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FUZZY SYSTEMS AND THEIR MATHEMATICS

Neutrosophic Riemann integration and its properties

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Abstract
In this article, some properties of neutrosophic numbers as well as neutrosophic functions are presented to develop neutrosophic integral calculus. This is the first time, the concept of neutrosophic Riemann integration has been introduced. The (α, β, γ)-level sets of the neutrosophic Riemann integral are presented. Some numerical examples have been given to verify the concept of neutrosophic Riemann integration, and the integration has been examined in terms of tables and figures. In one of the examples, it has been presented that how one can use trapezoidal rule to calculate neutrosophic Riemann integral and the numerical approximate result shows that here also, the numerical integral gives a neutrosophic number.

Keywords Neutrosophic Riemann integration · Closed neutrosophic number · Bounded neutrosophic number · Trapezoidal rule

1 Introduction

1.1 Neutrosophic set

In the history of neutrosophic mathematics, Smarandache (1999, 2003) introduced the neutrosophic set theory. Mainly, the idea of neutrosophic set comes from fuzzy set. After the invention of fuzzy set theory (Zadeh 1974), not only neutrosophic set theory, various kinds of generalization have been presented by various authors. In the following table, we have tried to give a comparison and make a conclusion about their importance and applicability.

Name	Properties	Remarks
Fuzzy sets (Zadeh 1974)	Every elements of this set has a membership value.	It is the generalization of classical set
Type 2 fuzzy sets (Zadeh 1975)	Its membership function is a fuzzy function	It is one kind of generalization of fuzzy sets
Intuitionistic fuzzy sets (Atanassov 1999)	Every element of this set has a membership value as well as a non-membership value	It is also one kind of a generalization of fuzzy sets. Its membership function is exactly same as the membership function of fuzzy sets
Neutrosophic set (Smarandache 2003, 1999)	Every element of this set has three different types of membership value: truth, indeterminacy and falsity membership value	It is also the generalization of fuzzy set. But it is not the generalization of intuitionistic fuzzy set, because for Intuitionistic fuzzy set, membership function and non-membership function are dependent on each other. But for neutrosophic set truth, falsity and indeterminacy membership functions may not always be dependent on each other

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In real-life application, when someone collects the data from a source which is not fully trustworthy but exactly how much trust one can put one the source is known, one can use fuzzy environment of type-1. If the source is not fully trustworthy and exactly how much trust one can put one the source is unknown, then one can use the fuzzy environment of type-2. But when the data source is very complex in nature that means the values of the trustworthiness and untrustworthiness of the source are not complement of each other, then we can use intuitionistic environment and for more complex environment where an indeterminacy is in play, one can use neutrosophic environment. Also, there are many researchers who have been working on the application of neutrosophic set. Ulucay et al. (2018) introduced some different types of similarity measures for bipolar neutrosophic set and they also developed a multicriteria decision-making method for bipolar neutrosophic set. They also compared their method in other existing methods. Then, Şahin et al. (2018a) introduced some different types of weighted arithmetic operators and geometric operators with single valued neutrosophic numbers and they have shown the application of these operators to multicriteria decision making problems. After that, Ulucay et al. (2019) studied a new distance-based similarity measure for refined neutrosophic sets and they have found the application of distance-based similarity measure in medical diagnosis of some diseases. Bakbak et al. (2019) introduced the concept of neutrosophic soft expert multisets and they also define an algorithm for neutrosophic soft expert multisets decision-making method. Recently, Uluçay (2021) has proposed the concept of Q-neutrosophic soft graphs. In the same article (Uluçay 2021), he introduced the concept of strong Q-neutrosophic soft graph, union and intersection of Q-neutrosophic soft graph and he also showed the application of Q-neutrosophic soft graph in communication network and decision-making problem. In recent times, many researchers are also working on the various types of application of neutrosophic set in different field of mathematics (Uluçay 2020; Sahin et al. 2017; Uluçay et al. 2018; Broumi et al. 2019; Uluçay and Şahin 2020; Uluçay et al. 2019) .

1.2 Neutrosophic calculus

As we have seen in the above, neutrosophic set theory has been presented as a general form of fuzzy set theory. Like fuzzy set theory, it also has huge scope of application in different fields of mathematics like bio-mathematics, optimization and inventory model in the complex environment where the trustworthy and untrustworthy values of the source are not complement of each other and an indeterminacy about the value of truthfulness and untruthfulness of the source is in play. The proper development of neutrosophic calculus has a very important role to play, and in this path, very few work has been done.

In the literature review, we have seen that Smarandache (2013) introduced the definition of neutrosophic derivative, which is the extension of fuzzy derivative. Then, Son Nguyen Thi Kim et al. (2020) introduced a new type of neutrosophic derivative, which is called granular derivative(gr-derivative). Also, Son Nguyen Thi Kim et al. (2020) gave an if and only if conditions for gr-derivative of a neutrosophic valued function. Again, they also introduced the gr-partial derivative of neutrosophic valued several variable function and they also gave the concept of neutrosophic granular fractional derivative and partial derivative. With the help of this definition, many researchers develop the concept of neutrosophic differential equation. But this definition has serious drawbacks, which was found by Moi et al. (2020) and they introduced the generalized neutrosophic derivative. Also, Moi et al. (2021) proposed a new method to find the solution of second order boundary value problem in neutrosophic environment.

The concept of neutrosophic integral is the extension of fuzzy integral. The concept of fuzzy integral was first introduced by Sugeno (1974). After that, many researchers have presented different types of fuzzy integrals. There are various works on fuzzy measures and fuzzy integrals in the literature (Ralescu and Adams 1980; Wang 1984; Zhang and Guo 1995; Congxin et al. 1993; Zhanga and Guo 1995; Guo et al. 1998; Biswas and Roy 2019, 2017; Biswas et al. 2021; Biswas and Roy 2018). The concept of fuzzy Riemann integration was presented by Wu (2000). Also, the improper fuzzy Riemann integral was introduced by Wu (1998). In recent time, Smarandache (2013) introduced the concept of neutrosophic measures and neutrosophic integration. There are lots of scopes to generalize and develop the concept of neutrosophic integration.

1.3 Motivation

In our literature review, it has been seen that the most of the works in neutrosophic set theory have been done on the application of neutrosophic set in different branches of science. But there is almost no work on the proper development of calculus in the neutrosophic environment. So, there is a conceptual gap between the development of core subject and its application. Researchers are applying neutrosophic set theory in different branches of science depending mostly on neutrosophic arithmetic. So, the proper development of neutrosophic calculus is required. So, that one can use neutrosophic environment in different branch of science with lot more accuracy and precise cases. This much importance of this topic motivates us to develop this article where we have concentrated to develop integral calculus involving the definition of neutrosophic Riemann integration.

1.4 Novelty

In order to build a bridge between theory and application of neutrosophic set theory and to apply neutrosophic set theory in different complex environment, neutrosophic Riemann integration and its properties are developed in this article.

The objectives of this article are presented as follows:

- To define some properties of neutrosophic number and neutrosophic function.
- To develop the neutrosophic integral calculus.
- To present some arithmetic properties of neutrosophic number like $'+'$ and $'\times'$.
- To define neutrosophic arithmetic in the sense of (α, β, γ)-cut.
- To define neutrosophic Riemann integration on closed and bounded interval.
- To present that the neutrosophic Riemann integration of neutrosophic function is a neutrosophic number.
- To prove that the addition and subtraction of two neutrosophic Riemann integrable functions is also Riemann integrable.
- To present the use of trapezoidal rule to calculate neutrosophic Riemann integration.

In this article, our mainly focus is on the theoretical development of neutrosophic calculus. In order to do that, we are going to introduce some properties of neutrosophic set, neutrosophic number and neutrosophic function in the form of several theorems, lemmas and propositions. Neutrosophic Riemann integral is defined on closed-bounded interval $[a_1, b_1]$. We are going to present a theorem to show that the neutrosophic Riemann integral of a neutrosophic function on $[a_1, b_1]$ is also a neutrosophic number. Later, we explain that how one can use trapezoidal rule to calculate neutrosophic Riemann integral with proper graph and tables.

1.5 Structure of the paper

The article has been organized as follows: In Sect. 2, the mathematical preliminaries are given, which is related to our article. Section 3 contains some definitions of neutrosophic number, theorems and propositions. The neutrosophic Riemann integration is given in Sect. 4. In Sect. 5, some test examples are examined. Finally, a brief conclusion about this article is given in Sect. 6.

2 Preliminaries

Definition 2.1 (Zadeh 1996) Let X be a space of points (objects), with a generic element of X denoted by x. Thus, $X = \{x\}$. A fuzzy set (class) A in X is characterized by a membership (characteristic) function $f_A(x)$ which associates with each point in X a real number in the interval [0, 1], with the value of $f_A(x)$ at x representing the "grade of membership" of x in A. Thus, the nearer the value of $f_A(x)$ to unity, the higher the grade of membership of x in A. When A is a set in the ordinary sense of the term, its membership function can take on only two values 0 and 1, with $f_A(x) = 1$ or 0 according as x does or does not belong to A. Thus, in this case $f_A(x)$ reduces to the familiar characteristic function of a set A. (When there is a need to differentiate between such sets and fuzzy sets, the sets with two-valued characteristic functions will be referred to as ordinary sets or simply sets.)

Fuzzy set is the generalization of classical set. The notion of union, intersection, inclusion, complement, relations, convexity, etc., of classical set is extended to the notion of fuzzy set.

Definition 2.2 (Deli and Şubaş 2017) An single-valued neutrosophic set (SVN-set) N over the universal set U is a neutrosophic set over U, but the truth, indeterminacy and falsity membership functions are, respectively, defined by $T_N : U \to [0, 1]$, $I_N : U \to [0, 1]$, $F_N : U \to [0, 1]$.

The neutrosophic set is the generalization of fuzzy set. For real-life applications, the neutrosophic set and set-theoretic operations need to be specified. Otherwise, it will be difficult to apply in the real applications. Single-valued neutrosophic set can be used for real-life applications. So, in the rest of this article we will use single-valued neutrosophic set.

Definition 2.3 (Sumathi and Sweety 2019) A neutrosophic set N over the universal set of real numbers $\mathbb{R}$ is said to be neutrosophic number if it has the following properties.

1. N is normal, i.e., there exists $x_0 \in \mathbb{R}$ such that $T_N(x_0) = 1.(I_N(x_0) = F_N(x_0) = 0)$.
2. N is convex for truth membership function $T_N(x)$, i.e., $T_N(\mu x_1 + (1-\mu)x_2) \geq min(T_N(x_1), T_N(x_2))$,$\forall x_1, x_2 \in \mathbb{R}$ and $\mu \in [0, 1]$.
3. N is concave for indeterminacy and falsity membership functions, $I_N(x)$ and $F_N(x)$, respectively, i.e., $I_N(\mu x_1 + (1-\mu)x_2) \geq \max(I_N(x_1), I_N(x_2))$, and $F_N(\mu x_1 + (1-\mu)x_2) \geq \max(F_N(x_1), F_N(x_2))$ $\forall x_1, x_2 \in \mathbb{R}$ and $\mu \in [0, 1]$.

