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Abstract

We present a neutrosophic set-based model for a time-dependent decision-support system (DSS) with multi-attribute criteria decision-making. We describe such a DSS as one that includes multiple conflicting objectives, having strategies spanning over several discrete time periods. In this paper, we utilize the concept of neutrosophic sets and some of its operations to present a computational model that captures decision trees with various imprecise preferences for a time-dependent DSS. Given a time-dependent DSS with N objectives spanning over discrete time periods t ranging from t_0 to t_n, we are able to use a set of m attributes, denoted by variables $a_1, \ldots, a_m$, where each variable a_k ($k = 1, \ldots, m$), for each $t \in [t_0, t_n]$, is described by a triplet variable $x_k(\tau_{k_t}, i_{k_t}, f_{k_t})$, where the terms τ_{k_t}, i_{k_t}, and f_{k_t} represent degrees of truthfulness membership, indeterminacy membership, and falsity membership for attribute a_k at time t, respectively. We then define a set of m time-dependent vectors of imprecise consequences S_q corresponding to a set of strategies derived from the membership of each attribute a_k. For each time t, we normalize the set of imprecise consequences to define the weighted values for each attribute. We proceed with an interpretation and a sensitivity analysis of the normalized imprecise consequences and derive a ranking process of strategies at each time t. As a result, the model presents the decision maker with a set of strategies ranked based on the neutrosophic values of each corresponding attribute at each time t.

Keywords: Neutrosophic set, neutrosophic logic, single-valued neutrosophic set, geometric operator score, set-theoretic model theory, decision theory.

1. Introduction

1.1. Computational modeling in decision-support systems

The challenges in decision-making of complex problems that are rapidly changing and not easily specified in advance has led to the development of model-driven decision-support systems (DSSs)[1]. Model-driven DSSs can be described as complex systems in which a set of specific required data (or attributes) are carefully studied and analyzed to develop multiple sets of strategies, which allows the decision maker to select the most efficient set of strategies that achieves a predefined set of objectives. Model-driven DSSs have evolved over the past three decades from simple model-oriented systems to advanced multi-function entities[2]. This stems from the continuous evolution of the necessary computational modeling required to

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solve most of these challenges. As a result, the growing need for decision support in such problems has shaped most DSS models to be robustly designed specifically to[3 4 5]:

- facilitate dynamic decision processes through the use of a series of attributes to reach a set of strategies.
  - Interdependency between each of the attributes and their consequent strategies is accounted for and supported.
  - Each strategy is then capable of affecting future attributes and/or strategies at some later time.
- support rather than just automate decision-making.
  - Some DSS models will make decisions based on predefined rules in the environment where the DSS operates. However, in the most complex situations, many of those rules may not be as relevant as originally thought. Thus, the best DSS models are also capable of presenting the decision maker with relevant information about each strategy and how it aligns with the corresponding objective(s). In the end, the decision maker will select the best set of strategies.
- be able to respond quickly to the changing needs of decision makers.
  - DSS models must support a body of knowledge for the DSS[4 5] that is capable of record keeping, presenting information on an ad hoc basis in both standardized and customized reports, selecting a desired subset of stored strategies for either presentation or for deriving new strategies, and interacting directly with a decision maker in such a way that the user has a flexible choice and sequence of knowledge-management activities[5 6].

In light of those stated specifications above, efficient model-driven DSSs serve as critical tools for the decision maker, in that they facilitate him/her with the ability to select the best strategies based on any given set of attributes in the present, foresee consequences in the future, and present ad hoc information on each strategy's alignment with each of the given objectives at any given time. Moreover, in more complex situations (e.g., when the problem contains multiple conflicting objectives spanning over several discrete time periods t ($t \in [t_0, t_n]$) and a predefined set of attributes along with their respective weights and uncertainty are allowed to change over time), model-driven DSSs have been developed to help identify optimal strategies by using *interactive multi-objective simulated annealing*[1] or *imprecise multi-attribute additive modeling*[7]. In both cases, each attribute is assigned an absolute weight value so that at each point in time, the decision maker is able to evaluate its importance or preference over other attributes at that point in time. In contrast, each attribute can also be set to a different weight value at each time t, ($t \in [t_0, t_n]$), as more data become available[8]. Either way, it becomes very difficult to assess each attribute (by the decision maker's preference, the attribute's importance, or the attribute's relevance, at time t) when presented with a sufficiently large number of attributes or time units within the $[t_0, t_n]$ interval. Consequently, since most multi-attribute DSS models use a finite but large number of attributes (time-dependent or not), we approach this problem using a neutrosophic set-based computational model, as described in more detail in Section 2. In the model, rather than restricting each individual attribute to a weighting value, we first define three categories of membership (truthfulness τ, indeterminacy i, and falsity f). Then, we assign each attribute a time-dependent triplet variable $x(\tau_t, i_t, f_t)$, denoting whether that attribute holds true, indeterminate, or false at time t. Finally, a strategy is developed based on the membership of that attribute at time t. This allows us to only specify what the membership of the attribute is at time t, and therefore enables us to bypass the need to evaluate each attribute individually by preference, importance, or relevance, at each time t, as mentioned above. Such a model is a neutrosophic set-based model, derived from the neutrosophic theory. Subsection 1.2 highlights the main properties of neutrosophic logic and derived neutrosophic sets necessary for the development of this model.

1.2. Neutrosophic sets and neutrosophy

Neutrosophic sets derive from *neutrosophy*. Neutrosophy and neutrosophic sets were both introduced by Florentin Smarandache in 1995[9 10] as a generalization of intuitionistic fuzzy logic[11 12] and sets, respectively. Smarandache defined neutrosophy as the study of origins, nature, and scope of neutralities, as well as their interactions with different ideational spectra[9]. In other words, neutrosophy is the study of ideas and notions which are not true, nor false, but are said to be between true and false, neutral, indeterminate, unclear, vague, ambiguous, incomplete, contradictory, and so forth[13]. Thus, we describe a neutrosophic set as one where each element of the universe has a degree of truth, indeterminacy, and falsity, respectively. The degrees of truth, indeterminacy, and falsity are called *neutrosophic components*. Each neutrosophic component lies between non-standard, infinitesimal unit (hyperreal) interval, denoted $]^{-}0, 1^{+}[$. Thus, unlike intuitionistic fuzzy sets, there are no constraint between the degree of truth, the degree of indeterminacy, and the degree of falsity in neutrosophic sets. Moreover, in neutrosophic sets, the sum of the scalar neutrosophic components (denoted $n_{n.s.}$) does not necessarily equal to 1. For example, assuming that $\tau, i, f \in]^{-}0, 1^{+}[$ are the degrees of truthfulness, indeterminacy, and falsity, respectively, thus $n_{n.s.} = \tau+i+f$ can be any number between 0 and 3. This leads to the two main distinctions between neutrosophic sets and intuitionistic fuzzy sets, which are (a) in neutrosophic sets, $n_{n.s.}$ can be any number in the range 0 and 3 in order to allow for the characterization of incomplete information while $n_{n.s.}$ is restricted and equal to 1 for intuitionistic fuzzy sets; and (b) in neutrosophic sets, we use the non-standard interval $]^{-}0, 1^{+}[$ for each of the neutrosophic components to differentiate between absolute membership (denoted by 1^{+}) and relative membership (denoted by 1), while the standard interval [0, 1] is used in intuitionistic fuzzy sets[9]. Since decision-making involves the analysis of a finite set of alternatives described in terms of evaluative criteria, neutrosophic sets can be useful in the development of DSS models. To this end, we define the general concepts and operations on neutrosophic sets.

