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On hesitant neutrosophic rough set over two universes and its application

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Abstract
As a further generalization of the concepts of fuzzy set, intuitionistic fuzzy set, single valued neutrosophic refined set, hesitant fuzzy set, and dual hesitant fuzzy set, Ye (J Intell Syst 24(1):23–36, 2015) proposed the concept of hesitant neutrosophic sets (also called single valued neutrosophic hesitant fuzzy sets). Following the idea of hesitant neutrosophic sets as introduced by Ye, in this paper, the model of hesitant neutrosophic rough sets is proposed, then the join semi-lattice structure of lower and upper hesitant neutrosophic rough approximation operators over two universes is given. In addition, an algorithm to handle decision making problem in medical diagnosis based on hesitant neutrosophic rough sets over two universes is provided. Finally, a numerical example is employed to demonstrate the validness of the proposed hesitant neutrosophic rough sets.

Keywords Single valued neutrosophic sets · Hesitant fuzzy sets · Hesitant neutrosophic sets · Hesitant neutrosophic rough set · Decision making

1 Introduction

In order to deal with imprecise information and inconsistent knowledge. Smarandache (1998, 1999) first introduced the notion of neutrosophic set by fusing a tri-component set and the non-standard analysis. A neutrosophic set consists of three membership functions (truth-membership function,indeterminacy membership function and

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falsity-membership function), where every function value is a real standard or non-standard subset of the nonstandard unit interval $]0^-, 1^+[$. Since then, many authors have been studied various aspects of neutrosophic sets from different point of view, for example, in order to apply the neutrosophic idea to logics. Smarandache (1998, 2002) proposed the neutrosophic logic, which is a generalization of fuzzy and intuitionistic fuzzy logic (see http://fs.unm.edu/eBook-Neutrosophics6.pdf, see Chapter 2, pp. 90–125).Guo and Cheng (2009) and Guo and Sengur (2015) obtained a good applications in image processing and cluster analysis by using neutrosophic sets. Salama and Broumi (2014) and Broumi and Smarandache (2014) first given a new hybrid mathematical structure called rough neutrosophic sets, handling incomplete and indeterminate information, and studied some operations and their properties. Zhang et al. (2017b, 2018a, b) successively studied a new inclusion relation of neutrosophic sets, some new operations of totally dependent-neutrosophic sets and totally dependent-neutrosophic soft sets, neutrosophic duplet semi-group and cancellable neutrosophic triplet groups.

Smarandache (1998) defined the single valued neutrosophic set, which is online at http://fs.unm.edu/eBooks-neutrosophics6.pdf (see p. 7). Single valued neutrosophic sets actually can also be viewed as an generalization of intuitionistic fuzzy sets (Atanassov 1986), in which three membership functions are unrelated and their function values belong to the unit closed interval. Single valued neutrosophic sets results in a new majorly research issue. Ye (2013, 2014), Ye and Ye (2014) proposed decision making based on correlation coefficients and weighted correlation coefficient of single valued neutrosophic sets, and illustrated the application of proposed methods. Majumdar and Samant (2014) studied distance, similarity and entropy of single valued neutrosophic sets from a theoretical aspect. Şahin and Küçük (2015) proposed a subsethood measure of single valued neutrosophic sets based on distance and showed its effectiveness by an example. We known that there's a certain connection between fuzzy relations and fuzzy rough approximation operators (resp., fuzzy topologies, information systems Li and Cui 2015a, b; Li et al. 2017). Hence, Yang et al. (2016) firstly proposed single valued neutrosophic relations and studied some kinds of kernels and closures of single valued neutrosophic relations, Subsequently they proposed single valued neutrosophic rough sets (Yang et al. 2017) by fusing rough sets (Pawlak 1982) and single valued neutrosophic sets, and they studied some properties of single value neutrosophic upper and lower approximation operators. As a generalization of single value neutrosophic rough sets, Bao and Yang (2017) introduced p-dimension single valued neutrosophic refined rough sets by combining single valued neutrosophic refined sets with rough sets, and they also gave some properties of p-dimension single valued neutrosophic upper and lower approximation operators. As another generalization of single value neutrosophic rough sets, Bo et al. (2018) proposed the concept of multi-granulation neutrosophic rough sets and obtained some basic properties of the pessimistic (optimistic) multigranulation neutrosophic rough approximation operators. However, the algebraical structures of those rough approximation operators in references (Bao and Yang 2017; Bo et al. 2018; Yang et al. 2017) were not comprehensively studied. Following this idea, Zhao and Zhang (2018a) gave the supremum and infimum of the p-dimension neutrosophic upper and lower approximation operators, but they did not study the relationship between the p-dimension neutrosophic upper approximation operators and the p-dimension neutrosophic lower approximation operators, especially in the one-dimensional case. Subsequently, they provide the lattice structure of the pessimistic multigranulation neutrosophic rough approximation operators (Zhao and Zhang 2018b). In the one-dimensional case, for special neutrosophic relations, the completely lattice isomorphic relationship between upper neutrosophic rough approximation operators and lower neutrosophic rough approximation operators was given.

Recently, Zhang et al. (2016, 2017a) proposed a general decision making framework based on the (interval-valued) hesitant fuzzy rough set model over two universes. However, the algebraical structures of the (interval-valued) hesitant fuzzy approximation operators was not well studied. As a further generalization of the concepts of fuzzy set,intuitionistic fuzzy set, hesitant fuzzy set, dual hesitant fuzzy set and single valued neutrosophic (refined) set, Ye (2015, 2016, 2018) proposed the concept of (interval) hesitant neutrosophic sets. At present, some papers on "hesitant neutrosophic set" have been published (e.g. Biswas et al. 2016; Khan et al. 2017; Guo and Sengur 2015; Li and Zhang 2018; Liu and Zhang 2017; Liu and Teng 2017; Mahmood et al. 2016; Şahin and Liu 2016). However, the study of hesitant neutrosophic rough sets based on hesitant neutrosophic sets is still a blank. In the present paper, we shall introduce the model of hesitant neutrosophic rough sets based on hesitant neutrosophic sets and explore the algebraical structures of lower and upper hesitant neutrosophic rough approximation operators over two universes. We also apply the new model to neutrosophic decision-making problems.

The structure of the article is as follows. In Sect. 2, some basic notions and operations are introduced. In Sect. 3, the model of hesitant neutrosophic rough sets over two universes is proposed. In Sect. 4, the join semi-lattice structure of lower and upper hesitant neutrosophic rough approximation operators over two universes is given. In Sect. 5, an algorithm to handle decision making problem in medical diagnosis based on hesitant neutrosophic rough sets over two universes is provided. And, a numerical example is employed to demonstrate the validness of the proposed hesitant neutrosophic rough sets. Finally, Sect. 6 concludes this paper.

2 Preliminaries

In this section, we briefly recall some basic definitions which will be used in the paper.

2.1 Neutrosophic sets and single valued neutrosophic sets

Smarandache first proposed the concept of a neutrosophic set as follows.

Definition 1 (Smarandache 1998) Let X be a space of points (objects), with a generic element in X denoted by a. A neutrosophic set A in X consists of three membership functions (truth-membership function T_A, indeterminacy membership function I_A and falsity-membership function F_A, where every function value is a real standard or non-standard subset of the nonstandard unit interval $]0^-, 1^+[$).

There is no restriction on the sum of $T_A(a), I_A(a)$ and $F_A(a)$, thus

$$0^- \leq \sup T_A(a) + \sup I_A(a) + \sup F_A(a) \leq 3^+.$$

In order to apply neutrosophic sets conveniently, Wang et al. proposed single valued neutrosophic sets as follows.

Definition 2 (Smarandache 1998) Let X be a space of points (objects), with a generic element in X denoted by a. A single valued neutrosophic set A in X consists of three membership functions (truth-membership function T_A, indeterminacy membership function I_A and falsity-membership function F_A, where every function value is a real standard subset of the unit interval [0, 1].

There is no restriction on the sum of $T_A(a), I_A(a)$ and $F_A(a)$, thus

$$0 \leq \sup T_A(a) + \sup I_A(a) + \sup F_A(a) \leq 3.$$

Definition 3 (Yang et al. 2016) Let A and B be two single valued neutrosophic sets in X, $T_A(a) \leq T_B(a), I_A(a) \geq I_B(a)$ and $F_A(a) \geq F_B(a)$ for each $a \in X$, then we called A is contained in B, i.e., $A \sqsubseteq B$. If $A \sqsubseteq B$ and $B \sqsubseteq A$, then we called A is equal to B, denoted by $A = B$.

Definition 4 (Yang et al. 2016, 2017) Let A and B be two single valued neutrosophic sets in X,

1. The union of A and B is a single valued neutrosophic set C, denoted by $A \sqcup B$, where $\forall x \in X$, $T_C(a) = \max\{T_A(a), T_B(a)\}$, $I_C(a) = \min\{I_A(a), I_B(a)\}$ and $F_C(a) = \min\{F_A(a), F_B(a)\}$.
2. The intersection of A and B is a single valued neutrosophic set D, denoted by $A \sqcap B$, where $\forall x \in X$, $T_D(a) = \min\{T_A(a), T_B(a)\}$, $I_D(a) = \max\{I_A(a), I_B(a)\}$ and $F_D(a) = \max\{F_A(a), F_B(a)\}$.

2.2 Hesitant fuzzy sets

Torra and Narukawa (2009) and Torra (2010) originally gave the definition of hesitant fuzzy sets as follows.