Definition 2.4 (Sun et al. 2015) An interval neutrosophic set N over the universal set U is described by a truth, indeterminacy and falsity-membership functions, T_N(x), I_N(x), and F_N(x), respectively. For all x $\in$ U, we have $T_N(x) = [\inf f T_N(x), \text{Sup} T_N(x)]$, $I_N(x) = [\inf f I_N(x), \text{Sup} I_N(x)]$, $F_N(x) = [\inf f F_N(x), \text{Sup} F_N(x)] \subseteq [0, 1]$ $\forall$ x $\in$ U. Here we consider sub-unitary interval [0, 1]. It is the sub class of a neutrosophic set. Therefore, all interval neutrosophic sets are clearly neutrosophic sets.

For convenience, let $\tilde{n} = \langle [T^L_{\tilde{n}}, T^U_{\tilde{n}}], [I^L_{\tilde{n}}, I^U_{\tilde{n}}], [F^L_{\tilde{n}}, F^U_{\tilde{n}}] \rangle$ denotes an interval neutrosophic number(INN).

Here, $T^L_{\tilde{n}}, T^U_{\tilde{n}}, I^L_{\tilde{n}}, I^U_{\tilde{n}}, F^L_{\tilde{n}}$ and $F^U_{\tilde{n}}$ denotes $\inf T_{\tilde{n}}(x)$, $\text{Sup}T_{\tilde{n}}(x)$, $\inf I_{\tilde{n}}(x)$, $\text{Sup}I_{\tilde{n}}(x)$, $\inf F_{\tilde{n}}(x)$ and $\text{Sup}F_{\tilde{n}}(x)$, respectively.

Definition 2.5 (Deli and Şubaş 2017) A single-valued triangular neutrosophic number (SVTN-number)

$N = \langle (p, q, r); \rho_N, \nu_N, \kappa_N \rangle$ is a special neutrosophic set on $\mathbb{R}$, whose truth, indeterminacy–membership and falsity–membership functions are defined by

$$T_N(x) = \begin{cases} \left(\dfrac{x-p}{q-p}\right)\rho_N & \text{for } p \le x \le q \\ \left(\dfrac{r-x}{r-q}\right)\rho_N & \text{for } q \le x \le r \\ 0 & \text{Otherwise} \end{cases}$$

$$I_N(x) = \begin{cases} \dfrac{(q-x+\nu_N(x-p))}{q-p} & \text{for } p \le x \le q \\ \dfrac{(x-q+\nu_N(r-x))}{r-q} & \text{for } q \le x \le r \\ 0 & \text{Otherwise} \end{cases}$$

$$F_N(x) = \begin{cases} \dfrac{(q-x+\kappa_N(x-p))}{q-p} & \text{for } p \le x \le q \\ \dfrac{(x-q+\kappa_N(r-x))}{r-q} & \text{for } q \le x \le r \\ 0 & \text{Otherwise,} \end{cases}$$

respectively.

Definition 2.6 (Deli and Şubaş 2017) A single-valued trapezoidal neutrosophic number (SVTrN-number)

$N = \langle (p, q, r, s); \rho_N, \nu_N, \kappa_N \rangle$ is a special neutrosophic set on $\mathbb{R}$, whose truth, indeterminacy and falsity–membership functions are defined by

$$T_N(x) = \begin{cases} \left(\dfrac{x-p}{q-p}\right)\rho_N & \text{for } p \le x \le q \\ \rho_N & \text{for } q \le x \le r \\ \left(\dfrac{s-x}{s-r}\right)\rho_N & \text{for } r \le x \le s \\ 0 & \text{Otherwise} \end{cases}$$

$$I_N(x) = \begin{cases} \dfrac{(q-x+\nu_N(x-p))}{q-p} & \text{for } p \le x \le q \\ \nu_N & \text{for } q \le x \le r \\ \dfrac{(x-r+\nu_N(s-x))}{s-r} & \text{for } r \le x \le s \\ 0 & \text{Otherwise} \end{cases}$$

$$F_N(x) = \begin{cases} \dfrac{(q-x+\kappa_N(x-p))}{q-p} & \text{for } p \le x \le q \\ \kappa_N & \text{for } q \le x \le r \\ \dfrac{(x-r+\kappa_N(s-x))}{s-r} & \text{for } r \le x \le s \\ 0 & \text{Otherwise,} \end{cases}$$

respectively.

Definition 2.7 (Sumathi and Sweety 2019) The (α, β, γ)-cut of a neutrosophic set N is denoted by $N_{(\alpha,\beta,\gamma)}$, where $\alpha, \beta, \gamma \in [0, 1]$. Then $N_{(\alpha,\beta,\gamma)} = \{\langle T_N(x), I_N(x), F_N(x) \rangle : x \in U, T_N(x) \ge \alpha, I_N(x) \le \beta, F_N(x) \le \gamma\}$.

3 Neutrosophic analysis

In this section, some properties of neutrosophic set and operations of neutrosophic number have been presented, which will be used in the rest of this article.

Definition 3.1 (Rudin 2006) Let $g(x)$ be a real valued function on a topological space. If $\{x : g(x) \ge \mu\}$ and $\{x : g(x) \le \mu\}$ are closed for every $\mu \in \mathbb{R}$, then $g(x)$ is said to be upper semi-continuous and lower semi-continuous, respectively.

Let us define some basic concept of neutrosophic number with the help of Definition 3.1.

Definition 3.2 (i) $\tilde{n}$ is called closed neutrosophic number, if $\tilde{n}$ is a neutrosophic number and the truth membership function is upper semi continuous, indeterminacy membership function and falsity membership function are lower semi-continuous.

(ii) $\tilde{n}$ is called a bounded neutrosophic number if $\tilde{n}$ is neutrosophic number and truth, indeterminacy and falsity membership functions has compact support.

(iii) Let $\tilde{m}$ and $\tilde{n}$ be the two neutrosophic number. Then, $\tilde{m}$ and $\tilde{n}$ are said to be equal, denoted as $\tilde{m} = \tilde{n}$ iff $\tilde{m}_{(\alpha,\beta,\gamma)} = \tilde{n}_{(\alpha,\beta,\gamma)}$. Here $\tilde{m}_{(\alpha,\beta,\gamma)}$ and $\tilde{n}_{(\alpha,\beta,\gamma)}$ denote the (α, β, γ)−cut of $\tilde{m}$ and $\tilde{n}$, respectively.

Definition 3.2 helps to show that the (α, β, γ)-cut of closed neutrosophic number is closed interval neutrosophic number and this result has been shown in the following proposition.

Proposition 3.1 *If $\tilde{n}$ is a closed neutrosophic number, then (α, β, γ)-cut of $\tilde{n}$ denoted as $\tilde{n}_{(\alpha,\beta,\gamma)} = \langle [\tilde{n}^L_\alpha, \tilde{n}^U_\alpha], [\tilde{n}^L_\beta, \tilde{n}^U_\beta], [\tilde{n}^L_\gamma, \tilde{n}^U_\gamma] \rangle$ is closed interval neutrosophic number(INN), where $[\tilde{n}^L_\alpha, \tilde{n}^U_\alpha]$, $[\tilde{n}^L_\beta, \tilde{n}^U_\beta]$ and $[\tilde{n}^L_\gamma, \tilde{n}^U_\gamma]$ are all closed intervals. Here, $\tilde{n}^L_\alpha, \tilde{n}^U_\alpha, \tilde{n}^L_\beta, \tilde{n}^U_\beta$, $\tilde{n}^L_\gamma$ and $\tilde{n}^U_\gamma$ denotes $\inf T_{\tilde{n}}(x)$, $\text{Sup}T_{\tilde{n}}(x)$, $\inf I_{\tilde{n}}(x)$, $\text{Sup}I_{\tilde{n}}(x)$, $\inf F_{\tilde{n}}(x)$ and $\text{Sup}F_{\tilde{n}}(x)$, respectively.*

Proof Since $T_{\tilde{n}}(x)$ is upper semi-continuous, $I_{\tilde{n}}(x)$ and $F_{\tilde{n}}(x)$ are lower semi-continuous. So, the (α, β, γ)-level set of $\tilde{n}$, i.e., $\tilde{n}_{(\alpha,\beta,\gamma)} = \{x : T_{\tilde{n}}(x) \geq \alpha, I_{\tilde{n}}(x) \leq \beta, F_{\tilde{n}}(x) \leq \gamma, x \in \mathbb{R}\}$ is a closed set. Then, from Definition 2.4, $\tilde{n}$ is interval neutrosophic number and intervals are closed intervals. □

Proposition 3.2 (Şahin et al. 2018) *Let $\tilde{m}$ and $\tilde{n}$ be the two neutrosophic number, then $\tilde{m}+\tilde{n}$, $\tilde{m}-\tilde{n}$, $\tilde{m}\times\tilde{n}$ and $\lambda\tilde{m}$ are also neutrosophic number, where $\lambda \neq 0$ is any real number.*

Proposition 3.3 *Let $\tilde{m}$ and $\tilde{n}$ be two closed neutrosophic numbers, then*

1. *$(\tilde{m} \odot \tilde{n})_{(\alpha,\beta,\gamma)} = \tilde{m}_{(\alpha,\beta,\gamma)} \odot \tilde{n}_{(\alpha,\beta,\gamma)}$, where $\odot$ denotes any binary operation $'+', '-'$ and $'\times'$.*
2. *$(\lambda\tilde{m})_{(\alpha,\beta,\gamma)} = \lambda\tilde{m}_{(\alpha,\beta,\gamma)}$, where $\lambda \neq 0$ be any real number.*

Proof 1. Since $\tilde{m}\odot\tilde{n} = \{z, \max_{z=x\odot y}\{\min(T_{\tilde{m}}(x), T_{\tilde{n}}(y))\}, \min_{z=x\odot y}\{\max(I_{\tilde{m}}(x), I_{\tilde{n}}(y))\}, \min_{z=x\odot y}\{\max(F_{\tilde{m}}(x), F_{\tilde{n}}(y))\}\}$ and $(\tilde{m} \odot \tilde{n})_{(\alpha,\beta,\gamma)} = \{z : T_{\tilde{m}\odot\tilde{n}}(z) \geq \alpha, I_{\tilde{m}\odot\tilde{n}}(z) \leq \beta, F_{\tilde{m}\odot\tilde{n}}(z) \leq \gamma\}$.
Let $z \in (\tilde{m}\odot\tilde{n})_{(\alpha,\beta,\gamma)}$, then there exist at least one $x \in \tilde{m}$ and $y \in \tilde{n}$ such that $z = x \odot y$. Then,

$$\begin{aligned}
&T_{\tilde{m}\odot\tilde{n}}(z) = \max\{\min(T_{\tilde{m}}(x), T_{\tilde{n}}(y))\} \geq \alpha\\
&\Rightarrow T_{\tilde{m}}(x) \geq \alpha \text{ and } T_{\tilde{n}}(y) \geq \alpha\\
&I_{\tilde{m}\odot\tilde{n}}(z) = \min\{\max(I_{\tilde{m}}(x), I_{\tilde{n}}(y))\} \leq \beta\\
&\Rightarrow I_{\tilde{m}}(x) \leq \beta \text{ and } I_{\tilde{n}}(y) \leq \beta\\
&F_{\tilde{m}\odot\tilde{n}}(z) = \min\{\max(F_{\tilde{m}}(x), F_{\tilde{n}}(y))\} \leq \gamma\\
&\Rightarrow F_{\tilde{m}}(x) \leq \gamma \text{ and } F_{\tilde{n}}(y) \leq \gamma
\end{aligned}$$