1.2.1. General concepts of neutrosophic sets

Definition 1.1. ***(Neutrosophic set)***[10] *Let A be a subset of a universe of discourse* $\mathbb{U}$. *Each element* $x \in \mathbb{U}$ *has neutrosophic components* $\tau \in]^{-}0, 1^{+}[$, $i \in]^{-}0, 1^{+}[$, *and* $f \in]^{-}0, 1^{+}[$ *with respect to A. The notation* $x(\tau, i, f) \in A$ *means that*

- *The neutrosophic component* τ *is the degree of truthfulness of x with respect to A;*
- *The neutrosophic component i is the degree of indeterminacy of x with respect to A;*
- *The neutrosophic component f is the degree of falsity of x with respect to A.*

A is a neutrosophic set, whereas τ, i, f *are the neutrosophic components of the element x with respect to A.*

Definition 1.2. ***(Complement)***[9] *Given a neutrosophic set A, we define the complement of A, denoted* c_A, *as the neutrosophic set with the property that* $x(1-\tau_A, 1-i_A, 1-f_A) \in c_A$ *if and only if* $x(\tau_A, i_A, f_A) \in A$.

Definition 1.3. ***(Containment)*** *Given neutrosophic sets* A_1 *and* A_2, *we write* $A_1 \subseteq A_2$ *if for all* $x(\tau_{A1}, i_{A1}, f_{A1}) \in A_1$ *and* $x(\tau_{A2}, i_{A2}, f_{A2}) \in A_2$,

$$\inf(\tau_{A_1}) \leq \inf(\tau_{A_2}), \sup(\tau_{A_1}) \leq \sup(\tau_{A_2}); \tag{1}$$

$$\inf(f_{A_1}) \geq \inf(f_{A_2}), \sup(f_{A_1}) \geq \sup(f_{A_2}). \tag{2}$$

Definition 1.4. ***(Union)*** *Given neutrosophic sets A_1 and A_2, for all $x(\tau_{A1}, i_{A1}, f_{A1}) \in A_1$ and $x(\tau_{A2}, i_{A2}, f_{A2}) \in A_2$, the neutrosophic components of x with respect to the union $A_3 = A_1 \cup A_2$ are defined by*

$$\tau_{A_3} = \tau_{A_1} + \tau_{A_2} - \tau_{A_1} \times \tau_{A_2}; \tag{3}$$
$$i_{A_3} = i_{A_1} + i_{A_2} - i_{A_1} \times i_{A_2}; \tag{4}$$
$$f_{A_3} = f_{A_1} + f_{A_2} - f_{A_1} \times f_{A_2}. \tag{5}$$

Definition 1.5. ***(Intersection)*** *Given neutrosophic sets A_1 and A_2, for all $x(\tau_{A1}, i_{A1}, f_{A1}) \in A_1$ and $x(\tau_{A2}, i_{A2}, f_{A2}) \in A_2$, the neutrosophic components of x with respect to the intersection $A_3 = A_1 \cap A_2$ are defined by*

$$\tau_{A_3} = \tau_{A_1} \times \tau_{A_2}; \tag{6}$$
$$i_{A_3} = i_{A_1} \times i_{A_2}; \tag{7}$$
$$f_{A_3} = f_{A_1} \times f_{A_2}. \tag{8}$$

Definition 1.6. ***(Single-valued neutrosophic set)***[9 14] *Let $u \subset \mathbb{U}$ be a topological space of points (or objects), given a neutrosophic set A in u, for all $x(\tau, i, f) \in A$, A is a single-valued neutrosophic set (SVNS) in u, denoted $A = \{\langle x : \tau, i, f \rangle, x \in u\}$, if and only if τ, i, and $f \in [0, 1]$.*

From Definition 1.6, there is no restriction on $n_{n.s.}$, which may be as low as 0 and as high as 3.

1.2.2. Set-theoretic operations on SVNSs

The following definitions highlight the set-theoretic operations on SVNSs.

Definition 1.7. [14] *Given a SVNS $A = \{\langle x : \tau_A, i_A, f_A \rangle, x \in u\}$; then,*

1. *the complement of A, denoted c_A, is given by*

$$c_A = \{\langle x : f_A, 1 - i_A, \tau_A \rangle, x \in u\}. \tag{9}$$

2. *for $\lambda > 0$, we have*

$$\lambda \times A = \{\langle x : 1 - (1 - \tau_A)^\lambda, i_A^\lambda, f_A^\lambda \rangle, x \in u\}; \tag{10}$$
$$A^\lambda = \{\langle x : \tau_A^\lambda, 1 - (1 - i_A)^\lambda, 1 - (1 - f_A)^\lambda \rangle, x \in u\}. \tag{11}$$

Definition 1.8. [9 14 15] *Given two SVNSs $A_1 = \{\langle x : \tau_{A_1}, i_{A_1}, f_{A_1} \rangle, x \in u\}$ and $A_2 = \{\langle x : \tau_{A_2}, i_{A_2}, f_{A_2} \rangle, x \in u\}$; then,*

1. *$A_1 \subseteq A_2$, if and only if*

$$\tau_{A_1} \le \tau_{A_2}, i_{A_1} \ge i_{A_2}, f_{A_1} \ge f_{A_2}. \tag{12}$$

2. *$A_1 = A_2$, if and only if*

$$\tau_{A_1} = \tau_{A_2}, i_{A_1} = i_{A_2}, f_{A_1} = f_{A_2}. \tag{13}$$

3. *$A_3 = A_1 \cup A_2$ is defined by*

$$A_3 = \{\langle x : \max(\tau_{A_1}, \tau_{A_2}), \min(i_{A_1}, i_{A_2}), \min(f_{A_1}, f_{A_2}) \rangle, x \in u\}. \tag{14}$$

4. $A_3 = A_1 \cap A_2$ *is defined by*

$$A_3 = \{\langle x : \min(\tau_{A_1}, \tau_{A_2}), \max(i_{A_1}, i_{A_2}), \max(f_{A_1}, f_{A_2})\rangle, x \in u\}. \tag{15}$$

5. $A_3 = A_1 + A_2$ *is defined by*

$$A_3 = \{\langle x : \tau_{A_1} + \tau_{A_2} - \tau_{A_1}\tau_{A_2}, i_{A_1}i_{A_2}, f_{A_1}f_{A_2}\rangle, x \in u\}. \tag{16}$$

6. $A_3 = A_1 \times A_2$ *is defined by*

$$A_3 = \{\langle x : \tau_{A_1}\tau_{A_2}, i_{A_1} + i_{A_2} - i_{A_1}i_{A_2}, f_{A_1} + f_{A_2} - f_{A_1}f_{A_2}\rangle, x \in u\}. \tag{17}$$

Definition 1.9. [14] *Given a SVNS* $A = \{\langle x : \tau_A, i_A, f_A\rangle, x \in u\}$, *then the score function* $\sigma : u \mapsto [-1, 1]$, *accuracy function* $\alpha : u \mapsto [-1, 1]$, *and certainty function* $\upsilon : u \mapsto [0, 1]$ *of A are defined by*

$$\sigma(A) = \frac{2 + \tau_A - i_A - f_A}{3}; \tag{18}$$

$$\alpha(A) = \tau_A - f_A; \tag{19}$$

$$\upsilon(A) = \tau_A. \tag{20}$$

Remark 1.1. *Given two SVNSs* $A_1 = \{\langle x : \tau_{A_1}, i_{A_1}, f_{A_1}\rangle, x \in u\}$ *and* $A_2 = \{\langle x : \tau_{A_2}, i_{A_2}, f_{A_2}\rangle, x \in u\}$, *the following conditions hold true:*

1. *If* $\sigma(A_1) > \sigma(A_2)$, *then* $A_1 > A_2$.
2. *If* $\sigma(A_1) = \sigma(A_2)$ *and* $\alpha(A_1) > \alpha(A_2)$, *then* $A_1 > A_2$.
3. *If* $\sigma(A_1) = \sigma(A_2)$, $\alpha(A_1) = \alpha(A_2)$, *and* $\upsilon(A_1) > \upsilon(A_2)$, *then* $A_1 > A_2$.
4. *If* $\sigma(A_1) = \sigma(A_2)$, $\alpha(A_1) = \alpha(A_2)$, *and* $\upsilon(A_1) = \upsilon(A_2)$, *then* $A_1 = A_2$.

