$ and $(d_F)_{\mathcal{G}_1+\mathcal{G}_2}(x) = (d_F)_{\mathcal{G}_1}(x) + \sum_{xy \in E'} \max\{F_{X_1}(x), F_{X_2}(y)\}$.

For any $x \in V_2$,

$$\begin{aligned}(d_T)_{\mathcal{G}_1+\mathcal{G}_2}(x) &= \sum_{xy \in E_2} T_{Y_2}(xy) + \sum_{xy \in E'} \min\{T_{X_1}(x), T_{X_2}(y)\} \\ &= (d_T)_{\mathcal{G}_2}(x) + \sum_{xy \in E'} \min\{T_{X_1}(x), T_{X_2}(y)\}.\end{aligned}$$

Similarly, $(d_I)_{\mathcal{G}_1+\mathcal{G}_2}(x) = (d_I)_{\mathcal{G}_2}(x) + \sum_{xy \in E'} \max \{I_{X_1}(x), I_{X_2}(y)\}$ and $(d_F)_{\mathcal{G}_1+\mathcal{G}_2}(x) = (d_F)_{\mathcal{G}_2}(x) + \sum_{xy \in E'} \max\{F_{X_1}(x), F_{X_2}(y)\}$.

4. Single valued neutrosophic digraph in travel time

In modern age, planning efficient routes is essential for industry and business, with applications as varied as product distribution, laying new fiber optic lines for broadband internet, and suggesting new friends within social network websites such as Facebook. When we visit a website like Google Maps and looking for directions from one city to another city in USA, we are usually asking for a shortest path between the two cities. These computer applications use representations of the road maps as graphs, with estimated travel times as edge weights. The travel time is a function of traffic density on the road or the length of the road. The traffic density is a fuzzy, while the length of a road is a crisp quantity. In a road network, crossings are represented by vertices, roads by edges and traffic density on the road is usually calculated between adjacent crossings. These factors can be represented as a SVNS. Any model of a road network can be represented as a SVNDG $\mathcal{D} = (\mathcal{C}, \mathcal{R})$, where $\mathcal{C}$ is a SVNS of crossings (vertices) at which the traffic density is calculated and connectivity conditions as truth-membership degree $T_{\mathcal{C}}(x)$, indeterminacy membership degree $I_{\mathcal{C}}(x)$ and falsity membership degree $F_{\mathcal{C}}(x)$,

$$\mathcal{C} = \Big\langle \Big(\frac{c_1}{0.6}, \frac{c_2}{0.5}, \frac{c_3}{0.1}, \frac{c_4}{0.8}, \frac{c_5}{0.4}, \frac{c_6}{0.7}, \frac{c_7}{0.4}\Big), \Big(\frac{c_1}{0.2}, \frac{c_2}{0.1}, \frac{c_3}{0.6}, \frac{c_4}{0.4}, \frac{c_5}{0.3}, \frac{c_6}{0.5}, \frac{c_7}{0.1}\Big), \Big(\frac{c_1}{0.5}, \frac{c_2}{0.3}, \frac{c_3}{0.2}, \frac{c_4}{0.2}, \frac{c_5}{0.2}, \frac{c_6}{0.3}, \frac{c_7}{0.4}\Big)\Big\rangle,$$

and $\mathcal{R}$ is a SVNS of roads (edges) between crossings, whose truth-membership degree $T_{\mathcal{R}}(xy)$, indeterminacy membership degree $I_{\mathcal{R}}(xy)$ and falsity membership degree $F_{\mathcal{R}}(xy)$ can be calculated as:

$$\begin{aligned}&T_{\mathcal{R}}(xy) \le \min\{T_{\mathcal{C}}(x), T_{\mathcal{C}}(y)\},\\ &I_{\mathcal{R}}(xy) \ge \max\{I_{\mathcal{C}}(x), I_{\mathcal{C}}(y)\} \text{ and}\\ &F_{\mathcal{R}}(xy) \ge \max\{F_{\mathcal{C}}(x), F_{\mathcal{C}}(y)\} \text{ for all } xy \in E.\end{aligned}$$

The SVNDG $\mathcal{D} = (\mathcal{C}, \mathcal{R})$ of the travel time is given in Fig. 5. The single valued neutrosophic out neighbourhoods are given in Table 1.

The final weights on edges can be calculated by finding the score function of single valued neutrosophic edges as $s_i = (T_{\mathcal{R}})_i + 1 - (I_{\mathcal{R}})_i + 1 - (F_{\mathcal{R}})_i$. The final weighted digraph given in Fig. 6, which can be used for finding the shortest/optimal path between two locations (vertices) by any of the known methods, like Djkastra and A star. Weighted relations are given in Table 2.

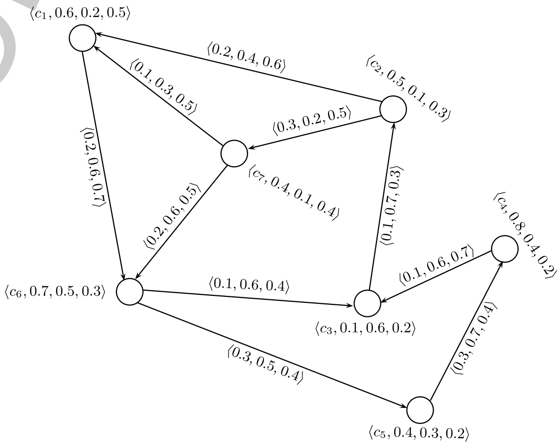

Fig. 5. SVNDG of a road network.