Definition 5 (Torra 2010; Torra and Narukawa 2009) Let X be a space of points (objects), a hesitant fuzzy set B in X is defined in terms of a function $h_B(x)$ that when applied to X returns a subset of [0, 1], that is,

$$B = \{\langle x, h_B(x)\rangle \mid x \in X\},$$

where $h_B(x)$ is a set of some different values in [0, 1], representing the possible membership degrees of element $x \in X$ to B. For convenience, we call $h_B(x)$ a hesitant fuzzy element (Xu and Xia 2011).

Definition 6 (Torra 2010; Torra and Narukawa 2009) Given a hesitant fuzzy element $h_B(x)$, its lower and upper bounds are defined as $h_B^-(x) = \min h_B(x)$ and $h_B^+(x) = \max h_B(x)$, respectively.

Definition 7 (Torra 2010) Let h, h_1, h_2 be three hesitant fuzzy elements, and let λ be a positive scale, defined some operations as follows:

1. $h^c = \bigcup_{\alpha \in h} \{1 - \alpha\}$,
2. $h_1 \widetilde{\vee} h_2 = \bigcup_{\alpha_1 \in h_1, \alpha_2 \in h_2} \max\{\alpha_1, \alpha_2\}$,
3. $h_1 \widetilde{\wedge} h_2 = \bigcup_{\alpha_1 \in h_1, \alpha_2 \in h_2} \min\{\alpha_1, \alpha_2\}$,
4. $h^\lambda = \bigcup_{\alpha \in h} \{\alpha^\lambda\}$,
5. $\lambda h = \bigcup_{\alpha \in h} \{1 - (1 - \alpha)^\lambda\}$,
6. $h_1 \bigoplus h_2 = \bigcup_{\alpha_1 \in h_1, \alpha_2 \in h_2} \{\alpha_1 + \alpha_2 - \alpha_1 \alpha_2\}$,
7. $h_1 \bigotimes h_2 = \bigcup_{\alpha_1 \in h_1, \alpha_2 \in h_2} \{\alpha_1 \alpha_2\}$.

Definition 8 (Xia and Xu 2011) Let h be a hesitant fuzzy element,

$$S(h) = \frac{1}{\sharp h} \sum_{\alpha \in h} \alpha$$

is called the score function of h, where $\sharp h$ is the number of the elements in h. For two hesitants h_1 and h_2, if $S(h_1) > S(h_2)$, then $h_1 > h_2$; if $S(h_1) = S(h_2)$, then $h_1 = h_2$.

2.3 Hesitant neutrosophic sets

As a further generalization of the concepts of fuzzy set, intuitionistic fuzzy set, single valued neutrosophic set, hesitant fuzzy set, and dual hesitant fuzzy set, Ye proposed hesitant neutrosophic sets as follows.

Definition 9 (Ye 2015, 2018) Let X be a space of points (objects), a hesitant neutrosophic set on X is defined as

$$\mathcal{N} = \{\langle a, t(a), i(a), f(a)\rangle \mid a \in X\},$$

in which $t(a)$, $i(a)$, and $f(a)$ are three sets of some values in [0, 1], denoting the possible truth-membership hesitant degrees, indeterminacy membership hesitant degrees, and falsity-membership hesitant degrees of the element $a \in X$ to the set $\mathcal{N}$, with the conditions $0 \leq \alpha, \beta, \gamma \leq 1$ and $0 \leq \alpha^+ + \beta^+ + \gamma^+ \leq 3$, where $\alpha \in t(a), \beta \in i(a), \gamma \in f(a)$, $\alpha^+ = \max t(a)$, $\beta^+ = \max i(a)$, and $\gamma^+ = \max f(a)$ for $a \in X$.

For convenience, the three tuple $\boldsymbol{n}(x) = \{t(x), i(x), f(x)\}$ is called a hesitant neutrosophic element, which is denoted by the simplified symbol $\boldsymbol{n} = \{t, i, f\}$, and the family of all hesitant neutrosophic sets in X will be denoted by $\mathbf{HNS}(X)$.

Definition 10 (Bao and Yang 2017) Let X be a space of points (objects), with a generic element in X denoted by x. A single valued neutrosophic refined set B in X is characterized by three membership functions: a truth-membership function T_B, an indeterminacy membership function I_B and a falsity-membership function F_B as follows:

$$B = \{\langle x, T_B(x), I_B(x), F_B(x)\rangle \mid x \in X\},$$

where

$$T_B(x) = \{T_{1B}(x), T_{2B}(x), \dots, T_{pB}(x)\},$$
$$I_B(x) = \{I_{1B}(x), I_{2B}(x), \dots, I_{pB}(x)\},$$
$$F_B(x) = \{F_{1B}(x), F_{2B}(x), \dots, F_{pB}(x)\},$$

p is a positive integer, $T_{iB}(x), I_{iB}(x), F_{iB}(x) \in [0, 1]$ and $0 \leq T_{iB}(x) + I_{iB}(x) + F_{iB}(x) \leq 3$ for $i = 1, 2, \dots, p$. Also, p is referred to as the dimension of B.

Remark 1 Clearly, a hesitant neutrosophic set is also a generalization of single valued neutrosophic refined set.

Definition 11 (Ye 2015) Let $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ be two hesitant neutrosophic elements in X, defined some operations in them as follows:

1. $\boldsymbol{n}_1 \widehat{\vee} \boldsymbol{n}_2 = \{\{\alpha \in t_1 \widetilde{\vee} t_2 \mid \alpha \geq \max(\alpha_1^-, \alpha_2^-)\}, \{\beta \in i_1 \widetilde{\wedge} i_2 \mid \beta \leq \min(\beta_1^+, \beta_2^+)\}, \{\gamma \in f_1 \widetilde{\wedge} f_2 \mid \gamma \leq \min(\gamma_1^+, \gamma_2^+)\}\},$
2. $\boldsymbol{n}_1 \widehat{\wedge} \boldsymbol{n}_2 = \{\{\alpha \in t_1 \widetilde{\wedge} t_2 \mid \alpha \leq \min(\alpha_1^+, \alpha_2^+)\}, \{\beta \in i_1 \widetilde{\vee} i_2 \mid \beta \geq \max(\beta_1^-, \beta_2^-)\}, \{\gamma \in f_1 \widetilde{\vee} f_2 \mid \gamma \geq \max(\gamma_1^-, \gamma_2^-)\}\},$
3. $\boldsymbol{n}_1 \widehat{\bigoplus} \boldsymbol{n}_2 = \bigcup_{\alpha_1 \in t_1, \beta_1 \in i_1, \gamma_1 \in f_1, \alpha_2 \in t_2, \beta_2 \in i_2, \gamma_2 \in f_2} \big\{\{\alpha_1 + \alpha_2 - \alpha_1\alpha_2\}, \{\beta_1\beta_2\}, \{\gamma_1\gamma_2\}\big\},$
4. $\boldsymbol{n}_1 \widehat{\bigotimes} \boldsymbol{n}_2 = \bigcup_{\alpha_1 \in t_1, \beta_1 \in i_1, \gamma_1 \in f_1, \alpha_2 \in t_2, \beta_2 \in i_2, \gamma_2 \in f_2} \big\{\{\alpha_1\alpha_2\}, \{\beta_1 + \beta_2 - \beta_1\beta_2\}, \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}\big\},$
5. $\lambda \boldsymbol{n}_1 = \bigcup_{\alpha_1 \in t_1, \beta_1 \in i_1, \gamma_1 \in f_1} \big\{\{1 - (1 - \alpha_1)^\lambda\}, \{\beta_1^\lambda\}, \{\gamma_1^\lambda\}\big\}, \quad \lambda > 0,$
6. $\boldsymbol{n}_1^\lambda = \bigcup_{\alpha_1 \in t_1, \beta_1 \in i_1, \gamma_1 \in f_1} \big\{\{\alpha_1^\lambda\}, \{1 - (1 - \beta_1)^\lambda\}, \{1 - (1 - \gamma_1)^\lambda\}\big\}, \quad \lambda > 0.$

Definition 12 (Ye 2015) Let $\boldsymbol{n}_1 = \{t_1, i_1, f_1\}$ and $\boldsymbol{n}_2 = \{t_2, i_2, f_2\}$ be two hesitant neutrosophic elements in X, then the cosine measure between $\boldsymbol{n}_j (j = 1, 2)$ and the ideal element $\boldsymbol{n}^\star = \{1, 0, 0\}$ is defined as follows:

$$\cos(\boldsymbol{n}_j, \boldsymbol{n}^\star) = \frac{\frac{1}{l_j} \sum_{\alpha_j \in t_j} \alpha_j}{\sqrt{\left[\frac{1}{l_j} \sum_{\alpha_j \in t_j} \alpha_j\right]^2 + \left[\frac{1}{m_j} \sum_{\beta_j \in i_j} \beta_j\right]^2 + \left[\frac{1}{n_j} \sum_{\gamma_j \in f_j} \gamma_j\right]^2}},$$

where l_j, m_j and n_j for $j = 1, 2$ are the numbers of the elements in t_j, i_j, f_j for $j = 1, 2$, respetively, and $\cos(\boldsymbol{n}_j, \boldsymbol{n}^\star) \in [0, 1]$ for $j = 1, 2$. Then, there are the following comparative laws based on the cosine measure:

1. If $\cos(\boldsymbol{n}_1, \boldsymbol{n}^\star) = \cos(\boldsymbol{n}_2, \boldsymbol{n}^\star)$, then $\boldsymbol{n}_1$ is equivalent to $\boldsymbol{n}_2$, denoted by $\boldsymbol{n}_1 \sim \boldsymbol{n}_2$,
2. If $\cos(\boldsymbol{n}_1, \boldsymbol{n}^\star) > \cos(\boldsymbol{n}_2, \boldsymbol{n}^\star)$, then $\boldsymbol{n}_1$ is superior to $\boldsymbol{n}_2$, denoted by $\boldsymbol{n}_1 \succ \boldsymbol{n}_2$.