Proof.

1. From equation (18), let $i_{A_1} + f_{A_1} = b_1$ for A_1 and $i_{A_2} + f_{A_2} = b_2$ for A_2, $\sigma(A_1) > \sigma(A_2)$ means that $\tau_{A_1} - b_1 > \tau_{A_2} - b_2$. We agree that $\tau_{A_1} \geq \tau_{A_2}$ and $b_1 \leq b_2$. It is easy to see that at a minimum, $A_1 \geq A_2$.
2. From equation (19), $\alpha(A_1) > \alpha(A_2)$ also means that $\tau_{A_1} - f_{A_1} > \tau_{A_2} - f_{A_2}$. Thus, $\tau_{A_1} \geq \tau_{A_2}$ and $f_{A_1} \leq f_{A_2}$ leads to $A_1 \geq A_2$, at least.
3. Let $i_{A_1} + f_{A_1} = b_1$ for A_1 and $i_{A_2} + f_{A_2} = b_2$ for A_2, we know that $b_1 = b_2$, and $\tau_{A_1} - f_{A_1} = \tau_{A_2} - f_{A_2}$. However, with $\upsilon(A_1) > \upsilon(A_2)$, then $\tau_{A_1} > \tau_{A_2}$. Thus, $A_1 > A_2$.
4. Let $i_{A_1} + f_{A_1} = b_1$ for A_1 and $i_{A_2} + f_{A_2} = b_2$ for A_2, having $b_1 = b_2$, $\tau_{A_1} - f_{A_1} = \tau_{A_2} - f_{A_2}$, and $\tau_{A_1} = \tau_{A_2}$, then $A_1 = A_2$.

□

Remark 1.2. *For a zero set, denoted* $0_{\mathbb{N}} = \{\langle x : 0, 1, 1\rangle, x \in u\}$, $\sigma(0_{\mathbb{N}}) = 0$, $\alpha(0_{\mathbb{N}}) = -1$, *and* $\upsilon(0_{\mathbb{N}}) = 0$.

Proof. The proof of this claim is easy to construct. □

Definition 1.10. ***(Truth- and falsity-favorite)***[9] *Given a SVNS* $A_1 = \{\langle x : \tau_{A_1}, i_{A_1}, f_{A_1}\rangle, x \in u\}$, *then the SVNS* A_2 *is a*

1. *truth-favorite of* A_1 *and is defined by* $A_2 = \Delta A_1$*, if and only if*

$$A_2 = \{\langle x : \min(\tau_{A_1} + i_{A_1}, 1), 0, f_{A_1}\rangle, x \in u\}. \tag{21}$$

2. *falsity-favorite of* A_1 *and is defined by* $A_2 = \nabla A_1$*, if and only if*

$$A_2 = \{\langle x : \tau_{A_1}, 0, \min(i_{A_1} + f_{A_1}, 1)\rangle, x \in u\}. \tag{22}$$

Remark 1.3. *The complement of a zero set, denoted* $c_{0_{\mathbb{N}}}$*, is defined by* $c_{0_{\mathbb{N}}} = \{\langle x : 1, 0, 0\rangle, x \in u\}$*. Thus,* $0_{\mathbb{N}} \subset c_{0_{\mathbb{N}}}$*. Furthermore,* $c_{O_{\mathbb{N}}}$ *is the complement of the false-favorite of* $0_{\mathbb{N}}$*, with* $\sigma(c_{0_{\mathbb{N}}}) = 1$*,* $\alpha(c_{0_{\mathbb{N}}}) = 1$*, and* $\upsilon(c_{0_{\mathbb{N}}}) = 1$.

Proof. The proof of this claim is easy to construct. □

Remark 1.4. *Given a SVNS* $A = \{\langle x : \tau_A, i_A, f_A\rangle, x \in u\}$*, then* $0_{\mathbb{N}} \subseteq A$.

Proof. This remark is a generalization of Remark 1.3. If A is a zero set, then $A = 0_{\mathbb{N}}$, which implies that $0_{\mathbb{N}} \subseteq A$. Otherwise, for all $x \in u$, it is clear that $0 \leq \tau_A$, $1 \geq i_A$, and $1 \geq f_A$ for all $\tau_A, i_A, f_A \in [0, 1]$ (See Definition 1.8). □

2. Developing a DSS computational model using SVNSs

2.1. Defining objectives and attributes

Using set-theoretic operations on SVNSs, we developed a model to support a finite number of predefined objectives over discrete time periods ranging from t_0 to t_n. The model supports the rearrangement of the given objectives into an objective tree with n objective levels matching the time interval $[t_0, t_n]$, as seen in figure 1. Previous studies have been able to determine that an objective's importance can change over time based on how the decision maker perceives each objective and its alignment with the strategies developed for each attribute[1]; thus, the model also needs to take this finding into account. As a result, the model is designed to support a finite number m time-dependent attributes, denoted, $a_1, \ldots, a_m$, spanning over each $t \in [t_0, t_n]$ (Figure 1). That is, each a_k, with $k \in [1, m]$, is a $(n + 1)$-tuple, thus

$$a_k(t) = \{a_k(t_0), \ldots, a_k(t_n)\}. \tag{23}$$

Each attribute a_k's value at time t, i.e., $a_k(t)$, is determined based on the neutrosophic values assigned to that attribute. That is, we bypass the need to determine the relevance of attribute a_k at time t, beforehand. The model solely asks, once known, whether a_k is true, indeterminate, or false at time t. Then, we define each $a_k(t)$ as a SVNS, whose neutrosophic component values are used to determining a set of strategies. The set of strategies is characterized by a vector of interval containing time-dependent consequence values for each $a_k(t)$. We describe the obtained time-dependent consequence values as *imprecise consequences*. This stems from the fact that the set of strategies developed are based on neutrosophic values that encompass unclear, ambiguous, or incomplete knowledge of each attribute in the future, thus at time t. The decision maker will then assess each attribute based on the imprecise consequence for that attribute. Last, it is worth noting that this model does not manipulate or make changes to the objectives; any manipulation/assessment other than rearrangement of attributes into an objective tree is left to the decision maker. To this end, the first definition for the model is as follows:

Definition 2.1. *Given a non-empty set of N predefined objectives and m time-dependent attributes, all spanning over* $[t_0, t_n]$*; each attribute* a_k*, at time t, is a SVNS, and is defined by*

$$a_k(t) = \Big\{\langle x : \tau_{k_t}, i_{k_t}, f_{k_t}\rangle, x \in [1, m] \times [t_0, t_n]\Big\}, \tag{24}$$

where $\tau_{k_t}, i_{k_t}, f_{k_t} \in [0, 1]$*,* $m \in \mathbb{N}^*$*,* $k \in [1, m]$*, and* $t \in [t_0, t_n]$*.*

Remark 2.1. *For* $k \in [1, m]$ *and* $t \in [t_0, t_n]$*,* $a_k(t)$ *is a non-zero set.*

Proof. We prove that the following conditions are met:

1. For $k \in [1, m]$ and all t, $0_{\mathbb{N}} \subseteq a_k$. (See Remark 1.4).

2. Let $x_{\text{val}} \in u$, where $u = [1, m] \times [t_0, t_n]$, for which we have 2 non-empty SVNSs $a_{\text{val}_1} = \{\langle x_{\text{val}} : \tau_{\text{val}_1}, i_{\text{val}_1}, f_{\text{val}_1}\rangle, x_{\text{val}} \in u\} \subset a_k$ and $a_{\text{val}_2} = \{\langle x_{\text{val}} : \tau_{\text{val}_2}, i_{\text{val}_2}, f_{\text{val}_2}\rangle, x_{\text{val}} \in u\} \subset a_k$, if there exist $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 \times a_{\text{val}_1} + \lambda_2 \times a_{\text{val}_2} \subset a_k$, then a_k is non-zero set. From Definition 1.8, we know that

$$\lambda_1 \times a_{\text{val}_1} = \Big\{\langle x_{\text{val}} : 1 - (1 - \tau_{\text{val}_1})^{\lambda_1}, i_{\text{val}_1}^{\lambda_1}, f_{\text{val}_1}^{\lambda_1}\rangle, x_{\text{val}} \in u\Big\}; \tag{25}$$

$$\lambda_2 \times a_{\text{val}_2} = \Big\{\langle x_{\text{val}} : 1 - (1 - \tau_{\text{val}_2})^{\lambda_2}, i_{\text{val}_2}^{\lambda_2}, f_{\text{val}_2}^{\lambda_2}\rangle, x_{\text{val}} \in u\Big\}. \tag{26}$$

   Thus, let $a_{\text{val}} = \lambda_1 \times a_{\text{val}_1} + \lambda_2 \times a_{\text{val}_2}$, we have

$$a_{\text{val}} = \Big\{\langle x_{\text{val}} : 1 - (1 - \tau_{\text{val}_1})^{\lambda_1}(1 - \tau_{\text{val}_2})^{\lambda_2}, i_{\text{val}_1}^{\lambda_1} i_{\text{val}_2}^{\lambda_2}, f_{\text{val}_1}^{\lambda_1} f_{\text{val}_2}^{\lambda_2}\rangle, x_{\text{val}} \in u\Big\} \tag{27}$$

   We know that $\tau_{\text{val}_1}, i_{\text{val}_1}, f_{\text{val}_1}, \tau_{\text{val}_2}, i_{\text{val}_2}, f_{\text{val}_2} \in [0, 1]$, thus for any $\lambda_1, \lambda_2 > 0$, $0 \leq i_{\text{val}_1}^{\lambda_1} i_{\text{val}_2}^{\lambda_2} \leq 1$ and $0 \leq f_{\text{val}_1}^{\lambda_1} f_{\text{val}_2}^{\lambda_2} \leq 1$. Furthermore, $1 - \tau_{\text{val}_1} \leq 1$ implies that $(1 - \tau_{\text{val}_1})^{\lambda_1} \leq 1$. The same applies for $(1 - \tau_{\text{val}_2})^{\lambda_2}$. As a result, $0 \leq 1 - (1 - \tau_{\text{val}_1})^{\lambda_1}(1 - \tau_{\text{val}_2})^{\lambda_2} \leq 1$ for any $\lambda_1, \lambda_2 > 0$. Now, since with $x_{\text{val}} \in u$, and $a_{\text{val}_1}, a_{\text{val}_2} \subset a_k$, then $a_{\text{val}} \subset a_k$ for all $x_{\text{val}} \in u$.

3. $a_{\text{val}_1} \cap a_{\text{val}_2} \subset a_k$. if $a_{\text{val}_1} \cap a_{\text{val}_2} = 0_{\mathbb{N}}$, then condition 1 applies. Otherwise,

$$a_{\text{val}_1} \cap a_{\text{val}_2} = \Big\{\langle x_{\text{val}} : \min(\tau_{\text{val}_1}, \tau_{\text{val}_2}), \max(i_{\text{val}_1}, i_{\text{val}_2}), \max(f_{\text{val}_1}, f_{\text{val}_2})\rangle, x_{\text{val}} \in u\Big\}. \tag{28}$$

   It becomes trivial that $a_{\text{val}_1} \cap a_{\text{val}_2} \subset a_k$, being that $a_{\text{val}_1}, a_{\text{val}_2}$ are non-empty sets.

All 3 conditions being met leads to the conclusion that $a_k(t)$ is a non-zero set. □

From Definition 2.1 and Remark 2.1, it is possible to rank attributes using the score function, compare attributes using the accuracy function, and determine the likelihood of an attribute using the certainty function (See Definition 1.9). In doing so, we are also able to derive strategies from attributes. The next subsection introduces a few more definitions and remarks.

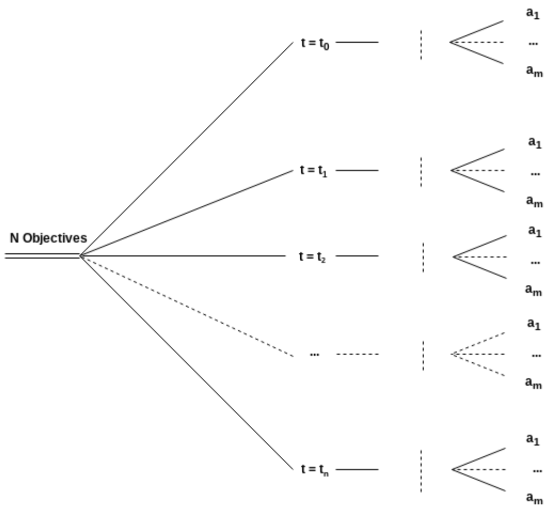

Figure 1: *Objective tree including time periods.*

2.2. Defining consequences of strategies

Definition 2.2. *Given a non-empty set of N predefined objectives and m time-dependent attributes represented by the SVNS $a_k(t) = \{\langle x : \tau_{k_t}, i_{k_t}, f_{k_t}\rangle, x \in [1, m] \times [t_0, t_n]\}$, let S be the set of available strategies derived from $a_k(t)$, the imprecise consequences of such strategies, denoted S_q, is a stream defined by a vector of intervals, and is defined by*

$$S_q = \{s_{q_1}(t), \ldots, s_{q_m}(t)\}; \tag{29}$$

with each $s_{q_k}(t)$, $k \in [1, m]$, defined by

$$s_{q_k}(t) \in \left[s_k^L(t), s_k^U(t)\right], \tag{30}$$

where $s_k^L(t)$ and $s_k^U(t)$ are (respectively) the lower and upper endpoints of the imprecise consequence for attribute a_k at time $t \in [t_0, t_n]$.

Assumption 2.1. *For all a_k, at time t, $S_q \subseteq S$.*

Assumption 2.1 is critical for this model. From equation (30), we can see that $s_{q_k}(t)$ is a value from the interval $[s_k^L(t), s_k^U(t)]$. Thus, for each $a_k(t)$, S contains all possible values within $[s_k^L(t), s_k^U(t)]$. Hence, the description of S being the set of all available strategies for each $a_k(t)$.

Assumption 2.2. *For all a_k, at time t, there is a continuous distribution between $s_k^L(t)$ and $s_k^U(t)$ endpoints.*

We will discuss the reason for Assumption 2.2 later in this subsection.