Table 1
Single valued neutrosophic out and in neighbourhoods of crossings (relations)

Crossings	$\mathcal{N}^+$(crossings)
c_1	$\{c_6\langle 0.2, 0.6, 0.7\rangle\}$
c_2	$\{c_1\langle 0.2, 0.4, 0.6\rangle, c_7\langle 0.3, 0.2, 0.5\rangle\}$
c_3	$\{c_2\langle 0.1, 0.7, 0.3\rangle\}$
c_4	$\{c_3\langle 0.1, 0.6, 0.7\rangle\}$
c_5	$\{c_4\langle 0.3, 0.7, 0.4\rangle\}$
c_6	$\{c_3\langle 0.1, 0.6, 0.4\rangle, c_5\langle 0.3, 0.5, 0.4\rangle\}$
c_7	$\{c_1\langle 0.1, 0.3, 0.5\rangle, c_6\langle 0.2, 0.6, 0.5\rangle\}$

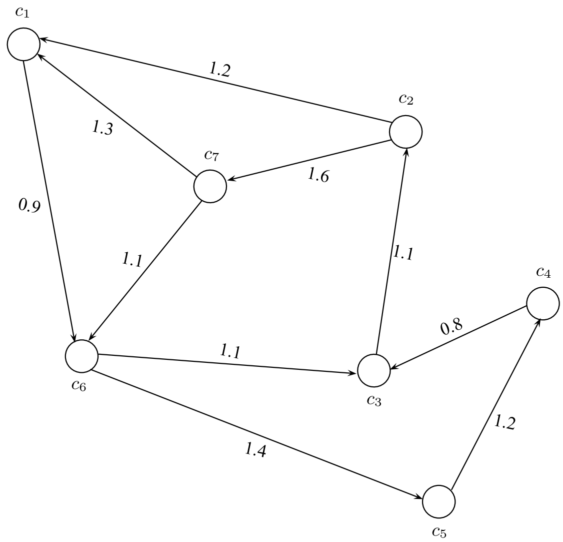

Fig. 6. Weighted digraph of a road network.

Table 2
Weighted out and in neighbourhoods of crossings (weighted relations)

Crossings	$\mathcal{N}^+$(crossings)
c_1	$\{c_6 0.9\}$
c_2	$\{c_1 1.2, c_7 1.6\}$
c_3	$\{c_2 1.1\}$
c_4	$\{c_3 0.8\}$
c_5	$\{c_4 1.2\}$
c_6	$\{c_3 1.1, c_5 1.4\}$
c_7	$\{c_1 1.3, c_6 1.1\}$

Algorithm given below generates the weighted digraph, WR, for given SVNDG and uses it to calculate the optimal path from a source vertex.

Algorithm

(1) void single valued neutrosophic shortest path(){
(2) $\mathcal{C}$ =SVNS of crossings;
(3) number of crossings=count($\mathcal{C}$);
(4) $\mathcal{R}$ =Empty SVNS of roads;
(5) for (int $c = 0; c < numberofcrossings; c++$){
(6) for (int $c' = 0; c' < numberofcrossings; c'++$){
(7) if ($\mathcal{C}(x)$ is adjacent to $\mathcal{C}(y)$){
(8) $T_{\mathcal{R}}(cc') \leq \min\{T_{\mathcal{C}}(c), T_{\mathcal{C}}(c')\}$;
(9) $I_{\mathcal{R}}(cc') \geq \max\{I_{\mathcal{C}}(c), I_{\mathcal{C}}(c')\}$;
(10) $F_{\mathcal{R}}(cc') \geq \max\{F_{\mathcal{C}}(c), F_{\mathcal{C}}(c')\}$;
(11) }
(12) }
(13) }
(14) $\mathcal{R}$ =SVNS of edges;
(15) R = Single valued neutrosophic relation;
(16) WR =Weighted relation;
(17) no of edges= $count(\mathcal{R})$;
(18) for (int $i = 0$;i <no of edges; $i++$){
(19) $s_i = (T_{\mathcal{R}})_i + 1 - (I_{\mathcal{R}})_i + 1 - (F_{\mathcal{R}})_i$;
(20) c =Adjacent from Node of $\mathcal{R}_i$;
(21) c' =Adjacent to Node of $\mathcal{R}_i$;
(22) $WR_{cc'} = s_i$;
(23) }
(24) print WR;
(25) Calculate optimal path using WR;
(26) }

5. Conclusions

Single valued neutrosophic models are more flexible and practical than fuzzy, interval-valued fuzzy, intuitionistic fuzzy and interval-valued intuitionistic fuzzy models. SVNGs can be used in computer technology, networking, communication, economics, genetics, linguistics, sociology etc, when the concept of indeterminacy is present. So, in this paper, we have defined the basic operations on SVNGs such as direct product, Cartesian product, semi-strong product, strong product, lexicographic product, union, ring sum and join, and investigated some of their properties. Moreover, the degree of a vertex in SVNGs formed by these operations in terms of the degree of vertices in the given SVNGs in some particular cases are determined. They will be helpful especially when the graphs are very large. We have also provided an application of SVNDG in travel time.

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