3 Hesitant neutrosophic rough sets over two universes

In this section, we will introduce notions of hesitant neutrosophic rough sets over two universes. To begin with, we give the definition of hesitant neutrosophic relations over two universes as follows.

Definition 13 Let X, Y be two nonempty and finite universes, a hesitant neutrosophic relation $\mathcal{R}$ on $X \times Y$ is defined as

$$\mathcal{R} = \{\langle (a, b), t(a, b), i(a, b), f(a, b)\rangle \mid (a, b) \in X \times Y\},$$

in which $t(a, b)$, $i(a, b)$, and $f(a, b)$ are three sets of some values in [0, 1], denoting the possible truth-membership hesitant degrees, indeterminacy membership hesitant degrees, and falsity-membership hesitant degrees of the element $(a, b) \in X \times Y$ to the set $\mathcal{R}$, with the conditions $0 \leq \alpha, \beta, \gamma \leq 1$ and $0 \leq \alpha^+ + \beta^+ + \gamma^+ \leq 3$, where $\alpha \in t(a, b), \beta \in i(a, b), \gamma \in f(a, b)$, $\alpha^+ = \max t(a, b)$, $\beta^+ = \max i(a, b)$, and $\gamma^+ = \max f(a, b)$ for each $(a, b) \in X \times Y$. If $X = Y$, then $\mathcal{R}$ degenerates to a hesitant neutrosophic relation on the same universe X, and the family of all hesitant neutrosophic relations on $X \times Y$ will be denoted by $\mathbf{HNR}(X, Y)$.

Example 1 Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$. Then $\mathcal{R} \in \mathbf{HNR}(X, Y)$ is a hesitant neutrosophic relation (see Table 1).

Table 1 A hesitant neutrosophic relation $\mathcal{R}$ on $X \times Y$

$\mathcal{R}$	x_1
y_1	$\langle\{1\}, \{0.2\}, \{0.2, 0.1\}\rangle$
y_2	$\langle\{0.7, 0.6\}, \{0.5\}, \{0.3\}\rangle$
y_3	$\langle\{0\}, \{0.1, 0.2\}, \{0.8, 0.7\}\rangle$
$\mathcal{R}$	x_2
y_1	$\langle\{0.1, 0.3\}, \{0.3, 0.4\}, \{0.8, 1\}\rangle$
y_2	$\langle\{0.2, 0\}, \{0.1, 0.2\}, \{1, 0.9, 0.8\}\rangle$
y_3	$\langle\{0.1, 0\}, \{0.2\}, \{0.8, 0.7\}\rangle$
$\mathcal{R}$	x_3
y_1	$\langle\{0.2, 0.5, 0\}, \{0.3, 0.2\}, \{0.8, 0.9, 1\}\rangle$
y_2	$\langle\{0.8, 1\}, \{0.2\}, \{0.1, 0.3, 0\}\rangle$
y_3	$\langle\{0.9, 0.8\}, \{0.1, 0.2\}, \{0.3, 0.2\}\rangle$

For each hesitant neutrosophic element $\boldsymbol{n} = \{t, i, f\}$. Suppose that $l(t)$, $l(i)$ and $l(f)$ stand for the number of values in t, i, and f, respectively. To operate correctly, we will follow the assumptions as introduced by Xu and Xia (2011).

1. All hesitant fuzzy elements are arranged in increasing order, and the $h^{\sigma(k)}$ is referred to as the k-th largest value in the hesitant fuzzy element h.
2. If, for two hesitant fuzzy elements $h_1, h_2, l(h_1) \neq l(h_2)$, then

$$l = \max\{l(h_1), l(h_2)\}.$$

To have a correct comparison, the two hesitant fuzzy elements h_1 and h_2 should have the same length l. If there are fewer elements in h_1 than in h_2, an extension of h_1 should be considered optimistically by repeating its maximum element until it has the same length with h_2.

In the following, we define some operations in hesitant neutrosophic relations.

Definition 14 Let X, Y be two nonempty and finite universes, and let $\mathcal{R}_1$ and $\mathcal{R}_2$ are two hesitant neutrosophic relations on $X \times Y$, defined the complement, the union, the intersection, the ring sum and the ring product in them as follows:

$$\begin{aligned}\mathcal{R}_1^c = \{\langle(a,b), &\{\gamma_1^{\sigma(k)}(a,b) \mid k = 1,2,\dots,l(f_1(a,b))\},\\ &\{1-\beta_1^{\sigma(k)}(a,b) \mid k = 1,2,\dots,l(i_1(a,b))\},\\ &\{\alpha_1^{\sigma(k)}(a,b) \mid k = 1,2,\dots,l(t_1(a,b))\}\rangle \mid (a,b) \in X \times Y\},\end{aligned}$$

$$\begin{aligned}\mathcal{R}_1 \sqcup \mathcal{R}_2 = \{\langle(a,b), &\{\alpha_1^{\sigma(k)}(a,b) \vee \alpha_2^{\sigma(k)}(a,b) \mid k = 1,2,\dots,l\},\\ &\{\beta_1^{\sigma(k)}(a,b) \wedge \beta_2^{\sigma(k)}(a,b) \mid k = 1,2,\dots,m\},\\ &\{\gamma_1^{\sigma(k)}(a,b) \wedge \gamma_2^{\sigma(k)}(a,b) \mid k = 1,2,\dots,n\}\rangle \mid (a,b) \in X \times Y\},\end{aligned}$$

$$\begin{aligned}\mathcal{R}_1 \sqcap \mathcal{R}_2 = \{\langle(a,b), &\{\alpha_1^{\sigma(k)}(a,b) \wedge \alpha_2^{\sigma(k)}(a,b) \mid k = 1,2,\dots,l\},\\ &\{\beta_1^{\sigma(k)}(a,b) \vee \beta_2^{\sigma(k)}(a,b) \mid k = 1,2,\dots,m\},\\ &\{\gamma_1^{\sigma(k)}(a,b) \vee \gamma_2^{\sigma(k)}(a,b) \mid k = 1,2,\dots,n\}\rangle \mid (a,b) \in X \times Y\},\end{aligned}$$

$$\begin{aligned}\mathcal{R}_1 \bigoplus \mathcal{R}_2 = \{\langle(a,b), &\{\alpha_1^{\sigma(k)}(a,b) + \alpha_2^{\sigma(k)}(a,b) - \alpha_1^{\sigma(k)}(a,b)\alpha_2^{\sigma(k)}(a,b) \mid k = 1,2,\dots,l\},\\ &\{\beta_1^{\sigma(k)}(a,b)\beta_2^{\sigma(k)}(a,b) \mid k = 1,2,\dots,m\},\\ &\{\gamma_1^{\sigma(k)}(a,b)\gamma_2^{\sigma(k)}(a,b) \mid k = 1,2,\dots,n\}\rangle \mid (a,b) \in X \times Y\},\end{aligned}$$

$$\begin{aligned}\mathcal{R}_1 \bigotimes \mathcal{R}_2 = \{\langle(a,b), &\{\alpha_1^{\sigma(k)}(a,b)\alpha_2^{\sigma(k)}(a,b) \mid k = 1,2,\dots,l\},\\ &\{\beta_1^{\sigma(k)}(a,b) + \beta_2^{\sigma(k)}(a,b) - \beta_1^{\sigma(k)}(a,b)\beta_2^{\sigma(k)}(a,b) \mid k = 1,2,\dots,m\},\\ &\{\gamma_1^{\sigma(k)}(a,b) + \gamma_2^{\sigma(k)}(a,b) - \gamma_1^{\sigma(k)}(a,b)\gamma_2^{\sigma(k)}(a,b) \mid k = 1,2,\dots,n\}\rangle \mid (a,b) \in X \times Y\},\end{aligned}$$

where $\alpha_j^{\sigma(k)}(a,b), \beta_j^{\sigma(k)}(a,b)$, and $\gamma_j^{\sigma(k)}(a,b)$ are, respectively, the k-th largest value in $t_j(a,b), i_j(a,b)$ and $f_j(a,b)$ for each $j \in \{1,2\}$, and

$$l = \max\{l(t_1(a,b)), l(t_2(a,b))\},$$
$$m = \max\{l(i_1(a,b)), l(i_2(a,b))\},$$
$$n = \max\{l(f_1(a,b)), l(f_2(a,b))\}.$$

Remark 2 Obviously, these operations in hesitant neutrosophic relations are also applicable to hesitant neutrosophic sets, including the notation $\sqsubseteq$ in the following theorem.

Theorem 1 *Defined a relation* $\sqsubseteq$ *on* $\mathbf{HNR}(X,Y)$ *as follows:* $\mathcal{R}_1 \sqsubseteq \mathcal{R}_2$ *for any* $(a,b) \in X \times Y$, $t_1(a,b) \preceq t_2(a,b), i_1(a,b) \succeq i_2(a,b), f_1(a,b) \succeq f_2(a,b) \iff \alpha_1^{\sigma(k)}(a,b) \leq \alpha_2^{\sigma(k)}(a,b)\ (k = 1,2,\dots,l)$, $\beta_1^{\sigma(k)}(a,b) \geq \beta_2^{\sigma(k)}(a,b)\ (k = 1,2,\dots,m)$, *and* $\gamma_1^{\sigma(k)}(a,b) \geq \gamma_2^{\sigma(k)}(a,b)\ (k = 1,2,\dots,n)$, *where*

$$l = \max\{l(t_1(a,b)), l(t_2(a,b))\},$$
$$m = \max\{l(i_1(a,b)), l(i_2(a,b))\},$$
$$n = \max\{l(f_1(a,b)), l(f_2(a,b))\}.$$

Then $(\mathbf{HNR}(X,Y), \sqsubseteq)$ *is a poset.*

Proof It can be easily verify that the notation $\sqsubseteq$ satisfies reflexive, transitive and antisymmetric on $\mathbf{HNR}(X,Y)$.