Definition 2.3. *Given a non-empty set of N predefined objectives and m time-dependent attributes represented by the SVNS $a_k(t) = \{\langle x : \tau_{k_t}, i_{k_t}, f_{k_t}\rangle, x \in [1, m] \times [t_0, t_n]\}$ with a derived set S of available strategies, at time t. Each imprecise consequence interval for $a_k(t)$ is restricted to the following conditions:*

1. *If $i_{k_t} = 0$, then*

$$s_{q_k}(t) = s_k^L(t) = s_k^U(t) = \tau_{k_t}. \tag{31}$$

2. *If* $i_{k_t} > 0$*, then*

$$s_k^L(t) = \min\left(\frac{2 + \tau_{k_t} - i^*_{k_t} - f_{k_t}}{3}\right), \tag{32}$$

$$s_k^U(t) = \max\left(\frac{2 + \tau_{k_t} - i^*_{k_t} - f_{k_t}}{3}\right), \tag{33}$$

and

$$s_k^L(t) \leq s_{q_k}(t) \leq s_k^U(t), \tag{34}$$

where $i^*_{k_t} = \{i_{k_t}, 1 - i_{k_t}\}$.

In Definition 2.3, we put an emphasis on the indeterminacy of $a_k(t)$. This stems from the fact that once we are able to precisely characterize attribute a_k at time t, i.e., $i_{k_t} = 0$, we agree that its certainty is based on the value of τ_{k_t}, i.e., $\upsilon[a_k(t)] = \tau_{k_t}$. We describe this scenario (i.e., $i_{k_t} = 0$) as having *precise knowledge* of $a_k(t)$. This is also applicable when $\tau_{k_t} = 0$, leading to $s_k^L(t) = s_k^U(t) = 0$, and both endpoints take the lowest possible value of 0. Conversely, with $i_{k_t} = 0$ and $\tau_{k_t} = 1$, then $s_k^L(t) = s_k^U(t) = 1$, and both endpoints take the highest possible value of 1. Now, if $i_{k_t} > 0$, we describe this scenario as having *imprecise knowledge* of $a_k(t)$. Therefore, having imprecise knowledge of the nature of attribute a_k at time t implies that its indeterminacy becomes prevalent, and is used in determining both endpoints of the imprecise consequence for $a_k(t)$. We do so by creating a masked indeterminacy value $i^*_{k_t}$ that takes both the value of i_{k_t} and the value of the complement $1 - i_{k_t}$, and substituting them in the score function of $a_k(t)$, as seen in equations (32) and (33).

A special case is where $i_{k_t} = 1 - i_{k_t}$, meaning $i_{k_t} = 0.5$, then $s_k^L(t) = s_k^U(t) = \sigma[a_k(t)]$. As a result, each $s_{q_k}(t)$ is expected to be within an interval with distinct endpoints, except for when $i_{k_t} \in \{0, 0.5\}$, in which case $s_{q_k}(t) = s_k^L(t) = s_k^U(t)$. We interpret this result as (a) having precise knowledge of $a_k(t)$ leads to a single strategy consequence that is solely based on the certainty of attribute a_k at time t, (b) having an imprecise knowledge of a_{k_t} at about 50% leads to a single strategy consequence that takes all neutrosophic components into consideration.

It is worth noting that, when $i_{k_t} \neq 0$, we have an interval of imprecise consequence values as seen in equation (30). This entails that we need some creative way to generate the values between $s_k^L(t)$ and $s_k^U(t)$ and estimate the best $s_{q_k}(t)$ value(s) from the interval for decision-making purposes. Obviously, this model is intended to empower the decision maker with a weighting tool in which attributes' impacts on the current and future strategies are accounted for. By presenting each potential strategy as a direct consequence of a particular attribute in the form of an interval, it is critical to predict which value(s) from the interval are most likely, based on i_{k_t}. Thus, we proceed with the following analysis:

First, it is clear that $0 \leq s_{q_k}(t) \leq 1$ regardless of the value of i_{k_t}. Second, let $\bar{i}_{k_t} = 1 - i_{k_t}$, if $i_{k_t} < 0.5$, we agree that $\bar{i}_{k_t} > i_{k_t}$, then,

$$s_k^L(t) = \frac{2 + \tau_{k_t} - \bar{i}_{k_t} - f_{k_t}}{3}. \tag{35}$$

Conversely, with $i_{k_t} > 0.5$, we have $\bar{i}_{k_t} < i_{k_t}$, and

$$s_k^U(t) = \frac{2 + \tau_{k_t} - \bar{i}_{k_t} - f_{k_t}}{3}. \tag{36}$$

Consequently, for all $i_{k_t} \neq 0$, it is safe to say that

$$\frac{2 + \tau_{k_t} - \max(\bar{i}_{k_t}, i_{k_t}) - f_{k_t}}{3} \leq s_{q_k}(t) \leq \frac{2 + \tau_{k_t} - \min(\bar{i}_{k_t}, i_{k_t}) - f_{k_t}}{3}. \tag{37}$$

Naturally, it is a more favorable scenario to have $\bar{i_{k_t}} > i_{k_t}$. We describe such scenario as *favorable indeterminacy*. This stems from the fact that precise knowledge about an attribute at time t is achieved only when $i_{k_t} \to 0$. We also describe the opposite scenario, i.e., $\bar{i_{k_t}} < i_{k_t}$, as *unfavorable indeterminacy*. Therefore, the smaller the indeterminacy, the larger the impact of the attribute's certainty at time t. Moreover, for $i_{k_t} \neq 0.5$, it is clear that there exists a $\zeta_t > 0$ such that $| i_{k_t} - \bar{i_{k_t}} |= \zeta_t$. Then, we provide the following definition and remarks.

Definition 2.4. *Given a non-empty set of N predefined objectives and m time-dependent attributes represented by the SVNS $a_k(t) = \{\langle x : \tau_{k_t}, i_{k_t}, f_{k_t}\rangle, x \in [1,m] \times [t_0, t_n]\}$ with a derived set S of available strategies, at time t. We define the reverse indeterminate SVNS of $a_k(t)$, denoted $a_{k_r}(t)$, as the SVNS with the property that $a_{k_r}(t) = \{\langle x : \tau_{k_t}, 1 - i_{k_t}, f_{k_t}\rangle, x \in [1,m] \times [t_0, t_n]\}$, and the imprecise consequence interval $s^L_{k_r}(t) \leq s_{q_{k_r}}(t) \leq s^U_{k_r}(t)$ is the reverse imprecise consequence interval of a_k at time t.*

Remark 2.2. *Given $a_k(t)$ and $a_{k_r}(t)$, let $\bar{i_{k_t}} = 1 - i_{k_t}$ and $i_{k_t} > 0$.*

1. *If $i_{k_t} = \bar{i_{k_t}}$, then $a_k = a_{k_r}$, and*
$$s_{q_{k_r}}(t) = s_{q_k}(t) = \sigma[a_k(t)]. \tag{38}$$

2. *If $i_{k_t} > \bar{i_{k_t}}$, then $s^L_k(t) \leq s_{q_{k_r}}(t) \leq s^U_{k_r}(t)$, where*
$$s^L_k(t) = \frac{2 + \tau_{k_t} - i_{k_t} - f_{k_t}}{3}, \tag{39}$$
$$s^U_{k_r}(t) = \frac{2 + \tau_{k_t} - \bar{i_{k_t}} - f_{k_t}}{3}. \tag{40}$$

3. *If $i_{k_t} < \bar{i_{k_t}}$, then $s^L_{k_r}(t) \leq s_{q_k}(t) \leq s^U_k(t)$, where*
$$s^L_{k_r}(t) = \frac{2 + \tau_{k_t} - \bar{i_{k_t}} - f_{k_t}}{3}, \tag{41}$$
$$s^U_k(t) = \frac{2 + \tau_{k_t} - i_{k_t} - f_{k_t}}{3}. \tag{42}$$

