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On Single-Valued Neutrosophic Closure Spaces

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Abstract: This paper aims to mark out new terms of single-valued neutrosophic notions in a Šostak sense called single-valued neutrosophic semi-closure spaces. To achieve this, notions such as $\beta\pounds$-closure operators and $\beta\pounds$-interior operators are first defined. More precisely, these proposed contributions involve different terms of single-valued neutrosophic continuous mappings called single-valued neutrosophic (almost $\beta\pounds$, faintly $\beta\pounds$, weakly $\beta\pounds$) and $\beta\pounds$-continuous. Finally, for the purpose of symmetry, we define the single-valued neutrosophic upper, single-valued neutrosophic lower and single-valued neutrosophic boundary sets of a rough single-valued neutrosophic set α_n in a single-valued neutrosophic approximation space $(\tilde{\mathcal{F}}, \delta)$. Based on α_n and δ, we also introduce the single-valued neutrosophic approximation interior operator $\mathrm{int}^{\delta}_{\alpha_n}$ and the single-valued neutrosophic approximation closure operator $\mathrm{Cl}^{\delta}_{\alpha_n}$.

Keywords: single-valued neutrosophic $\beta\pounds$-closure operators; $\beta\pounds$-interior operators; single-valued neutrosophic almost $\beta\pounds$-continuous; faintly $\beta\pounds$-continuous; weakly $\beta\pounds$-continuous; $\beta\pounds$-continuous; single-valued neutrosophic approximation space; approximation interior operator; approximation closure operator

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1. Introduction

Neutrosophic set theory has a very powerful influence given that is a recent section of philosophy that is presented as the study of origin, nature and scope of neutralities. The idea of neutrosophy is initiated by Smarandache [1] in 1999 as a new mathematical approach that corresponds to degree of indeterminacy (uncertainty, etc.). Moreover, the soft set theory was successfully applied to several directions, such as smoothness of functions and architecture-based neuro-linguistic programming (NLP) in the papers of Bakbak et al. [2] and Mishra et al. [3]. The concept of continuous mappings plays a crucial role in many branches of mathematics, such as, fuzzy set theory, algebra and quantum gravity (see [4]). El-Naschie also has shown that both string theory and ε^{∞} theory are kind of some applications in quantum particle physics especially in relation to heterotic strings and were influenced by the fuzzy topology in Šostak sense. [5].

In current times, the theory of neutrosophy has been recycled at various junctions of mathematics. More precisely, this theory has made an exceptional advancement in the field of topological spaces. Salama et al. [6–8] dispatched their works of neutrosophic topological spaces, following the method of Chang [9] in the situation of fuzzy topological spaces $(\tilde{\mathcal{F}}, \tau)$. Afterward, Hur et al. [10,11] presented NSet(H) and NCSet. Smarandache [12] defined the idea of neutrosophic topology on the non-standard interval. One can simply detect that the fuzzy topology familiarized by Chang is a crisp group of fuzzy subsets.

Šostak [13] determined that Chang's style is crisp in nature and so he redefined the idea of fuzzy topology, frequently mentioned as smooth fuzzy topology, as a mapping from the group of all fuzzy subsets of $\tilde{\mathcal{F}}$ to $[0, 1]$. Fang Jin-ming et al. and Zahran et al. [14,15] discussed the notion of foundation as a function from an appropriate collection of fuzzy subsets of X to $[0, 1]$. Saber et al. [16] found a parallel theory in the context fuzzy ideal topological space.

Wang [17], in 2010, established the idea of a single-valued neutrosophic set. In 2016, Gayyar [18] presented the notion of fuzzy neutrosophic topological spaces in a Šostak sense. The concept of the foundation for an ordinary single-valued neutrosophic topology was explored by Kim [19]. Several authors [20–25] posted their efforts for the idea of single-valued neutrosophic topological spaces $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$. Others focusing their works on single valued neutrosophic relations, see [26,27]. Last but not least, in the sense that not only the objects are fuzzified, but also the axiomatics, the single-valued neutrsophic ideal theory was introduced in 1985 by Šostak [13] as a generalization of classical topological structures and as an extension of both crisp topology and Changs fuzzy topology.

In this article, preliminaries of single-value neutrosophic sets and single-valued neutrosophic topology are reviewed in Section 2. In Section 3, we define the notions of a single-valued neutrosophic semi-closure space. Some of their characteristic properties are considered. Further, we present and explore the properties and characterizations of the single-valued neutrosophic operators, namely $\beta\pounds$-closure $(\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}})$ and $\beta\pounds$-interior $(\beta\pounds int_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}})$ in the single-valued neutrosophic ideal topological space $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, \pounds^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$. The concepts of single-valued neutrosophic (almost, faintly, weakly) $\beta\pounds$-continuous mappings are introduced and studied in Section 4. In Section 5, we introduce a new improved single-valued neutrosophic lower and single-valued neutrosophic upper sets by which we obtain a more reliable single-valued neutrosophic boundary region set of a single-valued neutrosophic set α_n. From these single-valued neutrosophic lower and fuzzy upper sets, we define new single-valued neutrosophic interior and single-valued neutrosophic closure operators associated with a specific single-valued neutrosophic set $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$.

2. Preliminaries

This section is devoted to bring a complete survey, some previous studies and important related notions to this work. Let us have a fixed universe $\tilde{\mathcal{F}}$ to be a finite set of objects and ζ a closed unit interval $[0,1]$. We will also let $\zeta^{\mathcal{F}}$ to denote the set of all single-valued neutrosophic subsets of $\tilde{\mathcal{F}}$.

Definition 1 ([12])**.** *Let $\tilde{\mathcal{F}}$ be a non-empty set. A neutrosophic set (briefly, $\mathcal{NS}$) in $\tilde{\mathcal{F}}$ is an object having the form*

$$\alpha_n = \{\langle v, \tilde{\varrho}_{\alpha_n}(v), \tilde{\sigma}_{\sigma_n}(v), \tilde{\varsigma}_{\alpha_n}(v)\rangle : v \in \tilde{\mathcal{F}}\}$$

where

$$\tilde{\varrho} : \tilde{\mathcal{F}} \to \rfloor^{-}0, 1^{+}\lfloor,\ \tilde{\sigma} : \tilde{\mathcal{F}} \to \rfloor^{-}0, 1^{+}\lfloor,\ \tilde{\varsigma} : \tilde{\mathcal{F}} \to \rfloor^{-}0, 1^{+}\lfloor,$$

and

$$^{-}0 \leq \tilde{\varrho}_{\alpha_n}(v) + \tilde{\sigma}_{\alpha_n}(v) + \tilde{\varsigma}_{\alpha_n}(v) \leq 3^{+}$$

represent the degree of membership ($\tilde{\varrho}_{\alpha_n}$), the degree of indeterminacy ($\tilde{\sigma}_{\alpha_n}$) and the degree of non-membership ($\tilde{\varsigma}_{\alpha_n}$), respectively, of any $v \in \tilde{\mathcal{F}}$ to the set α_n.

Definition 2 ([17])**.** *Suppose that $\tilde{\mathcal{F}}$ is a universal set a space of points (objects), with a generic element in $\tilde{\mathcal{F}}$ denoted by v. Then, α_n is called a single-valued neutrosophic set (briefly, $\mathcal{SVNS}$) in $\tilde{\mathcal{F}}$, if α_n has the form*

$$\alpha_n = \{\langle v, \tilde{\varrho}_{\alpha_n}(v), \tilde{\sigma}_{\alpha_n}(v), \tilde{\varsigma}_{\alpha_n}(v)\rangle : v \in \tilde{\mathcal{F}}\}.$$

Now, $\tilde{\varrho}_{\alpha_n}, \tilde{\sigma}_{\sigma_n}, \tilde{\varsigma}_{\alpha_n}$ indicate the degree of non-membership, the degree of indeterminacy and the degree of membership, respectively, of any element $v \in \tilde{\mathcal{F}}$ to the set α_n.

Definition 3 ([17])**.** *Let $\alpha_n = \{\langle v, \tilde{\varrho}_{\alpha_n}(v), \tilde{\sigma}_{\sigma_n}(v), \tilde{\varsigma}_{\alpha_n}(v)\rangle : v \in \tilde{\mathcal{F}}\}$ be an SVNS on $\tilde{\mathcal{F}}$. The complement of the set α_n (briefly α_n^c) defined as follows:*

$$\tilde{\varrho}_{\alpha_n^c}(v) = \tilde{\varsigma}_{\alpha_n}(v),\ \tilde{\varrho}_{\alpha_n^c}(v) = [\tilde{\varrho}_{\alpha_n}]^c(v),\ \tilde{\varsigma}_{\alpha_n^c}(v) = \tilde{\varrho}_{\alpha_n}(v).$$

Definition 4 ([9]). *Let $\tilde{\mathcal{F}}$ be a non-empty set, $\alpha_n, \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$ be in the form: $\alpha_n = \{\langle v, \tilde{\varrho}_{\alpha_n}(v), \tilde{\sigma}_{\alpha_n}(v), \tilde{\xi}_{\alpha_n}(v)\rangle : v \in \tilde{\mathcal{F}}\}$ and $\varepsilon_n = \{\langle v, \tilde{\varrho}_{\varepsilon_n}(v), \tilde{\sigma}_{\varepsilon_n}(v), \tilde{\xi}_{\varepsilon_n}(v)\rangle : v \in \tilde{\mathcal{F}}\}$ on $\tilde{\mathcal{F}}$ then,*

(a) $\alpha_n \subseteq \varepsilon_n$ for every $v \in \tilde{\mathcal{F}}$;

$$\tilde{\varrho}_{\alpha_n}(v) \leq \tilde{\varrho}_{\varepsilon_n}(v),\ \tilde{\sigma}_{\alpha_n}(v) \geq \tilde{\sigma}_{\varepsilon_n}(v),\ \tilde{\xi}_{\alpha_n}(v) \geq \tilde{\xi}_{\varepsilon_n}(v).$$

(b) $\alpha_n = \varepsilon_n$ iff $\sigma_n \subseteq \varepsilon_n$ and $\sigma_n \supseteq \varepsilon_n$.
(c) $\tilde{0} = \langle 0, 1, 1\rangle$ and $\tilde{1} = \langle 1, 0, 0\rangle$.

Definition 5 ([26]). *Let $\alpha_n, \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$. Then,*

(a) $\alpha_n \cap \varepsilon_n$ is an SVNS, if for every $v \in \tilde{\mathcal{F}}$,

$$\alpha_n \cap \varepsilon_n = \langle(\tilde{\varrho}_{\alpha_n} \cap \tilde{\varrho}_{\varepsilon_n})(v),\ (\tilde{\sigma}_{\alpha_n} \cup \tilde{\sigma}_{\varepsilon_n})(v),\ (\tilde{\xi}_{\alpha_n} \cup \tilde{\xi}_{\varepsilon_n})(v)\rangle$$

where, $(\tilde{\varrho}_{\alpha_n} \cap \tilde{\varrho}_{\varepsilon_n})(v) = \tilde{\varrho}_{\alpha_n}(v) \cap 5\tilde{\varrho}_{\varepsilon_n}(v)$ and $(\tilde{\xi}_{\alpha_n} \cup \tilde{\xi}_{\varepsilon_n})(v) = \tilde{\xi}_{\alpha_n}(v) \cup \tilde{\xi}_{\varepsilon_n}(v)$, for all $v \in \tilde{\mathcal{F}}$,
(b) $\alpha_n \cup \varepsilon_n$ is an SVNS, if for every $v \in \tilde{\mathcal{F}}$,

$$\alpha_n \cup \varepsilon_n = \langle(\tilde{\varrho}_{\alpha_n} \cup \tilde{\varrho}_{\varepsilon_n})(v),\ (\tilde{\sigma}_{\alpha_n} \cap \tilde{\sigma}_{\varepsilon_n})(v),\ (\tilde{\xi}_{\alpha_n} \cap \tilde{\xi}_{\varepsilon_n})(v)\rangle$$

Definition 6 ([6]). *For any arbitrary family $\{\alpha_n\}_{i \in j} \in \zeta^{\tilde{\mathcal{F}}}$ of SVNS the union and intersection are given by*

(a) $\bigcap_{i \in j}[\alpha_n]_i = \langle \cap_{i \in j}\tilde{\varrho}_{[\alpha_n]_i}(v),\ \cup_{i \in j}\tilde{\sigma}_{[\alpha_n]_i}(v),\ \cup_{i \in j}\tilde{\xi}_{[\alpha_n]_i}(v)\rangle$,
(b) $\bigcup_{i \in j}[\alpha_n]_i = \langle \cup_{i \in j}\tilde{\varrho}_{[\alpha_n]_i}(v),\ \cap_{i \in j}\tilde{\sigma}_{[\alpha_n]_i}(v),\ \cap_{i \in j}\tilde{\xi}_{[\alpha_n]_i}(v)\rangle$.

Definition 7 ([18]). *A single-valued neutrosophic topological spaces is an ordered $(\tilde{\mathcal{F}}, \tilde{\tau}^{\tilde{\varrho}}, \tilde{\tau}^{\tilde{\sigma}}, \tilde{\tau}^{\tilde{\xi}})$ where $\tilde{\tau}^{\tilde{\varrho}}, \tilde{\tau}^{\tilde{\sigma}}, \tilde{\tau}^{\tilde{\xi}} : \zeta^{\tilde{\mathcal{F}}} \to \zeta$ is a mapping satisfying the following axioms:*

(SVNT1) $\tilde{\tau}^{\tilde{\varrho}}(\tilde{0}) = \tilde{\tau}^{\tilde{\varrho}}(\tilde{1}) = \tilde{\tau}^{\tilde{\sigma}}(\tilde{0}) = \tilde{\tau}^{\tilde{\sigma}}(\tilde{1}) = 0$ *and* $\tilde{\tau}^{\tilde{\xi}}(\tilde{0}) = \tilde{\tau}^{\tilde{\xi}}(\tilde{1}) = 1$.
(SVNT2) $\tilde{\tau}^{\tilde{\varrho}}(\alpha_n \cap \varepsilon_n) \geq \tilde{\tau}^{\tilde{\varrho}}(\alpha_n) \cap \tilde{\tau}^{\tilde{\varrho}}(\varepsilon_n)$, $\quad \tilde{\tau}^{\tilde{\sigma}}(\alpha_n \cap \varepsilon_n) \leq \tau^{\tilde{\sigma}}(\alpha_n) \cup \tilde{\tau}^{\tilde{\sigma}}(\varepsilon_n)$,
$\tilde{\tau}^{\tilde{\xi}}(\alpha_n \cap \varepsilon_n) \leq \tilde{\tau}^{\tilde{\xi}}(\alpha_n) \cup \tilde{\tau}^{\tilde{\xi}}(\varepsilon_n)$, *for every* $\alpha_n, \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$,
(SVNT3) $\tilde{\tau}^{\tilde{\varrho}}(\cup_{j \in \Gamma}[\alpha_n]_j) \geq \cap_{j \in \Gamma}\tilde{\tau}^{\tilde{\varrho}}([\alpha_n]_j)$, $\quad \tilde{\tau}^{\tilde{\sigma}}(\cup_{i \in \Gamma}[\alpha_n]_j) \leq \cup_{j \in \Gamma}\tilde{\tau}^{\tilde{\sigma}}([\alpha_n]_j)$,
$\tilde{\tau}^{\tilde{\xi}}(\cup_{j \in \Gamma}[\alpha_n]_j) \leq \cup_{j \in \Gamma}\tilde{\tau}^{\tilde{\xi}}([\alpha_n]_j)$, *for every* $[\alpha_n]_j \in \zeta^{\tilde{\mathcal{F}}}$.

The quadruple $(\tilde{\mathcal{F}}, \tilde{\tau}^{\tilde{\varrho}}, \tilde{\tau}^{\tilde{\sigma}}, \tilde{\tau}^{\tilde{\xi}})$ is called a single-valued neutrosophic topological space (briefly, *SVNT*, for short). Occasionally we write $\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}$ for $(\tilde{\tau}^{\tilde{\varrho}}, \tilde{\tau}^{\tilde{\sigma}}, \tilde{\tau}^{\tilde{\xi}})$ and it will cause no ambiguity.