By Definition 14 and Theorem 1, we have the following theorem:

Theorem 2 $(\mathbf{HNR}(X,Y), \sqsubseteq, \sqcup, \sqcap)$ *is a bounded lattice,* $\mathbb{X} \times \mathbb{Y}$ *and* $\emptyset$ *are its top element and bottom element, respectively, where* $\mathbb{X} \times \mathbb{Y}$ *and* $\emptyset$ *are two hesitant neutrosophic relations on* $X \times Y$ *and defined as follows:* $\forall (a,b) \in X \times Y$,

$$t_{\mathbb{X}\times\mathbb{Y}}(a,b) = \{1\}, \quad i_{\mathbb{X}\times\mathbb{Y}}(a,b) = \{0\}, \quad f_{\mathbb{X}\times\mathbb{Y}}(a,b) = \{0\},$$

and

$$t_{\emptyset}(a,b) = \{0\}, \quad i_{\emptyset}(a,b) = \{1\}, \quad f_{\emptyset}(a,b) = \{1\}.$$

Now, we give the notion of hesitant neutrosophic rough sets as follows.

Definition 15 Let X, Y be two nonempty and finite universes, and let $\mathcal{R}$ be a hesitant neutrosophic relation from X to Y, the tuple $(X, Y, \mathcal{R})$ is termed as a hesitant neutrosophic approximation space over two universes. $\forall \mathcal{N} \in \mathbf{HNS}(Y)$, the lower and upper approximations of $\mathcal{N}$ with respect to $(X, Y, \mathcal{R})$ are two hesitant neutrosophic sets in X, denoted by

$$\underline{\mathcal{R}}(\mathcal{N}) = \{\langle a, t_{\underline{\mathcal{R}}(\mathcal{N})}(a), i_{\underline{\mathcal{R}}(\mathcal{N})}(a), f_{\underline{\mathcal{R}}(\mathcal{N})}(a)\rangle \mid a \in X\},$$

and

$$\overline{\mathcal{R}}(\mathcal{N}) = \{\langle a, t_{\overline{\mathcal{R}}(\mathcal{N})}(a), i_{\overline{\mathcal{R}}(\mathcal{N})}(a), f_{\overline{\mathcal{R}}(\mathcal{N})}(a)\rangle \mid a \in X\},$$

where

$$t_{\underline{\mathcal{R}}(\mathcal{N})}(a) = \widetilde{\wedge}_{b\in Y}\big(f_{\mathcal{R}}(a,b)\widetilde{\vee}t_{\mathcal{N}}(b)\big),$$
$$i_{\underline{\mathcal{R}}(\mathcal{N})}(a) = \widetilde{\vee}_{b\in Y}(i^c_{\mathcal{R}}(a,b)\widetilde{\wedge}i_{\mathcal{N}}(b)),$$

$$f_{\underline{\mathcal{R}}(\mathcal{N})}(a) = \widetilde{\vee}_{b\in Y}(t_{\mathcal{R}}(a,b)\widetilde{\wedge}f_{\mathcal{N}}(b));$$
$$t_{\overline{\mathcal{R}}(\mathcal{N})}(a) = \widetilde{\vee}_{b\in Y}(t_{\mathcal{R}}(a,b)\widetilde{\wedge}t_{\mathcal{N}}(b)),$$

$$i_{\overline{\mathcal{R}}(\mathcal{N})}(a) = \widetilde{\wedge}_{b\in Y}(i_{\mathcal{R}}(a,b)\widetilde{\vee}i_{\mathcal{N}}(b)),$$
$$f_{\overline{\mathcal{R}}(\mathcal{N})}(a) = \widetilde{\wedge}_{b\in Y}(f_{\mathcal{R}}(a,b)\widetilde{\vee}f_{\mathcal{N}}(b)).$$

The pair $(\underline{\mathcal{R}}(\mathcal{N}), \overline{\mathcal{R}}(\mathcal{N}))$ is termed as the hesitant neutrosophic rough set of $\mathcal{N}$ with respect to $(X, Y, \mathcal{R})$, and $\underline{\mathcal{R}}, \overline{\mathcal{R}} : \mathbf{HNS}(Y) \to \mathbf{HNS}(X)$ are referred to as lower and upper hesitant neutrosophic rough approximation operators, respectively.

Remark 3 Notice that the hesitant neutrosophic set is a generalization of the concepts single valued neutrosophic refined set and hesitant fuzzy set. Thus, the hesitant neutrosophic rough set is also a generalization of single valued neutrosophic refined rough set (Bao and Yang 2017) and hesitant fuzzy rough set (Zhang et al. 2017a).

Example 2 Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$. $\mathcal{R} \in \mathbf{HNR}(X, Y)$ is a hesitant neutrosophic relation given in Table 1. Assume

$$\mathcal{N} = \{\langle y, t(y), i(y), f(y)\rangle \mid y \in Y\} \in \mathbf{HNS}(Y)$$

is given as follows:

$$\begin{aligned}\mathcal{N} = \{&\langle y_1, \{0.9, 0.8\}, \{0.1, 0.2\}, \{0.1, 0.3, 0\}\rangle,\\ &\langle y_2, \{0.7, 0.6\}, \{0.3, 0.1, 0\}, \{0.2, 0.1\}\rangle,\\ &\langle y_3, \{0.5\}, \{0.2, 0.1\}, \{0.5, 0.3\}\rangle\}.\end{aligned}$$

By Definitions 14 and 15 (notice that the equivalences of the formulas), we can obtain the upper and lower approximations of A with respect to $(X, Y, \mathcal{R})$ as follows:

$$\begin{aligned}\overline{\mathcal{R}}(\mathcal{N}) = \{&\langle x_1, \{0.8, 0.9\}, \{0.1, 0.2, 0.2\}, \{0.1, 0.2, 0, 3\}\rangle,\\ &\langle x_2, \{0.1, 0.3\}, \{0.1, 0.2, 0.2\}, \{0.7, 0.8, 0.8\}\rangle,\\ &\langle x_3, \{0.6, 0.7, 0.7\}, \{0.1, 0.2, 0.2\}, \{0.1, 0.2, 0.3\}\rangle\},\end{aligned}$$

and

$$\underline{\mathcal{R}}(\mathcal{N}) = \{\langle x_1, \{0.6, 0.7\}, \{0.1, 0.2, 0.3\}, \{0.1, 0.2, 0.3\}\rangle,$$
$$\langle x_2, \{0.7, 0.8, 0.8\}, \{0.1, 0.2, 0.3\}, \{0, 0.2, 0.3\}\rangle,$$
$$\langle x_3, \{0.5, 0.5, 0.5\}, \{0.1, 0.2, 0.3\}, \{0.3, 0.5, 0.5\}\rangle\}.$$

In general, $\underline{\mathcal{R}}(\mathcal{N}) \sqsubseteq \overline{\mathcal{R}}(\mathcal{N})$ can not hold.

4 The join semi-lattice structure of lower and upper hesitant neutrosophic rough approximation operators over two universes

In this section, let X, Y be two nonempty and finite universes, we take

$$H(X, Y) = \{\overline{\mathcal{R}} : \mathbf{HNS}(Y) \rightarrow \mathbf{HNS}(X) \mid \mathcal{R} \in \mathbf{HNR}(X, Y)\}$$

and

$$L(X, Y) = \{\underline{\mathcal{R}} : \mathbf{HNS}(Y) \rightarrow \mathbf{HNS}(X) \mid \mathcal{R} \in \mathbf{HNR}(X, Y)\}$$

be the family of hesitant neutrosophic rough upper and lower approximation operators on X, respectively. We will study the lattice structure of hesitant neutrosophic rough approximation operators.

Theorem 3 *Let $(X, Y, \mathcal{R}_1)$ and $(X, Y, \mathcal{R}_2)$ be two hesitant neutrosophic approximation spaces over two universes. Then for any $\mathcal{N} \in \mathbf{HNS}(Y)$,*

1. $$\underline{\mathcal{R}_1}(\mathcal{N}^c) = (\overline{\mathcal{R}_1}(\mathcal{N}))^c,$$
$$\overline{\mathcal{R}_1}(\mathcal{N}^c) = (\underline{\mathcal{R}_1}(\mathcal{N}))^c,$$

2. $$\underline{\mathcal{R}_1 \sqcup \mathcal{R}_2}(\mathcal{N}) = \underline{\mathcal{R}_1}(\mathcal{N}) \sqcap \underline{\mathcal{R}_2}(\mathcal{N}),$$
$$\underline{\mathcal{R}_1}(\mathcal{N}) \sqcup \underline{\mathcal{R}_2}(\mathcal{N}) \sqsubseteq \underline{\mathcal{R}_1 \sqcap \mathcal{R}_2}(\mathcal{N}).$$

3. $$\overline{\mathcal{R}_1 \sqcup \mathcal{R}_2}(\mathcal{N}) = \overline{\mathcal{R}_1}(\mathcal{N}) \sqcup \overline{\mathcal{R}_2}(\mathcal{N}),$$
$$\overline{\mathcal{R}_1 \sqcap \mathcal{R}_2}(\mathcal{N}) \sqsubseteq \overline{\mathcal{R}_1}(\mathcal{N}) \sqcap \overline{\mathcal{R}_2}(\mathcal{N}).$$