Proof. The proof of this claim is easy to construct. □

Remark 2.3. *For distinct $k, l \in [1,m]$, assuming that $[s^L_k(t), s^U_k(t)] \subseteq [s^L_l(t), s^U_l(t)]$; if $i_{k_t} = \min(i_{k_t}, \bar{i_{k_t}})$ and $i_{l_t} = \min(i_{l_t}, \bar{i_{l_t}})$, then $a_k \leq a_l$.*

Proof. We are aware that each interval is a continuous distribution (See Assumption 2.2), but we also know that $[s^L_k(t), s^U_k(t)] \subseteq [s^L_l(t), s^U_l(t)]$ is equivalent to $s^L_l(t) \leq s^L_k(t) \leq s^U_k(t) \leq s^U_l(t)$. That said, from equation (37), we agree that

$$\frac{2 + \tau_{l_t} - \max(\bar{i_{l_t}}, i_{l_t}) - f_{l_t}}{3} \leq \frac{2 + \tau_{k_t} - \max(\bar{i_{k_t}}, i_{k_t}) - f_{k_t}}{3}; \tag{43}$$
$$\frac{2 + \tau_{k_t} - \min(\bar{i_{k_t}}, i_{k_t}) - f_{k_t}}{3} \leq \frac{2 + \tau_{l_t} - \min(\bar{i_{l_t}}, i_{l_t}) - f_{l_t}}{3}. \tag{44}$$

Then, we can deduce that
$$| i_{k_t} - \bar{i_{k_t}} | \leq | i_{l_t} - \bar{i_{l_t}} | . \tag{45}$$

From Remark 2.2, we know that if

1. $i_{k_t} > \bar{i_{k_t}}$, then $s_k^L(t) \leq s_{q_{k_r}}(t) \leq s_{k_r}^U(t)$;

2. $i_{k_t} < \bar{i_{k_t}}$, then $s_{k_r}^L(t) \leq s_{q_k}(t) \leq s_k^U(t)$.

Doing the same for i_{l_t} and $\bar{i_{l_t}}$; when $i_{k_t} = \min(i_{k_t}, \bar{i_{k_t}})$ and $i_{l_t} = \min(i_{l_t}, \bar{i_{l_t}})$, we can see that $s_k^U(t) \leq s_l^U(t)$. As a result, $a_k \leq a_l$ (See Remark 1.1). □

In Assumption 2.2, we assumed that there is a continuous distribution between $s_k^L(t)$ and $s_k^U(t)$ endpoints for each $a_k(t)$, thus we agree that the probability that a particular value between $s_k^L(t)$ and $s_k^U(t)$ endpoints is assumed is 0; which is fine, as the intent here is not to handpick a value from that interval. Assuming that the continuous distribution between $s_k^L(t)$ and $s_k^U(t)$ endpoints is uniform, if $\bar{s_{q_t}}$ is the distribution mean, then we can imply that, for any values $\epsilon, y > 0$,

$$\Pr\left[\bar{s_{q_t}} - \epsilon \leq y \leq \bar{s_{q_t}}\right] = \Pr\left[\bar{s_{q_t}} \leq y \leq \bar{s_{q_t}} + \epsilon\right]. \tag{46}$$

Then, it is easy to see that the probability density function is

$$f(y) = \begin{cases} \dfrac{1}{s_k^U(t) - s_k^L(t)}, & \text{if } s_k^L(t) \leq y \leq s_k^U(t); \\ \\ 0, & \text{otherwise;} \end{cases} \tag{47}$$

and that any value within the $[\min(i_{k_t}, \bar{i_{k_t}}), \max(i_{k_t}, \bar{i_{k_t}})]$ yields equally probable real y values, with the condition that $s_k^L(t) \leq y \leq s_k^U(t)$.

Now, we try a more complex distribution. Assuming that the continuous distribution is normal, using the *central limit theorem* (*CLT*)[16], we know that the density of the sum of two or more independent variables within the $s_k^L(t) \leq y \leq s_k^U(t)$ interval is the convolution of their densities[17]. That is, as we add more independent variables to the sum, the density of the sum tends to converge towards the normal density. If $\bar{s_{q_t}}$ is the distribution mean at time t, let n_s be the number of real values within $[s_k^L(t), s_k^U(t)]$, we expect $n_s \to \infty$. Thus, the classical CLT states that as n_s gets sufficiently large, the distribution gets close to the normal distribution with mean $\bar{s_{q_t}}$ and variance δ^2. As a result, within each closed set $[s_k^L(t), s_k^U(t)]$, we want $s_{q_k}(t)$ to be as close to the mean $\bar{s_{q_t}}$ as possible. In the end, each attribute $a_k(t)$ has a consequence of strategy that is either presented as a single value within $[0, 1]$ or as a continuous distribution in $[s_k^L(t), s_k^U(t)]$, dependent on whether $i_{k_t} = 0$ or not. This gives the decision maker the freedom to (a) develop a problem-solving approach on approximation and conduct sensitivity analysis as needed, or (b) perform discounting on each attribute over time once more data about each attribute become available and perform further sensitivity (or any other) analysis as needed. Since this model solely relies on current data based on the neutrosophic values of each attribute, we use an example in which we apply (a) and leave (b) for further discussions on this topic.

3. Computing example

3.1. Model example input and computation

We use an example in which we apply the definitions and remarks from Section 2. We use an objective tree over time that consists of $N = 100$ objectives, $n = 10$ objective levels, and 10 time periods. We are also given 10 attributes, defined by the time-dependent set $a = \{a_1, a_2, \ldots, a_{10}\}$, as seen in Table 1. We describe the time periods as a $(n + 1)$-tuple, i.e., $t = \{0, 1, \ldots, 10\}$, in order to account for the initial knowledge or characteristics of attributes at time $t = 0$. At time $t = 0$, it is expected that the decision maker's knowledge of each attribute and its relevance to objectives is precise, and therefore there are no indeterminacy and $i_k = 0$, with $k \in [1, 10]$. It is also assumed that each attribute $a_k(t)$ is a SVNS, therefore each τ_{k_t}, i_{k_t}, and f_{k_t} are in $[0, 1]$. Also, each $a_k(t)$ is a 11-tuple in the form of equations (23) and (24). All simulations are run using the input values recorded in Tables 1 and 2. Neutrosophic components for each a_k at times $t = 1$ through $t = 10$ are recorded in Table 2.

Neutrosophic Component	**Attributes** ($a_k(t = 0)$)									
	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
τ_k	0.581	0.74	0.149	0.258	0.97	0.515	0.565	0.144	0.925	0.634
i_k	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
f_k	0.419	0.26	0.851	0.742	0.03	0.485	0.435	0.856	0.075	0.366

Table 1: Initial neutrosophic values for all attributes, at time $t = 0$.