Definition 8 ([21]). *Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}})$ be an SVNTS . Then, for every $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$ and $r \in \zeta_0$. Then the single-valued neutrosophic closure and single-valued neutrosophic interior of α_n are defined by:*

$$C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r) = \bigcap\{\varepsilon_n \in \zeta^{\tilde{\mathcal{F}}} : \alpha_n \leq \varepsilon_n,\ \tau^{\tilde{\varrho}}([\varepsilon_n]^c) \geq r,\ \tau^{\tilde{\sigma}}([\varepsilon_n]^c) \leq 1 - r,\ \tau^{\tilde{\xi}}([\varepsilon_n]^c) \leq 1 - r\},$$

$$\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r) = \bigcup\{\varepsilon_n \in \zeta^{\tilde{\mathcal{F}}} : \alpha_n \geq \varepsilon_n,\ \tau^{\tilde{\varrho}}(\varepsilon_n) \geq r,\ \tau^{\tilde{\sigma}}(\varepsilon_n) \leq 1 - r,\ \tau^{\tilde{\xi}}(\varepsilon_n) \leq 1 - r\}.$$

Definition 9 ([24,25]). *Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}})$ be an* SVNTS *and $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$, $r \in \zeta_0$. Then,*

(1) α_n is said to be r-single-valued neutrosophic semi-open (briefly, r-SVNSO) iff

$$\alpha_n \leq C_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{int}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}([\alpha_n]_r, r), r),$$

(2) α_n is said to be r-single-valued neutrosophic β-open (briefly, r-SVNβO) iff

$$\alpha_n \leq C_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{int}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(C_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}([\alpha_n]_r, r), r), r).$$

(3) α_n *is said to be r-single-valued neutrosophic regular open (briefly, r-SVNRO) iff*

$$\alpha_n = \mathrm{int}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{C}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}([\alpha_n]_r, r), r),$$

The complement of r-SVNSO *(resp,* r-SVNβO*) are said to be* r-SVNSC *(resp,* r-SVNβC*)), respectively.*

Definition 10 ([21])**.** *Let* $(\tilde{\mathcal{F}})$ *be a non-empty set and* $v \in \tilde{\mathcal{F}}$*, let* $s \in (0,1]$*,* $t \in [0,1)$ *and* $k \in [0,1)$*, then the single-valued neutrosophic point* $x_{s,t,k}$ *in* $\tilde{\mathcal{F}}$ *given by*

$$x_{s,t,k}(v) = \begin{cases} (s,t,k), & \text{if } x = v, \\ (0,1,1), & \text{otherwise}. \end{cases}$$

We say that, $x_{s,t,p} \in \alpha_n$ *iff* $s < \tilde{\varrho}_{\alpha_n}(v)$*,* $t \geq \tilde{\sigma}_{\alpha_n}(v)$ *and* $k \geq \tilde{\tilde{\xi}}_{\alpha_n}(v)$*. We indicate the set of all single-valued neutrosophic points in* $\tilde{\mathcal{F}}$ *as* $\mathrm{P}_{x_{s,t,k}}(\tilde{\mathcal{F}})$*. A single-valued neutrosophic set* α_n *is said to be quasi-coincident with another single-valued neutrosophic set* ε_n*, denoted by* $\alpha_n q \varepsilon_n$*, if there exists an element* $v \in \tilde{\mathcal{F}}$ *such that*

$$\tilde{\varrho}_{\alpha_n}(v) + \tilde{\varrho}_{\varepsilon_n}(v) > 1, \quad \tilde{\sigma}_{\alpha_n}(v) + \tilde{\sigma}_{\varepsilon_n}(v) \leq 1, \quad \tilde{\varsigma}_{\alpha_n}(v) + \tilde{\varsigma}_{\varepsilon_n}(v) \leq 1.$$

Definition 11 ([21])**.** *A mapping* $\pounds^{\tilde{\varrho}}, \pounds^{\tilde{\sigma}}, \pounds^{\tilde{\varsigma}} : \zeta^{\tilde{\mathcal{F}}} \to \zeta$ *is called single-valued neutrosophic ideal (*SVNI*) on* $\tilde{\mathcal{F}}$ *if it satisfies the following conditions:*

($\pounds_1$) $\pounds^{\tilde{\varrho}}(\tilde{0}) = 1$ *and* $\pounds^{\tilde{\sigma}}(\tilde{0}) = \pounds^{\tilde{\varsigma}}(\tilde{0}) = 0$.
($\pounds_2$) *If* $\sigma_n \leq \gamma_n$*, then* $\pounds^{\tilde{\varrho}}(\varepsilon_n) \leq \pounds^{\tilde{\varrho}}(\alpha_n)$*,* $\pounds^{\tilde{\sigma}}(\varepsilon_n) \geq \pounds^{\tilde{\sigma}}(\alpha_n)$ *and* $\pounds^{\tilde{\varsigma}}(\varepsilon_n) \geq \pounds^{\tilde{\varsigma}}(\alpha_n)$*, for every* $\varepsilon_n, \alpha_n \in \zeta^{\tilde{\mathcal{F}}}$.
($\pounds_3$) $\pounds^{\tilde{\varrho}}(\alpha_n \cup \varepsilon_n) \geq \pounds^{\tilde{\varrho}}(\alpha_n) \cap \pounds^{\tilde{\varrho}}(\varepsilon_n)$*,* $\pounds^{\tilde{\sigma}}(\alpha_n \cup \varepsilon_n) \leq \pounds^{\tilde{\sigma}}(\alpha_n) \cup \pounds^{\tilde{\sigma}}(\varepsilon_n)$ *and* $\pounds^{\tilde{\varsigma}}(\alpha_n \cup \varepsilon_n) \leq \pounds^{\tilde{\varsigma}}(\alpha_n) \cup \pounds^{\tilde{\varsigma}}(\varepsilon_n)$*, for every* $\alpha_n, \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$.

The triable $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, \pounds^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ is called a single-valued neutrosophic ideal topological space in the Šostak sense (briefly, *SVNITS*).

Definition 12 ([21])**.** *Let* $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, \pounds^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ *be an SVNITS for each* $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$*. Then, the single-valued neutrosophic ideal open local function* $[\alpha_n]_r^{\odot}(\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, \pounds^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ *of* α_n *is the union of all single-valued neutrosophic points* $x_{s,t,k}$ *such that if* $\varepsilon_n \in Q_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(x_{s,t,k}, r)$ *and* $\pounds^{\tilde{\varrho}}(\omega_n) \geq r$*,* $\pounds^{\tilde{\sigma}}(\omega_n) \leq 1 - r$*,* $\pounds^{\tilde{\varsigma}}(\omega_n) \leq 1 - r$*, then there is at least one* $v \in \tilde{\mathcal{F}}$ *for which*

$$\tilde{\varrho}_{\alpha_n}(v) + \tilde{\varrho}_{\varepsilon_n}(\nu) - 1 > \tilde{\varrho}_{\omega_n}(v), \quad \tilde{\sigma}_{\alpha_n}(v) + \tilde{\sigma}_{\varepsilon_n}(v) - 1 \leq \tilde{\sigma}_{\omega_n}(v), \quad \tilde{\varsigma}_{\alpha_n}(v) + \tilde{\varsigma}_{\varepsilon_n}(v) - 1 \leq \tilde{\varsigma}_{\omega_n}(v)$$

Occasionally, we will write $[\alpha_n]_r^{\odot}$ for $[\alpha_n]_r^{\odot}(\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, \pounds^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ and it will have no ambiguity.

Remark 1 ([21])**.** *Let* $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, \pounds^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ *be an SVNITS and* $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$*, we can define*

$$\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) = \alpha_n \cup [\alpha_n]_r^{\odot}, \qquad \mathrm{int}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) = \alpha_n \cap [(\alpha_n^c)_r^{\odot}]^c.$$

Clearly, $\mathrm{CI}^{\odot}_{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}$ *is a single-valued neutrosophic closure operator and* $(\tau^{\tilde{\varrho}^{\odot}}(\pounds), \tau^{\tilde{\sigma}^{\odot}}(\pounds), \tau^{\tilde{\varsigma}^{\odot}}(\pounds))$ *is the single-valued neutrosophic topology generated by* $\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}$*, i.e.,*

$$\tau^{\odot}(\mathcal{I})(\alpha_n) = \bigcup\{r \mid \mathrm{CI}^{\odot}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n^c, r) = \alpha_n^c\}.$$

Definition 13 (25)**.** *An SVNS* δ *in* $\delta : \tilde{\mathcal{F}} \times \tilde{\mathcal{F}}$ *is called a single-valued neutrosophic relation (SVNR) in* $\tilde{\mathcal{F}}$*, denoted by* $\delta = \{\langle(\omega, \nu), \tilde{\varrho}_\delta(\omega, \nu), \tilde{\sigma}_\delta(\omega, \nu), \tilde{\varsigma}_\delta(\omega, \nu)\rangle | (\omega, \nu) \in \tilde{\mathcal{F}} \times \tilde{\mathcal{F}}\}$*, where* $\tilde{\varrho}_\delta : \tilde{\mathcal{F}} \times \tilde{\mathcal{F}} \Rightarrow [0,1]$*,* $\tilde{\sigma}_\delta : \tilde{\mathcal{F}} \times \tilde{\mathcal{F}} \Rightarrow [0,1]$ *and* $\tilde{\varsigma}_\delta : \tilde{\mathcal{F}} \times \tilde{\mathcal{F}} \Rightarrow [0,1]$ *denote the truth-membership function, indeterminacy membership function and falsity-membership function of* δ*, respectively. In what follows,* SVNR$(\tilde{\mathcal{F}})$ *will denote the family of all single-valued neutrosophic relations in* $\tilde{\mathcal{F}}$*.*

3. Single-Valued Neutrosophic Semi-Closure Spaces in Šostak Sense

We begin this section by defining the notion of single-valued neutrosophic semi-closure space. Some of its characteristic properties are considered. Further, we present and explore the properties and characterizations of the single-valued neutrosophic operators, namely $\beta£$-closure ($\beta£C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}$) and $\beta£$-interior ($\beta£\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}$) in the single-valued neutrosophic ideal topological space $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$.

Definition 14. *A mapping $\mathbb{SC} : \zeta^{\tilde{\mathcal{F}}} \times \zeta_0 \to \zeta^{\mathcal{F}}$ is called a single-valued neutrosophic semi-closure operator on $\mathcal{F}$ if, for every $\alpha_n, \varepsilon_n \in \zeta^{\mathcal{F}}$ and $r, s \in \zeta_0$, the following axioms are satisfied:*

($\mathbb{SC}_1$) $\mathbb{SC}(\tilde{0}), r) = \tilde{0}$,
($\mathbb{SC}_2$) $\alpha_n \leq \mathbb{SC}(\alpha_n, r)$,
($\mathbb{SC}_3$) $\mathbb{SC}(\alpha_n, r) \vee \mathbb{SC}(\varepsilon_n, r) = \mathbb{SC}(\alpha_n \vee \varepsilon_n, r)$,
($\mathbb{SC}_4$) $\mathbb{SC}(\alpha_n, s) \leq \mathbb{SC}(\alpha_n, r)$ *if* $s \leq r$,
($\mathbb{SC}_5$) $\mathbb{SC}(\mathbb{SC}(\alpha_n, s), r) = \mathbb{SC}(\alpha_n, r)$.

The pair $(X, \mathbb{SC})$ is a single-valued neutrosophic semi-closure space ($\mathcal{SVNSCS}$, $for\ short$).

If $\mathbb{SC}_1$ and $\mathbb{SC}_2$ are single-valued neutrosophic closure operators on $\mathcal{F}$. Then, $\mathbb{SC}_1$ is finer than $\mathbb{C}_2$, denoted by $\mathbb{SC}_2 \leq \mathbb{SC}_1$ iff $\mathbb{SC}_1(\alpha_n, r) \leq \mathbb{SC}_2(\alpha_n, r)$, for every $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$ and $r \in \zeta_0$.

Theorem 1. *Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNTS . Then, for any $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$ and $r \in \zeta_0$, we define an operator $\mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}} : \zeta^{\tilde{\mathcal{F}}} \times \zeta_0 \to \zeta^{\tilde{\mathcal{F}}}$ as follows:*

$$\mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, s) = \bigwedge\{\varepsilon_n \in \zeta^{\tilde{\mathcal{F}}} : \alpha_n \leq \varepsilon_n, \quad \varepsilon_n \textit{ is } \text{r-SVNSC}\}.$$

Then, $(\mathcal{F}, \mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}})$ is an $\mathcal{SVNSCS}$.

Proof. Suppose that $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ is an *SVNTS*. Then, ($\mathbb{SC}_1$), ($\mathbb{SC}_2$) and ($\mathbb{SC}_4$) follows directly from the definition of $\mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}$.

($\mathbb{SC}_3$) Since $\alpha_n, \varepsilon_n \leq \alpha_n \cup \varepsilon_n$ we obtain $\mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r) \leq \mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n \cup \varepsilon_n, r)$ and $\mathbb{SC}_{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}$ $(\alpha_n, r) \leq \mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n \cup \varepsilon_n, r)$, therefore,

$$\mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, s) \cup \mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r) \leq \mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n \cup \varepsilon_n, r).$$

Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an *SVNTS*. From ($\mathbb{SC}_2$), we have

$$\begin{aligned} \alpha_n \leq \mathbb{SC}_{\tau^{\tilde{\varrho}}}(\alpha_n, r), \quad & [\mathbb{SC}_{\tau^{\tilde{\varrho}}}(\alpha_n, r)]^c \leq C_{\tau^{\tilde{\varrho}}}(\mathrm{int}_{\tau^{\tilde{\varrho}}}([\mathbb{SC}_{\tau^{\tilde{\varrho}}}(\alpha_n, r)]^c, r), r) \\ & [\mathbb{SC}_{\tau^{\tilde{\sigma}}}(\alpha_n, r)]^c \geq C_{\tau^{\tilde{\sigma}}}(\mathrm{int}_{\tau^{\tilde{\sigma}}}([\mathbb{SC}_{\tau^{\tilde{\sigma}}}(\alpha_n, r)]^c, r), r), \\ & [\mathbb{SC}_{\tau^{\tilde{\varsigma}}}(\alpha_n, r)]^c \geq C_{\tau^{\tilde{\varsigma}}}(\mathrm{int}_{\tau^{\tilde{\varsigma}}}([\mathbb{SC}_{\tau^{\tilde{\varsigma}}}(\alpha_n, r)]^c, r), r) \end{aligned}$$

$$\begin{aligned} \varepsilon_n \leq \mathbb{SC}_{\tau^{\tilde{\varrho}}}(\varepsilon_n, r), \quad & [\mathbb{SC}_{\tau^{\tilde{\varrho}}}(\varepsilon_n, r)]^c \leq C_{\tau^{\tilde{\varrho}}}(\mathrm{int}_{\tau^{\tilde{\varrho}}}([\mathbb{SC}_{\tau^{\tilde{\varrho}}}(\varepsilon_n, r)]^c, r), r) \\ & [\mathbb{SC}_{\tau^{\tilde{\sigma}}}(\varepsilon_n, r)]^c \geq C_{\tau^{\tilde{\sigma}}}(\mathrm{int}_{\tau^{\tilde{\sigma}}}([\mathbb{SC}_{\tau^{\tilde{\sigma}}}(\varepsilon_n, r)]^c, r), r), \\ & [\mathbb{SC}_{\tau^{\tilde{\varsigma}}}(\varepsilon_n, r)]^c \geq C_{\tau^{\tilde{\varsigma}}}(\mathrm{int}_{\tau^{\tilde{\varsigma}}}([\mathbb{SC}_{\tau^{\tilde{\varsigma}}}(\varepsilon_n, r)]^c, r), r) \end{aligned}$$

It implies that $\alpha_n \cup \varepsilon_n \leq \mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) \cup \mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r)$ and

$$\begin{aligned} [\mathbb{SC}_{\tau^{\tilde{\varrho}}}(\alpha_n, r) \cup \mathbb{SC}_{\tau^{\tilde{\varrho}}}(\varepsilon_n, r)]^c \quad & = [\mathbb{SC}_{\tau^{\tilde{\varrho}}}(\alpha_n]^c \cap [\mathbb{SC}_{\tau^{\tilde{\varrho}}}(\varepsilon_n, r)]^c \\ & \leq C_{\tau^{\tilde{\varrho}}}(\mathrm{int}_{\tau^{\tilde{\varrho}}}([\mathbb{SC}_{\tau^{\tilde{\varrho}}}(\alpha_n, r)]^c, r), r) \cap C_{\tau^{\tilde{\varrho}}}(\mathrm{int}_{\tau^{\tilde{\varrho}}}([\mathbb{SC}_{\tau^{\tilde{\varrho}}}(\varepsilon_n, r)]^c, r), r) \\ & = C_{\tau^{\tilde{\varrho}}}(\mathrm{int}_{\tau^{\tilde{\varrho}}}([\mathbb{SC}_{\tau^{\tilde{\varrho}}}(\alpha_n, r)]^c \cap [\mathbb{SC}_{\tau^{\tilde{\varrho}}}(\varepsilon_n, r)]^c, r), r) \\ & = C_{\tau^{\tilde{\varrho}}}(\mathrm{int}_{\tau^{\tilde{\varrho}}}([\mathbb{SC}_{\tau^{\tilde{\varrho}}}(\alpha_n, r) \cup \mathbb{SC}_{\tau^{\tilde{\varrho}}}(\varepsilon_n, r)]^c, r), r), \end{aligned}$$

$$\begin{aligned}[\mathbb{SC}_{\tau^{\tilde{\sigma}}}(\alpha_n, r) \cup \mathbb{SC}_{\tau^{\tilde{\sigma}}}(\varepsilon_n, r)]^c &= [\mathbb{SC}_{\tau^{\tilde{\sigma}}}(\alpha_n]^c \cap [\mathbb{SC}_{\tau^{\tilde{\sigma}}}(\varepsilon_n, r)]^c \\ &\geq \mathrm{C}_{\tau^{\tilde{\sigma}}}(\mathrm{int}_{\tau^{\tilde{\sigma}}}([\mathbb{SC}_{\tau^{\tilde{\sigma}}}(\alpha_n, r)]^c, r), r) \cap \mathrm{C}_{\tau^{\tilde{\sigma}}}(\mathrm{int}_{\tau^{\tilde{\sigma}}}([\mathbb{SC}_{\tau^{\tilde{\sigma}}}(\varepsilon_n, r)]^c, r), r) \\ &= \mathrm{C}_{\tau^{\tilde{\sigma}}}(\mathrm{int}_{\tau^{\tilde{\sigma}}}([\mathbb{SC}_{\tau^{\tilde{\sigma}}}(\alpha_n, r)]^c \cap [\mathbb{SC}_{\tau^{\tilde{\sigma}}}(\varepsilon_n, r)]^c, r), r) \\ &= \mathrm{C}_{\tau^{\tilde{\sigma}}}(\mathrm{int}_{\tau^{\tilde{\sigma}}}([\mathbb{SC}_{\tau^{\tilde{\sigma}}}(\alpha_n, r) \cup \mathbb{SC}_{\tau^{\tilde{\sigma}}}(\varepsilon_n, r)]^c, r), r)\end{aligned}$$

$$\begin{aligned}[\mathbb{SC}_{\tau^{\tilde{\xi}}}(\alpha_n, r) \cup \mathbb{SC}_{\tau^{\tilde{\xi}}}(\varepsilon_n, r)]^c &= [\mathbb{SC}_{\tau^{\tilde{\xi}}}(\alpha_n]^c \cap [\mathbb{SC}_{\tau^{\tilde{\xi}}}(\varepsilon_n, r)]^c \\ &\geq \mathrm{C}_{\tau^{\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\xi}}}([\mathbb{SC}_{\tau^{\tilde{\xi}}}(\alpha_n, r)]^c, r), r) \cap \mathrm{C}_{\tau^{\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\xi}}}([\mathbb{SC}_{\tau^{\tilde{\xi}}}(\varepsilon_n, r)]^c, r), r) \\ &= \mathrm{C}_{\tau^{\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\sigma}}}([\mathbb{SC}_{\tau^{\tilde{\xi}}}(\alpha_n, r)]^c \cap [\mathbb{SC}_{\tau^{\tilde{\xi}}}(\varepsilon_n, r)]^c, r), r) \\ &= \mathrm{C}_{\tau^{\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\xi}}}([\mathbb{SC}_{\tau^{\tilde{\xi}}}(\alpha_n, r) \cup \mathbb{SC}_{\tau^{\tilde{\sigma}}}(\varepsilon_n, r)]^c, r), r)\end{aligned}$$