This implies $x \in \tilde{m}(\alpha,\beta,\gamma)$ and $y \in \tilde{n}(\alpha,\beta,\gamma)$. Therefore, $z = x \odot y \in \tilde{m}_{(\alpha,\beta,\gamma)} \odot \tilde{n}_{(\alpha,\beta,\gamma)}$.
Again let $z^* \in \tilde{m}_{(\alpha,\beta,\gamma)} \odot \tilde{n}_{(\alpha,\beta,\gamma)}$. Then, there exist at least one $x^* \in \tilde{m}_{(\alpha,\beta,\gamma)}$ and $y^* \in \tilde{n}_{(\alpha,\beta,\gamma)}$ such that $z^* = x^* \odot y^*$. Then, we have,

$$\begin{aligned}
&T_{\tilde{m}}(x^*) \geq \alpha \text{ and } T_{\tilde{n}}(y^*) \geq \alpha\\
&I_{\tilde{m}}(x^*) \leq \beta \text{ and } I_{\tilde{n}}(y^*) \leq \beta\\
&F_{\tilde{m}}(x^*) \leq \gamma \text{ and } F_{\tilde{n}}(y^*) \leq \gamma
\end{aligned}$$

This implies that $\min(T_{\tilde{m}}(x^*), T_{\tilde{n}}(y^*)) \geq \alpha$, $\max(I_{\tilde{m}}(x^*)$ $I_{\tilde{n}}(y^*)) \leq \beta$ and $\max(F_{\tilde{m}}(x^*), F_{\tilde{n}}(y^*)) \leq \gamma$. Then, we have

$$\begin{aligned}
&\max_{z^*=x^*\odot y^*}\left\{\min(T_{\tilde{m}}(x^*), T_{\tilde{n}}(y^*))\right\} \geq \alpha \Rightarrow T_{\tilde{m}\odot\tilde{n}}(x^*\odot y^*) = T_{\tilde{m}\odot\tilde{n}}(z^*) \geq \alpha\\
&\min_{z^*=x^*\odot y^*}\left\{\max(I_{\tilde{m}}(x^*), I_{\tilde{n}}(y^*))\right\} \leq \beta \Rightarrow I_{\tilde{m}\odot\tilde{n}}(x^*\odot y^*) = I_{\tilde{m}\odot\tilde{n}}(z^*) \leq \beta\\
&\min_{z^*=x^*\odot y^*}\left\{\max(F_{\tilde{m}}(x^*), F_{\tilde{n}}(y^*))\right\} \leq \gamma \Rightarrow F_{\tilde{m}\odot\tilde{n}}(x^*\odot y^*) = F_{\tilde{m}\odot\tilde{n}}(z^*) \leq \gamma
\end{aligned}$$

Therefore, $z^* \in (\tilde{m}\odot\tilde{n})_{(\alpha,\beta,\gamma)}$.

2. Since $\lambda\tilde{m} = \{z, \max_{z=\lambda x} T_{\tilde{m}}(x), \min_{z=\lambda x} I_{\tilde{m}}(x), \min_{z=\lambda x} F_{\tilde{m}}(x)\}$ and $(\lambda\tilde{m})_{(\alpha,\beta,\gamma)} = \{z : T_{\lambda\tilde{m}}(x) \geq \alpha, I_{\lambda\tilde{m}}(x) \leq \beta, F_{\lambda\tilde{m}}(x) \leq \gamma\}$.
Let $z \in (\lambda\tilde{m})_{(\alpha,\beta,\gamma)}$, then there exists $x \in \tilde{m}$ such that $z = \lambda x$. Then, $T_{\tilde{m}}(x) \geq \alpha$, $I_{\tilde{m}}(x) \leq \beta$ and $F_{\tilde{m}}(x) \leq \gamma$. This implies that $x \in \tilde{m}_{(\alpha,\beta,\gamma)}$, then $z = \lambda x \in \lambda\tilde{m}_{(\alpha,\beta,\gamma)}$.
Again let $z^* \in \lambda\tilde{m}_{(\alpha,\beta,\gamma)}$. Then, there exists $x^* \in \tilde{m}$ such that $z^* = \lambda x^*$. This implies $T_{\tilde{m}}(x^*) \geq \alpha$, $I_{\tilde{m}}(x^*) \leq \beta$ and $F_{\tilde{m}}(x^*) \leq \gamma$. Then, we have

$$\begin{aligned}
&\max_{z^*=\lambda x^*} T_{\tilde{m}}(x^*) \geq \alpha \Rightarrow T_{\lambda\tilde{m}}(\lambda x^*) = T_{\lambda\tilde{m}}(z^*) \geq \alpha\\
&\min_{z^*=\lambda x^*} I_{\tilde{m}}(x^*) \leq \beta \Rightarrow I_{\lambda\tilde{m}}(\lambda x^*) = I_{\lambda\tilde{m}}(z^*) \leq \beta\\
&\min_{z^*=\lambda x^*} F_{\tilde{m}}(x^*) \leq \gamma \Rightarrow F_{\lambda\tilde{m}}(\lambda x^*) = F_{\lambda\tilde{m}}(z^*) \leq \gamma
\end{aligned}$$

Therefore $z^* \in (\lambda\tilde{m})_{(\alpha,\beta,\gamma)}$. □

Proposition 3.4 *Let $\tilde{m}$ and $\tilde{n}$ be two closed neutrosophic number then $\tilde{m}+\tilde{n}$, $\tilde{m}-\tilde{n}$, $\tilde{m}\times\tilde{n}$ and $\lambda\tilde{m}$ are also closed neutrosophic number, where $\lambda \neq 0$ is any real number.*

Proof From Proposition 3.2, we can show that $T_{\tilde{m}\odot\tilde{n}}$, $T_{\lambda\tilde{m}}$ are upper semi-continuous and $I_{\tilde{m}\odot\tilde{n}}$, $I_{\lambda\tilde{m}}$ and $F_{\tilde{m}\odot\tilde{n}}$, $F_{\lambda\tilde{m}}$ are lower semi-continuous. Again from Proposition 3.3, we can show that $(\tilde{m}\odot\tilde{n})_{(\alpha,\beta,\gamma)}$ and $(\lambda\tilde{m})_{(\alpha,\beta,\gamma)}$ are closed set for all (α,β,γ), where $\odot$ denotes the binary operation $'+', '-'$ and $'\times'$. □

Proposition 3.5 *Let $\tilde{m}$ and $\tilde{n}$ be two closed neutrosophic number then*

1. $(\tilde{m}+\tilde{n})_{(\alpha,\beta,\gamma)} = \langle[m_\alpha^L + n_\alpha^L, m_\alpha^U + n_\alpha^U], [m_\beta^L + n_\beta^L, m_\beta^U + n_\beta^U], [m_\gamma^L + n_\gamma^L, m_\gamma^U + n_\gamma^U]\rangle$.
2. $(\tilde{m}-\tilde{n})_{(\alpha,\beta,\gamma)} = \langle[m_\alpha^L - n_\alpha^U, m_\alpha^U - n_\alpha^L], [m_\beta^L - n_\beta^U, m_\beta^U - n_\beta^L], [m_\gamma^L - n_\gamma^U, m_\gamma^U - n_\gamma^L]\rangle$.
3. $(\lambda\tilde{m})_{(\alpha,\beta,\gamma)} = \langle[\lambda m_\alpha^L, \lambda m_\alpha^U], [\lambda m_\beta^L, \lambda m_\beta^U], [\lambda m_\gamma^L, \lambda m_\gamma^U]\rangle$, *for* $\lambda > 0$
$= \langle[\lambda m_\alpha^U, \lambda m_\alpha^L], [\lambda m_\beta^U, \lambda m_\beta^L], [\lambda m_\gamma^U, \lambda m_\gamma^L]\rangle$, *for* $\lambda < 0$.

Proof

$$
\begin{aligned}
1.\ Since\ (\tilde{m}+\tilde{n})_{(\alpha,\beta,\gamma)} &= \tilde{m}_{(\alpha,\beta,\gamma)}+\tilde{n}_{(\alpha,\beta,\gamma)} \quad (By\ Proposition\ 3.3)\\
&= \left\langle \left[m_\alpha^L, m_\alpha^U\right], \left[m_\beta^L, m_\beta^U\right], \left[m_\gamma^L, m_\gamma^U\right] \right\rangle + \left\langle \left[n_\alpha^L, n_\alpha^U\right], \left[n_\beta^L, n_\beta^U\right], \left[n_\gamma^L, n_\gamma^U\right] \right\rangle\\
&= \left\langle \left[m_\alpha^L+n_\alpha^L, m_\alpha^U+n_\alpha^U\right], \left[m_\beta^L+n_\beta^L, m_\beta^U+n_\beta^U\right], \left[m_\gamma^L+n_\gamma^L, m_\gamma^U+n_\gamma^U\right] \right\rangle.
\end{aligned}
$$

The proof of the second and third parts of this proposition is similar to the first part. □

Theorem 3.1 *Let N be a neutrosophic set and* $N_{(\alpha,\beta,\gamma)} = \{x : T_N(x) \geq \alpha, I_N(x) \leq \beta, F_N(x) \leq \gamma\}$. *Then*

1. $N_{(\alpha_2,\beta_2,\gamma_2)} \subseteq N_{(\alpha_1,\beta_1,\gamma_1)}$ *where* $\alpha_1 < \alpha_2, \beta_1 > \beta_2$ *and* $\gamma_1 > \gamma_2$.
2. $\bigcap_{n=1}^{\infty} N_{(\alpha_n,\beta_n,\gamma_n)} = N_{(\alpha,\beta,\gamma)}$ *for* $\alpha_n \uparrow \alpha, \beta_n \downarrow \beta$ *and* $\gamma_n \downarrow \gamma$.

Proof 1. Let $x \in N_{(\alpha_2,\beta_2,\gamma_2)}$, then we have $T_N(x) \geq \alpha_2, I_N(x) \leq \beta_2, F_N(x) \leq \gamma_2$.

Now, $T_N(x) \geq \alpha_2 > \alpha_1, I_N(x) \leq \beta_2 < \beta_1, F_N(x) \leq \gamma_2 < \gamma_1$. This implies that

$T_N(x) \geq \alpha_1, I_N(x) \leq \beta_1, F_N(x) \leq \gamma_1$. Therefore, $x \in N_{(\alpha_1,\beta_1,\gamma_1)} \Rightarrow N_{(\alpha_2,\beta_2,\gamma_2)} \subseteq N_{(\alpha_1,\beta_1,\gamma_1)}$

2. Let $x \in \bigcap_{n=1}^{\infty} N_{(\alpha_n,\beta_n,\gamma_n)} \Rightarrow x \in N_{(\alpha_n,\beta_n,\gamma_n)}$. Then, $T_N(x) \geq \alpha_n, I_N(x) \leq \beta_n, F_N(x) \leq \gamma_n$.

Now, $T_N(x) \geq \lim_{n\to\infty} \alpha_n = \alpha, I_N(x) \leq \lim_{n\to\infty} \beta_n = \beta, F_N(x) \leq \lim_{n\to\infty} \gamma_n = \gamma$.