Neutrosophic Component	**Attributes** ($a_k(t > 0)$)									
	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
	$a_k(t = 1)$									
τ_k	0.913	0.636	0.624	0.374	0.504	0.989	0.471	0.009	0.35	0.477
i_k	0.434	0.861	0.749	0.829	0.001	0.29	0.981	0.867	0.755	0.512
f_k	0.087	0.364	0.376	0.626	0.496	0.011	0.529	0.991	0.65	0.523
	$a_k(t = 2)$									
τ_k	0.553	0.77	0.519	0.293	0.246	0.701	0.532	0.276	0.094	0.903
i_k	0.966	0.126	0.951	0.797	0.544	0.373	0.782	0.596	0.482	0.033
f_k	0.447	0.23	0.481	0.707	0.754	0.299	0.468	0.724	0.906	0.097
	$a_k(t = 3)$									
τ_k	0.223	0.256	0.014	0.898	0.026	0.994	0.851	0.887	0.704	0.257
i_k	0.965	0.619	0.537	0.066	0.22	0.111	0.06	0.125	0.252	0.535
f_k	0.777	0.744	0.986	0.102	0.974	0.006	0.149	0.113	0.296	0.743
	$a_k(t = 4)$									
τ_k	0.227	0.505	0.254	0.872	0.116	0.634	0.274	0.614	0.904	0.1
i_k	0.4	0.548	0.012	0.39	0.843	0.129	0.558	0.305	0.198	0.574
f_k	0.773	0.495	0.746	0.128	0.884	0.366	0.726	0.386	0.096	0.9
	$a_k(t = 5)$									
τ_k	0.345	0.103	0.481	0.036	0.01	0.692	0.479	0.85	0.495	0.187
i_k	0.048	0.66	0.083	0.333	0.306	0.065	0.568	0.354	0.349	0.379
f_k	0.655	0.897	0.519	0.964	0.99	0.308	0.521	0.15	0.505	0.813
	$a_k(t = 6)$									
τ_k	0.421	0.236	0.379	0.072	0.582	0.598	0.794	0.837	0.553	0.041
i_k	0.115	0.947	0.764	0.556	0.805	0.948	0.426	0.043	0.408	0.503
f_k	0.579	0.764	0.621	0.928	0.418	0.402	0.206	0.163	0.447	0.959
	$a_k(t = 7)$									
τ_k	0.431	0.498	0.436	0.27	0.235	0.004	0.468	0.334	0.564	0.568

i_k	0.197	0.046	0.041	0.591	0.143	0.081	0.875	0.682	0.014	0.159
f_k	0.569	0.502	0.564	0.73	0.765	0.996	0.532	0.666	0.436	0.432
	$a_k(t = 8)$									
τ_k	0.767	0.571	0.761	0.344	0.032	0.168	0.239	0.807	0.359	0.051
i_k	0.265	0.164	0.436	0.68	0.054	0.778	0.514	0.228	0.855	0.846
f_k	0.233	0.429	0.239	0.656	0.968	0.832	0.761	0.193	0.641	0.949
	$a_k(t = 9)$									
τ_k	0.005	0.318	0.816	0.064	0.286	0.337	0.622	0.457	0.09	0.554
i_k	0.028	0.749	0.314	0.722	0.915	0.475	0.687	0.37	0.297	0.473
f_k	0.995	0.682	0.184	0.936	0.714	0.663	0.378	0.543	0.91	0.446
	$a_k(t = 10)$									
τ_k	0.738	0.788	0.46	0.017	0.401	0.304	0.657	0.921	0.367	0.925
i_k	0.447	0.399	0.973	0.962	0.626	0.191	0.379	0.201	0.957	0.162
f_k	0.262	0.212	0.54	0.983	0.599	0.696	0.343	0.079	0.633	0.075

Table 2: Neutrosophic values for all attributes, at $1 \leq t \leq 10$.

We calculate the score, accuracy, and certainty values for each a_k for all t using equations (18)-(20), as seen in Tables 3. Using Definition 2.3 and equations (31) through (33), we then determine the imprecise consequences intervals for each a_k at all t in Table 4. As expected, since the neutrosophic values used for $t = 0$ have no indeterminacy, i.e., $i_{k_t} = 0$, the imprecise consequences intervals only takes one value, which is the value of τ_{k_t}. In this case, the term *imprecise* is an oxymoron since technically, the consequence for each a_k is precise at $t = 0$ and refers to the certainty of that a_k. At times $t > 0$, we have determined the imprecise consequence intervals with both $s_k^L(t)$ and $s_k^U(t)$ endpoints for each attributes a_k. From Assumption 2.2, there is a continuous distribution between $s_k^L(t)$ and $s_k^U(t)$. This is also taken into account in our analysis.

Function	**Attributes** (a_k)									
	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
	$a_k(t = 0)$									
$\sigma(a_k)$	0.721	0.827	0.433	0.505	0.98	0.677	0.71	0.429	0.95	0.756
$\alpha(a_k)$	0.162	0.48	-0.702	-0.484	0.94	0.03	0.13	-0.712	0.85	0.268
$\upsilon(a_k)$	0.581	0.74	0.149	0.258	0.97	0.515	0.565	0.144	0.925	0.634
	$a_k(t = 1)$									
$\sigma(a_k)$	0.797	0.47	0.5	0.306	0.669	0.896	0.32	0.05	0.315	0.481
$\alpha(a_k)$	0.826	0.272	0.248	-0.252	0.008	0.978	-0.058	-0.982	-0.3	-0.046
$\upsilon(a_k)$	0.913	0.636	0.624	0.374	0.504	0.989	0.471	0.009	0.35	0.477
	$a_k(t = 2)$									
$\sigma(a_k)$	0.38	0.805	0.362	0.263	0.316	0.676	0.427	0.319	0.235	0.924
$\alpha(a_k)$	0.106	0.54	0.038	-0.414	-0.508	0.402	0.064	-0.448	-0.812	0.806
$\upsilon(a_k)$	0.553	0.77	0.519	0.293	0.246	0.701	0.532	0.276	0.094	0.903
	$a_k(t = 3)$									
$\sigma(a_k)$	0.16	0.298	0.164	0.91	0.277	0.959	0.881	0.883	0.719	0.326
$\alpha(a_k)$	-0.554	-0.488	-0.972	0.796	-0.948	0.988	0.702	0.774	0.408	-0.486
$\upsilon(a_k)$	0.223	0.256	0.014	0.898	0.026	0.994	0.851	0.887	0.704	0.257

	$a_k(t=4)$									
$\sigma(a_k)$	0.351	0.487	0.499	0.785	0.13	0.713	0.33	0.641	0.87	0.209
$\alpha(a_k)$	-0.546	0.01	-0.492	0.744	-0.768	0.268	-0.452	0.228	0.808	-0.8
$\upsilon(a_k)$	0.227	0.505	0.254	0.872	0.116	0.634	0.274	0.614	0.904	0.1
	$a_k(t=5)$									
$\sigma(a_k)$	0.547	0.182	0.626	0.246	0.238	0.773	0.463	0.782	0.547	0.332
$\alpha(a_k)$	-0.31	-0.794	-0.038	-0.928	-0.98	0.384	-0.042	0.7	-0.01	-0.626
$\upsilon(a_k)$	0.345	0.103	0.481	0.036	0.01	0.692	0.479	0.85	0.495	0.187
	$a_k(t=6)$									
$\sigma(a_k)$	0.576	0.175	0.331	0.196	0.453	0.416	0.721	0.877	0.566	0.193
$\alpha(a_k)$	-0.158	-0.528	-0.242	-0.856	0.164	0.196	0.588	0.674	0.106	-0.918
$\upsilon(a_k)$	0.421	0.236	0.379	0.072	0.582	0.598	0.794	0.837	0.553	0.041
	$a_k(t=7)$									
$\sigma(a_k)$	0.555	0.65	0.61	0.316	0.442	0.309	0.354	0.329	0.705	0.659
$\alpha(a_k)$	-0.138	-0.004	-0.128	-0.46	-0.53	-0.992	-0.064	-0.332	0.128	0.136
$\upsilon(a_k)$	0.431	0.498	0.436	0.27	0.235	0.004	0.468	0.334	0.564	0.568
	$a_k(t=8)$									
$\sigma(a_k)$	0.756	0.659	0.695	0.336	0.337	0.186	0.321	0.795	0.288	0.085
$\alpha(a_k)$	0.534	0.142	0.522	-0.312	-0.936	-0.664	-0.522	0.614	-0.282	-0.898
$\upsilon(a_k)$	0.767	0.571	0.761	0.344	0.032	0.168	0.239	0.807	0.359	0.051
	$a_k(t=9)$									
$\sigma(a_k)$	0.327	0.296	0.773	0.135	0.219	0.4	0.519	0.515	0.294	0.545
$\alpha(a_k)$	-0.99	-0.364	0.632	-0.872	-0.428	-0.326	0.244	-0.086	-0.82	0.108
$\upsilon(a_k)$	0.005	0.318	0.816	0.064	0.286	0.337	0.622	0.457	0.09	0.554
	$a_k(t=10)$									
$\sigma(a_k)$	0.676	0.726	0.316	0.024	0.392	0.472	0.645	0.88	0.259	0.896
$\alpha(a_k)$	0.476	0.576	-0.08	-0.966	-0.198	-0.392	0.314	0.842	-0.266	0.85
$\upsilon(a_k)$	0.738	0.788	0.46	0.017	0.401	0.304	0.657	0.921	0.367	0.925

Table 3: Score (α), accuracy (σ), and certainty (υ) values for all attributes' SVNSs, at time $t \in [0, 10]$.