Hence, $\mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, s) \cup \mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\varepsilon_n, r) \geq \mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n \cup \varepsilon_n, r)$; therefore,

$$\mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, s) \cup \mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\varepsilon_n, r) = \mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n \cup \varepsilon_n, r).$$

($\mathbb{SC}_5$) Suppose that there exists $r \in \zeta_0$, $\alpha_n \in \zeta^{\mathcal{F}}$ and $\kappa \in \mathcal{F}$ such that

$$\begin{aligned}&\tau^{\tilde{\varrho}}_{\mathbb{SC}_{\tau^{\tilde{\varrho}}}(\mathbb{SC}_{\tau^{\tilde{\varrho}}}(\alpha_n,r),r)}(v) > \tau^{\tilde{\varrho}}_{\mathbb{SC}_{\tau^{\tilde{\varrho}}}(\alpha_n,r)}(v).\\ &\tau^{\tilde{\sigma}}_{\mathbb{SC}_{\tau^{\tilde{\sigma}}}(\mathbb{SC}_{\tau^{\tilde{\sigma}}}(\alpha_n,r),r)}(v) \leq \tau^{\tilde{\sigma}}_{\mathbb{SC}_{\tau^{\tilde{\sigma}}}(\alpha_n,r)}(v).\\ &\tau^{\tilde{\xi}}_{\mathbb{SC}_{\tau^{\tilde{\xi}}}(\mathbb{SC}_{\tau^{\tilde{\xi}}}(\alpha_n,r),r)}(v) \leq \tau^{\tilde{\xi}}_{\mathbb{SC}_{\tau^{\tilde{\xi}}}(\alpha_n,r)}(v).\end{aligned}$$

By the definition of $\mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}$, there exists an $\alpha_n \in \zeta^{\mathcal{F}}$ with $\pi_n \geq \alpha_n$ and π_n that is *r-SVNSC* such that

$$\begin{aligned}&\tau^{\tilde{\varrho}}_{\mathbb{SC}_{\tau^{\tilde{\varrho}}}(\mathbb{SC}_{\tau^{\tilde{\varrho}}}(\alpha_n,r),r)}(\kappa) > \tau^{\tilde{\varrho}}_{\pi_n}(\kappa) \geq \tau^{\tilde{\varrho}}_{\mathbb{SC}_{\tau^{\tilde{\varrho}}}(\alpha_n,r)}(\kappa).\\ &\tau^{\tilde{\sigma}}_{\mathbb{SC}_{\tau^{\tilde{\sigma}}}(\mathbb{SC}_{\tau^{\tilde{\sigma}}}(\alpha_n,r),r)}(\kappa) \leq \tau^{\tilde{\sigma}}_{\pi_n}(\kappa) < \tau^{\tilde{\sigma}}_{\mathbb{SC}_{\tau^{\tilde{\sigma}}}(\alpha_n,r)}(\kappa).\\ &\tau^{\tilde{\xi}}_{\mathbb{SC}_{\tau^{\tilde{\xi}}}(\mathbb{SC}_{\tau^{\tilde{\xi}}}(\alpha_n,r),r)}(\kappa) \leq \tau^{\tilde{\xi}}_{\pi_n}(\kappa) < \tau^{\tilde{\xi}}_{\mathbb{SC}_{\tau^{\tilde{\xi}}}(\alpha_n,r)}(\kappa).\end{aligned}$$

Since $\mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, s) \leq \pi_n$ and π_n is *r-SVNSC*, by the definition of $\mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}})$, we have

$$\tau^{\tilde{\varrho}}_{\mathbb{SC}_{\tau^{\tilde{\varrho}}}(\mathbb{SC}_{\tau^{\tilde{\varrho}}}(\alpha_n,r),r)}(\kappa) \leq \tau^{\tilde{\varrho}}_{\pi_n}(\kappa), \qquad \tau^{\tilde{\sigma}}_{\mathbb{SC}_{\tau^{\tilde{\sigma}}}(\mathbb{SC}_{\tau^{\tilde{\sigma}}}(\alpha_n,r),r)}(\kappa) > \tau^{\tilde{\sigma}}_{\pi_n}(\kappa),$$

$$\tau^{\tilde{\xi}}_{\mathbb{SC}_{\tau^{\tilde{\sigma}}}(\mathbb{SC}_{\tau^{\tilde{\xi}}}(\alpha_n,r),r)}(\kappa) > \tau^{\tilde{\xi}}_{\pi_n}(\kappa).$$

It is a contradiction. Thus, $\mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r), r) = \mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r)$. Hence, $\mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}$ is a single-valued neutrosophic semi-closure operator on $\mathcal{F}$. □

Theorem 2. *Let $(\mathcal{F}, \mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}})$ be an SVNSCS and $\alpha_n \in \zeta^{\mathcal{F}}$. Define the mapping $\tau_{\mathbb{SC}}^{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}} : \zeta^{\mathcal{F}} \to \zeta$ on $\mathcal{F}$ by*

$$\begin{aligned}\tau^{\tilde{\varrho}}_{\mathbb{SC}}(\alpha_n) &= \bigcup\{r \in \zeta_0 \mid \mathbb{SC}_{\tau^{\varrho}}([\alpha_n]^c, r) = [\alpha_n]^c\},\\ \tau^{\tilde{\sigma}}_{\mathbb{SC}}(\alpha_n) &= \bigcap\{1 - r \in \zeta_0 \mid \mathbb{SC}_{\tau^{\sigma}}([\alpha_n]^c, r) = [\alpha_n]^c\},\\ \tau^{\tilde{\xi}}_{\mathbb{SC}}(\alpha_n) &= \bigcap\{1 - r \in \zeta_0 \mid \mathbb{SC}_{\tau^{\xi}}([\alpha_n]^c, r) = [\alpha_n]^c\},\end{aligned}$$

Then,

(1) $\tau_{\mathbb{SC}}^{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}$ is an SVNTS on $\mathcal{F}$;

(2) $\mathbb{SC}_{\mathbb{SC}}^{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}$ is finer than $\mathbb{SC}$.

Proof. (SVNT1) Let $(\mathcal{F}, \mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}})$ be an *SVNSCS*. Since $\mathbb{SC}(\tilde{0}, r) = \tilde{0}$ and $\mathbb{SC}(\tilde{1}, r) = \tilde{1}$ for every $r \in \zeta_0$,

(SVNT2) Let $(\mathcal{F}, \mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}})$ be an *SVNSCS*. Suppose that there exists $[\alpha_n]_1, [\alpha_n]_2 \in \zeta^{\mathcal{F}}$ such that

$$\tau^{\tilde{\varrho}}_{\mathbb{SC}}([\alpha_n]_1 \cap [\alpha_n]_2) < \tau^{\tilde{\varrho}}_{\mathbb{SC}}([\alpha_n]_1) \cap \tau^{\tilde{\varrho}}_{\mathbb{SC}}([\alpha_n]_2), \quad \tau^{\tilde{\sigma}}_{\mathbb{SC}}([\alpha_n]_1 \cap [\alpha_n]_2) > \tau^{\tilde{\sigma}}_{\mathbb{SC}}([\alpha_n]_1) \cup \tau^{\tilde{\sigma}}_{\mathbb{SC}}([\alpha_n]_2),$$
$$\tau^{\tilde{\xi}}_{\mathbb{SC}}([\alpha_n]_1 \cap [\alpha_n]_2) > \tau^{\tilde{\xi}}_{\mathbb{SC}}([\alpha_n]_1) \cup \tau^{\tilde{\xi}}_{\mathbb{SC}}([\alpha_n]_2)$$

There exists $r \in \zeta_0$ such that

$$\tau^{\tilde{\varrho}}_{\mathbb{SC}}([\alpha_n]_1 \cap [\alpha_n]_2) < r < \tau^{\tilde{\varrho}}_{\mathbb{SC}}([\alpha_n]_1) \cap \tau^{\tilde{\varrho}}_{\mathbb{SC}}([\alpha_n]_2), \quad \tau^{\tilde{\sigma}}_{\mathbb{SC}}([\alpha_n]_1 \cap [\alpha_n]_2) > 1 - r > \tau^{\tilde{\sigma}}_{\mathbb{SC}}([\alpha_n]_1) \cup \tau^{\tilde{\sigma}}_{\mathbb{SC}}([\alpha_n]_2),$$
$$\tau^{\tilde{\xi}}_{\mathbb{SC}}([\alpha_n]_1 \cap [\alpha_n]_2) > 1 - r > \tau^{\tilde{\xi}}_{\mathbb{SC}}([\alpha_n]_1) \cup \tau^{\tilde{\xi}}_{\mathbb{SC}}([\alpha_n]_2)$$

For each $i \in \{1, 2\}$, there exists $r \in \zeta_0$ with $\mathbb{SC}([\alpha_n]_i, r_i) = [\alpha_n]_i$ such that

$$r < r_i \leq \tau^{\tilde{\varrho}}_{\mathbb{SC}}([\alpha_n]_i), \quad \tau^{\tilde{\sigma}}_{\mathbb{SC}}([\alpha_n]_i) \leq 1 - r_i < 1 - r, \quad \tau^{\tilde{\xi}}_{\mathbb{SC}}([\alpha_n]_i) \leq 1 - r_i < 1 - r.$$

In addition, since $\mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}([\alpha_n]_i, r) = [\alpha_n]_i$ by $\mathbb{SC}_2$ and $\mathbb{SC}_4$ of Definition 13, for any $i \in \{1, 2\}$,

$$\mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}([\alpha_n]_1 \cup [\alpha_n]_2, r) = [\alpha_n]_1 \cup [\alpha_n]_2$$

It follows that $\tau^{\tilde{\varrho}}_{\mathbb{SC}}([\alpha_n]_1 \cap [\alpha_n]_2) \geq r$, $\tau^{\tilde{\sigma}}_{\mathbb{SC}}([\alpha_n]_1 \cap [\alpha_n]_2) \leq 1 - r$ and $\tau^{\tilde{\xi}}_{\mathbb{SC}}([\alpha_n]_1 \cap [\alpha_n]_2) \leq 1 - r$. It is a contradiction. Thus, for every $\alpha_n, \varepsilon_n \in \zeta^{\mathcal{F}}$, $\tau^{\tilde{\varrho}}_{\mathbb{SC}}(\alpha_n \cap \varepsilon_n) \geq \tau^{\tilde{\varrho}}_{\mathbb{SC}}(\alpha_n) \cap \tau^{\tilde{\varrho}}_{\mathbb{SC}}(\varepsilon_n)$, $\tau^{\tilde{\sigma}}_{\mathbb{SC}}(\alpha_n \cap \varepsilon_n) \leq \tau^{\tilde{\sigma}}_{\mathbb{SC}}(\alpha_n) \cup \tau^{\tilde{\sigma}}_{\mathbb{SC}}(\varepsilon_n)$ and $\tau^{\tilde{\xi}}_{\mathbb{SC}}(\alpha_n \cap \varepsilon_n) \leq \tau^{\tilde{\xi}}_{\mathbb{SC}}(\alpha_n) \cup \tau^{\tilde{\xi}}_{\mathbb{SC}}(\varepsilon_n)$.

(SVNT3) Suppose that there exists $\alpha_n = \bigcup_{i \in \zeta}, [\alpha_n]_i \in \zeta^{\mathcal{F}}$ such that

$$\tau^{\tilde{\varrho}}_{\mathbb{SC}}(\alpha_n) < \bigcup_{i \in I} \tau^{\tilde{\varrho}}_{\mathbb{SC}}([\alpha_n]_i), \quad \tau^{\tilde{\sigma}}_{\mathbb{SC}}(\alpha_n) > \bigcup_{i \in I} \tau^{\tilde{\sigma}}_{\mathbb{SC}}([\alpha_n]_i), \quad \tau^{\tilde{\xi}}_{\mathbb{SC}}(\alpha_n) > \bigcup_{i \in I} \tau^{\tilde{\xi}}_{\mathbb{SC}}([\alpha_n]_i).$$

There exists $r_0 \in \zeta_0$ such that

$$\tau^{\tilde{\varrho}}_{\mathbb{SC}}(\alpha_n) < r_0 < \bigcup_{i \in I} \tau^{\tilde{\varrho}}_{\mathbb{SC}}([\alpha_n]_i), \quad \tau^{\tilde{\sigma}}_{\mathbb{SC}}(\alpha_n) > 1 - r_0 > \bigcup_{i \in I} \tau^{\tilde{\sigma}}_{\mathbb{SC}}([\alpha_n]_i), \quad \tau^{\tilde{\xi}}_{\mathbb{SC}}(\alpha_n)) > 1 - r_0 > \bigcup_{i \in I} \tau^{\tilde{\xi}}_{\mathbb{SC}}([\alpha_n]_i).$$

For every $i \in I$, there exists $\mathbb{SC}([\alpha_n]^c_i, r_i) = [\alpha_n]^c_i$ and $r_i \in \zeta_0$ such that

$$r_0 < r_i \leq \tau^{\tilde{\varrho}}_{\mathbb{SC}}([\alpha_n]_i), \quad 1 - r_0 > 1 - r_i \geq \tau^{\tilde{\sigma}}_{\mathbb{SC}}([\alpha_n]_i), \quad 1 - r_i > 1 - r_0 \geq \tau^{\tilde{\xi}}_{\mathbb{SC}}([\alpha_n]_i).$$

In addition, since $\mathbb{SC}([\alpha_n]^c, r_0) \leq \mathbb{SC}([\alpha_n]^c_i, r_i) = [\alpha_n]^c_i$, by $\mathbb{SC}_2$ of Definition 13,

$$\mathbb{SC}([\alpha_n]^c_i, r_0) = [\alpha_n]^c_i.$$

It implies, for all $i \in I$,

$$\mathbb{SC}(\alpha_n]^c, s_0) \leq \mathbb{SC}([\alpha_n]^c_i, s_0) = [\alpha_n]^c_i.$$

It follows that

$$\mathbb{SC}([\alpha_n]^c, r_0) \leq \bigcap_{i \in J} ([\alpha_n]^c_i) = [\alpha_n]^c.$$

Thus, $\mathbb{SC}([\alpha_n]^c, r_0) = [\alpha_n]^c$, that is, $\tau^{\tilde{\varrho}}_{\mathbb{SC}}(\alpha_n) \geq r_0$, $\tau^{\tilde{\sigma}}_{\mathbb{SC}}(\alpha_n) \leq 1 - r_0$ and $\tau^{\tilde{\xi}}_{\mathbb{SC}}(\alpha_n) \leq 1 - r_0$. It is a contradiction. Hence, $\tau^{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}_{\mathbb{SC}}$ is an *SVNTS* on $\mathcal{F}$.