Proof

1. The proofs are straightforward from Definitions 14 and 15.
2. $\forall a \in X$, by Definitions 14 and 15, we have

$$
\begin{aligned}
& t_{\underline{\mathcal{R}_1 \sqcup \mathcal{R}_2}(\mathcal{N})}(a) \\
&\quad = \widetilde{\wedge}_{b\in Y}\big(f_{\mathcal{R}_1 \sqcup \mathcal{R}_2}(a,b)\widetilde{\vee} t_{\mathcal{N}}(b)\big), \\
&\quad = \widetilde{\wedge}_{b\in Y}\big((f_{\mathcal{R}_1}(a,b)\widetilde{\wedge} f_{\mathcal{R}_2}(a,b))\widetilde{\vee} t_{\mathcal{N}}(b)\big) \\
&\quad = \widetilde{\wedge}_{b\in Y}\{[\gamma^{\sigma(k)}_{\mathcal{R}_1}(a,b) \wedge \gamma^{\sigma(k)}_{\mathcal{R}_2}(a,b)] \vee \alpha^{\sigma(k)}_{\mathcal{N}}(b) \mid k = 1,2,\dots,l\} \\
&\quad = \widetilde{\wedge}_{b\in Y}\{(\gamma^{\sigma(k)}_{\mathcal{R}_1}(a,b) \vee \alpha^{\sigma(k)}_{\mathcal{N}}(b)) \wedge (\gamma^{\sigma(k)}_{\mathcal{R}_2}(a,b) \vee \alpha^{\sigma(k)}_{\mathcal{N}}(b)) \mid k = 1,2,\dots,l\} \\
&\quad = \widetilde{\wedge}_{b\in Y}\{\gamma^{\sigma(k)}_{\mathcal{R}_1}(a,b) \vee \alpha^{\sigma(k)}_{\mathcal{N}}(b) \mid k = 1,2,\dots,m\} \\
&\qquad \widetilde{\wedge}\ \widetilde{\wedge}_{b\in Y}\{\gamma^{\sigma(k)}_{\mathcal{R}_2}(a,b) \vee \alpha^{\sigma(k)}_{\mathcal{N}}(b) \mid k = 1,2,\dots,n\} \\
&\quad = \widetilde{\wedge}_{b\in Y}\big(f_{\mathcal{R}_1}(a,b)\widetilde{\vee} t_{\mathcal{N}}(b)\big)\widetilde{\wedge}\ \widetilde{\wedge}_{b\in Y}\big(f_{\mathcal{R}_2}(a,b)\widetilde{\vee} t_{\mathcal{N}}(b)\big) \\
&\quad = t_{\underline{\mathcal{R}_1}(\mathcal{N})}(a)\widetilde{\wedge}\ t_{\underline{\mathcal{R}_2}(\mathcal{N})}(a) \\
&\quad = t_{\underline{\mathcal{R}_1}(\mathcal{N}) \sqcap \underline{\mathcal{R}_2}(\mathcal{N})}(a),
\end{aligned}
$$

where

$$
\begin{aligned}
m &= \max\{l(f_{\mathcal{R}_1}(a,b)), l(t_{\mathcal{N}}(b))\}, n = \max\{l(f_{\mathcal{R}_2}(a,b)), l(t_{\mathcal{N}}(b))\}, \\
l &= \max\{l(f_{\mathcal{R}_1}(a,b)), l(f_{\mathcal{R}_2}(a,b)), l(t_{\mathcal{N}}(b))\}.
\end{aligned}
$$

$$
\begin{aligned}
& i_{\underline{\mathcal{R}_1 \sqcup \mathcal{R}_2}(\mathcal{N})}(a) \\
&\quad = \widetilde{\vee}_{b\in Y}(i^c_{\mathcal{R}_1 \sqcup \mathcal{R}_2}(a,b)\widetilde{\wedge} i_{\mathcal{N}}(b)) \\
&\quad = \widetilde{\vee}_{b\in Y}\Big((i^c_{\mathcal{R}_1}(a,b)\widetilde{\vee} i^c_{\mathcal{R}_2}(a,b))\widetilde{\wedge} i_{\mathcal{N}}(b)\Big) \\
&\quad = \widetilde{\vee}_{b\in Y}\{([1-\beta^{\sigma(l+1-k)}_{\mathcal{R}_1}(a,b)] \vee [1-\beta^{\sigma(l+1-k)}_{\mathcal{R}_2}(a,b)]) \wedge \beta^{\sigma(k)}_{\mathcal{N}}(b) \mid k = 1,2,\dots,l\} \\
&\quad = \widetilde{\vee}_{b\in Y}\{[1-\beta^{\sigma(m+1-k)}_{\mathcal{R}_1}(a,b)] \wedge \beta^{\sigma(k)}_{\mathcal{N}}(b) \mid k = 1,2,\dots,m\}\widetilde{\vee} \\
&\qquad \widetilde{\vee}_{b\in Y}\{[1-\beta^{\sigma(n+1-k)}_{\mathcal{R}_2}(a,b)] \wedge \beta^{\sigma(k)}_{\mathcal{N}}(b) \mid k = 1,2,\dots,n\} \\
&\quad = \widetilde{\vee}_{b\in Y}(i^c_{\mathcal{R}_1}(a,b)\widetilde{\wedge} i_{\mathcal{N}}(b))\widetilde{\vee}\ \widetilde{\vee}_{b\in Y}(i^c_{\mathcal{R}_2}(a,b)\widetilde{\wedge} i_{\mathcal{N}}(b)) \\
&\quad = i_{\underline{\mathcal{R}_1}(\mathcal{N})}(a)\widetilde{\vee}\pi_{\underline{\mathcal{R}_2}(\mathcal{N})}(a) \\
&\quad = i_{\underline{\mathcal{R}_1}(\mathcal{N}) \sqcap \underline{\mathcal{R}_2}(\mathcal{N})}(a)
\end{aligned}
$$

where

$$
\begin{aligned}
m &= \max\{l(i_{\mathcal{R}_1}(a,b)), l(i_{\mathcal{N}}(b))\}, n = \max\{l(i_{\mathcal{R}_2}(a,b)), l(i_{\mathcal{N}}(b))\}, \\
l &= \max\{l(i_{\mathcal{R}_1}(a,b)), l(i_{\mathcal{R}_2}(a,b)), l(i_{\mathcal{N}}(b))\}.
\end{aligned}
$$

$$\begin{aligned}
& f_{\underline{\mathcal{R}_1 \sqcup \mathcal{R}_2}(\mathcal{N})}(a) \\
&\quad = \widetilde{\vee}_{b\in Y}(t_{\mathcal{R}_1\sqcup\mathcal{R}_2}(a,b)\widetilde{\wedge}f_{\mathcal{N}}(b)) \\
&\quad = \widetilde{\vee}_{b\in Y}\big((t_{\mathcal{R}_1}(a,b)\widetilde{\vee}t_{\mathcal{R}_2}(a,b))\widetilde{\wedge}f_{\mathcal{N}}(b)\big) \\
&\quad = \widetilde{\vee}_{b\in Y}\{[\alpha^{\sigma(k)}_{\mathcal{R}_1}(a,b)\vee\alpha^{\sigma(k)}_{\mathcal{R}_1}(a,b)]\wedge\gamma^{\sigma(k)}_{\mathcal{N}}(b)\mid k=1,2,\dots,l\} \\
&\quad = \widetilde{\vee}_{b\in Y}\{\alpha^{\sigma(k)}_{\mathcal{R}_1}(a,b)\wedge\gamma^{\sigma(k)}_{\mathcal{N}}(b)\mid k=1,2,\dots,m\}\widetilde{\vee} \\
&\qquad \widetilde{\vee}_{b\in Y}\{\alpha^{\sigma(k)}_{\mathcal{R}_2}(a,b)\wedge\gamma^{\sigma(k)}_{\mathcal{N}}(b)\mid k=1,2,\dots,n\} \\
&\quad = \widetilde{\vee}_{b\in Y}(t_{\mathcal{R}_1}(a,b)\widetilde{\wedge}f_{\mathcal{N}}(b))\ \widetilde{\vee}\ \widetilde{\vee}_{b\in Y}(t_{\mathcal{R}_2}(a,b)\widetilde{\wedge}f_{\mathcal{N}}(b)) \\
&\quad = f_{\underline{\mathcal{R}_1}(\mathcal{N})}(a)\widetilde{\vee}f_{\underline{\mathcal{R}_2}(\mathcal{N})}(a) \\
&\quad = f_{\underline{\mathcal{R}_1}(\mathcal{N})\sqcap\underline{\mathcal{R}_2}(\mathcal{N})}(a),
\end{aligned}$$

where

$$\begin{aligned}
m &= \max\{l(t_{\mathcal{R}_1}(a,b)),l(f_{\mathcal{N}}(b))\}, n=\max\{l(t_{\mathcal{R}_2}(a,b)),l(f_{\mathcal{N}}(b))\}, \\
l &= \max\{l(t_{\mathcal{R}_1}(a,b)),l(t_{\mathcal{R}_2}(a,b)),l(f_{\mathcal{N}}(b))\}.
\end{aligned}$$