So, $T_N(x) \geq \alpha, I_N(x) \leq \beta, F_N(x) \leq \gamma$. Therefore, $x \in N_{(\alpha,\beta,\gamma)} \Rightarrow \bigcap_{n=1}^{\infty} N_{(\alpha_n,\beta_n,\gamma_n)} \subseteq N_{(\alpha,\beta,\gamma)}$

Again let $x \in N_{(\alpha,\beta,\gamma)}$. Then, $T_N(x) \geq \alpha, I_N(x) \leq \beta, F_N(x) \leq \gamma$. Since $\alpha_n \uparrow \alpha, \beta_n \downarrow \beta$ and $\gamma_n \downarrow \gamma$.

Then, we have $T_N(x) \geq \alpha \geq \alpha_n, I_N(x) \leq \beta \leq \beta_n, F_N(x) \leq \gamma \leq \gamma_n$ for all n.

This implies $x \in \bigcap_{n=1}^{\infty} N_{(\alpha_n,\beta_n,\gamma_n)} \Rightarrow N_{(\alpha,\beta,\gamma)} \subseteq \bigcap_{n=1}^{\infty} N_{(\alpha_n,\beta_n,\gamma_n)}$.

Therefore, $\bigcap_{n=1}^{\infty} N_{(\alpha_n,\beta_n,\gamma_n)} = N_{(\alpha,\beta,\gamma)}$. □

Proposition 3.6 *Let* $\{N_{(\alpha,\beta,\gamma)} = \langle [l_\alpha, u_\alpha], [l_\beta, u_\beta], [l_\gamma, u_\gamma] \rangle : 0 \leq \alpha \leq 1, 0 \leq \beta \leq 1, 0 \leq \gamma \leq 1\}$ *be a family of interval neutrosophic set and each intervals* $[l_\alpha, u_\alpha], [l_\beta, u_\beta]$ *and* $[l_\gamma, u_\gamma]$ *are closed intervals. Suppose* $N_{(\alpha,\beta,\gamma)}$ *is decreasing with respect to* (α, β, γ) *and* $\tilde{n}$ *is a closed neutrosophic number. Then,* $\{N_{(\alpha,\beta,\gamma)}\}$ *can induce* $\tilde{n}$ *and* $\tilde{n}_{(\alpha,\beta,\gamma)} = N_{(\alpha,\beta,\gamma)}$, *where* $\tilde{n}_{(\alpha,\beta,\gamma)}$ *and* $N_{(\alpha,\beta,\gamma)}$ *denotes* $(\alpha, \beta, \gamma)-$*cut of* $\tilde{n}$ *and N, respectively.*

Proof Since $\tilde{n}$ is a closed neutrosophic number and $N_{(\alpha,\beta,\gamma)}$ is decreasing with respect to (α, β, γ). From Theorem 3.1, we have $\bigcap_{n=1}^{\infty} N_{(\alpha_n,\beta_n,\gamma_n)} = N_{(\alpha,\beta,\gamma)}$ for $\alpha_n \uparrow \alpha, \beta_n \downarrow \beta$ and $\gamma_n \downarrow \gamma$.

Now, from Definitions 2.3 and 2.4 it can be said that $\{N_{(\alpha,\beta,\gamma)}\}$ can induce $\tilde{n}$.

Now from Definition 2.7, $\{x : T_{\tilde{n}}(x) \geq \alpha, I_{\tilde{n}}(x) \leq \beta, F_{\tilde{n}}(x) \leq \gamma\} = N_{(\alpha,\beta,\gamma)} = \langle [l_\alpha, u_\alpha], [l_\beta, u_\beta], [l_\gamma, u_\gamma] \rangle$ is a closed neutrosophic number. □

Proposition 3.7 *If* $\tilde{n}$ *is a closed neutrosophic number then,* $\tilde{n}_{\alpha_n}^L \uparrow \tilde{n}_\alpha^L$ *and* $\tilde{n}_{\alpha_n}^U \downarrow \tilde{n}_\alpha^U$ *for* $\alpha_n \uparrow \alpha$, $\tilde{n}_{\beta_n}^L \uparrow \tilde{n}_\beta^L$ *and* $\tilde{n}_{\beta_n}^U \downarrow \tilde{n}_\beta^U$ *for* $\beta_n \downarrow \beta$, $\tilde{n}_{\gamma_n}^L \uparrow \tilde{n}_\gamma^L$ *and* $\tilde{n}_{\gamma_n}^U \downarrow \tilde{n}_\gamma^U$ *for* $\gamma_n \downarrow \gamma$. □

Proof Since $N_{(\alpha,\beta,\gamma)} = \tilde{n}_{(\alpha,\beta,\gamma)} = \langle [\tilde{n}_\alpha^L, \tilde{n}_\alpha^U], [\tilde{n}_\beta^L, \tilde{n}_\beta^U], [\tilde{n}_\gamma^L, \tilde{n}_\gamma^U] \rangle$ and by Theorem 3.1, we know that $N_{(\alpha,\beta,\gamma)}$ is decreasing with respect to (α, β, γ). Then, we have,

$$
\lim \tilde{n}_{\alpha_n}^L \leq \tilde{n}_\alpha^L, \lim \tilde{n}_{\alpha_n}^U \geq \tilde{n}_\alpha^U, \lim \tilde{n}_{\beta_n}^L \leq \tilde{n}_\beta^L, \lim \tilde{n}_{\beta_n}^U \geq \tilde{n}_\beta^U, \lim \tilde{n}_{\gamma_n}^L \leq \tilde{n}_\gamma^L \ and\ \lim \tilde{n}_{\gamma_n}^U \geq \tilde{n}_\gamma^U.
$$

Since $\tilde{n}_{\alpha_n}^L \leq \lim \tilde{n}_{\alpha_n}^L, \tilde{n}_{\alpha_n}^U \geq \lim \tilde{n}_{\alpha_n}^U, \tilde{n}_{\beta_n}^L \leq \lim \tilde{n}_{\beta_n}^L, \tilde{n}_{\beta_n}^U \geq \lim \tilde{n}_{\beta_n}^U, \tilde{n}_{\gamma_n}^L \leq \lim \tilde{n}_{\gamma_n}^L$ and $\tilde{n}_{\gamma_n}^U \geq \lim \tilde{n}_{\gamma_n}^U$. Then,

$$
\begin{aligned}
&\lim \tilde{n}_{\alpha_n}^L \leq x \leq \lim \tilde{n}_{\alpha_n}^U \Rightarrow \tilde{n}_{\alpha_n}^L \leq x \leq \tilde{n}_{\alpha_n}^U\\
&\lim \tilde{n}_{\beta_n}^L \leq x \leq \lim \tilde{n}_{\beta_n}^U \Rightarrow \tilde{n}_{\beta_n}^L \leq x \leq \tilde{n}_{\beta_n}^U\\
&\lim \tilde{n}_{\gamma_n}^L \leq x \leq \lim \tilde{n}_{\gamma_n}^U \Rightarrow \tilde{n}_{\gamma_n}^L \leq x \leq \tilde{n}_{\gamma_n}^U
\end{aligned}
$$

This implies that $x \in \bigcap_{n=1}^{\infty} \langle [\tilde{n}_{\alpha_n}^L, \tilde{n}_{\alpha_n}^U], [\tilde{n}_{\beta_n}^L, \tilde{n}_{\beta_n}^U], [\tilde{n}_{\gamma_n}^L, \tilde{n}_{\gamma_n}^U] \rangle = \langle [\tilde{n}_\alpha^L, \tilde{n}_\alpha^U], [\tilde{n}_\beta^L, \tilde{n}_\beta^U], [\tilde{n}_\gamma^L, \tilde{n}_\gamma^U] \rangle$ (by Theorem 3.1).

Therefore, $\langle [\lim \tilde{n}_{\alpha_n}^L, \lim \tilde{n}_{\alpha_n}^U], [\lim \tilde{n}_{\beta_n}^L, \lim \tilde{n}_{\beta_n}^U], [\lim \tilde{n}_{\gamma_n}^L, \lim \tilde{n}_{\gamma_n}^U] \rangle \subseteq \langle [\tilde{n}_\alpha^L, \tilde{n}_\alpha^U], [\tilde{n}_\beta^L, \tilde{n}_\beta^U], [\tilde{n}_\gamma^L, \tilde{n}_\gamma^U] \rangle$, i.e.,

$\lim \tilde{n}_{\alpha_n}^L \geq \tilde{n}_\alpha^L, \lim \tilde{n}_{\alpha_n}^U \leq \tilde{n}_\alpha^U, \lim \tilde{n}_{\beta_n}^L \geq \tilde{n}_\beta^L, \lim \tilde{n}_{\beta_n}^U \leq \tilde{n}_\beta^U, \lim \tilde{n}_{\gamma_n}^L \geq \tilde{n}_\gamma^L$ and $\lim \tilde{n}_{\gamma_n}^U \leq \tilde{n}_\gamma^U$. □

Note Propositions 3.2 and 3.3 have helped to proof of Propositions 3.4 and 3.5. Propositions 3.4 and 3.5 will help to define some basic properties of neutrosophic function. Also, Propositions 3.6, 3.7 will help to develop the concept of neutrosophic Riemann integration and in the next section we will discuss neutrosophic Riemann integration and its properties.

4 Neutrosophic Riemann integration

Here, we shall discuss neutrosophic Riemann integration and some basic concept of neutrosophic function. Before discussing neutrosophic Riemann integration, it is necessary to define closed neutrosophic function and bounded neutrosophic function.

Definition 4.1 Let $\mathcal{N}$ be the set of all neutrosophic numbers, $\mathcal{N}_{cl}$ be the set of all closed neutrosophic numbers and $\mathcal{N}_b$ denote set of all bounded neutrosophic numbers. Then,

1. $\tilde{f}_{\mathcal{N}}(x)$ is a neutrosophic valued function, if $\tilde{f}_{\mathcal{N}} : X \rightarrow \mathcal{N}$.
2. $\tilde{f}_{\mathcal{N}}(x)$ is a closed neutrosophic valued function, if $\tilde{f}_{\mathcal{N}} : X \rightarrow \mathcal{N}_{cl}$.
3. $\tilde{f}_{\mathcal{N}}(x)$ is a bounded neutrosophic valued function, if $\tilde{f}_{\mathcal{N}} : X \rightarrow \mathcal{N}_b$.

Now, let us define the concept of neutrosophic Riemann integration.