Since $\alpha_n \leq \mathbb{SC}(\alpha_n, r)$,

$$\tau^{\tilde{\varrho}}_{\mathbb{SC}}([\mathbb{SC}(\alpha_n, r)]^c) \geq r,\ \tau^{\tilde{\sigma}}_{\mathbb{SC}}([\mathbb{SC}(\alpha_n, r)]^c)) \leq 1 - r,\ \tau^{F}_{\mathbb{SC}}([\mathbb{SC}(\alpha_n, r)]^c) \leq 1 - r.$$

From $\mathbb{SC}_5$ of Definition 9, we have $\mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}_{\mathbb{SC}}}(\alpha_n, r) \leq \mathbb{SC}(\alpha_n, r)$. Thus, $\mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}_{\mathbb{SC}}}$ is finer than $\mathbb{SC}$. □

Example 1. *Let $\mathcal{F} = \{a, b\}$. Define $\varepsilon_n, \pi_n \in \zeta^{\mathcal{F}}$ as follows:*

$$\varepsilon_n = \langle (0.1, 0.1), (0.3, 0.3), (0.3, 0.3) \rangle; \pi_n = \langle (0.4, 0.4), (0.1, 0.1), (0.1, 0.1) \rangle.$$

We define the mapping $\mathbb{SC} : \zeta^{\mathcal{F}} \times \zeta_0 \to \zeta^{\mathcal{F}}$ as follows:

$$\mathbb{SC}(\alpha_n, s) = \begin{cases} \tilde{0}, & \textit{if} \quad \alpha_n = \tilde{0}, \quad s \in \zeta_0, \\ \varepsilon_n \cap \pi_n, & \textit{if} \quad 0 \neq \alpha_n \leq \varepsilon_n \cap \pi_n, \ \ 0 < r < \frac{1}{2}, \\ \varepsilon_n, & \textit{if} \quad \alpha_n \leq \varepsilon_n, \alpha_n \nleq \pi_n, \quad 0 < r < \frac{1}{2}, \\ & \quad \textit{or}\ 0 \neq \alpha_n \leq \varepsilon_n \quad \frac{1}{2} < r < \frac{2}{3}, \\ \pi_n, & \textit{if} \quad \alpha_n \leq \pi_n, \alpha_n \nleq \varepsilon_n, \ \ 0 < r < \frac{1}{2}, \\ \varepsilon_n \cup \pi_n, & \textit{if} \quad 0 \neq \alpha_n \leq \varepsilon_n \cup \pi_n, \ 0 < r < \frac{1}{2}, \\ \tilde{1}, & \quad \textit{otherwise}. \end{cases}$$

Then, $\mathbb{SC}$ is a single-valued neutrosophic closure operator.
From Theorem 2, we have a single-valued neutrosophic topology $(\tau^{\tilde{\varrho}}_{\mathbb{SC}}, \tau^{\tilde{\sigma}}_{\mathbb{SC}}, \tau^{\tilde{\varsigma}}_{\mathbb{SC}})$ on $\mathcal{F}$ as follows:

$$\tau^{\tilde{\varrho}}_{\mathbb{SC}}(\alpha_n) = \begin{cases} 1, & \textit{if} \ \ \alpha_n = \tilde{1} \ \textit{or} \ \tilde{0}, \\ \frac{2}{3}, & \textit{if} \ \ \alpha_n = [\varepsilon_n]^c, \\ \frac{1}{2}, & \textit{if} \ \ \alpha_n = [\pi_n]^c, \\ \frac{1}{2}, & \textit{if} \ \ \alpha_n = [\varepsilon_n]^c \cup [\pi_n]^c, \\ \frac{1}{2}, & \textit{if} \ \ \alpha_n = [\varepsilon_n]^c \cap [\pi_n]^c, \\ 0, & \quad \textit{otherwise}. \end{cases}$$

$$\tau^{\tilde{\sigma}}_{\mathbb{SC}}(\alpha_n) = \begin{cases} 0, & \textit{if} \ \ \alpha_n = \tilde{1} \ \textit{or} \ \tilde{0}, \\ \frac{1}{3}, & \textit{if} \ \ \alpha_n = [\varepsilon_n]^c, \\ \frac{1}{2}, & \textit{if} \ \ \alpha_n = [\pi_n]^c, \\ \frac{1}{2}, & \textit{if} \ \ \alpha_n = [\varepsilon_n]^c \cup [\pi_n]^c, \\ \frac{1}{2}, & \textit{if} \ \ \alpha_n = [\varepsilon_n]^c \cap [\pi_n]^c, \\ 1, & \quad \textit{otherwise}. \end{cases}$$

$$\tau^{\tilde{\varsigma}}_{\mathbb{SC}}(\alpha_n) = \begin{cases} 0, & \textit{if} \ \ \alpha_n = \tilde{1} \ \textit{or} \ \tilde{0}, \\ \frac{1}{3}, & \textit{if} \ \ \alpha_n = [\varepsilon_n]^c, \\ \frac{1}{2}, & \textit{if} \ \ \alpha_n = [\pi_n]^c, \\ \frac{1}{2}, & \textit{if} \ \ \alpha_n = [\varepsilon_n]^c \cup [\pi_n]^c, \\ \frac{1}{2}, & \textit{if} \ \ \alpha_n = [\varepsilon_n]^c \cap [\pi_n]^c, \\ 1, & \quad \textit{otherwise}. \end{cases}$$

Thus, the $\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}_{\mathbb{SC}}$ is a single-valued neutrosophic topology on $\mathcal{F}$.

Theorem 3. *Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNTS. Then, for any $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$ and $r \in \zeta_0$, we define an operator $\mathbb{SINT}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}} : \zeta^{\tilde{\mathcal{F}}} \times \zeta_0 \to \zeta^{\tilde{\mathcal{F}}}$ as follows:*

$$\mathbb{SINT}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, s) = \bigwedge \{\varepsilon_n \in \zeta^{\tilde{\mathcal{F}}} : \alpha_n \geq \varepsilon_n, \quad \varepsilon_n \ \textit{is}\ \text{r-SVNSO}\}.$$

For each $\alpha_n, \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$ and $r, s \in \zeta_0$ the operator $\mathbb{SINT}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}$ satisfies the following conditions:

(1) $\mathbb{SINT}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\tilde{1}, r) = \tilde{1}$,
(2) $\mathbb{SINT}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) \leq \alpha_n$,
(3) $\mathbb{SINT}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) \wedge \mathbb{SINT}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) = \mathbb{SINT}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n \wedge \varepsilon_n, r)$,
(4) $\mathbb{SINT}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) \geq \mathbb{SINT}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, s)$ *if* $r \leq s$,
(5) $\mathbb{SINT}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathbb{SINT}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r) = \mathbb{SINT}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)$.

Proof. From the Definition of $\mathbb{SINT}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}$, the proof can be performed □

Definition 15. *Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNTS, $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$, $x_{s,t,k} \in \mathrm{P}_{s,t,k}(\tilde{\mathcal{F}})$ and $r \in \zeta_0$. Then,*

(1) α_n is called r-single-valued neutrosophic $Q_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}$-neighborhood of $x_{s,t,k}$ if $x_{s,t,k} q \alpha_n$ with

$$\tau^{\tilde{\varrho}}(\alpha_n) \geq r,\ \tau^{\tilde{\sigma}}(\alpha_n) \leq 1 - r,\ \tau^{\tilde{\varsigma}}(\alpha_n) \leq 1 - r,$$

*(2) $x_{s,t,k}$ α_n is called r-single-valued neutrosophic θ-cluster point (r-θ-*cluster point*) of α_n if for any $\varepsilon_n \in Q_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(x_{s,t,k}, r)$, we have $\alpha_n q \mathrm{C}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r)$,*

(3) r-θ-closure operator is a mapping $\mathrm{CI}^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}} : \zeta^{\tilde{\mathcal{F}}} \times \zeta_0 \to \zeta^{\tilde{\mathcal{F}}}$ defined as:

$$\mathrm{C}^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) = \bigvee\{x_{s,t,k} \in P_{s,t,k}(\tilde{\mathcal{F}}) : x_{s,t,k}\ is\ an\ r\text{-}\theta\text{-}cluster\ point\ of\ \alpha_n\},$$

(4) α_n is said to be r-θ-closed iff $\mathrm{CI}^{\theta\pounds}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) = \alpha_n$. We define

$$\Theta^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) = \bigwedge\{\varepsilon_n |\ \alpha_n \leq \varepsilon_n,\ \varepsilon_n = \mathrm{CI}^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r)\}.$$

Theorem 4. *Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNTS. For $r \in \zeta_0$ and $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$. The following properties hold:*

(1) If $\alpha_n \leq \mathrm{C}^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)$,
(2) If $\alpha_n \leq \varepsilon_n$, then $\mathrm{C}^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) \leq \mathrm{C}^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r)$
(3) $\mathrm{C}^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) = \bigwedge\{\varepsilon_n |\ \alpha_n \leq \mathrm{int}^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r),\ \tau^{\tilde{\varrho}}([\varepsilon_n]^c) \geq r,\ \tau^{\tilde{\sigma}}([\varepsilon_n]^c) \leq 1 - r,\ \tau^{\tilde{\varsigma}}([\varepsilon_n]^c) \leq 1 - r\}$.
(4) $\Theta^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) = \mathrm{CI}^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\Theta^{\delta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r)$,
(5) $\Theta^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)$ is r-θ-closed,
(6) $\mathrm{CI}^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) \leq \Theta^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)$.

Proof. (1) and (2) are easily proved from Definition 14.

(3) $\gamma = \bigwedge\{\varepsilon_n |\ \alpha_n \leq \mathrm{int}^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r),\ \tau^{\tilde{\varrho}}([\varepsilon_n]^c) \geq r,\ \tau^{\tilde{\sigma}}([\varepsilon_n]^c) \leq 1 - r,\ \tau^{\tilde{\varsigma}}([\varepsilon_n]^c) \leq 1 - r\}$. Suppose that $\mathrm{C}^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) \not\geq \gamma$, then there exists $v \in \mathcal{F}$ and $s, t, k \in \zeta_0$ such that

$$\begin{aligned} &\tau^{\tilde{\varrho}}_{\mathrm{C}^{\theta}_{\tilde{\tau}^{\tilde{\varrho}}}(\alpha_n, r)}(v) < s \leq \tau^{\tilde{\varrho}}_{\gamma}(v). \\ &\tau^{\tilde{\sigma}}_{\mathrm{C}^{\theta}_{\tilde{\tau}^{\tilde{\sigma}}}(\alpha_n, r)}(v) > t \geq \tau^{\tilde{\sigma}}_{\gamma}(v) \\ &\tau^{\tilde{\xi}}_{\mathrm{C}^{\theta}_{\tilde{\tau}^{\tilde{\xi}}}(\alpha_n, r)}(v) > k \geq \tau^{\tilde{\xi}}_{\gamma}(v). \end{aligned} \tag{1}$$

Then $x_{S,t,K}$ is not *r-θ-cluster point* of α_n. So, there exists $\varepsilon_n \in Q_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(x_{s,t,k}, r)$, and $\alpha_n \leq [\mathrm{C}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}([\varepsilon_n, r)]^c$. Thus, $\alpha_n \leq [\mathrm{C}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r)]^c = \mathrm{int}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}([\varepsilon_n]^c, r)$ and $\tau^{\tilde{\varrho}}(\varepsilon_n) \geq r$, $\tau^{\tilde{\sigma}}(\varepsilon_n) \leq 1 - r$, $\tau^{\tilde{\varsigma}}(\varepsilon_n) \leq 1 - r$. Hence,

$$\tau^{\tilde{\varrho}}_{\gamma}(v) \leq \tau^{\tilde{\varrho}}_{[\varepsilon_n)]^c}(v) < s,\ \ \tau^{\tilde{\sigma}}_{\gamma}(v) \geq \tau^{\tilde{\varrho}}_{[\varepsilon_n)]^c}(v) > t,\ \ \tau^{\tilde{\xi}}_{\gamma}(v) \geq \tau^{\tilde{\xi}}_{[\varepsilon_n)]^c}(v) > k.$$

It is a contradiction for Equation (1). Thus $\mathrm{C}^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) \geq \gamma$.

Suppose that $C^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) \not\leq \gamma$, then there exists *r-$\theta$-cluster point* of $y_{s,t,k} \in P_{x_{s,t,k}}(\tilde{\mathcal{F}})$ of α_n such that

$$\tau^{\tilde{\varrho}}_{C^{\theta}_{\tilde{\tau}^{\tilde{\varrho}}}(\alpha_n, r)}(y) > s \geq \tau^{\tilde{\varrho}}_{\gamma}(y).$$

$$\tau^{\tilde{\sigma}}_{C^{\theta}_{\tilde{\tau}^{\tilde{\sigma}}}(\alpha_n, r)}(y) < t \leq \tau^{\tilde{\sigma}}_{\gamma}(y) \tag{2}$$

$$\tau^{\tilde{\varsigma}}_{C^{\theta}_{\tilde{\tau}^{\tilde{\varsigma}}}(\alpha_n, r)}(y) < k \leq \tau^{\tilde{\varsigma}}_{\gamma}(v).$$

By definition of γ, there exists $\varepsilon_n \in \zeta^{tilde\mathcal{F}}$ with $\alpha_n \leq \mathrm{int}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r)$ and $\tau^{\tilde{\varrho}}([\varepsilon_n]^c) \geq r$, $\tau^{\tilde{\sigma}}([\varepsilon_n]^c) \leq 1 - r$, $\tau^{\tilde{\varsigma}}([\varepsilon_n]^c) \leq 1 - r$ such that

$$\tau^{\tilde{\varrho}}_{C^{\theta}_{\tilde{\tau}^{\tilde{\varrho}}}(\alpha_n, r)}(y) > s > \tau^{\tilde{\varrho}}_{\varepsilon_n}(y) \geq \tau^{\tilde{\varrho}}_{\gamma}(y),$$

$$\tau^{\tilde{\sigma}}_{C^{\theta}_{\tilde{\tau}^{\tilde{\sigma}}}(\alpha_n, r)}(y) < t < \tau^{\tilde{\sigma}}_{\varepsilon_n}(y) \leq \tau^{\tilde{\sigma}}_{\gamma}(y),$$

$$\tau^{\tilde{\varsigma}}_{C^{\theta}_{\tilde{\tau}^{\tilde{\varsigma}}}(\alpha_n, r)}(y) < k < \tau^{\tilde{\varsigma}}_{\varepsilon_n}(y) \leq \tau^{\tilde{\varsigma}}_{\gamma}(v).$$

Then $[\varepsilon_n]^c \in Q_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(y_{s,t,k}, r)$. Furthermore, $\alpha_n \leq \mathrm{int}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r) = [C_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}([\varepsilon_n]^c, r)]^c$ which implies $\alpha_n \overline{q} C_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}([\varepsilon_n]^c, r)$. Hence $y_{s,t,k}$ is not an *r-θ-cluster point* of α_n. It is a contradiction for Equation (2). Thus $C^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) \leq \gamma$.

(4) Let $\alpha_n \leq [\varepsilon_n]_i = CI^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}([\varepsilon_n]_i, r)$ for each $i \in \Gamma$. Then

$$\bigwedge_{i\in\Gamma} [\varepsilon_n]_i \leq C^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\bigwedge_{i\in\Gamma} [\varepsilon_n]_i, r) \leq C^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}([\varepsilon_n]_i, r) = [\varepsilon_n]_i.$$

So, $\bigwedge_{i\in\Gamma} [\varepsilon_n]_i \leq C^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\bigwedge_{i\in\Gamma} [\varepsilon_n]_i, r)$. Hence, $\Theta^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) = CI^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\Theta^{\theta\pounds}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r)$.

(5) It is directly obtained from (4).

(6) Since $\alpha_n \leq \Theta^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)$, by (5), we have $C^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) \leq C^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\Theta^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r) = \Theta^{\theta}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)$. □

Definition 16. *Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, \pounds^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNITS and $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$, $r \in \zeta_0$. Then, α_n is said to be r-single-valued neutrosophic £β-open (briefly, r-SVN£βO) iff $\alpha_n \leq C_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(C_{\tilde{\tau}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}([\alpha_n]_r, r), r), r)$. The complement of r-SVNβO is said to be r-SVNβC.*

Remark 2. *Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, \pounds^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNITS. For $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$, $x_{s,t,k} \in P_{x_{s,t,k}}(\tilde{\mathcal{F}})$ and $r \in \zeta_0$. Then, α_n is called r-open $Q_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}_{\pounds\beta}}$-neighborhood of $x_{s,t,k}$ if $x_{s,t,k} q \alpha_n$ with α_n is r-SVN£βO set, denoted as:*

$$Q_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}_{\pounds\beta}}(x_{s,t,k}, r) = \{\alpha_n \in \zeta^{\mathcal{F}} | x_{s,t,k} q \alpha_n \leq C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(CI^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r), r)\}.$$

Definition 17. *Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, \pounds^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNITS. For each $\alpha_n, \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$ and $r \in \zeta_0$, we define the operators $\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}, \beta\pounds\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}} : \zeta^{\mathcal{F}} \times \zeta_0 \to \zeta^{\mathcal{F}}$ as follows:*

$$\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) = \bigwedge\{\varepsilon_n \in \zeta^{\mathcal{F}} | \alpha_n \leq \varepsilon_n, \varepsilon_n \textit{ is } \text{r-SVN£}\beta\text{C}\},$$
$$\beta\pounds\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) = \bigvee\{\varepsilon_n \in \zeta^{\mathcal{F}} | \varepsilon_n \leq \alpha_n, \varepsilon_n \textit{ is } \text{r-SVN£}\beta\text{O}\}.$$

Theorem 5. *Let $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, \pounds^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be an SVNITS and $\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}, \beta\pounds\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}} : \zeta^{\mathcal{F}} \times \zeta_0 \to \zeta^{\mathcal{F}}$. Then,*

(1) $\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\tilde{0}, r) = \tilde{0}$,
(2) $\alpha_n \leq \beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r)$,
(3) $\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}([\alpha_n^c, r) = [\beta\pounds \mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r)]^c$,
(4) $\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r) \leq \beta\pounds\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r) \leq \alpha_n \leq \beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r) \leq C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r)$,
(5) $\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r) \vee \beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\varepsilon_n, r) \leq \beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n \vee \varepsilon_n, r)$,
(6) α_n *is* r-SVN£βC *iff* $\alpha_n = \beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r)$,
(7) $\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r), r) = \beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r)$,
(8) $\beta\pounds\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r) = \alpha_n \wedge C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r), r), r)$.