From which it follows that $\underline{\mathcal{R}_1\sqcup\mathcal{R}_2}(\mathcal{N})=\underline{\mathcal{R}_1}(\mathcal{N})\sqcap\underline{\mathcal{R}_2}(\mathcal{N})$. Similarly,

$$\begin{aligned}
& t_{\underline{\mathcal{R}_1}(\mathcal{N})\sqcup\underline{\mathcal{R}_2}(\mathcal{N})}(a) \\
&\quad = t_{\underline{\mathcal{R}_1}(\mathcal{N})}(a)\widetilde{\vee}t_{\underline{\mathcal{R}_2}(\mathcal{N})}(a) \\
&\quad = \widetilde{\wedge}_{b\in Y}\big(f_{\mathcal{R}_1}(a,b)\widetilde{\vee}t_{\mathcal{N}}(b)\big)\widetilde{\vee}\widetilde{\wedge}_{b\in Y}\big(f_{\mathcal{R}_2}(a,b)\widetilde{\vee}t_{\mathcal{N}}(b)\big) \\
&\quad = \widetilde{\wedge}_{b\in Y}\{\gamma^{\sigma(k)}_{\mathcal{R}_1}(a,b)\vee\alpha^{\sigma(k)}_{\mathcal{N}}(b)\mid k=1,2,\dots,m\}\widetilde{\vee} \\
&\qquad \widetilde{\wedge}_{b\in Y}\{\gamma^{\sigma(k)}_{\mathcal{R}_2}(a,b)\vee\alpha^{\sigma(k)}_{\mathcal{N}}(b)\mid k=1,2,\dots,n\} \\
&\quad \preceq \widetilde{\wedge}_{b\in Y}\{(\gamma^{\sigma(k)}_{\mathcal{R}_1}(a,b)\vee\gamma^{\sigma(k)}_{\mathcal{R}_2}(a,b))\vee\alpha^{\sigma(k)}_{\mathcal{N}}(b)\mid k=1,2,\dots,l\} \\
&\quad = \widetilde{\wedge}_{b\in Y}\big((f_{\mathcal{R}_1}(a,b)\widetilde{\vee}f_{\mathcal{R}_2}(a,b))\widetilde{\vee}t_{\mathcal{N}}(b)\big) \\
&\quad = \widetilde{\wedge}_{b\in Y}\big(f_{\mathcal{R}_1\sqcap\mathcal{R}_2}(a,b)\widetilde{\vee}t_{\mathcal{N}}(b)\big) \\
&\quad = t_{\underline{\mathcal{R}_1\sqcap\mathcal{R}_2}(\mathcal{N})}(a),
\end{aligned}$$

where

$$\begin{aligned}
m &= \max\{l(f_{\mathcal{R}_1}(a,b)),l(t_{\mathcal{N}}(b))\}, n=\max\{l(f_{\mathcal{R}_2}(a,b)),l(t_{\mathcal{N}}(b))\}, \\
l &= \max\{l(f_{\mathcal{R}_1}(a,b)),l(f_{\mathcal{R}_2}(a,b)),l(t_{\mathcal{N}}(b))\}.
\end{aligned}$$

$$
\begin{aligned}
& i_{\underline{\mathcal{R}_1}(\mathcal{N})\sqcup\underline{\mathcal{R}_2}(\mathcal{N})}(a) \\
&\quad = i_{\underline{\mathcal{R}_1}(\mathcal{N})}(a)\widetilde{\wedge} i_{\underline{\mathcal{R}_2}(\mathcal{N})}(a) \\
&\quad = \widetilde{\vee}_{b\in Y}(i^c_{\mathcal{R}_1}(a,b)\widetilde{\wedge} i_{\mathcal{N}}(b))\widetilde{\wedge}\widetilde{\vee}_{b\in Y}(i^c_{\mathcal{R}_2}(a,b)\widetilde{\wedge} i_{\mathcal{N}}(b)) \\
&\quad = \widetilde{\vee}_{b\in Y}\{[1-\beta^{\sigma(m+1-k)}_{\mathcal{R}_1}(a,b)] \wedge \beta^{\sigma(k)}_{\mathcal{N}}(b) \mid k=1,2,\dots,m\}\widetilde{\wedge} \\
&\quad\quad \widetilde{\vee}_{b\in Y}\{[1-\beta^{\sigma(n+1-k)}_{\mathcal{R}_2}(a,b)] \wedge \beta^{\sigma(k)}_{\mathcal{N}}(b) \mid k=1,2,\dots,n\} \\
&\quad \geq \widetilde{\vee}_{b\in Y}\{[1-\beta^{\sigma(l+1-k)}_{\mathcal{R}_1}(a,b) \vee \beta^{\sigma(l+1-k)}_{\mathcal{R}_2}(a,b)] \wedge \beta^{\sigma(k)}_{\mathcal{N}}(b) \mid k=1,2,\dots,l\} \\
&\quad = \widetilde{\vee}_{b\in Y}\left(i^c_{\mathcal{R}_1\sqcap\mathcal{R}_2}(a,b)\widetilde{\wedge} i_{\mathcal{N}}(b)\right) \\
&\quad = i_{\underline{\mathcal{R}_1\sqcap\mathcal{R}_2}(\mathcal{N})}(a)
\end{aligned}
$$

where

$$
\begin{aligned}
m &= \max\{l(i_{\mathcal{R}_1}(a,b)), l(i_{\mathcal{N}}(b))\}, n = \max\{l(i_{\mathcal{R}_2}(a,b)), l(i_{\mathcal{N}}(b))\}, \\
l &= \max\{l(i_{\mathcal{R}_1}(a,b)), l(i_{\mathcal{R}_2}(a,b)), l(i_{\mathcal{N}}(b))\}.
\end{aligned}
$$

$$
\begin{aligned}
& f_{\underline{\mathcal{R}_1}(\mathcal{N})\sqcup\underline{\mathcal{R}_2}(\mathcal{N})}(a) \\
&\quad = f_{\underline{\mathcal{R}_1}(\mathcal{N})}(a)\widetilde{\wedge} t_{\underline{\mathcal{R}_2}(\mathcal{N})}(a) \\
&\quad = \widetilde{\vee}_{b\in Y}\left(t_{\mathcal{R}_1}(a,b)\widetilde{\wedge} f_{\mathcal{N}}(b)\right)\widetilde{\wedge}\widetilde{\vee}_{b\in Y}\left(t_{\mathcal{R}_2}(a,b)\widetilde{\wedge} f_{\mathcal{N}}(b)\right) \\
&\quad = \widetilde{\vee}_{b\in Y}\{\alpha^{\sigma(k)}_{\mathcal{R}_1}(a,b) \wedge \gamma^{\sigma(k)}_{\mathcal{N}}(b) \mid k=1,2,\dots,m\}\widetilde{\wedge} \\
&\quad\quad \widetilde{\vee}_{b\in Y}\{\alpha^{\sigma(k)}_{\mathcal{R}_2}(a,b) \wedge \gamma^{\sigma(k)}_{\mathcal{N}}(b) \mid k=1,2,\dots,n\} \\
&\quad \geq \widetilde{\vee}_{b\in Y}\{(\alpha^{\sigma(k)}_{\mathcal{R}_1}(a,b) \wedge \alpha^{\sigma(k)}_{\mathcal{R}_2}(a,b)) \wedge \gamma^{\sigma(k)}_{\mathcal{N}}(b) \mid k=1,2,\dots,l\} \\
&\quad = \widetilde{\vee}_{b\in Y}\left((t_{\mathcal{R}_1}(a,b)\widetilde{\wedge} t_{\mathcal{R}_2}(a,b))\widetilde{\wedge} f_{\mathcal{N}}(b)\right) \\
&\quad = \widetilde{\vee}_{b\in Y}\left(t_{\mathcal{R}_1\sqcap\mathcal{R}_2}(a,b)\widetilde{\wedge} f_{\mathcal{N}}(b)\right) \\
&\quad = f_{\underline{\mathcal{R}_1\sqcap\mathcal{R}_2}(\mathcal{N})}(a),
\end{aligned}
$$

where

$$
\begin{aligned}
m &= \max\{l(t_{\mathcal{R}_1}(a,b)), l(f_{\mathcal{N}}(b))\}, n = \max\{l(t_{\mathcal{R}_2}(a,b)), l(f_{\mathcal{N}}(b))\}, \\
l &= \max\{l(t_{\mathcal{R}_1}(a,b)), l(t_{\mathcal{R}_2}(a,b)), l(f_{\mathcal{N}}(b))\}.
\end{aligned}
$$

From which it follows that $\underline{\mathcal{R}_1}(\mathcal{N}) \sqcup \underline{\mathcal{R}_2}(\mathcal{N}) \sqsubseteq \underline{\mathcal{R}_1 \sqcap \mathcal{R}_2}(\mathcal{N})$.

3. It follows immediately from the above results (1) and (2).

By Theorem 3, we have the following corollary:

Corollary 1 *Let* $(X, Y, \mathcal{R}_1)$ *and* $(X, Y, \mathcal{R}_2)$ *be two hesitant neutrosophic approximation spaces over two universes.* $\forall \mathcal{N} \in \mathbf{HNS}(Y)$. *If* $\mathcal{R}_1 \sqsubseteq \mathcal{R}_2$, *then* $\underline{\mathcal{R}_2}(\mathcal{N}) \sqsubseteq \underline{\mathcal{R}_1}(\mathcal{N})$ *and* $\overline{R_1}(\mathcal{N}) \sqsubseteq \overline{\mathcal{R}_2}(\mathcal{N})$.