Definition 4.2 Let $\tilde{f}_{\mathcal{N}}(x)$ be a closed-bounded neutrosophic valued function on a closed-bounded interval $[a_1, b_1]$. Let $\tilde{f}^L_{\mathcal{N}\alpha}(x)$, $\tilde{f}^U_{\mathcal{N}\alpha}(x)$, $\tilde{f}^L_{\mathcal{N}\beta}(x)$, $\tilde{f}^U_{\mathcal{N}\beta}(x)$, $\tilde{f}^L_{\mathcal{N}\gamma}(x)$ and $\tilde{f}^U_{\mathcal{N}\gamma}(x)$ are all Riemann integrable on $[a_1, b_1]$, for all (α, β, γ). Let

$$N_{(\alpha,\beta,\gamma)} = \Bigg\langle \left[\int_{a_1}^{b_1} \tilde{f}^L_{\mathcal{N}\alpha}(x)\mathrm{d}x, \int_{a_1}^{b_1} \tilde{f}^U_{\mathcal{N}\alpha}(x)\mathrm{d}x\right], \left[\int_{a_1}^{b_1} \tilde{f}^L_{\mathcal{N}\beta}(x)\mathrm{d}x, \int_{a_1}^{b_1} \tilde{f}^U_{\mathcal{N}\beta}(x)\mathrm{d}x\right], \left[\int_{a_1}^{b_1} \tilde{f}^L_{\mathcal{N}\gamma}(x)\mathrm{d}x, \int_{a_1}^{b_1} \tilde{f}^U_{\mathcal{N}\gamma}(x)\mathrm{d}x\right]\Bigg\rangle$$

Then, $\tilde{f}_{\mathcal{N}}(x)$ is called neutrosophic Riemann integrable on closed-bounded interval $[a_1, b_1]$, denoted as $\tilde{f}_{\mathcal{N}}(x) \in \mathcal{N}_{RI}$ on closed-bounded interval $[a_1, b_1]$, where $\mathcal{N}_{RI}$ is the set of all neutrosophic Riemann integrable functions. Here, $[\tilde{f}^L_{\mathcal{N}\alpha}(x), \tilde{f}^U_{\mathcal{N}\alpha}(x)]$, $[\tilde{f}^L_{\mathcal{N}\beta}(x), \tilde{f}^U_{\mathcal{N}\beta}(x)]$ and $[\tilde{f}^L_{\mathcal{N}\gamma}(x), \tilde{f}^U_{\mathcal{N}\gamma}(x)]$ denote $(\alpha, \beta, \gamma)-$cut of $\tilde{f}_{\mathcal{N}}(x)$, respectively.

Proposition 4.1 (Royden and Fitzpatrick 1988) *$g(x)$ is a bounded function defined on $[a_1, b_1]$. If $g(x)$ is Riemann integrable over $[a_1, b_1]$, then $g(x)$ is also Lebesgue integrable over $[a_1, b_1]$ and the two integrals are identical.*

Proposition 4.2 (Royden and Fitzpatrick 1988) *$g(x)$ is a bounded function on $[a_1, b_1]$. Then, $g(x)$ is Riemann integrable on $[a_1, b_1]$ if and only if $g(x)$ is continuous almost everywhere on closed-bounded interval $[a_1, b_1]$.*

Proposition 4.3 (Royden and Fitzpatrick 1988) *If $g(x)$ is Riemann integrable on $[a_1, b_1]$ and $\mu \in \mathbb{R}$, then $\mu g(x)$ is also Riemann integrable on $[a_1, b_1]$ and $\int_{a_1}^{b_1} \mu g(x)\mathrm{d}x = \mu \int_{a_1}^{b_1} g(x)\mathrm{d}x$.*

Propositions 4.1, 4.2 and 4.3 will help to prove the following theorems. Also, in the following theorems we have shown some basic properties and operations of neutrosophic Riemann integrable functions.

Theorem 4.1 *Let $\tilde{f}_{\mathcal{N}}(x)$ be a closed-bounded neutrosophic valued function on closed-bounded interval $[a_1, b_1]$. If $\tilde{f}_{\mathcal{N}}(x) \in \mathcal{N}_{RI}$ on closed-bounded interval $[a_1, b_1]$, then the neutrosophic Riemann integral $\int_{a_1}^{b_1} \tilde{f}_{\mathcal{N}}(x)\mathrm{d}x$ is a closed neutrosophic number. Then, (α, β, γ)-cut of $\int_{a_1}^{b_1} \tilde{f}_{\mathcal{N}}(x)\mathrm{d}x$ is*

$$\left(\int_{a_1}^{b_1} \tilde{f}_{\mathcal{N}}(x)\mathrm{d}x\right)_{(\alpha,\beta,\gamma)} = \Bigg\langle \left[\int_{a_1}^{b_1} \tilde{f}^L_{\mathcal{N}\alpha}(x)\mathrm{d}x, \int_{a_1}^{b_1} \tilde{f}^U_{\mathcal{N}\alpha}(x)\mathrm{d}x\right], \left[\int_{a_1}^{b_1} \tilde{f}^L_{\mathcal{N}\beta}(x)\mathrm{d}x, \int_{a_1}^{b_1} \tilde{f}^U_{\mathcal{N}\beta}(x)\mathrm{d}x\right], \left[\int_{a_1}^{b_1} \tilde{f}^L_{\mathcal{N}\gamma}(x)\mathrm{d}x, \int_{a_1}^{b_1} \tilde{f}^U_{\mathcal{N}\gamma}(x)\mathrm{d}x\right]\Bigg\rangle$$

Proof Let $N_{(\alpha,\beta,\gamma)} = \langle [\int_{a_1}^{b_1} \tilde{f}^L_{\mathcal{N}\alpha}(x)\mathrm{d}x, \int_{a_1}^{b_1} \tilde{f}^U_{\mathcal{N}\alpha}(x)\mathrm{d}x], [\int_{a_1}^{b_1} \tilde{f}^L_{\mathcal{N}\beta}(x)\mathrm{d}x, \int_{a_1}^{b_1} \tilde{f}^U_{\mathcal{N}\beta}(x)\mathrm{d}x], [\int_{a_1}^{b_1} \tilde{f}^L_{\mathcal{N}\gamma}(x)\mathrm{d}x, \int_{a_1}^{b_1} \tilde{f}^U_{\mathcal{N}\gamma}(x)\mathrm{d}x]\rangle$

For $\alpha_1 < \alpha_2$, $\beta_1 > \beta_2$ and $\gamma_1 > \gamma_2$, we have

$$\tilde{f}^L_{\mathcal{N}\alpha_1}(x) \le \tilde{f}^L_{\mathcal{N}\alpha_2}(x),\ \tilde{f}^U_{\mathcal{N}\alpha_1}(x) \ge \tilde{f}^U_{\mathcal{N}\alpha_2}(x),\ \tilde{f}^L_{\mathcal{N}\beta_1}(x) \ge \tilde{f}^L_{\mathcal{N}\beta_2}(x),\ \tilde{f}^U_{\mathcal{N}\beta_1}(x) \le \tilde{f}^U_{\mathcal{N}\beta_2}(x),\ \tilde{f}^L_{\mathcal{N}\gamma_1}(x) \ge \tilde{f}^L_{\mathcal{N}\gamma_2}(x)\ \textit{and}\ \tilde{f}^U_{\mathcal{N}\gamma_1}(x) \le \tilde{f}^U_{\mathcal{N}\gamma_2}(x)$$

Then, $N_{(\alpha_2,\beta_2,\gamma_2)} \subseteq N_{(\alpha_1,\beta_1,\gamma_1)}$.

For $0 \le \alpha_{1n} \le 1$, we get that $\tilde{f}^L_{\mathcal{N}0}(x) \le \tilde{f}_{\mathcal{N}\alpha_{1n}}(x) \le \tilde{f}^L_{\mathcal{N}1}(x)$. Then, $|\tilde{f}^L_{\mathcal{N}\alpha_{1n}}(x)| \le \max\{|\tilde{f}^L_{\mathcal{N}0}(x)|, |\tilde{f}^L_{\mathcal{N}1}(x)|\} = h(x)$.

Since $\tilde{f}^L_{\mathcal{N}0}(x)$ and $\tilde{f}^L_{\mathcal{N}1}(x)$ both are Riemann-integrable on $[a_1, b_1]$, $|\tilde{f}^L_{\mathcal{N}0}(x)|, |\tilde{f}^L_{\mathcal{N}1}(x)|$ and $h(x)$ are also Riemann integrable on $[a_1, b_1]$. Since $\tilde{f}_{\mathcal{N}}(x)$ is bounded on $[a_1, b_1]$, $\tilde{f}^L_{\mathcal{N}\alpha_1}(x)$ and $\tilde{f}^U_{\mathcal{N}\alpha_1}(x)$ are also bounded on closed-bounded interval $[a_1, b_1]$ $\forall \alpha_1$. Then $h(x)$ is also bounded on $[a_1, b_1]$. Now, from Proposition 4.1 it can be said that $h(x)$ is Lebesgue integrable on $[a_1, b_1]$. Now we should apply Lebesgue dominated convergence theorem. For $\alpha_{1n} \uparrow \alpha_1$, we have

$$\lim_{n\to\infty} \int_{a_1}^{b_1} \tilde{f}^L_{\mathcal{N}\alpha_{1n}}(x)\mathrm{d}x = \int_{a_1}^{b_1} \tilde{f}^L_{\mathcal{N}\alpha_1}(x)\mathrm{d}x$$

Since $\tilde{f}_{\mathcal{N}}(x)$ is a closed neutrosophic valued function, we have $\lim_{n\to\infty} \tilde{f}^L_{\mathcal{N}\alpha_{1n}}(x) = \tilde{f}^L_{\mathcal{N}\alpha_1}(x)$ by Proposition 3.7. Again, for $0 \le \alpha_{1n} \le 1$, we have $\tilde{f}^U_{\mathcal{N}1}(x) \le \tilde{f}_{\mathcal{N}\alpha_{1n}}(x) \le \tilde{f}^U_{\mathcal{N}0}(x)$. Then $|\tilde{f}^U_{\mathcal{N}\alpha_{1n}}(x)| \le \max\{|\tilde{f}^U_{\mathcal{N}1}(x)|, |\tilde{f}^U_{\mathcal{N}0}(x)|\} = g(x)$.

Then, by similar argument we can say that

$$\lim_{n\to\infty} \int_{a_1}^{b_1} \tilde{f}^U_{\mathcal{N}\alpha_{1n}}(x)\mathrm{d}x = \int_{a_1}^{b_1} \tilde{f}^U_{\mathcal{N}\alpha_1}(x)\mathrm{d}x$$

Since $\tilde{f}_{\mathcal{N}}(x)$ is a closed neutrosophic valued function, then by Proposition 3.7 we have $\lim_{n\to\infty} \tilde{f}^U_{\mathcal{N}\alpha_{1n}}(x) = \tilde{f}^U_{\mathcal{N}\alpha_1}(x)$.