Proof. (1), (2), (3), (4), (5) and (6) are easily proved from the definitions of $\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}$ and $\beta\pounds\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}$.

(7) From (4) we only show $\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r), r) \geq \beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r)$. Suppose that

$$\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r), r) \not\geq \beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r),$$

there exist $v \in \mathcal{F}$ and $s, t, k \in \zeta_0$ such that

$$\begin{aligned}
&\tau^{\tilde{\varrho}}_{\beta\pounds C_{\tau^{\tilde{\varrho}}}(\alpha_n,r)}(v) < s < \tau^{\tilde{\varrho}}_{\beta\pounds C_{\tau^{\tilde{\varrho}}}(\beta\pounds C_{\tau^{\tilde{\varrho}}}(\alpha_n,r),r)}(v),\\
&\tau^{\tilde{\sigma}}_{\beta\pounds C_{\tau^{\tilde{\sigma}}}(\alpha_n,r)}(v) \geq t \geq \tau^{\tilde{\sigma}}_{\beta\pounds C_{\tau^{\tilde{\sigma}}}(\beta\pounds C_{\tau^{\tilde{\sigma}}}(\alpha_n,r),r)}(v),\\
&\tau^{\tilde{\xi}}_{\beta\pounds C_{\tau^{\tilde{\xi}}}(\alpha_n,r)}(v) \geq k \geq \tau^{\tilde{\xi}}_{\beta\pounds C_{\tau^{\tilde{\xi}}}(\beta\pounds C_{\tau^{\tilde{\xi}}}(\alpha_n,r),r)}(v),
\end{aligned} \tag{3}$$

Since $\tau^{\tilde{\varrho}}_{\beta\pounds C_{\tau^{\tilde{\varrho}}}(\alpha_n,r)}(v) < s$, $\tau^{\tilde{\sigma}}_{\beta\pounds C_{\tau^{\tilde{\sigma}}}(\alpha_n,r)}(v) \geq t$ and $\tau^{\tilde{\xi}}_{\beta\pounds C_{\tau^{\tilde{\xi}}}(\alpha_n,r)}(v) \geq k$ by definition $\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}$ there exists *r-SVN£βC* set $\varepsilon_n \in \zeta^{\mathcal{F}}$ with $\alpha_n \leq \varepsilon_n$ such that

$$\begin{aligned}
&\tau^{\tilde{\varrho}}_{\beta\pounds C_{\tau^{\tilde{\varrho}}}(\alpha_n,r)}(v) \leq \tau^{\tilde{\varrho}}_{\varepsilon_n}(v) < s,\\
&\tau^{\tilde{\sigma}}_{\beta\pounds C_{\tau^{\tilde{\sigma}}}(\alpha_n,r)}(v) \geq \tau^{\tilde{\sigma}}_{\varepsilon_n}(v) \geq t,\\
&\tau^{\tilde{\xi}}_{\beta\pounds C_{\tau^{\tilde{\xi}}}(\alpha_n,r)}(v) \geq \tau^{\tilde{\xi}}_{\varepsilon_n}(v) \geq k.
\end{aligned}$$

Since $\alpha_n \leq \varepsilon_n$, $\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r) \leq \varepsilon_n$. Again, by the definition of $\beta\mathcal{I}C_\tau$, we have $\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r), r) \leq \varepsilon_n$. Hence

$$\begin{aligned}
&\tau^{\tilde{\varrho}}_{\beta\pounds C_{\tau^{\tilde{\varrho}}}(\beta\pounds C_{\tau^{\tilde{\varrho}}}(\alpha_n,r),r)}(v) \leq \tau^{\tilde{\varrho}}_{\varepsilon_n}(v) < s,\\
&\tau^{\tilde{\sigma}}_{\beta\pounds C_{\tau^{\tilde{\sigma}}}(\beta\pounds C_{\tau^{\tilde{\sigma}}}(\alpha_n,r),r)}(v) \geq \tau^{\tilde{\sigma}}_{\varepsilon_n}(v) \geq t,\\
&\tau^{\tilde{\xi}}_{\beta\pounds C_{\tau^{\tilde{\xi}}}(\beta\pounds C_{\tau^{\tilde{\xi}}}(\alpha_n,r),r)}(v) \geq \tau^{\tilde{\xi}}_{\varepsilon_n}(v) \geq k.
\end{aligned}$$

It is a contradiction for Equation (1).

(8) Since

$$\begin{aligned}
\alpha_n \quad &\wedge C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r), r), r) \leq C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r), r), r)\\
&= C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}([\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r), r) \wedge \mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r), r)], r)\\
&= C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}([\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r) \wedge \mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r), r)], r), r)\\
&\leq C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}([\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r) \wedge C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r), r), r)], r), r)\\
&= C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}([\alpha_n \wedge C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r), r), r)], r), r), r).
\end{aligned}$$

Hence, $\alpha_n \wedge C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r), r), r)$ is an *r-SVN£βO* set contained in α_n and so $\alpha_n \wedge C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r), r), r) \leq \beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r)$. Since $\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r)$ is an *r-SVN£βO* set, we have

$$\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r) \leq C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r), r), r), r) \leq C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r), r), r).$$

So, $\alpha_n \wedge C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r), r), r) \geq \beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r)$. Hence

$$\beta\pounds\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r) = \alpha_n \wedge C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r), r), r).$$

□

4. New Terms of Single-Valued Neutrosophic Continuity

In this section, we introduce and characterize new classes of mappings called single-valued neutrosophic almost β£, faintly β£, weakly β£ and β£-continuous mappings. These findings lead to many theorems and consequences. Using these different attributes, we provide an example at the end of this section to show the difference between these kinds of mappings.

Definition 18. *Let $f : (\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}, \pounds^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}) \to (\tilde{\mathcal{G}}, \varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}})$ be a mapping. Then, f is called single-valued neutrosophic almost β£-continuous mapping (SVNAβ£C, for short) iff for each $\alpha_n \in Q_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(f(x_{s,t,k}), r)$ there exists $\varepsilon_n \in Q_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}_{\pounds\beta}}(x_{s,t,k}, r)$ such that $f(\varepsilon_n) \leq \mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r), r)$.*

Lemma 1. *For any single-valued neutrosophic set $\alpha_n \in \zeta^{\mathcal{F}}$ in an SVNTS $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}})$ and $r \in \zeta_0$, if $\tau^{\tilde{\varrho}}(\alpha_n) \geq r, \tau^{\tilde{\sigma}}(\alpha_n) \leq 1 - r, \tau^{\tilde{\xi}}(\alpha_n) \leq 1 - r$, then $\mathbb{SC}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r) = \mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r), r)$.*

Lemma 2. *For any single-valued neutrosophic set $\alpha_n \in \zeta^{\mathcal{F}}$ in an SVNTS $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}})$ and $r \in \zeta_0$, if α_n is r-SVNβO, then $C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r) = C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_N, r), r), r)$.*

Theorem 6. *Let $f : (\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}, \pounds^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}) \to (\tilde{\mathcal{G}}, \varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}})$ be a mapping for each $\alpha_n \in \zeta^{\tilde{\mathcal{G}}}$ and $r \in \zeta_0$. Then the following statements are equivalent:*

(1) f is SVNAβ£C,
(2) For every $x_{s,t,k} q C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r)), r), r), r)$,
(3) $f^{-1}(\alpha_n) \leq \beta\pounds\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r)), r)$ for every $\varphi^{\tilde{\varrho}}(\alpha_n) \geq r$, $\varphi^{\tilde{\sigma}}(\alpha_n) \leq 1 - r$, $\varphi^{\tilde{\xi}}(\alpha_n) \leq 1 - r$.

Proof. (1)⇒(2): Let $\alpha_n \in Q_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(f(x_{s,t,k}), r)$. Then by (1), there exists $\varepsilon_n \in Q_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}_{\pounds\beta}}(x_{s,t,k}, r)$ such that $f(\varepsilon_n) \leq \mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r), r)$. Since, $\varphi^{\tilde{\varrho}}(\alpha_n) \geq r$, $\varphi^{\tilde{\sigma}}(\alpha_n) \leq 1 - r$, $\varphi^{\tilde{\xi}}(\alpha_n) \leq 1 - r$. by Lemma 1, $\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r) = \mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r), r)$. Hence, $f(\varepsilon_n) \leq S\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r)$. Since ε_n is *r-SVN£βO* set,

$$\varepsilon_n \leq C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\varepsilon_n, r), r), r) \leq C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r)), r), r), r).$$

Since, $x_{s,t,k} q \varepsilon_n$ we obtain $x_{s,t,k} q C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r)), r), r), r)$.

(2)⇒(3): Let $x_{s,t,k} q f^{-1}(\alpha_n)$ and $\varphi^{\tilde{\varrho}}(\alpha_n) \geq r$, $\varphi^{\tilde{\sigma}}(\alpha_n) \leq 1 - r$, $\varphi^{\tilde{\xi}}(\alpha_n) \leq 1 - r$, by (2), we have

$$x_{s,t,k} q C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r)), r), r), r).$$

Since $x_{s,t,k} q f^{-1}(\alpha_n)$ we obtain $x_{s,t,k} q f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r))$. Hence, by Theorem 5(8),

$$\begin{aligned} x_{S,t,K} q f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r)) \quad & \wedge C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r)), r), r), r) \\ & = \beta\pounds\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r)), r). \end{aligned}$$

Thus, $x_{S,t,K} q f^{-1}(\alpha_n)$ implies $x_{S,t,K} q \beta\pounds\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)), r)$. Hence

$$f^{-1}(\alpha_n) \leq \beta\pounds\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)), r).$$

(3)⇒(1): Let $\alpha_n \in Q_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f(x_{s,t,k}), r)$. and since $f^{-1}(\alpha_n) \leq \beta\pounds\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)), r)$. Then by (3), $x_{s,t,k} q \beta\pounds\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)), r)$. Since, $\beta\pounds\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)), r)$ is *r-SVN£βO* set, $\beta\pounds\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)), r) \in Q_{\tau_{\pounds\beta}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(x_{s,t,k}, r)$. Moreover,

$$f(\beta\pounds\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)), r)) \leq f(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r))) \leq \mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r),$$

by Lemma 1, $f(\beta\pounds\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)), r)) \leq \mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r)$. □

Theorem 7. *Let $f : (\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, \pounds^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}) \to (\tilde{\mathcal{G}}, \varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ be a mapping for each $\alpha_n \in \zeta^{\tilde{\mathcal{G}}}$, $\varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$ and $r \in \zeta_0$. Then the following statements are equivalent:*

(1) f is SVNAβ£C,
(2) $f^{-1}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r))$ is an r-SVN£βO set in $\tilde{\mathcal{F}}$, for every $\varphi^{\tilde{\varrho}}(\alpha_n) \geq r$, $\varphi^{\tilde{\sigma}}(\alpha_n) \leq 1 - r$, $\varphi^{\tilde{\varsigma}}(\alpha_n) \leq 1 - r$,
(3) $f^{-1}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r))$ is an r-SVN£βC set in $\tilde{\mathcal{F}}$, for each $\varphi^{\tilde{\varrho}}([\alpha_n]^c) \geq r$, $\varphi^{\tilde{\sigma}}([\alpha_n]^c) \leq 1 - r$, $\varphi^{\tilde{\varsigma}}([\alpha_n]^c) \leq 1 - r$,
(4) $f^{-1}(\alpha_n)$ is an r-SVN£βO set in $\tilde{\mathcal{F}}$, for each r-SVNRO set $\alpha_n \in \zeta^{\tilde{\mathcal{G}}}$,
(5) $f^{-1}(\alpha_n)$ is an r-SVN£βC set in $\tilde{\mathcal{F}}$, for each r-SVNRC set $\alpha_n \in \zeta^{\tilde{\mathcal{G}}}$,
(6) For each $\alpha_n \in Q_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f(x_{s,t,k}), r)$ there exists $\varepsilon_n \in Q_{\tau_{\pounds\beta}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(x_{s,t,k}, r)$ such that $f(\varepsilon_n) \leq \mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)$,
(7) $f^{-1}(\alpha_n) \leq \beta\pounds\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)), r)$ for each $\varphi^{\tilde{\varrho}}(\alpha_n) \geq r$, $\varphi^{\tilde{\sigma}}(\alpha_n) \leq 1 - r$, $\varphi^{\tilde{\varsigma}}(\alpha_n) \leq 1 - r$,
(8) $\beta\pounds\mathrm{C}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathbb{SINT}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)), r) \leq f^{-1}(\alpha_n)$, for each $\varphi^{\tilde{\varrho}}([\alpha_n]^c) \geq r$, $\varphi^{\tilde{\sigma}}([\alpha_n]^c) \leq 1 - r$, $\varphi^{\tilde{\varsigma}}([\alpha_n]^c) \leq 1 - r$,
(9) $\beta\pounds\mathrm{C}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r)), r) \leq f^{-1}(\alpha_n)$ for each $\varphi^{\tilde{\varrho}}([\alpha_n]^c) \geq r$, $\varphi^{\tilde{\sigma}}([\alpha_n]^c) \leq 1 - r$, $\varphi^{\tilde{\varsigma}}([\alpha_n]^c) \leq 1 - r$,
(10) $\beta\pounds\mathrm{C}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\alpha_n), r) \leq f^{-1}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)$ for each r-SVNβO set $\alpha_n \in \zeta^{\tilde{\mathcal{G}}}$,
(11) $\beta\pounds\mathrm{C}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\alpha_n), r) \leq f^{-1}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)$ for each r-SVNSO set $\alpha_n \in \zeta^{\tilde{\mathcal{G}}}$.

Proof. (1)⇒(2): Let $x_{S,t,K} q f^{-1}(\alpha_n)$ and α_n is *r-SVNRO* set in $\tilde{\mathcal{G}}$. Then $\alpha_n \in Q_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f(x_{s,t,k}), r)$. Since f is *SVNAβ£C*, then there exists $\varepsilon_n \in Q_{\tau_{\pounds\beta}^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(x_{s,t,k}, r)$ such that $f(\varepsilon_n) \leq \mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r)$. So,

$$x_{s,t,k} q \varepsilon_n \leq \mathrm{C}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r), r), r) \leq \mathrm{C}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r)), r), r), r).$$

Thus, $x_{s,t.,k} q f^{-1}(\alpha_n)$ implies $x_{s,t,k} q \mathrm{C}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r)), r), r), r)$, Hence,

$$f^{-1}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r)) \leq \mathrm{C}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r)), r), r), r).$$

Therefore $f^{-1}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r))$ is an *r-SVN£βO* set in $\tilde{\mathcal{F}}$.

(2)⇒(3): Let $\varphi^{\tilde{\varrho}}([\alpha_n]^c) \geq r$, $\varphi^{\tilde{\sigma}}([\alpha_n]^c) \leq 1 - r$, $\varphi^{\tilde{\varsigma}}([\alpha_n]^c) \leq 1 - r$. Then, by (2), $f^{-1}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}([\alpha_n]^C, r), r)) = [f^{-1}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r))]^c$ is *r-SVN£βO* set. This yields $f^{-1}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r))$ to be an *r-SVN£βC* set in $\tilde{\mathcal{F}}$.

(3)⇒(4): Let α_n be an *r-SVNRO* set in $\tilde{\mathcal{G}}$. Then, $\varphi^{\tilde{\varrho}}(\alpha_n) \geq r$, $\varphi^{\tilde{\sigma}}(\alpha_n) \leq 1 - r$, $\varphi^{\tilde{\varsigma}}(\alpha_n) \leq 1 - r$. From (3), we have $f^{-1}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}([\alpha_n]^C, r), r)) = [f^{-1}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r))]^c$ is *r-SVN£βC* set. Hence, $f^{-1}(\alpha_n)$ is *r-SVN£βO* set in $\tilde{\mathcal{F}}$.

(4)⇒(5): It is easily proved from (4) and the fact that $f^{-1}([\alpha_n]^c) = [f^{-1}(\alpha_n)]^c$.