Theorem 4 *Let* $(X, Y, \mathcal{R}_1)$ *and* $(X, Y, \mathcal{R}_2)$ *be two hesitant neutrosophic approximation spaces over two universes, defined a relation* $\leqq$ *on H(X, Y) as follows:* $\overline{\mathcal{R}_1} \leqq \overline{\mathcal{R}_2}$ *if and only if* $\overline{R_1}(\mathcal{N}) \sqsubseteq \overline{\mathcal{R}_2}(\mathcal{N})$ *for each* $\mathcal{N} \in \mathbf{HNS}(Y)$. *Then* $(H(X, Y), \leqq)$ *is a join semi-lattice.*

Proof By Remark 2 and Theorem 1, we can easily obtain that $(H(X, Y), \leqq)$ is a poset.
$\forall \{\overline{\mathcal{R}_i}\}_{i\in\{1,2\}} \subseteq (H(X, Y), \leqq)$, we can define union of $\overline{\mathcal{R}_i}$ as follows:

$$\overline{\mathcal{R}_1} \hat{\vee} \overline{\mathcal{R}_2} = \overline{\mathcal{R}_1 \sqcup \mathcal{R}_2}.$$

Then $\overline{\mathcal{R}_1} \hat{\vee} \overline{\mathcal{R}_2}$ is the supremum of $\{\overline{\mathcal{R}_i}\}_{i\in\{1,2\}}$.

In fact, let $\mathcal{R} = \mathcal{R}_1 \sqcup \mathcal{R}_2$, then $\mathcal{R}_i \sqsubseteq \mathcal{R}$ for each $i \in \{1, 2\}$. By Corollary 1, we have $\overline{\mathcal{R}_i} \leqq \overline{\mathcal{R}}$. If $\mathcal{R}'$ is a hesitant neutrosophic relation such that $\overline{\mathcal{R}_i} \leqq \overline{\mathcal{R}'}$ for each $i \in \{1, 2\}$, then $\forall \mathcal{N} \in \mathbf{HNS}(Y)$, $\overline{\mathcal{R}_i}(\mathcal{N}) \sqsubseteq \overline{\mathcal{R}'}(\mathcal{N})$. Moreover, by Theorem 3, we have $\overline{\mathcal{R}}(\mathcal{N}) = \overline{\mathcal{R}_1 \sqcup \mathcal{R}_2}(\mathcal{N}) = \overline{\mathcal{R}_1}(\mathcal{N}) \sqcup \overline{\mathcal{R}_2}(\mathcal{N}) \sqsubseteq \overline{\mathcal{R}'}(\mathcal{N})$. This is equivalent to $\overline{\mathcal{R}} \leqq \overline{\mathcal{R}'}$. So $\overline{\mathcal{R}_1} \hat{\vee} \overline{\mathcal{R}_2} = \overline{\mathcal{R}_1 \sqcup \mathcal{R}_2}$ is the supremum of $\{\overline{\mathcal{R}_i}\}_{i\in\{1,2\}}$. Thus, $(H(X, Y), \leqq)$ is a join semi-lattice.

Theorem 5 *Let* $(X, Y, \mathcal{R}_1)$ *and* $(X, Y, \mathcal{R}_2)$ *be two hesitant neutrosophic approximation spaces over two universes. defined a relation* $\leqq$ *on L (X, Y) as follows:* $\underline{\mathcal{R}_1} \leqq \underline{\mathcal{R}_2}$ *if and only if* $\underline{\mathcal{R}_2}(\mathcal{N}) \sqsubseteq \underline{\mathcal{R}_1}(\mathcal{N})$ *for each* $\forall \mathcal{N} \in \mathbf{HNS}(Y)$. *Then* $(L(X, Y), \leqq)$ *is a join semi-lattice.*

Proof By Remark 2 and Theorem 1, we can easily obtain that $(L(X, Y), \leqq)$ is a poset.

$\forall \{\underline{\mathcal{R}_j}\}_{j\in\{1,2\}} \subseteq (L(X, Y), \leqq)$, we can define union of $\underline{\mathcal{R}_j}$ as follows:

$$\underline{\mathcal{R}_1} \hat{\vee} \underline{\mathcal{R}_2} = \underline{\mathcal{R}_1 \sqcup \mathcal{R}_2}.$$

Then $\underline{\mathcal{R}_1} \hat{\vee} \underline{\mathcal{R}_2}$ is the supremum of $\{\underline{\mathcal{R}_j}\}_{j\in\{1,2\}}$.

Let $\mathcal{R} = \mathcal{R}_1 \sqcup \mathcal{R}_2$, then $\mathcal{R}_j \sqsubseteq \mathcal{R}$ for each $j \in \{1, 2\}$. By Corollary 1, we have $\underline{\mathcal{R}_j} \leqq \underline{\mathcal{R}}$. If $\mathcal{R}'$ is a hesitant neutrosophic relation such that $\underline{\mathcal{R}_j} \leqq \underline{\mathcal{R}'}$ for each $j \in \{1, 2\}$, then $\forall \mathcal{N} \in \mathbf{HNS}(Y)$, $\underline{\mathcal{R}'}(\mathcal{N}) \sqsubseteq \underline{\mathcal{R}_j}(\mathcal{N})$. Moreover, by Theorem 3, we have $\underline{\mathcal{R}'}(\mathcal{N}) \sqsubseteq \underline{\mathcal{R}_1}(\mathcal{N}) \sqcap \underline{\mathcal{R}_2}(\mathcal{N}) = \underline{\mathcal{R}_1 \sqcup \mathcal{R}_2}(\mathcal{N}) = \underline{\mathcal{R}}(\mathcal{N})$. This is equivalent to $\underline{\mathcal{R}} \leqq \underline{\mathcal{R}'}$. So $\underline{\mathcal{R}_1} \hat{\vee} \underline{\mathcal{R}_2} = \underline{\mathcal{R}_1 \sqcup \mathcal{R}_2}$ is the supremum of $\{\underline{\mathcal{R}_j}\}_{j\in\{1,2\}}$. Thus, $(L(X, Y), \leqq)$ is a join semi-lattice.

5 An application of hesitant neutrosophic rough sets over two universes

In this section, a general framework is presented for the decision making based on hesitant neutrosophic rough sets over two universes. we will consider medical diagnosis problem based on hesitant neutrosophic rough sets over two universes.

Suppose that the universe $X = \{x_1, x_2, \dots, x_n\}$ denotes a set of diseases, and the universe $Y = \{y_1, y_2, \dots, y_m\}$ represents a set of symptoms. Let $\mathcal{R} \in \mathbf{HNR}(X, Y)$ be a hesitant neutrosophic relation from X to Y, where $\forall (x_i, y_j) \in X \times Y$, $\mathcal{R}(x_i, y_j)$ denoted the degree that the disease x_i has the symptom y_i. $\mathcal{R}$ is a medical knowledge statistic data is obtained from plenty of clinical experience in advance.

Given a patient $\mathcal{N}$, symptoms of the patient are described by a hesitant neutrosophic set $\mathcal{N}$ in the universe Y according to different doctors. In the following, we propose an algorithm to diagnose which kind of disease the patient $\mathcal{N}$ is suffering from.

Algorithm

Step 1. By Definition 15, we calculate the hesitant neutrosophic rough set $(\underline{\mathcal{R}}(\mathcal{N}), \overline{\mathcal{R}}(\mathcal{N}))$ of $\mathcal{N}$.

Step 2. By Definition 14 and Remark 2, we get $\underline{\mathcal{R}}(\mathcal{N}) \bigoplus \overline{\mathcal{R}}(\mathcal{N})$.

Step 3. By Definition 12, we compute the cosine measure between each hesitant neutrosophic element $\boldsymbol{n}(x_j)$ $(j = 1, 2, \dots, n)$ in $\underline{\mathcal{R}}(\mathcal{N}) \bigoplus \overline{\mathcal{R}}(\mathcal{N})$ and the ideal element $\boldsymbol{n}^\star = \{1, 0, 0\}$, i.e., $\cos(\boldsymbol{n}(x_j), \boldsymbol{n}^\star)$ $(j = 1, 2, \dots, n)$.

Step 4. The optimal decision is to select $\boldsymbol{n}(x_k)$ if

$$\cos(\boldsymbol{n}(x_k), \boldsymbol{n}^\star) = \max_{j \in \{1,2,\dots,n\}} \cos(\boldsymbol{n}(x_j), \boldsymbol{n}^\star).$$

Step 5. If k has more than one value, then each $\boldsymbol{n}(x_k)$ will be the optimal decision. In this case, the patient may suffer more than one disease and each $\boldsymbol{n}(x_k)$ will be chosen as the most possible disease, or we need other methods to make a further decision.