By similar process, we have,

$$\lim_{n\to\infty} \int_{a_1}^{b_1} \tilde{f}^L_{\mathcal{N}\beta_{1n}}(x)\mathrm{d}x = \int_{a_1}^{b_1} \tilde{f}^L_{\mathcal{N}\beta_1}(x)\mathrm{d}x$$
$$\lim_{n\to\infty} \int_{a_1}^{b_1} \tilde{f}^U_{\mathcal{N}\beta_{1n}}(x)\mathrm{d}x = \int_{a_1}^{b_1} \tilde{f}^U_{\mathcal{N}\beta_1}(x)\mathrm{d}x$$
$$\lim_{n\to\infty} \int_{a_1}^{b_1} \tilde{f}^L_{\mathcal{N}\gamma_{1n}}(x)\mathrm{d}x = \int_{a_1}^{b_1} \tilde{f}^L_{\mathcal{N}\gamma_1}(x)\mathrm{d}x$$
$$\lim_{n\to\infty} \int_{a_1}^{b_1} \tilde{f}^U_{\mathcal{N}\gamma_{1n}}(x)\mathrm{d}x = \int_{a_1}^{b_1} \tilde{f}^U_{\mathcal{N}\gamma_1}(x)\mathrm{d}x$$

From Proposition 3.6, the proof of this theorem is complete. □

Theorem 4.2 *If $\tilde{f}_{\mathcal{N}}(x)$ is closed-bounded neutrosophic valued function on closed-bounded interval $[a_1, b_1]$ and $\tilde{f}^L_{\mathcal{N}\alpha}(x)$, $\tilde{f}^U_{\mathcal{N}\alpha}(x)$, $\tilde{f}^L_{\mathcal{N}\beta}(x)$, $\tilde{f}^U_{\mathcal{N}\beta}(x)$, $\tilde{f}^L_{\mathcal{N}\gamma}(x)$, $\tilde{f}^U_{\mathcal{N}\gamma}(x)$ are all continuous on $[a_1, b_1]$, then $\tilde{f}_{\mathcal{N}}(x) \in \mathcal{N}_{RI}$ on closed-bounded interval $[a_1, b_1]$ and*

$$\left(\int_{a_1}^{b_1} \tilde{f}_{\mathcal{N}}(x)\mathrm{d}x\right)_{(\alpha,\beta,\gamma)} = \Big\langle \left[\int_{a_1}^{b_1} \tilde{f}^L_{\mathcal{N}\alpha}(x)\mathrm{d}x, \int_{a_1}^{b_1} \tilde{f}^U_{\mathcal{N}\alpha}(x)\mathrm{d}x\right], \left[\int_{a_1}^{b_1} \tilde{f}^L_{\mathcal{N}\beta}(x)\mathrm{d}x, \int_{a_1}^{b_1} \tilde{f}^U_{\mathcal{N}\beta}(x)\mathrm{d}x\right], \left[\int_{a_1}^{b_1} \tilde{f}^L_{\mathcal{N}\gamma}(x)\mathrm{d}x, \int_{a_1}^{b_1} \tilde{f}^U_{\mathcal{N}\gamma}(x)\mathrm{d}x\right]\Big\rangle$$

Proof From Proposition 4.2, $\tilde{f}^L_{\mathcal{N}\alpha}(x)$, $\tilde{f}^U_{\mathcal{N}\alpha}(x)$, $\tilde{f}^L_{\mathcal{N}\beta}(x)$, $\tilde{f}^U_{\mathcal{N}\beta}(x)$, $\tilde{f}^L_{\mathcal{N}\gamma}(x)$, and $\tilde{f}^U_{\mathcal{N}\gamma}(x)$ are all Riemann integrable on $[a_1, b_1]$. Then, $\tilde{f}_{\mathcal{N}}(x)$ is Riemann-integrable on $[a_1, b_1]$.

From Theorem 4.1, we have

$$\left(\int_{a_1}^{b_1} \tilde{f}_{\mathcal{N}}(x)\mathrm{d}x\right)_{(\alpha,\beta,\gamma)} = \Big\langle \left[\int_{a_1}^{b_1} \tilde{f}^L_{\mathcal{N}\alpha}(x)\mathrm{d}x, \int_{a_1}^{b_1} \tilde{f}^U_{\mathcal{N}\alpha}(x)\mathrm{d}x\right], \left[\int_{a_1}^{b_1} \tilde{f}^L_{\mathcal{N}\beta}(x)\mathrm{d}x, \int_{a_1}^{b_1} \tilde{f}^U_{\mathcal{N}\beta}(x)\mathrm{d}x\right], \left[\int_{a_1}^{b_1} \tilde{f}^L_{\mathcal{N}\gamma}(x)\mathrm{d}x, \int_{a_1}^{b_1} \tilde{f}^U_{\mathcal{N}\gamma}(x)\mathrm{d}x\right]\Big\rangle$$

This completes the proof. □

Theorem 4.3 *Let $\tilde{f}_{\mathcal{N}}(x)$ and $\tilde{g}_{\mathcal{N}}(x)$ be closed-bounded neutrosophic valued function on $[a_1, b_1]$. If $\tilde{f}_{\mathcal{N}}(x), \tilde{g}_{\mathcal{N}}(x) \in \mathcal{N}_{RI}$, then $\tilde{f}_{\mathcal{N}}(x) + \tilde{g}_{\mathcal{N}}(x) \in \mathcal{N}_{RI}$ and $\tilde{f}_{\mathcal{N}}(x) - \tilde{g}_{\mathcal{N}}(x) \in \mathcal{N}_{RI}$. Moreover, we have*

$$\int_{a_1}^{b_1} (\tilde{f}_{\mathcal{N}}(x) + \tilde{g}_{\mathcal{N}}(x))\mathrm{d}x = \int_{a_1}^{b_1} \tilde{f}_{\mathcal{N}}(x)\mathrm{d}x + \int_{a_1}^{b_1} \tilde{g}_{\mathcal{N}}(x)\mathrm{d}x$$
$$\int_{a_1}^{b_1} (\tilde{f}_{\mathcal{N}}(x) - \tilde{g}_{\mathcal{N}}(x))\mathrm{d}x = \int_{a_1}^{b_1} \tilde{f}_{\mathcal{N}}(x)\mathrm{d}x - \int_{a_1}^{b_1} \tilde{g}_{\mathcal{N}}(x)\mathrm{d}x$$

Proof Let $\tilde{h}_{\mathcal{N}}(x) = \tilde{f}_{\mathcal{N}}(x) + \tilde{g}_{\mathcal{N}}(x)$. From Proposition 3.4, we can say that $\tilde{h}_{\mathcal{N}}(x)$ is closed neutrosophic valued function.

Then,

$$\left(\int_{a_1}^{b_1} \tilde{f}_{\mathcal{N}}(x)\mathrm{d}x + \int_{a_1}^{b_1} \tilde{g}_{\mathcal{N}}(x)\mathrm{d}x\right)_{(\alpha,\beta,\gamma)}$$
$$= \left\langle \left[\int_{a_1}^{b_1} \tilde{f}^L_{\mathcal{N}\alpha}(x)\mathrm{d}x + \int_{a_1}^{b_1} \tilde{g}^L_{\mathcal{N}\alpha}(x)\mathrm{d}x, \int_{a_1}^{b_1} \tilde{f}^U_{\mathcal{N}\alpha}(x)\mathrm{d}x + \int_{a_1}^{b_1} \tilde{g}^U_{\mathcal{N}\alpha}(x)\mathrm{d}x\right], \right.$$
$$\left[\int_{a_1}^{b_1} \tilde{f}^L_{\mathcal{N}\beta}(x)\mathrm{d}x + \int_{a_1}^{b_1} \tilde{g}^L_{\mathcal{N}\beta}(x)\mathrm{d}x, \int_{a_1}^{b_1} \tilde{f}^U_{\mathcal{N}\beta}(x)\mathrm{d}x + \int_{a_1}^{b_1} \tilde{g}^U_{\mathcal{N}\beta}(x)\mathrm{d}x\right],$$
$$\left.\left[\int_{a_1}^{b_1} \tilde{f}^L_{\mathcal{N}\gamma}(x)\mathrm{d}x + \int_{a_1}^{b_1} \tilde{g}^L_{\mathcal{N}\gamma}(x)\mathrm{d}x, \int_{a_1}^{b_1} \tilde{f}^U_{\mathcal{N}\gamma}(x)\mathrm{d}x + \int_{a_1}^{b_1} \tilde{g}^U_{\mathcal{N}\gamma}(x)\mathrm{d}x\right]\right\rangle$$

Now, from Theorem 4.1 and Proposition 3.5 we have,

$$\left(\int_{a_1}^{b_1} \tilde{f}_{\mathcal{N}}(x)\mathrm{d}x + \int_{a_1}^{b_1} \tilde{g}_{\mathcal{N}}(x)\mathrm{d}x\right)_{(\alpha,\beta,\gamma)}$$
$$= \left\langle \left[\int_{a_1}^{b_1} (\tilde{f}^L_{\mathcal{N}\alpha}(x) + \tilde{g}^L_{\mathcal{N}\alpha}(x))\mathrm{d}x, \int_{a_1}^{b_1} (\tilde{f}^U_{\mathcal{N}\alpha}(x) + \tilde{g}^U_{\mathcal{N}\alpha}(x))\mathrm{d}x\right], \right.$$
$$\left[\int_{a_1}^{b_1} (\tilde{f}^L_{\mathcal{N}\beta}(x) + \tilde{g}^L_{\mathcal{N}\beta}(x))\mathrm{d}x, \int_{a_1}^{b_1} (\tilde{f}^U_{\mathcal{N}\beta}(x) + \tilde{g}^U_{\mathcal{N}\beta}(x))\mathrm{d}x\right],$$
$$\left.\left[\int_{a_1}^{b_1} (\tilde{f}^L_{\mathcal{N}\gamma}(x) + \tilde{g}^L_{\mathcal{N}\gamma}(x))\mathrm{d}x, \int_{a_1}^{b_1} (\tilde{f}^U_{\mathcal{N}\gamma}(x) + \tilde{g}^U_{\mathcal{N}\gamma}(x))\mathrm{d}x\right]\right\rangle$$
$$= \left\langle \left[\int_{a_1}^{b_1} (\tilde{f}_{\mathcal{N}}(x) + \tilde{g}_{\mathcal{N}}(x))^L_{\alpha}\mathrm{d}x, \int_{a_1}^{b_1} (\tilde{f}_{\mathcal{N}}(x) + \tilde{g}_{\mathcal{N}}(x))^U_{\alpha}\mathrm{d}x\right], \right.$$
$$\left[\int_{a_1}^{b_1} (\tilde{f}_{\mathcal{N}}(x) + \tilde{g}_{\mathcal{N}}(x))^L_{\beta}\mathrm{d}x, \int_{a_1}^{b_1} (\tilde{f}_{\mathcal{N}}(x) + \tilde{g}_{\mathcal{N}}(x))^L_{\beta}\mathrm{d}x\right],$$
$$\left.\left[\int_{a_1}^{b_1} (\tilde{f}_{\mathcal{N}}(x) + \tilde{g}_{\mathcal{N}}(x))^L_{\gamma}\mathrm{d}x, \int_{a_1}^{b_1} (\tilde{f}_{\mathcal{N}}(x) + \tilde{g}_{\mathcal{N}}(x))^U_{\gamma}\mathrm{d}x\right]\right\rangle$$
$$= \left(\int_{a_1}^{b_1} (\tilde{f}_{\mathcal{N}}(x) + \tilde{g}_{\mathcal{N}}(x))\mathrm{d}x\right)_{(\alpha,\beta,\gamma)} \qquad [By\ Theorem\ 4.1]$$

Therefore from Definition 3.2 (iii), we have $\int_{a_1}^{b_1} (\tilde{f}_{\mathcal{N}}(x) + \tilde{g}_{\mathcal{N}}(x))\mathrm{d}x = \int_{a_1}^{b_1} \tilde{f}_{\mathcal{N}}(x)\mathrm{d}x + \int_{a_1}^{b_1} \tilde{g}_{\mathcal{N}}(x)\mathrm{d}x$

This completes the proof of the first part of this theorem and the proof of the second part of this theorem is similar to the first part. □

Theorem 4.4 *Let $\tilde{f}_{\mathcal{N}}(x)$ be a closed-bounded neutrosophic valued function on closed-bounded interval $[a_1, b_1]$. If $\tilde{f}_{\mathcal{N}}(x) \in \mathcal{N}_{RI}$ then $\lambda \tilde{f}_{\mathcal{N}}(x) \in \mathcal{N}_{RI}$. Moreover, $\int_{a_1}^{b_1} \lambda \tilde{f}_{\mathcal{N}}(x)\mathrm{d}x = \lambda \int_{a_1}^{b_1} \tilde{f}_{\mathcal{N}}(x)\mathrm{d}x$, where $\lambda \neq 0$ be any real number.*