(5)⇒(6): Let $\alpha_n \in Q_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f(x_{s,t,k}), r)$. Then $x_{s,t,k} q f^{-1}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r))$ and $\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r)$ is *r-SVNRO*, which implies that $\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}([\alpha_n]^c, r), r)$ is *r-SVNRC*. By (5), $f^{-1}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}([\alpha_n]^c, r), r))$ is an *r-SVN£βC* set. Then, $f^{-1}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r))$ is *r-SVN£βO*. Put $\varepsilon_n = f^{-1}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r))$. Then $\varepsilon_n \in Q_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}_{£\beta}}(x_{s,t,k}, r)$ and

$$f(\varepsilon_n) \leq f(f^{-1}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r))) = \mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r).$$

Since $\varphi^{\tilde{\varrho}}(\alpha_n) \geq r$, $\varphi^{\tilde{\sigma}}(\alpha_n) \leq 1 - r$, $\varphi^{\tilde{\varsigma}}(\alpha_n) \leq 1 - r$ and by Lemma 1, we have

$$f(\varepsilon_n) \leq \mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r) = \mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r).$$

(6)⇒(7): Let $x_{S,t,K} q f^{-1}(\alpha_n)$ and $\varphi^{\tilde{\varrho}}(\alpha_n) \geq r$, $\varphi^{\tilde{\sigma}}(\alpha_n) \leq 1 - r$, $\varphi^{\tilde{\varsigma}}(\alpha_n) \leq 1 - r$. Then, $\alpha_n \in Q_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f(x_{s,t,k}), r)$ by (6), there exists $\varepsilon_n \in Q_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}_{£\beta}}(x_{s,t,k}, r)$ such that $f(\varepsilon_n) \leq \mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)$. Thus, $\mathcal{B} \leq f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r))$. Since ε_n is *r-SVN£βO* set,

$$x_{s,t,k} q \varepsilon_n = \beta£\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r) \leq \beta£\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)), r).$$

Thus, $x_t q f^{-1}(\alpha_n)$ implies $x_{s,t,k} q \beta£\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)), r)$. Hence,

$$f^{-1}(\alpha_n) \leq \beta£\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)), r).$$

(7)⇒(8): Let $\varphi^{\tilde{\varrho}}([\alpha_n]^c) \geq r$, $\varphi^{\tilde{\sigma}}([\alpha_n]^c) \leq 1 - r$, $\varphi^{\tilde{\varsigma}}([\alpha_n]^c) \leq 1 - r$. Then by (7),

$$f^{-1}([\alpha_n]^c) \leq \beta£\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}([\alpha_n]^c, r)), r) = [\beta£\mathrm{C}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathbb{SINT}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)), r)]^c.$$

Then, $\beta£\mathrm{C}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathbb{SINT}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)), r) \leq f^{-1}(\alpha_n)$.

(8)⇒(9): Since $\varphi^{\tilde{\varrho}}([\alpha_n]^c) \geq r$, $\varphi^{\tilde{\sigma}}([\alpha_n]^c) \leq 1 - r$, $\varphi^{\tilde{\varsigma}}([\alpha_n]^c) \leq 1 - r$, by (8) and Lemma 1, we have

$$\beta£\mathrm{C}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathbb{SINT}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)), r) = \beta£\mathrm{C}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r)), r) \leq f^{-1}(\alpha_n).$$

(9)⇒(10): Let α_n be an *r-SVNβO* set in $\mathcal{G}$ Then by Lemma 2,

$$\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) = \mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}((\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r), r)$$

and hence $\varphi^{\tilde{\varrho}}([\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n]^c, r)) \geq r$, $\varphi^{\tilde{\sigma}}([\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n]^c, r)) \leq 1 - r$, $\varphi^{\tilde{\varsigma}}([\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n]^c, r)) \leq 1 - r$ by (9), we have

$$\beta£\mathrm{C}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r), r)), r) \leq f^{-1}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)).$$

Since $\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r) = \mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}((\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r), r)$ we obtain

$$\beta£\mathrm{C}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\alpha_n), r) \leq f^{-1}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r).$$

(10)⇒(11): It is easily proved from Definition 9.

(11)⇒(1): Obvious. □

Theorem 8. *Let* $f : (\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}) \to (\tilde{\mathcal{G}}, \varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ *be a mapping for each* $\alpha_n \in \zeta^{\tilde{\mathcal{G}}}$, $\varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$ *and* $r \in \zeta_0$. *Then the following statements are equivalent:*

(1) f *is SVNAβ£C,*

(2) $\beta£\mathrm{C}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r), r), r)), r) \leq f^{-1}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r))$,

(3) $\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r),r)),r) \leq f^{-1}(\alpha_n)$, *for every* $\varphi^{\tilde{\varrho}}([\alpha_n]^c) \geq r$, $\varphi^{\tilde{\sigma}}([\alpha_n]^c) \leq 1-r$, $\varphi^{\tilde{\varsigma}}([\alpha_n]^c) \leq 1-r$,

(4) $\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r)),r) \leq f^{-1}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r))$, *for every* $\varphi^{\tilde{\varrho}}(\alpha_n) \geq r$, $\varphi^{\tilde{\sigma}}(\alpha_n) \leq 1-r$, $\varphi^{\tilde{\varsigma}}(\alpha_n]) \leq 1-r$,

(5) $f^{-1}(\alpha_n) \leq \beta\pounds\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r)),r)$, *for every* $\varphi^{\tilde{\varrho}}(\alpha_n) \geq r$, $\varphi^{\tilde{\sigma}}(\alpha_n) \leq 1-r$, $\varphi^{\tilde{\varsigma}}(\alpha_n]) \leq 1-r$,

(6) $f^{-1}(\alpha_n) \leq C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(CI^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r)),r),r),r)$, *for every* $\varphi^{\tilde{\varrho}}(\alpha_n) \geq r$, $\varphi^{\tilde{\sigma}}(\alpha_n) \leq 1-r$, $\varphi^{\tilde{\varsigma}}(\alpha_n) \leq 1-r$,

(7) $f^{-1}(\alpha_n) \leq C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(CI^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}((C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r),r)),r),r),r)$, *for every* $\varphi^{\tilde{\varrho}}(\alpha_n) \geq r$, $\varphi^{\tilde{\sigma}}(\alpha_n) \leq 1-r$, $\varphi^{\tilde{\varsigma}}(\alpha_n) \leq 1-r$.

Proof. (1)⇒(2): Let $\alpha_n \in \zeta^{\tilde{\mathcal{G}}}$ and $x_{s,t,k}q[f^{-1}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r))]^c$. Then, $f(x_{s,t,k})q[C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r))]^c$ and since $\varphi^{\tilde{\varrho}}([C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r)]^c) \geq r$, $\varphi^{\tilde{\sigma}}([C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r)]^c) \leq 1-r$, $\varphi^{\tilde{\varsigma}}([C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r)]^c) \leq 1-r$. Then, $[C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r))]^c \in Q_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f(x_{s,t,k}),r)$, By *SVNAβ£C* of $\tilde{\mathcal{F}}$, there exists $\varepsilon_n \in Q_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}_{\pounds\beta}}(x_{s,t,k},r)$ such that

$$f(\varepsilon_n) \leq \mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}([C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r)]^c,r),r) = [C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}([C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r),r),r)]^c.$$

It implies that $\varepsilon_n \leq [f^{-1}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}([C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r),r),r))]^c$. Since ε_n is *SVNAβ£O* set in $\tilde{\mathcal{F}}$

$$\begin{aligned} x_{s,t,k}q\varepsilon_n &\leq \beta\pounds\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}([C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}([C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r),r),r)]^c),r) \\ &= [\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}([C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r),r),r)),r)]^c. \end{aligned}$$

Thus, $x_{s,t,k}q[f^{-1}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r))]^c$ implies $x_{s,t,k}q[\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r),r),r)),r)]^c$. Hence,

$$[f^{-1}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r))]^c \leq [\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r),r),r)),r)]^c.$$

Thus,

$$\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}([C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r),r),r)),r) \leq f^{-1}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r)).$$

(2)⇒(3): It is trivial.

(3)⇒(4): Since $\varphi^{\tilde{\varrho}}(\alpha_n) \geq r$, $\varphi^{\tilde{\sigma}}(\alpha_n) \leq 1-r$, $\varphi^{\tilde{\varsigma}}(\alpha_n) \leq 1-r$, we have $C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r) = C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r),r),r)$ By (3), we have

$$\begin{aligned} \beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r)) &= \beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r),r)),r) \\ &\leq f^{-1}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r)). \end{aligned}$$

(4)⇒(5): Since $\varphi^{\tilde{\varrho}}(\alpha_n) \geq r$, $\varphi^{\tilde{\sigma}}(\alpha_n) \leq 1-r$, $\varphi^{\tilde{\varsigma}}(\alpha_n) \leq 1-r$, we have $\varphi^{\tilde{\varrho}}([C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r)]^c) \geq r$, $\varphi^{\tilde{\sigma}}([C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r)]^c) \leq 1-r$, $\varphi^{\tilde{\varsigma}}([C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r)]^c) \leq 1-r$. and by Lemma 1, we have $\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r) = \mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r),r)$. From (4), we have

$$\begin{aligned} [\beta\pounds\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r),r)),r)]^c &= \beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}([C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r)]^n,r)),r) \\ &\leq f^{-1}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}([C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r)]^c,r)) \\ &\leq [f^{-1}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r),r))]^c. \end{aligned}$$

It implies that $f^{-1}(\alpha_n) \leq \beta\pounds\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n,r)),r)$.

(5)⇒(6): Let $\varphi^{\tilde{\varrho}}(\alpha_n) \geq r$, $\varphi^{\tilde{\sigma}}(\alpha_n) \leq 1-r$, $\varphi^{\tilde{\xi}}(\alpha_n) \leq 1-r$. Then by (5), we have

$$\begin{aligned} f^{-1}(\alpha_n) &\leq \beta\pounds\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r)), r) \\ &\leq \mathrm{C}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\beta\pounds\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r)), r), r), r), r) \\ &\leq \mathrm{C}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r)), r), r), r). \end{aligned}$$

(6)⇒(7): It is easily proved from Lemma 1.

(7)⇒(1): Let $\alpha_n \in Q_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(f(x_{s,t,k}), r)$, then $x_{s,t,k} q f^{-1}(\alpha_n)$ and $\varphi^{\tilde{\varrho}}(\alpha_n) \geq r$, $\varphi^{\tilde{\sigma}}(\alpha_n) \leq 1-r$, $\varphi^{\tilde{\xi}}(\alpha_n) \leq 1-r$. From (7), $x_{s,t,k} q \mathrm{C}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(f^{-1}(\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r), r)), r), r), r)$. Since $\varphi^{\tilde{\varrho}}(\alpha_n) \geq r$, $\varphi^{\tilde{\sigma}}(\alpha_n) \leq 1-r$, $\varphi^{\tilde{\xi}}(\alpha_n) \leq 1-r$, $\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r) = i\mathrm{int}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r), r)$ and $x_{s,t,k} q \mathrm{C}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\mathrm{CI}^{\odot}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(f^{-1}(\mathbb{SC}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r)), r), r), r)$. By Theorem 6(2), we have f is *SVNAβ£C*. □

Definition 19. *Let $f : (\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}, \pounds^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}) \to (\tilde{\mathcal{G}}, \varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}})$ be a mapping. Then,*

(1) f is called single-valued neutrosophic faintly β£-continuous (SVNFβ£C, for short) iff for every $\alpha_n \in Q_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(f(x_{s,t,k}), r)$, there exists $\varepsilon_n \in Q_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}_{\pounds\beta}}(x_{s,t,k}, r)$ such that $f(\varepsilon_n) \leq \alpha_n$,

(2) f is called single-valued neutrosophic weakly β£-continuous (SVNWβ£C, for short) iff for every $\alpha_n \in Q_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(f(x_{s,t,k}), r)$, there exists $\varepsilon_n \in Q_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}_{\pounds\beta}}(x_{s,t,k}, r)$ such that $f(\varepsilon_n) \leq \mathrm{C}_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r)$,

(3) f is called single-valued neutrosophic β£-continuous (SVNβ£C, for short) iff $f^{-1}(\alpha_n)$ is r-SVN£βO, for every $\varphi^{\tilde{\varrho}}(\alpha_n) \geq r$, $\varphi^{\tilde{\sigma}}(\alpha_n) \leq 1-r$, $\varphi^{\tilde{\xi}}(\alpha_n) \leq 1-r$.

Remark 3. *From the above definition we obtain the following diagram:*

$$\begin{array}{ccc} SVN\beta\pounds\mathrm{C} & \Rightarrow & SVNA\beta\pounds\mathrm{C} \\ & & \Downarrow \\ SVNF\beta\pounds\mathrm{C} & \Leftarrow & SVNW\beta\pounds\mathrm{C} \end{array}$$

Some supporting examples will be shown after the following two theorems.

Theorem 9. *Let $f : (\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}, \pounds^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}) \to (\tilde{\mathcal{G}}, \varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}})$ be a mapping. Then the following statements are equivalent:*

(1) f is SVNWβ£C,
(2) $f(\beta\pounds\mathrm{C}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r)) \leq \Theta_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(f(\alpha_n), r)$, for each $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$, $r \in \zeta_0$,
(3) $\beta\pounds\mathrm{C}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(f^{-1}(\varepsilon_n), r) \leq f^{-1}(\Theta_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\varepsilon_n, r))$, for each $\varepsilon_n \in \zeta^{\tilde{\mathcal{G}}}$, $r \in \zeta_0$,
(4) $f^{-1}(\varepsilon_n)$ is r-SVNβ£C set in $\tilde{\mathcal{F}}$ for each r-θ-closed set,
(5) $f^{-1}(\varepsilon_n)$ isr-SVNβ£O set in $\tilde{\mathcal{F}}$ for each r-θ-open set.

Proof. (1)⇒(2) Suppose there exists $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$ and $r \in \zeta_0$ such that $f(\beta\pounds\mathrm{C}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r)) \not\leq \Theta_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(f(\alpha_n), r)$. Then there exists $v \in \tilde{\mathcal{G}}$ and $s, t, k \in \zeta_0$ such that

$$\varphi^{\tilde{\varrho}}_{f(\beta\pounds\mathrm{C}_{\tau^{\tilde{\varrho}}}(\alpha_n, r))}(v) > s > \varphi^{\tilde{\varrho}}_{\Theta_{\varphi^{\tilde{\varrho}}}(f(\alpha_n), r)}(v)$$

$$\varphi^{\tilde{\sigma}}_{f(\beta\pounds\mathrm{C}_{\tau^{\tilde{\sigma}}}(\alpha_n, r))}(v) \leq t \leq \varphi^{\tilde{\sigma}}_{\Theta_{\varphi^{\tilde{\sigma}}}(f(\alpha_n), r)}(v)$$

$$\varphi^{\tilde{\xi}}_{f(\beta\pounds\mathrm{C}_{\tau^{\tilde{\xi}}}(\alpha_n, r))}(v) \leq k \leq \varphi^{\tilde{\xi}}_{\Theta_{\varphi^{\tilde{\xi}}}(f(\alpha_n), r)}(v)$$

If $f^{-1}(\{v\}) = \emptyset$, provides a contradiction that $f(\beta£C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)) = 0$. If $f^{-1}(\{v\}) \neq \emptyset$, there exists $\omega \in f^{-1}(\{v\})$ such that

$$\begin{aligned}
&\varphi^{\tilde{\varrho}}_{f(\beta£C_{\tau^{\tilde{\varrho}}}(\alpha_n,r))}(v) \geq \tau^{\tilde{\varrho}}_{\beta£C_{\tau^{\tilde{\varrho}}}(\alpha_n,r)}(\omega) > s > \varphi^{\tilde{\varrho}}_{\Theta_{\varphi^{\tilde{\varrho}}}(f(\alpha_n),r)}(f(\omega)) \\
&\varphi^{\tilde{\sigma}}_{f(\beta£C_{\tau^{\tilde{\sigma}}}(\alpha_n,r))}(v) \leq \tau^{\tilde{\sigma}}_{\beta£C_{\tau^{\tilde{\sigma}}}(\alpha_n,r)}(\omega) \leq t \leq \varphi^{\tilde{\sigma}}_{\Theta_{\varphi^{\tilde{\sigma}}}(f(\alpha_n),r)}(f(\omega)) \qquad (4) \\
&\varphi^{\tilde{\varsigma}}_{f(\beta£C_{\tau^{\tilde{\varsigma}}}(\alpha_n,r))}(v) \leq \tau^{\tilde{\varsigma}}_{\beta£C_{\tau^{\tilde{\varsigma}}}(\alpha_n,r)}(\omega) \leq k \leq \varphi^{\tilde{\varsigma}}_{\Theta_{\varphi^{\tilde{\varsigma}}}(f(\alpha_n),r)}(f(\omega)).
\end{aligned}$$

Since $\varphi^{\tilde{\varrho}}_{\Theta_{\varphi^{\tilde{\varrho}}}(f(\alpha_n),r)}(f(\omega)) < s$, $\varphi^{\tilde{\sigma}}_{\Theta_{\varphi^{\tilde{\sigma}}}(f(\alpha_n),r)}(f(\omega)) \geq t$ and $\varphi^{\tilde{\varsigma}}_{\Theta_{\varphi^{\tilde{\varsigma}}}(f(\alpha_n),r)}(f(\omega)) \geq k$. Then, $f(x)_{s,.t,k}$ is not *r-θ-cluster point* of $f(\alpha_n)$, there exists $\varepsilon_n \in Q_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f(x_{s,t,k}), r)$ such that $f(\alpha_n) \leq [C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r)]^n$. By *SVNWβ£C* of f, there exists $\pi_n \in Q_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}_{£\beta}}(x_{s,t,k}, r)$ such that $f(\pi_n) \leq C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\varepsilon_n, r)$. Thus, $f(\alpha_n) \leq [f(\pi_n)]^c$ implies $\alpha_n \leq [\pi_n]^c$. Hence

$$\begin{aligned}
&\tau^{\tilde{\varrho}}_{\beta£C_{\tau^{\tilde{\varrho}}}(\alpha_n,r)}(\omega) \leq \tau^{\tilde{\varrho}}_{[\pi_n]^c}(\omega) < s, \\
&\tau^{\tilde{\sigma}}_{\beta£C_{\tau^{\tilde{\sigma}}}(\alpha_n,r)}(\omega) \geq \tau^{\tilde{\sigma}}_{[\pi_n]^c}(\omega) \geq t, \\
&\tau^{\tilde{\varsigma}}_{\beta£C_{\tau^{\tilde{\varsigma}}}(\alpha_n,r)}(\omega) \geq \tau^{\tilde{\varsigma}}_{[\pi_n]^c}(\omega) \geq t.
\end{aligned}$$

It is a contradiction for Equation (2).