Table 2 The hesitant neutrosophic relation $\mathcal{R}$ on $X \times Y$

$\mathcal{R}$	x_1
y_1	$\langle\{0.4, 0.5, 0.2\}, \{0.2, 0.3, 0.8\}, \{0.3, 0.4, 0.2\}\rangle$
y_2	$\langle\{0.5, 0.6, 0.6\}, \{0.3, 0.4, 0.2\}, \{0.2, 0.3, 0.1\}\rangle$
y_3	$\langle\{0, 0.1, 0\}, \{0.1, 0.2, 0.2\}, \{0.8, 0.9, 0.8\}\rangle$
y_4	$\langle\{0.7, 0.8, 0.8\}, \{0.3, 0.4, 0.3\}, \{0.2, 0.3, 0.1\}\rangle$
y_5	$\langle\{0.4, 0.5, 0.4\}, \{0.5, 0.6, 0.6\}, \{0.6, 0.8, 0.7\}\rangle$
$\mathcal{R}$	x_2
y_1	$\langle\{0.8, 0.9, 0.9\}, \{0.1, 0.2, 0.2\}, \{0, 0.1, 0.1\}\rangle$
y_2	$\langle\{0.8, 0.9, 0.8\}, \{0.2, 0.3, 0.2\}, \{0, 0.1, 0.1\}\rangle$
y_3	$\langle\{0.1, 0.2, 0\}, \{0.2, 0.3, 0.1\}, \{0.8, 0.7, 0.9\}\rangle$
y_4	$\langle\{0, 0.1, 0.1\}, \{0.1, 0.2, 0\}, \{0.8, 0.9, 1\}\rangle$
y_5	$\langle\{0, 0.1, 0.2\}, \{0.1, 0.2, 0.3\}, \{0.9, 1, 0.8\}\rangle$
$\mathcal{R}$	x_3
y_1	$\langle\{0.8, 1, 0.9\}, \{0.4, 0.2, 0.3\}, \{0, 0.1, 0.1\}\rangle$
y_2	$\langle\{0.9, 1, 1\}, \{0.3, 0.1, 0.1\}, \{0, 0.1, 0\}\rangle$
y_3	$\langle\{0.7, 0.9, 0.8\}, \{0.4, 0.6, 0.5\}, \{0.3, 0.1, 0.2\}\rangle$
y_4	$\langle\{0, 0.1, 0.1\}, \{0.3, 0.2, 0.4\}, \{0.8, 0.7, 0.9\}\rangle$
y_5	$\langle\{0, 0.2, 0.1\}, \{0.3, 0.4, 0.2\}, \{0.6, 0.7, 1\}\rangle$
$\mathcal{R}$	x_4
y_1	$\langle\{0.2, 0.3, 0.1\}, \{0.3, 0.4, 0.1\}, \{0.8, 1, 0.9\}\rangle$
y_2	$\langle\{0.1, 0.2, 0\}, \{0.2, 0.3, 0.1\}, \{0.8, 1, 0.9\}\rangle$
y_3	$\langle\{0.9, 0.9, 1\}, \{0.4, 0.5, 0.6\}, \{0.1, 0.3, 0.2\}\rangle$
y_4	$\langle\{0, 0.1, 0.2\}, \{0.1, 0.2, 0.2\}, \{0.8, 0.7, 0.9\}\rangle$
y_5	$\langle\{0.1, 0.2, 0.4\}, \{0.4, 0.2, 0.3\}, \{0.8, 0.7, 0.6\}\rangle$

In what follows, we give a numerical example to illustrate the application of hesitant neutrosophic rough sets over two universes by us of the algorithm above.

Example 3 Let $Y = \{y_1, y_2, y_3, y_4, y_5\}$ be five symptoms in clinic, where y_i $(i = 1, 2, 3, 4)$ stand for "stomach pain", "temperature", "headache", "cough" and "chest-pain", respectively, and $X = \{x_1, x_2, x_3, x_4\}$ be a set of four diseases, x_j $(j = 1, 2, 3, 4)$ represents "Viral fever", "Stomach disease" "Typhoid", and "Malaria" respectively. $\mathcal{R}$ be a hesitant neutrosophic relation on $X \times Y$ which is actually a medical knowledge statistic data of the relationship between the symptom y_i and the disease x_j, The statistic date is provided in Table 2.

In clinical practice, a patient can see different docttors and may get different diagnoses. To decrease the risk of misdiagnosis, we should carefully consider all the doctor's comments. So the symptoms of a patient are described by a hesitant neutrosophic set. In this example, a patient $\mathcal{N}$ are illustrated by a hesitant neutrosophic set in the universe Y which are obtained from three different doctors as follows.

$$\begin{aligned} \mathcal{N} = \{ & \langle y_1, \{0.9, 0.8, 1\}, \{0.3, 0.2, 0\}, \{0.1, 0.3, 0\}\rangle, \\ & \langle y_2, \{0.7, 0.8, 0.9\}, \{0.1, 0.2, 0.1\}, \{0.2, 0.1, 0.2\}\rangle, \\ & \langle y_3, \{0.8, 0.8, 0.7\}, \{0.2, 0.3, 0.4\}, \{0.1, 0.2, 0, 3\}\rangle, \\ & \langle y_4, \{0.1, 0.2, 0.1\}, \{0.3, 0.4, 0.2\}, \{0.8, 0.7, 0.9\}\rangle, \\ & \langle y_5, \{0.1, 0, 0\}, \{0.2, 0.1, 0.3\}, \{0.8, 0.9, 1\}\rangle\}. \end{aligned}$$

By Definition 15, we calculate the hesitant neutrosophic rough set

$$(\underline{\mathcal{R}}(\mathcal{N}), \overline{\mathcal{R}}(\mathcal{N}))$$

of $\mathcal{N}$ as follows:

$$\begin{aligned} \underline{\mathcal{R}}(\mathcal{N}) = \{ & \langle x_1, \{0.1, 0.2, 0.3\}, \{0.2, 0.3, 0.4\}, \{0.7, 0.8, 0.8\}\rangle, \\ & \langle x_2, \{0.7, 0.8, 0.9\}, \{0.2, 0.3, 0.4\}, \{0.1, 0.2, 0.3\}\rangle, \\ & \langle x_3, \{0.6, 0.7, 0.8\}, \{0.2, 0.3, 0.4\}, \{0.1, 0.2, 0.3\}\rangle, \\ & \langle x_4, \{0.6, 0.7, 0.8\}, \{0.2, 0.3, 0.4\}, \{0.1, 0.2, 0.4\}\rangle\}, \end{aligned}$$

and

$$\begin{aligned} \overline{\mathcal{R}}(\mathcal{N}) = \{ & \langle x_1, \{0.5, 0.6, 0.6\}, \{0.2, 0.3, 0.4\}, \{0.1, 0.2, 0.3\}\rangle, \\ & \langle x_2, \{0.8, 0.9, 0.9\}, \{0.1, 0.2, 0.3\}, \{0, 0.1, 0.2\}\rangle, \\ & \langle x_3, \{0.8, 0.9, 1\}, \{0.1, 0.1, 0.3\}, \{0, 0.1, 0.2\}\rangle, \\ & \langle x_4, \{0.7, 0.8, 0.8\}, \{0.1, 0.2, 0.3\}, \{0.1, 0.2, 0.3\}\rangle\}. \end{aligned}$$

By Definition 14 and Remark 2, we have

$$
\begin{aligned}
&\underline{\mathcal{R}}(\mathcal{N}) \bigoplus \overline{\mathcal{R}}(\mathcal{N}) \\
&\quad = \{\langle x_1, \{0.55, 0.68, 0.72\}, \{0.04, 0.09, 0.16\}, \{0.07, 0.16, 0.24\}\rangle, \\
&\quad\ \langle x_2, \{0.94, 0.98, 0.99\}, \{0.02, 0.06, 0.12\}, \{0, 0.02, 0.06\}\rangle, \\
&\quad\ \langle x_3, \{0.92, 0.97, 1\}, \{0.02, 0.03, 0.12\}, \{0, 0.02, 0.06\}\rangle, \\
&\quad\ \langle x_4, \{0.88, 0.94, 0.96\}, \{0.02, 0.06, 0.12\}, \{0.01, 0.04, 0.12\}\rangle\}.
\end{aligned}
$$

Moreover, by Definition 12, we compute the cosine measure between each hesitant neutrosophic element $\boldsymbol{n}(x_j)$ $(j = 1, 2, \ldots, 4)$ in $\underline{\mathcal{R}}(\mathcal{N}) \bigoplus \overline{\mathcal{R}}(\mathcal{N})$ and the ideal element $\boldsymbol{n}^\star = \{1, 0, 0\}$ as follows:

$$
\begin{aligned}
\cos(\boldsymbol{n}(x_1), \boldsymbol{n}^\star) &= \frac{1.95}{\sqrt{4.1075}} \approx 0.96216, \\
\cos(\boldsymbol{n}(x_2), \boldsymbol{n}^\star) &= \frac{2.91}{\sqrt{8.5145}} \approx 0.99727, \\
\cos(\boldsymbol{n}(x_3), \boldsymbol{n}^\star) &= \frac{2.89}{\sqrt{8.3874}} \approx 0.99789, \\
\cos(\boldsymbol{n}(x_4), \boldsymbol{n}^\star) &= \frac{2.78}{\sqrt{7.7973}} \approx 0.99557.
\end{aligned}
$$

Then, we have

$$
\cos(\boldsymbol{n}(x_3), \boldsymbol{n}^\star) > \cos(\boldsymbol{n}(x_2), \boldsymbol{n}^\star) > \cos(\boldsymbol{n}(x_4), \boldsymbol{n}^\star) > \cos(\boldsymbol{n}(x_1), \boldsymbol{n}^\star).
$$

So, the optimal decision is to select x_3. That is, we can conclude that the patient $\mathcal{N}$ is suffering from "Typhoid" x_3.

Compared with the models proposed in Bao and Yang (2017) and Zhang et al. (2017a), the model in this paper can deal with information which come from hesitant neutrosophic information providers in the process of decision making. For hesitant neutrosophic sets is a generalization of fuzzy sets, intuitionistic fuzzy sets, single valued neutrosophic (refined) sets, hesitant fuzzy sets, and dual hesitant fuzzy sets, the algorithm based on hesitant neutrosophic rough set over two universes suits more general decision-making environment.

6 Conclusion

In this paper, we propose the model of hesitant neutrosophic rough sets. Specifically, we investigate the join semi-lattice structure of lower and upper hesitant neutrosophic rough approximation operators over two universes. In addition, we provide an algorithm to handle decision making problem in medical diagnosis based on hesitant neutrosophic rough sets over two universes. Finally, a numerical example is employed to demonstrate the validness of the proposed hesitant neutrosophic rough sets.

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