Proof For $\lambda > 0$, let $\tilde{g}_{\mathcal{N}}(x) = \lambda \tilde{f}_{\mathcal{N}}(x)$. From Proposition 3.4, $\tilde{g}_{\mathcal{N}}(x)$ is a closed neutrosophic valued function. From Proposition 3.5, we have,

$$\left(\lambda\int_{a_1}^{b_1}\tilde{f}_{\mathcal{N}}(x)\mathrm{d}x\right)_{(\alpha,\beta,\gamma)}$$
$$=\left\langle\left[\lambda\int_{a_1}^{b_1}\tilde{f}^L_{\mathcal{N}\alpha}(x)\mathrm{d}x,\lambda\int_{a_1}^{b_1}\tilde{f}^U_{\mathcal{N}\alpha}(x)\mathrm{d}x\right],\left[\lambda\int_{a_1}^{b_1}\tilde{f}^L_{\mathcal{N}\beta}(x)\mathrm{d}x,\lambda\int_{a_1}^{b_1}\tilde{f}^U_{\mathcal{N}\beta}(x)\mathrm{d}x\right],\left[\lambda\int_{a_1}^{b_1}\tilde{f}^L_{\mathcal{N}\gamma}(x)\mathrm{d}x,\lambda\int_{a_1}^{b_1}\tilde{f}^U_{\mathcal{N}\gamma}(x)\mathrm{d}x\right]\right\rangle$$
$$=\left\langle\left[\int_{a_1}^{b_1}\lambda\tilde{f}^L_{\mathcal{N}\alpha}(x)\mathrm{d}x,\int_{a_1}^{b_1}\lambda\tilde{f}^U_{\mathcal{N}\alpha}(x)\mathrm{d}x\right],\left[\int_{a_1}^{b_1}\lambda\tilde{f}^L_{\mathcal{N}\beta}(x)\mathrm{d}x,\int_{a_1}^{b_1}\lambda\tilde{f}^U_{\mathcal{N}\beta}(x)\mathrm{d}x\right],\left[\int_{a_1}^{b_1}\lambda\tilde{f}^L_{\mathcal{N}\gamma}(x)\mathrm{d}x,\int_{a_1}^{b_1}\lambda\tilde{f}^U_{\mathcal{N}\gamma}(x)\mathrm{d}x\right]\right\rangle$$
$$=\left\langle\left[\int_{a_1}^{b_1}(\lambda\tilde{f}_{\mathcal{N}}(x))^L_\alpha\mathrm{d}x,\int_{a_1}^{b_1}(\lambda\tilde{f}_{\mathcal{N}}(x))^U_\alpha\mathrm{d}x\right],\left[\int_{a_1}^{b_1}(\lambda\tilde{f}_{\mathcal{N}}(x))^L_\beta\mathrm{d}x,\int_{a_1}^{b_1}(\lambda\tilde{f}_{\mathcal{N}}(x))^U_\beta\mathrm{d}x\right],\left[\int_{a_1}^{b_1}(\lambda\tilde{f}_{\mathcal{N}}(x))^L_\gamma\mathrm{d}x,\int_{a_1}^{b_1}(\lambda\tilde{f}_{\mathcal{N}}(x))^U_\gamma\mathrm{d}x\right]\right\rangle\ (By\ Proposition\ 4.3)$$
$$=\left(\int_{a_1}^{b_1}\lambda\tilde{f}_{\mathcal{N}}(x)\mathrm{d}x\right)_{(\alpha,\beta,\gamma)}$$

Therefore from Definition 3.2 (iii) we have, $\int_{a_1}^{b_1}\lambda\tilde{f}_{\mathcal{N}}(x)\mathrm{d}x=\lambda\int_{a_1}^{b_1}\tilde{f}_{\mathcal{N}}(x)\mathrm{d}x$.

The proof is similar for $\lambda<0$.

This completes the proof. □

5 Examples

In this section, we shall discuss some test problems and their numerical results. Tables 1 and 2 are calculated by using Wolfram Mathematica 9.0. Here, the figures have been drawn by using MATLAB R2014a.

Example 1 Let us consider the neutrosophic valued function $\tilde{f}_{\mathcal{N}}(x)=\tilde{n}x^2$ on [0, 1], where $\tilde{n}=\langle(0,1,2);0.8,0.6,0.4\rangle$ is a single valued triangular neutrosophic number. Now we integrate the neutrosophic valued function on [0, 1], i.e., we try to find $\int_0^1\tilde{f}_{\mathcal{N}}(x)\mathrm{d}x$.

Now, we take (α,β,γ)-cut of the integral $\int_0^1\tilde{f}_{\mathcal{N}}(x)\mathrm{d}x$. Then, we have,

$$\left(\int_0^1\tilde{f}_{\mathcal{N}}(x)\mathrm{d}x\right)_{(\alpha,\beta,\gamma)}$$
$$=\left(\int_0^1\tilde{n}x^2\mathrm{d}x\right)_{(\alpha,\beta,\gamma)}$$
$$=\left\langle\left[\int_0^1\frac{5\alpha}{4}x^2\mathrm{d}x,\int_0^1\frac{8-5\alpha}{4}x^2\mathrm{d}x\right],\left[\int_0^1\frac{5}{2}(1-\beta)x^2\mathrm{d}x,\int_0^1\frac{1}{2}(5\beta-1)x^2\mathrm{d}x\right],\left[\int_0^1\frac{5}{3}(1-\gamma)x^2\mathrm{d}x,\int_0^1\frac{1}{3}(5\gamma+1)x^2\mathrm{d}x\right]\right\rangle$$
$$=\left\langle\left[\frac{5\alpha}{4}\int_0^1x^2\mathrm{d}x,\frac{8-5\alpha}{4}\int_0^1x^2\mathrm{d}x\right],\left[\frac{5}{2}(1-\beta)\int_0^1x^2\mathrm{d}x,\frac{1}{2}(5\beta-1)\int_0^1x^2\mathrm{d}x\right],\left[\frac{5}{3}(1-\gamma)\int_0^1x^2\mathrm{d}x,\frac{1}{3}(5\gamma+1)\int_0^1x^2\mathrm{d}x\right]\right\rangle$$
$$=\left\langle\left[\frac{5\alpha}{12},\frac{8-5\alpha}{12}\right],\left[\frac{5}{6}(1-\beta),\frac{1}{6}(5\beta-1)\right],\left[\frac{5}{9}(1-\gamma),\frac{1}{9}(5\gamma+1)\right]\right\rangle$$

where $\int_0^1\tilde{f}^L_{\mathcal{N}\alpha}(x)\mathrm{d}x=\frac{5\alpha}{12}$, $\int_0^1\tilde{f}^U_{\mathcal{N}\alpha}(x)\mathrm{d}x=\frac{8-5\alpha}{12}$, $\int_0^1\tilde{f}^L_{\mathcal{N}\beta}(x)\mathrm{d}x=\frac{5}{6}(1-\beta)$, $\int_0^1\tilde{f}^U_{\mathcal{N}\beta}(x)\mathrm{d}x=\frac{1}{6}(5\beta-1)$, $\int_0^1\tilde{f}^L_{\mathcal{N}\gamma}(x)\mathrm{d}x=\frac{5}{9}(1-\gamma)$ and $\int_0^1\tilde{f}^U_{\mathcal{N}\gamma}(x)\mathrm{d}x=\frac{1}{9}(5\gamma+1)$ for $\alpha\in[0,0.8]$, $\beta\in[0.6,1]$ and $\gamma\in[0.4,1]$.

In Table 1, it is seen that when the value of α increases, the value of $\int_0^1\tilde{f}^L_{\mathcal{N}\alpha}(x)\mathrm{d}x$ increases and the value of $\int_0^1\tilde{f}^U_{\mathcal{N}\alpha}(x)\mathrm{d}x$ decreases. At $\alpha=0.8$, $\int_0^1\tilde{f}^L_{\mathcal{N}\alpha}(x)\mathrm{d}x$ and

Table 1 Solutions for the different values of α, β and γ for Example 1

α	$\int_0^1 \tilde{f}^L_{\mathcal{N}\alpha}(x)\mathrm{d}x$	$\int_0^1 \tilde{f}^U_{\mathcal{N}\alpha}(x)\mathrm{d}x$	β	$\int_0^1 \tilde{f}^L_{\mathcal{N}\beta}(x)\mathrm{d}x$	$\int_0^1 \tilde{f}^U_{\mathcal{N}\beta}(x)\mathrm{d}x$	γ	$\int_0^1 \tilde{f}^L_{\mathcal{N}\gamma}(x)\mathrm{d}x$	$\int_0^1 \tilde{f}^U_{\mathcal{N}\gamma}(x)\mathrm{d}x$
0	0	0.66667	0.6	0.33333	0.33333	0.4	0.33333	0.33333
0.4	0.16667	0.5	0.8	1.66667	0.5	0.6	0.22222	0.44444
0.6	0.25	0.41667	0.9	0.08333	0.58333	0.8	0.11111	0.66667
0.8	0.33333	0.33333	1	0	0.66667	1	0	0.66667

Table 2 Approximate value of the integral in Example 2 by trapezoidal rule for the different values of α, β and γ

α	$\int_0^1 \tilde{f}^L_{\mathcal{N}\alpha}(x)\mathrm{d}x$	$\int_0^1 \tilde{f}^U_{\mathcal{N}\alpha}(x)\mathrm{d}x$	β	$\int_0^1 \tilde{f}^L_{\mathcal{N}\beta}(x)\mathrm{d}x$	$\int_0^1 \tilde{f}^U_{\mathcal{N}\beta}(x)\mathrm{d}x$	γ	$\int_0^1 \tilde{f}^L_{\mathcal{N}\gamma}(x)\mathrm{d}x$	$\int_0^1 \tilde{f}^U_{\mathcal{N}\gamma}(x)\mathrm{d}x$
0	0	1.03125	0.6	0.34375	0.6875	0.4	0.34375	0.6875
0.4	0.171875	0.859375	0.8	0.171875	0.859375	0.6	0.229167	0.802083
0.6	0.257813	0.773438	0.9	0.0859375	0.945313	0.8	0.114583	0.916667
0.8	0.34375	0.6875	1	0	1.03125	1	0	1.03125

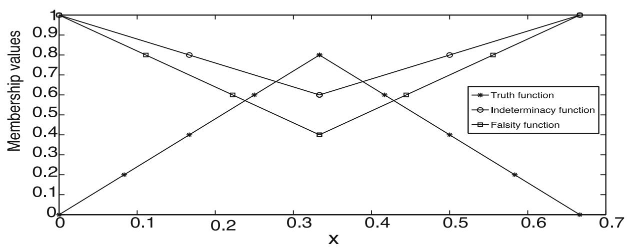

Fig. 1 The truth, indeterminacy and falsity membership functions of the integral in Example 1

$\int_0^1 \tilde{f}^U_{\mathcal{N}\alpha}(x)\mathrm{d}x$ gives the same solution. Again, when the value of β increases, the value of $\int_0^1 \tilde{f}^L_{\mathcal{N}\beta}(x)\mathrm{d}x$ decreases and the value of $\int_0^1 \tilde{f}^U_{\mathcal{N}\beta}(x)\mathrm{d}x$ increases. At $\beta = 0.6$, $\int_0^1 \tilde{f}^L_{\mathcal{N}\beta}(x)\mathrm{d}x$ and $\int_0^1 \tilde{f}^U_{\mathcal{N}\beta}(x)\mathrm{d}x$ gives the same solution and similarly when γ increases, the value of $\int_0^1 \tilde{f}^L_{\mathcal{N}\gamma}(x)\mathrm{d}x$ decreases and the value of $\int_0^1 \tilde{f}^U_{\mathcal{N}\gamma}(x)\mathrm{d}x$ increases. This follows that the approximate solution in Table 1 gives a neutrosophic number. From Fig. 1, it is seen that the solution gives a triangular neutrosophic number.