(2)⇒(3), (3)⇒(4) and (4)⇒(5): are obvious.

(5)⇒(1): Let $\alpha_n \in Q_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(f(x_{s,t,k}), r)$. Then $x_{s,t,k} q f^{-1}(\alpha_n)$ and α_n is *r*- *θ*-open set. By (5), we have $f^{-1}(\alpha_n)$ is *r-SVNβ£O* set in $\tilde{\mathcal{F}}$. Since $x_{s,t,k} q f^{-1}(\alpha_n)$ we obtain $f^{-1}(\alpha_n) \in Q_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}_{£\beta}}(x_{s,t,k}, r)$ and hence from $f(f^{-1}(\alpha_n)) \leq \alpha_n \leq C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}(\alpha_n, r)$. Then, f is *SVNWβ£C*. □

Theorem 10. *A mapping* $f : (\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}) \to (\tilde{\mathcal{G}}, \varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}})$ *is SVNFβ£C iff for each r-θ-closed set* $\alpha_n \in \zeta^{\tilde{\mathcal{G}}}$, $f^{-1}(\alpha_n)$ *is r-SVNβ£C.*

Proof. Obvious. □

Example 2. *Let* $\mathcal{F} = \{a, b\}$. *Define* $\varepsilon_n, \pi_n \in \zeta^{\mathcal{F}}$ *as follows:*

$$\varepsilon_n = \langle(0.3, 0.3), (0.7, 0.7), (0.3, 0.3)\rangle; \pi_n = \langle(0.3, 0.3), (0.3, 0.3), (0.3, 0.3)\rangle.$$

We define the mapping $\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, \varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}, £^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}} : \zeta^{\mathcal{F}} \times \zeta_0 \to \zeta^{\mathcal{F}}$ *as follows:*

$$\tau^{\tilde{\varrho}}(\alpha_n) = \begin{cases} 1, & \textit{if } \alpha_n = \tilde{1} \textit{ or } \tilde{0}, \\ \frac{2}{3}, & \textit{if } \alpha_n = \varepsilon_n \textit{ or } \pi_n \\ 0, & \textit{otherwise}, \end{cases} \quad \varphi^{\tilde{\varrho}}(\alpha_n) = \begin{cases} 1, & \textit{if } \alpha_n = \tilde{1} \textit{ or } \tilde{0}, \\ \frac{2}{3}, & \textit{if } \alpha_n = \pi_n \\ 0, & \textit{otherwise}, \end{cases}$$

$$\tau^{\tilde{\sigma}}(\alpha_n) = \begin{cases} 0, & \textit{if } \alpha_n = \tilde{1} \textit{ or } \tilde{0}, \\ \frac{1}{3}, & \textit{if } \alpha_n = \varepsilon_n \textit{ or } \pi_n \\ 1, & \textit{otherwise}, \end{cases} \quad \varphi^{\tilde{\sigma}}(\alpha_n) = \begin{cases} 0, & \textit{if } \alpha_n = \tilde{1} \textit{ or } \tilde{0}, \\ \frac{1}{3}, & \textit{if } \alpha_n = \pi_n \\ 0, & \textit{otherwise}, \end{cases}$$

$$\tau^{\tilde{\varsigma}}(\alpha_n) = \begin{cases} 0, & \textit{if } \alpha_n = \tilde{1} \textit{ or } \tilde{0}, \\ \frac{1}{3}, & \textit{if } \alpha_n = \varepsilon_n \textit{ or } \pi_n \\ 1, & \textit{otherwise}, \end{cases} \quad \varphi^{\tilde{\varsigma}}(\alpha_n) = \begin{cases} 0, & \textit{if } \alpha_n = \tilde{1} \textit{ or } \tilde{0}, \\ \frac{1}{3}, & \textit{if } \alpha_n = \pi_n \\ 1, & \textit{otherwise}, \end{cases}$$

$$\pounds^{\tilde{\varrho}}(\alpha_n) = \begin{cases} 1, & \text{if } \alpha_n = \tilde{0}, \\ \frac{1}{2}, & \text{if } \alpha_n = \pi_n, \\ \frac{2}{3}, & \text{if } \alpha_n = 0 < \alpha_n < \pi_n, \\ 0, & \text{otherwise}, \end{cases} \quad \pounds^{\tilde{\sigma}}(\alpha_n) = \begin{cases} 0, & \text{if } \alpha_n = \tilde{0}, \\ \frac{1}{2}, & \text{if } \alpha_n = \pi_n, \\ \frac{1}{3}, & \text{if } \alpha_n = 0 < \alpha_n < \pi_n, \\ 1, & \text{otherwise}, \end{cases}$$

$$\pounds^{\tilde{\xi}}(\alpha_n) = \begin{cases} 0, & \text{if } \alpha_n = \tilde{0}, \\ \frac{1}{2}, & \text{if } \alpha_n = \pi_n, \\ \frac{1}{3}, & \text{if } \alpha_n = 0 < \alpha_n < \pi_n, \\ 1, & \text{otherwise}, \end{cases}$$

From Theorems 4 and 5, we obtain $\beta\mathcal{I}C_\tau, T_\tau : I^X \times I_0 \to I^X$ as follows:

$$\Theta^{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r) = \begin{cases} \tilde{0}, & \text{if } \alpha_n = \tilde{0}, \\ \varepsilon_n, & \text{if } \tilde{0} \neq \alpha_n \leq \pi_n, \ 0 < r \leq \frac{2}{3}, \\ \tilde{1}, & \text{otherwise}, \end{cases}$$

$$\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\alpha_n, r) = \begin{cases} \tilde{0}, & \text{if } \mathcal{B} = \underline{0}, \\ \pi_n, & \text{if } \tilde{0} \neq \alpha_n \leq \pi_n, \ 0 < r \leq \frac{2}{3}, \\ \tilde{1}, & \text{otherwise}, \end{cases}$$

By Theorem 9(2), the identity mapping $id_X :: (\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}, \pounds^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}) \to (\tilde{\mathcal{G}}, \varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}})$ is SVNW$\beta\pounds$C but is not SVNA$\beta\pounds$C, because by Theorem 8(5), for each $\varphi^{\tilde{\varrho}}(\pi_n) \geq \frac{2}{3}$, $\varphi^{\tilde{\sigma}}(\pi_n) \leq \frac{1}{3}$, $\varphi^{\tilde{\xi}}(\pi_n) \leq \frac{1}{3}$ and $\varphi^{\tilde{\xi}}(\pi_n) \leq \frac{1}{3}$,

$$\tilde{1} = \beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\pi_n, \frac{2}{3})) \not\leq C_{\varphi^{\tilde{\varrho}\tilde{\sigma}\tilde{\xi}}}(\pi_n, \frac{2}{3}) = \varepsilon_n.$$

5. Single-Valued Neutrosophic Approximation Space

In this section, and for symmetrical purposes, we establish the definition of the single-valued neutrosophic upper, single-valued neutrosophic lower and single-valued neutrosophic boundary sets of a rough single-valued neutrosophic set α_n in a single-valued neutrosophic approximation space $(\tilde{\mathcal{F}}, \delta)$. Based on α_n and δ, we introduce the single-valued neutrosophic approximation interior operator $\text{int}^{\delta}_{\alpha_n}$ and the single-valued neutrosophic approximation closure operator $\text{Cl}^{\delta}_{\alpha_n}$.

Definition 20. *Assume that an SVNR δ is defined so that $\tilde{\varrho}_\delta(\omega, \omega) = 1, \tilde{\sigma}_\delta(\omega, \omega) = 0, \tilde{\xi}_\delta(\omega, \omega) = 0$ for every $\omega \in \tilde{\mathcal{F}}$, $\tilde{\varrho}_\delta(\omega, \nu) = \tilde{\varrho}_{\alpha_n}(\nu, \omega)$, $\tilde{\sigma}_\delta(\omega, \nu) = \tilde{\sigma}_\delta(\nu, \omega)$, $\tilde{\xi}_\delta(\omega, \nu) = \tilde{\xi}_{\alpha_n}(\nu, \omega)$ for every $\omega, \nu \in \tilde{\mathcal{F}}$ and $\tilde{\varrho}_\delta(\omega, \nu) \geq (\tilde{\varrho}_\delta(\omega, \mu) \wedge \tilde{\varrho}_\delta(\mu, \nu))$, $\tilde{\sigma}_\delta(\omega, \nu) \leq (\tilde{\sigma}_\delta(\omega, \mu) \vee \tilde{\sigma}_\delta(\mu, \nu))$ and $\tilde{\xi}_\delta(\omega, \nu) \leq (\tilde{\xi}_\delta(\omega, \mu) \vee \tilde{\xi}_\delta(\mu, \nu))$ for every $\omega, \nu, \mu \in \tilde{\mathcal{F}}$. That is, δ is a single-valued neutrosophic equivalence relation on $\tilde{\mathcal{F}}$. Then $(\tilde{\mathcal{F}}, \delta)$ is called a single-valued neutrosophic approximation space based on the single-valued neutrosophic equivalence relation (briefly, SVN-equivalence relation) δ on $\tilde{\mathcal{F}}$.*

Definition 21. *For each $\omega \in \tilde{\mathcal{F}}$, define a single-valued neutrosophic coset $[\omega] : \tilde{\mathcal{F}} \Rightarrow [0,1]$ by:*

$$\tilde{\varrho}_{[\omega]}(\nu) = \tilde{\sigma}_\delta(\omega, \nu), \quad \tilde{\sigma}_{[\omega]}(\nu) = \tilde{\sigma}_\delta(\omega, \nu), \quad \tilde{\xi}_{[\omega]}(\nu) = \tilde{\xi}_\delta(\omega, \nu) \quad \forall\ \nu \in .\tilde{\mathcal{F}}.$$

All elements $\nu \in \tilde{\mathcal{F}}$ with SVNR value $\tilde{\varrho}_\delta(\omega, \nu) > 0, \tilde{\sigma}_\delta(\omega, \nu) \leq 1, \tilde{\xi}_\delta(\omega, \nu) \leq 1$ are elements having a membership value in the single-valued neutrosophic coset $[\omega]$, and any element $\nu \in .\tilde{\mathcal{F}}$ with $\tilde{\varrho}_\delta(\omega, \omega) = 1, \tilde{\sigma}_\delta(\omega, \omega) = 0, \tilde{\xi}_\delta(\omega, \omega) = 0$ is not included in the single-valued neutrosophic coset $[\omega]$. Any single-valued neutrosophic coset $[\omega]$ surely include the element $\omega \in \tilde{\mathcal{F}}$, and consequently

$$\tilde{\varrho}_{\bigvee_{\mu \in \tilde{\mathcal{F}}}}([\omega](\mu)) = 1, \quad \tilde{\sigma}_{\bigwedge_{\mu \in \tilde{\mathcal{F}}}}([\omega](\mu)) = 0, \quad \tilde{\xi}_{\bigwedge_{\mu \in \tilde{\mathcal{F}}}}([\omega](\mu)) = 0, \quad \forall\ \omega \in \tilde{\mathcal{F}}$$

Further,

$$\tilde{\varrho}_{\bigvee_{\mu\in\tilde{\mathcal{F}}}}([\omega](\nu)) = 1, \quad \tilde{\sigma}_{\bigwedge_{\mu\in\tilde{\mathcal{F}}}}([\omega](\nu)) = 0, \quad \tilde{\varsigma}_{\bigwedge_{\mu\in\tilde{\mathcal{F}}}}([\omega](\nu)) = 0, \quad \forall\, \nu \in \tilde{\mathcal{F}},$$

such that $\bigvee_{\nu\in\tilde{\mathcal{F}}}([\nu]) = \langle 0,1,1\rangle$. *Clearly, if* $\tilde{\varrho}_{\delta}(\omega,\nu) > 0, \tilde{\sigma}_{\delta}(\omega,\nu) \leq 1, \tilde{\varsigma}_{\delta}(\omega,\nu) \leq 1$, *then the single-valued neutrosophic cosets* $[\omega], [\nu]$ *(as SVNSs) are containing the same elements of* $\tilde{\mathcal{F}}$ *with some non-zero membership values, and moreover, if* $\tilde{\varrho}_{[\nu]}(\mu) = 0, \quad \tilde{\sigma}_{[\nu]}(\mu) = 1$ *and* $\tilde{\varsigma}_{[\omega]}(\mu) = 1$, *then it must be that* $\tilde{\varrho}_{[\omega]}(\mu) = 0, \quad \tilde{\sigma}_{[\omega]}(\mu) = 1$ *and* $\tilde{\varsigma}_{[\nu]}(\mu) = 1$ *whenever* $\tilde{\varrho}_{\delta}(\omega,\nu) > 0, \tilde{\sigma}_{\delta}(\omega,\nu) \leq 1, \tilde{\varsigma}_{\delta}(\omega,\nu) \leq 1$. *That is, any two single-valued neutrosophic cosets are either two single-valued neutrosophic sets containing the same elements of* $\tilde{\mathcal{F}}$ *with some non-zero membership values or containing completely different elements of* $\tilde{\mathcal{F}}$ *with some non-zero membership values.*

Definition 22. *Let* $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$ *and* δ *be SVN-equivalence relation on* $\tilde{\mathcal{F}}$ *and the single-valued neutrosophic cosets. Then, the single-valued neutrosophic lower set (briefly, SVN-lower)* $(\alpha_n)^{\delta}$, *the single-valued neutrosophic upper set (briefly, SVN-upper)* $(\alpha_n)_{\delta}$ *and the single-valued neutrosophic boundary region set (briefly, SVN-boundary region)* $(\alpha_n)^{B}$ *are defined as follows: for* $\omega \in \tilde{\mathcal{F}}$,

$$\tilde{\varrho}_{(\alpha_n)_{\delta}}(\omega) = \tilde{\varrho}_{\alpha_n}(\omega) \vee \bigwedge_{(\alpha_n)^c(\mu)>0,\mu\neq\omega} \tilde{\varsigma}_{[\omega]}(\mu), \quad \tilde{\sigma}_{(\alpha_n)_{\delta}}(\omega) = \tilde{\sigma}_{\alpha_n}(\omega) \wedge \bigvee_{(\alpha_n)^c(\mu)>0,\mu\neq\omega} (1-\tilde{\sigma}_{[\omega]})(\mu) \tag{5}$$

$$\tilde{\varsigma}_{(\alpha_n)_{\delta}}(\omega) = \tilde{\varsigma}_{\alpha_n}(\omega) \wedge \bigvee_{(\alpha_n)^c(\mu)>0,\mu\neq\omega} \tilde{\varrho}_{[\omega]}(\mu)$$

$$\tilde{\varrho}_{(\alpha_n)^{\delta}}(\omega) = \tilde{\varrho}_{\alpha_n}(\omega) \wedge \bigvee_{\alpha_n(\mu)>0,\mu\neq\omega} \tilde{\varrho}_{[\omega]}(\mu), \quad \tilde{\sigma}_{(\alpha_n)^{\delta}}(\omega) = \tilde{\sigma}_{\alpha_n}(\omega) \vee \bigwedge_{\alpha_n(\mu)>0,\mu\neq\omega} (\tilde{\sigma}_{[\omega]}(\mu) \tag{6}$$

$$\tilde{\varsigma}_{(\alpha_n)^{\delta}}(\omega) = \tilde{\varsigma}_{\alpha_n}(\omega) \vee \bigwedge_{\alpha_n(\mu)>0,\mu\neq\omega} \tilde{\varsigma}_{[\omega]}(\mu)$$

$$\begin{aligned}
\tilde{\varrho}_{(\alpha_n)^B}(\omega) &= \tilde{\varrho}_{(\alpha_n)^{\delta}} \overline{\wedge} \tilde{\varrho}_{(\alpha_n)_{\delta}} = \begin{cases} \tilde{0}, & \textit{if } (\alpha_n)^{\delta} \leq (\alpha_n)_{\delta}, \\ (\alpha_n)^{\delta} \wedge [(\alpha_n)_{\delta}]^c, & \textit{otherwise,} \end{cases} \\
\tilde{\sigma}_{(\alpha_n)^B}(\omega) &= \tilde{\sigma}_{(\alpha_n)^{\delta}} \overline{\vee} \tilde{\sigma}_{(\alpha_n)_{\delta}} = \begin{cases} \tilde{1}, & \textit{if } (\alpha_n)^{\delta} \geq (\alpha_n)_{\delta}, \\ (\alpha_n)^{\delta} \vee [(\alpha_n)_{\delta}]^c, & \textit{otherwise,} \end{cases} \\
\tilde{\varsigma}_{(\alpha_n)^B}(\omega) &= \tilde{\varsigma}_{(\alpha_n)^{\delta}} \overline{\vee} \tilde{\varsigma}_{(\alpha_n)_{\delta}} = \begin{cases} \tilde{1}, & \textit{if } (\alpha_n)^{\delta} \geq (\alpha_n)_{\delta}, \\ (\alpha_n)^{\delta} \vee [(\alpha_n)_{\delta}]^c, & \textit{otherwise,} \end{cases}
\end{aligned} \tag{7}$$

$(\alpha_n)^{\delta}$, $(\alpha_n)_{\delta}$ *and* $(\alpha_n)^{B}$ *are then called SVN-lower, SVN-upper and SVN-boundary region sets associated with the SVNS* $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$ *and based on the SVN-equivalence relation* δ *in a single-valued neutrosophic approximation space* $(\tilde{\mathcal{F}}, \delta)$.