In the following example, we are going to show that how one can use the existing numerical integration methods to solve the neutrosophic integral. So, in the following example, we consider the same neutrosophic function, but the parameter was taken in the form of trapezoidal neutrosophic number.

Example 2 Let us consider the neutrosophic valued function $\tilde{f}_{\mathcal{N}}(x) = \tilde{n}x^2$ on $[0, 1]$, where $\tilde{n} = \langle (0, 1, 2, 3); 0.8, 0.6, 0.4\rangle$ is a single-valued trapezoidal neutrosophic number. Now we integrate the neutrosophic valued function on $[0, 1]$, i.e., we try to find $\int_0^1 \tilde{f}_{\mathcal{N}}(x)\mathrm{d}x$.

Now, we take (α, β, γ)-cut of the integral $\int_0^1 \tilde{f}_{\mathcal{N}}(x)\mathrm{d}x$. Then, we have,

$$\left(\int_0^1 \tilde{f}_{\mathcal{N}}(x)\mathrm{d}x\right)_{(\alpha,\beta,\gamma)} = \left\langle \left[\int_0^1 \tilde{f}^L_{\mathcal{N}\alpha}(x)\mathrm{d}x, \int_0^1 \tilde{f}^U_{\mathcal{N}\alpha}(x)\mathrm{d}x\right], \left[\int_0^1 \tilde{f}^L_{\mathcal{N}\beta}(x)\mathrm{d}x, \int_0^1 \tilde{f}^U_{\mathcal{N}\beta}(x)\mathrm{d}x\right], \left[\int_0^1 \tilde{f}^L_{\mathcal{N}\gamma}(x)\mathrm{d}x, \int_0^1 \tilde{f}^U_{\mathcal{N}\gamma}(x)\mathrm{d}x\right]\right\rangle$$

Since each of the integral $\int_0^1 \tilde{f}^L_{\mathcal{N}\alpha}(x)\mathrm{d}x$, $\int_0^1 \tilde{f}^U_{\mathcal{N}\alpha}(x)\mathrm{d}x$, $\int_0^1 \tilde{f}^L_{\mathcal{N}\beta}(x)\mathrm{d}x$, $\int_0^1 \tilde{f}^U_{\mathcal{N}\beta}(x)\mathrm{d}x$, $\int_0^1 \tilde{f}^L_{\mathcal{N}\gamma}(x)\mathrm{d}x$ and $\int_0^1 \tilde{f}^U_{\mathcal{N}\gamma}(x)\mathrm{d}x$ are Riemann integrable on $[0, 1]$. Then, we can use trapezoidal rule to find the approximate value of the each integral.

Now we find the approximate value of the integral $\int_0^1 x^2\mathrm{d}x$ with the help of trapezoidal rule.

Let $P = \{0.\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ be the set of all elements in the endpoints of the sub-interval and $\Delta x = \frac{1-0}{4} = \frac{1}{4}$.

Then,

$$\begin{aligned}\int_0^1 x^2\mathrm{d}x &\approx \frac{1}{4} \times \frac{1}{4}[f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) \\ &\quad + 2f\left(\frac{3}{4}\right) + f(1) \\ &= \frac{1}{8}\left(0 + \frac{1}{8} + \frac{1}{2} + \frac{9}{8} + 1\right) \\ &= \frac{11}{32}\end{aligned}$$

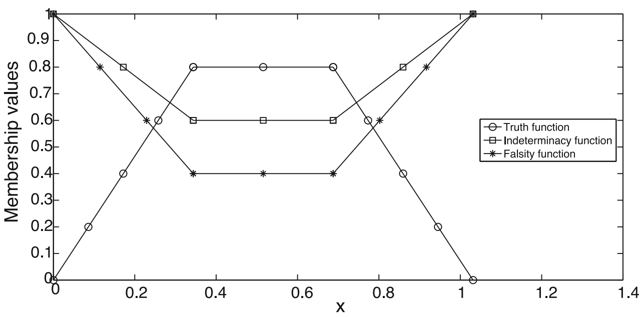

Fig. 2 The truth, indeterminacy and falsity membership functions of the integral in Example 2 by trapezoidal rule

Therefore,

$$\left(\int_0^1 \tilde{f}_{\mathcal{N}}(x)\mathrm{d}x\right)_{(\alpha,\beta,\gamma)}$$
$$= \left(\int_0^1 \tilde{n}x^2\mathrm{d}x\right)_{(\alpha,\beta,\gamma)}$$
$$= \left\langle\left[\int_0^1 \frac{5\alpha}{4}x^2\mathrm{d}x, \int_0^1 \frac{12-5\alpha}{4}x^2\mathrm{d}x\right], \left[\int_0^1 \frac{5}{2}(1-\beta)x^2\mathrm{d}x, \int_0^1 \frac{1}{2}(5\beta+1)x^2\mathrm{d}x\right], \left[\int_0^1 \frac{5}{3}(1-\gamma)x^2\mathrm{d}x, \int_0^1 \frac{1}{3}(5\gamma+4)x^2\mathrm{d}x\right]\right\rangle$$
$$= \left\langle\left[\frac{5\alpha}{4}\int_0^1 x^2\mathrm{d}x, \frac{12-5\alpha}{4}\int_0^1 x^2\mathrm{d}x\right], \left[\frac{5}{2}(1-\beta)\int_0^1 x^2\mathrm{d}x, \frac{1}{2}(5\beta+1)\int_0^1 x^2\mathrm{d}x\right], \left[\frac{5}{3}(1-\gamma)\int_0^1 x^2\mathrm{d}x, \frac{1}{3}(5\gamma+4)\int_0^1 x^2\mathrm{d}x\right]\right\rangle$$

$$= \left\langle\left[\frac{55\alpha}{128}, \frac{132-55\alpha}{128}\right], \left[\frac{55}{64}(1-\beta), \frac{11}{64}(5\beta+1)\right], \left[\frac{55}{96}(1-\gamma), \frac{11}{96}(5\gamma+4)\right]\right\rangle$$

where $\int_0^1 \tilde{f}^L_{\mathcal{N}\alpha}(x)\mathrm{d}x = \frac{55\alpha}{128}$, $\int_0^1 \tilde{f}^U_{\mathcal{N}\alpha}(x)\mathrm{d}x = \frac{132-55\alpha}{128}$, $\int_0^1 \tilde{f}^L_{\mathcal{N}\beta}(x)\mathrm{d}x = \frac{55}{64}(1-\beta)$, $\int_0^1 \tilde{f}^U_{\mathcal{N}\beta}(x)\mathrm{d}x = \frac{11}{64}(5\beta+1)$, $\int_0^1 \tilde{f}^L_{\mathcal{N}\gamma}(x)\mathrm{d}x = \frac{55}{96}(1-\gamma)$ and $\int_0^1 \tilde{f}^U_{\mathcal{N}\gamma}(x)\mathrm{d}x = \frac{11}{96}(5\gamma+4)$ for $\alpha \in [0, 0.8]$, $\beta \in [0.6, 1]$ and $\gamma \in [0.4, 1]$.

In Table 2, it is seen that when the value of α increases, the value of $\int_0^1 \tilde{f}^L_{\mathcal{N}\alpha}(x)\mathrm{d}x$ increases and the value of $\int_0^1 \tilde{f}^U_{\mathcal{N}\alpha}(x)\mathrm{d}x$ decreases. Again, when the value of β increases, the value of $\int_0^1 \tilde{f}^L_{\mathcal{N}\beta}(x)\mathrm{d}x$ decreases and the value of $\int_0^1 \tilde{f}^U_{\mathcal{N}\beta}(x)\mathrm{d}x$ increases and similarly when γ increases, the value of $\int_0^1 \tilde{f}^L_{\mathcal{N}\gamma}(x)\mathrm{d}x$ decreases and the value of $\int_0^1 \tilde{f}^U_{\mathcal{N}\gamma}(x)\mathrm{d}x$ increases. This follows that the approximate solution in Table 2 gives a neutrosophic number. From Fig. 2, it is seen that the solution gives a trapezoidal neutrosophic number.

6 Conclusion

In the beginning of this article basically, we have defined some properties of neutrosophic set, neutrosophic number and neutrosophic function. In this article, we have mainly focused on the theoretical development of neutrosophic calculus. In Proposition 3.1, it has been seen that the (α, β, γ)-cut of a closed neutrosophic number is a closed interval neutrosophic number. From Proposition 3.6, it has been shown that the (α, β, γ)-cut of a neutrosophic set can induce a closed neutrosophic number. Some properties of neutrosophic number like $'+'$, $'-'$ and $'\times'$ have been presented here, and we have redefined neutrosophic arithmetic in the sense of (α, β, γ)-cut in Proposition 3.3 which will help to solve different types of neutrosophic differential, integral and integro-differential equation.

We have also defined some properties of neutrosophic function which are necessary to define neutrosophic Riemann integration over closed and bounded interval $[a_1, b_1]$. In Definition 4.2, we have defined neutrosophic Riemann integral on closed-bounded interval $[a_1, b_1]$. In Theorem 4.1, it has been seen that the neutrosophic Riemann integral of a neutrosophic function on $[a_1, b_1]$ is also a neutrosophic number. In Theorems 4.3 and 4.4, it has been seen that if $\tilde{f}_{\mathcal{N}}(x), \tilde{g}_{\mathcal{N}}(x) \in \mathcal{N}_{RI}$ and $\lambda \in \mathbb{R}$, then $\tilde{f}_{\mathcal{N}}(x) + \tilde{g}_{\mathcal{N}}(x) \in \mathcal{N}_{RI}$, $\tilde{f}_{\mathcal{N}}(x) - \tilde{g}_{\mathcal{N}}(x) \in \mathcal{N}_{RI}$ and $\lambda \tilde{f}_{\mathcal{N}}(x) \in \mathcal{N}_{RI}$. In the second example, it has been shown that the existing numerical integration method like trapezoidal rule can be used to solve the neutrosophic Riemann integration. In our article, it has been studied that we can apply neutrosophic Riemann integral only for bounded neutrosophic function. Also, the interval where the function is defined, it must be bounded. So, these are some of the limitations of the neutrosophic Riemann integration method. The study of improper integral in the neutrosophic environment will help to overcome from these limitations of the neutrosophic Riemann integration method. This article will help to develop the concept of neutrosophic improper integral method. Also, in the future, this article will be the pillar of the neutrosophic integral calculus and it has probable scope to develop the neutrosophic integral calculus in the future.

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Declarations

Conflict of interest Authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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