From (5) and (6), we obtain that $\tilde{\varrho}_{(\alpha_n)_{\delta}}(\omega) \leq \tilde{\varrho}_{\alpha_n}(\omega) \leq \tilde{\varrho}_{(\alpha_n)^{\delta}}(\omega)$, $\tilde{\sigma}_{(\alpha_n)_{\delta}}(\omega) \geq \tilde{\sigma}_{\alpha_n}(\omega) \geq \tilde{\sigma}_{(\alpha_n)^{\delta}}(\omega)$ and $\tilde{\varsigma}_{(\alpha_n)_{\delta}}(\omega) \geq \tilde{\varsigma}_{\alpha_n}(\omega) \geq \tilde{\varsigma}_{(\alpha_n)^{\delta}}(\omega)$ for each $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$. Whenever $(\alpha_n)^{\delta}$ so that $\tilde{\varrho}_{(\alpha_n)^{\delta}}(\omega) \leq \tilde{\varrho}_{(\alpha_n)_{\delta}(\omega)}$, $\tilde{\sigma}_{(\alpha_n)^{\delta}}(\omega) \geq \tilde{\sigma}_{(\alpha_n)_{\delta}(\omega)}$ and $\tilde{\varsigma}_{(\alpha_n)^{\delta}}(\omega) \geq \tilde{\varsigma}_{(\alpha_n)_{\delta}(\omega)}$ we obtain that $\tilde{\varrho}_{\alpha_n}(\omega) = \tilde{\varrho}_{(\alpha_n)_{\delta}}(\omega) = \tilde{\varrho}_{(\alpha_n)^{\delta}}(\omega)$, $\tilde{\sigma}_{\alpha_n}(\omega) = \tilde{\sigma}_{(\alpha_n)_{\delta}}(\omega) = \tilde{\sigma}_{(\alpha_n)^{\delta}}(\omega)$ and $\tilde{\varsigma}_{\alpha_n}(\omega) = \tilde{\varsigma}_{(\alpha_n)_{\delta}}(\omega) = \tilde{\varsigma}_{(\alpha_n)^{\delta}}(\omega)$, and then from (7), we obtain $\tilde{\varrho}_{(\alpha_n)^B}(\omega) = 0$, $\tilde{\sigma}_{(\alpha_n)^B}(\omega) = 1$ and $\tilde{\varsigma}_{(\alpha_n)^B}(\omega) = 1$. Otherwise, $\tilde{\varrho}_{(\alpha_n)^B}(\omega) = (\alpha_n)^{\delta} \wedge [(\alpha_n)_{\delta}]^c$, $\tilde{\sigma}_{(\alpha_n)^B}(\omega) = (\alpha_n)^{\delta} \vee [(\alpha_n)_{\delta}]^c$ and $\tilde{\varsigma}_{(\alpha_n)^B}(\omega) = (\alpha_n)^{\delta} \vee [(\alpha_n)_{\delta}]^c$. .

Theorem 11. *For any SVNS* $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$ *we find that*
(1) $\tilde{0}_{\delta} = \tilde{0}^{\delta} = \tilde{0}$ *and* $\tilde{1}_{\delta} = \tilde{1}^{\delta} = \tilde{1}$,

(2) $(\alpha_n \vee \varepsilon_n)_\delta \geq (\alpha_n)_\delta \vee (\varepsilon_n)_\delta$,
(3) $(\alpha_n \wedge \varepsilon_n)^\delta \leq (\alpha_n)^\delta \wedge (\varepsilon_n)^\delta$,
(4) $\alpha_n \leq \varepsilon_n$, *implies that* $(\alpha_n)^\delta \leq (\varepsilon_n)^\delta$ *and* $(\alpha_n)_\delta \leq (\varepsilon_n)_\delta$,
(5) $(\alpha_n \vee \varepsilon_n)^\delta = (\alpha_n)^\delta \vee (\varepsilon_n)^\delta$,
(6) $(\alpha_n \wedge \varepsilon_n)_\delta = (\alpha_n)_\delta \wedge (\varepsilon_n)_\delta$,
(7) $[(\alpha_n)^\delta]^c = ([\alpha_n]^c)_\delta$ *and* $[(\alpha_n)_\delta]^c = ([\alpha_n]^c)^\delta$,
(8) $[(\alpha_n)_\delta]^\delta \geq (\alpha_n)_\delta = [(\alpha_n)_\delta]_\delta$,
(9) $[(\alpha_n)^\delta]_\delta \leq (\alpha_n)^\delta = [(\alpha_n)^\delta]^\delta$.

Proof. Obvious. □

Example 3. *Let δ be an SVNR on a set $\tilde{\mathcal{F}} = \{\omega_1, \omega_2, \omega_3\}$ as shown below.*

α_n	ω_1	ω_2	ω_3
ω_1	$\langle 0, 0, 1\rangle$	$\langle 0.3, 0.1, 0.6\rangle$	$\langle 0.1, 0, 0.4\rangle$
ω_2	$\langle 0, 0.2, 0.4\rangle$	$\langle 0.6, 0.5, 1\rangle$	$\langle 0.6, 0, 1\rangle$
ω_3	$\langle 1, 0, 1\rangle$	$\langle 1, 0.5, 1\rangle$	$\langle 1, 1, 0\rangle$

Assume that $\alpha_n = \langle (0.3, 0.4, 1), (0.5, 0.2, 0.2), (0.3, 0.5, 0.7)\rangle$. Then, the single-valued neutrosophic cosets are as follows:

$$\tilde{\varrho}_{(\alpha_n)_\delta}(\omega_1) = \tilde{\varrho}_{\alpha_n}(\omega_1) \vee \bigwedge_{(\alpha_n)^c(\mu)>0, \mu \neq \omega_1} \tilde{\varsigma}_{[\omega]}(\mu) = 0.4$$
$$\tilde{\sigma}_{(\alpha_n)_\delta}(\omega_1) = \tilde{\sigma}_{\alpha_n}(\omega_1) \wedge \bigvee_{(\alpha_n)^c(\mu)>0, \mu \neq \omega_1} (1 - \tilde{\sigma}_{[\omega_1]})(\mu) = 0.4$$
$$\tilde{\varsigma}_{(\alpha_n)_\delta}(\omega_1) = \tilde{\varsigma}_{\alpha_n}(\omega_1) \wedge \bigvee_{(\alpha_n)^c(\mu)>0, \mu \neq \omega_1} \tilde{\varrho}_{[\omega_1]}(\mu) = 0$$

Hence, $(\alpha_n)_\delta(\omega_1) = (0.4, 0.4, 0)$ and

$$\tilde{\varrho}_{(\alpha_n)_\delta}(\omega_2) = \tilde{\varrho}_{\alpha_n}(\omega_2) \vee \bigwedge_{(\alpha_n)^c(\mu)>0, \mu \neq \omega_2} \tilde{\varsigma}_{[\omega]}(\mu) = 0.6$$
$$\tilde{\sigma}_{(\alpha_n)_\delta}(\omega_2) = \tilde{\sigma}_{\alpha_n}(\omega_2) \wedge \bigvee_{(\alpha_n)^c(\mu)>0, \mu \neq \omega_2} (1 - \tilde{\sigma}_{[\omega_2]})(\mu) = 0.2$$
$$\tilde{\varsigma}_{(\alpha_n)_\delta}(\omega_2) = \tilde{\varsigma}_{\alpha_n}(\omega_2) \wedge \bigvee_{(\alpha_n)^c(\mu)>0, \mu \neq \omega_2} \tilde{\varrho}_{[\omega_2]}(\mu) = 0.2$$

Hence, $(\alpha_n)_\delta(\omega_2) = (0.6, 0.2, 0.2)$. Similarly, we can obtain $(\alpha_n)_\delta(\omega_2) = (0.4, 0.5, 0.7)$; therefore, $(\alpha_n)_\delta = \langle (0.4, 0.4, 0), (0.6, 0.2, 0.2), (0.4, 0.5, 0.7)\rangle$, $(\alpha_n)^\delta = \langle (0.3, 0.4, 0.4), (0.5, 0.5, 0.6), (0.3, 0.5, 0.7)\rangle$ and then $(\alpha_n)^B = \langle (0, 0.6, 0.4), (0.2, 0.8, 0.6), (0.3, 0.5, 0.7)\rangle$.

Definition 23. *The single-valued neutrosophic approximation interior operator* $\mathrm{int}_\delta^{\alpha_n} : \zeta^{\tilde{\mathcal{F}}} \to \zeta^{\tilde{\mathcal{F}}}$ *is defined as follows:*

$$\mathrm{int}_\delta^{\alpha_n}(\varepsilon_n) = (\alpha_n)_\delta \wedge (\varepsilon_n)_\delta, \quad \forall\ \varepsilon_n \neq \tilde{1} \text{ and } \mathrm{int}_\delta^{\alpha_n}(\tilde{1}) = \tilde{1}.$$

That is associated with an SVNS $\alpha_n \in \zeta^{\tilde{\mathcal{F}}}$ *in a single-valued neutrosophic approximation space $(\tilde{\mathcal{F}}, \delta)$.*

Theorem 12. *The following conditions are satisfied*
(1) $\mathrm{int}_\delta^{\alpha_n}(\tilde{0}) = \tilde{0}$,
(2) $\mathrm{int}_\delta^{\alpha_n}(\varepsilon_n) \leq \varepsilon_n\ \forall\ \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$,

(3) $\varepsilon_n \leq \pi_n \Rightarrow \mathrm{int}_\delta^{\alpha_n}(\varepsilon_n) \leq \mathrm{int}_\delta^{\alpha_n}(\pi_n), \forall\, \varepsilon_n, \pi_n \in \zeta^{\tilde{\mathcal{F}}}$,
(4) $\mathrm{int}_\delta^{\alpha_n}(\varepsilon_n \wedge \pi_n) = \mathrm{int}_\delta^{\alpha_n}(\varepsilon_n) \wedge \mathrm{int}_\delta^{\alpha_n}(\pi_n)\ \forall\, \varepsilon_n, \pi_n \in \zeta^{\tilde{\mathcal{F}}}$,
(5) $\mathrm{int}_\delta^{\alpha_n}(\mathrm{int}_\delta^{\alpha_n}(\varepsilon_n)) = \mathrm{int}_\delta^{\alpha_n}(\varepsilon_n)\ \forall\, \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$.

Proof. For (1): $\mathrm{int}_\delta^{\alpha_n}(\varepsilon_n) = (\alpha_n)_\delta \wedge (\tilde{0})_\delta = \tilde{0}$.
For (2): $\mathrm{int}_\delta^{\alpha_n}(\varepsilon_n) = (\alpha_n)_\delta \wedge (\varepsilon_n)_\delta \leq (\alpha_n)_\delta \wedge \varepsilon_n \leq \varepsilon_n$.
For (3): $\varepsilon_n \leq \pi_n$ then $(\varepsilon_n)_\delta \leq (\pi_n)_\delta \Rightarrow \mathrm{int}_\delta^{\alpha_n}(\varepsilon_n) \leq \mathrm{int}_\delta^{\alpha_n}(\pi_n)$.
For (4): $\mathrm{int}_\delta^{\alpha_n}(\varepsilon_n \wedge \pi_n) = (\alpha_n)_\delta \wedge (\varepsilon_n \wedge \pi_n)_\delta = (\alpha_n)_\delta \wedge (\varepsilon_n)_\delta \wedge (\alpha_n)_\delta \wedge (\pi_n)_\delta = \mathrm{int}_\delta^{\alpha_n}(\varepsilon_n) \wedge \mathrm{int}_\delta^{\alpha_n}(\pi_n)$.
For (5): Similarly to (4). □

Thus, this is called a single-valued neutrosophic interior associated with α_n in the single-valued neutrosophic approximation space $(\tilde{\mathcal{F}}, \delta)$ generating a single-valued neutrosophic topology defined by:

$$\varpi_\delta^{\alpha_n} = \{\varepsilon_n \in \zeta^{\tilde{\mathcal{F}}} : \varepsilon_n = \mathrm{int}_\delta^{\alpha_n}(\varepsilon_n)\}. \tag{8}$$

Definition 24. *The single-valued neutrosophic approximation closure operator* $\mathrm{Cl}_\delta^{\alpha_n} : \zeta^{\tilde{\mathcal{F}}} \to \zeta^{\tilde{\mathcal{F}}}$ *is defined as follows:*

$$\mathrm{Cl}_\delta^{\alpha_n}(\varepsilon_n) = [(\alpha_n)_\delta]^c \vee (\varepsilon_n)^\delta, \quad \forall\, \varepsilon_n \neq \tilde{0} \textit{ and } \mathrm{Cl}_\delta^{\alpha_n}(\tilde{0}) = \tilde{0}.$$

Theorem 13. *The single-valued neutrosophic approximation closure operator satisfies the following conditions:*
(1) $\mathrm{Cl}_\delta^{\alpha_n}(\tilde{1}) = \tilde{1}\ \forall\, \alpha_n \in \zeta^{\tilde{\mathcal{F}}}$,
(2) $\mathrm{Cl}_\delta^{\alpha_n}(\varepsilon_n) \geq \varepsilon_n\ \forall\, \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$,
(3) $\varepsilon_n \leq \pi_n \Rightarrow \mathrm{Cl}_\delta^{\alpha_n}(\varepsilon_n) \leq \mathrm{Cl}_\delta^{\alpha_n}(\pi_n), \forall\, \varepsilon_n, \pi_n \in \zeta^{\tilde{\mathcal{F}}}$,
(4) $\mathrm{Cl}_\delta^{\alpha_n}(\varepsilon_n \wedge \pi_n) = \mathrm{Cl}_\delta^{\alpha_n}(\varepsilon_n) \wedge \mathrm{Cl}_\delta^{\alpha_n}(\pi_n)$ *and* $\mathrm{Cl}_\delta^{\alpha_n}(\varepsilon_n \vee \pi_n) = \mathrm{Cl}_\delta^{\alpha_n}(\varepsilon_n) \vee \mathrm{Cl}_\delta^{\alpha_n}(\pi_n)\ \forall\, \varepsilon_n, \pi_n \in \zeta^{\tilde{\mathcal{F}}}$,
(5) $\mathrm{Cl}_\delta^{\alpha_n}(\mathrm{Cl}_\delta^{\alpha_n}(\varepsilon_n)) = \mathrm{Cl}_\delta^{\alpha_n}(\varepsilon_n)\ \forall\, \varepsilon_n \in \zeta^{\tilde{\mathcal{F}}}$.

Proof. Similar to the proof of Theorem 12. □

6. Conclusions

In this paper, we defined the single-valued neutrosophic semi-closure space (SVNSCS). It has been proven that every $\tau_{\mathbb{SC}}^{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}$ is a single-valued neutrosophic ideal topological space on $\mathcal{F}$. It has also been proven that every $\mathbb{SC}_{\mathbb{SC}}^{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}$ is finer than $\mathbb{SC}$ (see Theorem 2). In addition, single-valued neutrosophic operators, namely $\beta\pounds C_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}$ and $\beta\pounds\mathrm{int}_{\tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}}$, are constructed from a single-valued neutrosophic topological space $(\tilde{\mathcal{F}}, \tau^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}))$ (see Theorem 5). Next, the concepts of a single-valued neutrosophic almost $\beta\pounds$-continuous, single-valued neutrosophic faintly $\beta\pounds$-continuous and single-valued neutrosophic weakly $\beta\pounds$-continuous based on a single-valued neutrosophic ideal $\pounds^{\tilde{\varrho}\tilde{\sigma}\tilde{\varsigma}}$ were introduced and studied (see Theorems 6–8). Finally, we introduced the single-valued neutrosophic sets $(\alpha_n)_\delta$, $(\alpha_n)^\delta$, $(\alpha_n)^B$ for a single-valued neutrosophic set α_n that explains the single-valued neutrosophic roughness of the single-valued neutrosophic set α_n. We introduced the notion of single-valued neutrosophic approximation space and the related single-valued neutrosophic topology.

7. Discussion for Further Works

The theories that were used in this article could be extended to study some similar notions in the neutrosophic metric topological spaces.

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