Imprecise Consequences Set (S_q) of Strategies Set S				
s_{q_1} s_{q_6}	s_{q_2} s_{q_7}	s_{q_3} s_{q_8}	s_{q_4} s_{q_9}	s_{q_5} $s_{q_{10}}$
$t=0$				
0.581 0.515	0.74 0.565	0.149 0.144	0.258 0.925	0.97 0.634
$t=1$				
$[0.753, 0.797]$ $[0.756, 0.896]$	$[0.47, 0.711]$ $[0.32, 0.641]$	$[0.5, 0.666]$ $[0.05, 0.295]$	$[0.306, 0.526]$ $[0.315, 0.485]$	$[0.336, 0.669]$ $[0.481, 0.489]$
$t=2$				
$[0.38, 0.691]$ $[0.592, 0.676]$	$[0.555, 0.805]$ $[0.427, 0.615]$	$[0.362, 0.663]$ $[0.319, 0.383]$	$[0.263, 0.461]$ $[0.223, 0.235]$	$[0.316, 0.345]$ $[0.613, 0.924]$
$t=3$				

[0.16, 0.47] [0.7, 0.959]	[0.298, 0.377] [0.587, 0.881]	[0.164, 0.188] [0.633, 0.883]	[0.621, 0.91] [0.553, 0.719]	[0.091, 0.277] [0.326, 0.35]
$t = 4$				
[0.285, 0.351] [0.466, 0.713]	[0.487, 0.519] [0.33, 0.369]	[0.173, 0.499] [0.511, 0.641]	[0.711, 0.785] [0.669, 0.87]	[0.13, 0.358] [0.209, 0.258]
$t = 5$				
[0.246, 0.547] [0.483, 0.773]	[0.182, 0.289] [0.463, 0.509]	[0.348, 0.626] [0.685, 0.782]	[0.135, 0.246] [0.446, 0.547]	[0.109, 0.238] [0.251, 0.332]
$t = 6$				
[0.319, 0.576] [0.416, 0.715]	[0.175, 0.473] [0.671, 0.721]	[0.331, 0.507] [0.572, 0.877]	[0.196, 0.233] [0.505, 0.566]	[0.453, 0.656] [0.193, 0.195]
$t = 7$				
[0.353, 0.555] [0.03, 0.309]	[0.347, 0.65] [0.354, 0.604]	[0.304, 0.61] [0.329, 0.45]	[0.316, 0.377] [0.381, 0.705]	[0.204, 0.442] [0.432, 0.659]
$t = 8$				
[0.6, 0.756] [0.186, 0.371]	[0.435, 0.659] [0.321, 0.331]	[0.653, 0.695] [0.614, 0.795]	[0.336, 0.456] [0.288, 0.524]	[0.039, 0.337] [0.085, 0.316]
$t = 9$				
[0.013, 0.327] [0.383, 0.4]	[0.296, 0.462] [0.519, 0.644]	[0.649, 0.773] [0.428, 0.515]	[0.135, 0.283] [0.159, 0.294]	[0.219, 0.496] [0.527, 0.545]
$t = 10$				
[0.641, 0.676] [0.266, 0.472]	[0.658, 0.726] [0.564, 0.645]	[0.316, 0.631] [0.681, 0.88]	[0.024, 0.332] [0.259, 0.564]	[0.392, 0.476] [0.671, 0.896]

Table 4: Imprecise consequence intervals for all attributes' SVNSs, at time $t \in [0, 10]$.

3.2. Approximation-based approach

In Definition 2.4, and remarks 2.2 and 2.3, we introduced the terms *reverse indeterminate SVNSs* and *reverse imprecise consequence intervals* for each attribute when presented with the scenario in which $i_{k_t} > \bar{i_{k_t}}$. This is critical in determining one of the two endpoints in the imprecise intervals for each a_k at time t when presented with an indeterminacy that is larger than 0.5. We also determined that, for distinct $k, l \in [1, m]$, having $[s_k^L(t), s_k^U(t)] \subseteq [s_l^L(t), s_l^U(t)]$ where $i_{k_t} = \min(i_{k_t}, \bar{i_{k_t}})$ and $i_{l_t} = \min(i_{l_t}, \bar{i_{l_t}})$ leads to the conclusion that $a_k \leq a_l$ from an attribute scoring or ranking standpoint. Depending on the number of objectives and attributes being present, drawing such conclusion can be a tough task for a fairly large number of attributes. Thus, we describe the following approximation-based approach to help determine the most relevant attribute(s) at time t, using equation (48). We assume that each imprecise consequence interval is a normal distribution, and we look at the median of each distribution at time t, denoted $\bar{s_{q_k}}$, as seen in Table 5. Once again, at time $t = 0$, we had both endpoints being equal (See Table 4); therefore, for $t = 0$, $\bar{s_{q_k}} = \tau_{k_t}$. For each t, the $\bar{s_{q_k}}$ values from Table 5 are then normalized to generate each attribute weight value, denoted $\hat{s_{q_k}}$ (See Table 6). The results from Table 6 allow the decision maker to rank each attribute based on the imprecise consequence values at each time t. As a result, to obtain the $\hat{s_{q_k}}$ values in Table 6, at each t, with $m = 10$ and $k \in [1, m]$, we use

$$\hat{s_{q_k}} = \frac{\bar{s_{q_k}}}{\sum_{k=1}^{m} \bar{s_{q_k}}}, \tag{48}$$

and

$$\sum_{k=1}^{m} \hat{s_{q_k}} = 1. \tag{49}$$

Time (t)	**Imprecise Consequence Median Values** ($\bar{s_{q_k}}$) **for** a_k **at Time** t									
	$\bar{s_{q_1}}$	$\bar{s_{q_2}}$	$\bar{s_{q_3}}$	$\bar{s_{q_4}}$	$\bar{s_{q_5}}$	$\bar{s_{q_6}}$	$\bar{s_{q_7}}$	$\bar{s_{q_8}}$	$\bar{s_{q_9}}$	$\bar{s_{q_{10}}}$
$t = 0$	0.581	0.74	0.149	0.258	0.97	0.515	0.565	0.144	0.925	0.634
$t = 1$	0.775	0.59	0.583	0.416	0.503	0.826	0.481	0.172	0.4	0.485
$t = 2$	0.536	0.68	0.512	0.362	0.33	0.634	0.521	0.351	0.229	0.768
$t = 3$	0.315	0.338	0.176	0.766	0.184	0.829	0.734	0.758	0.636	0.338
$t = 4$	0.318	0.503	0.336	0.748	0.244	0.59	0.35	0.576	0.77	0.233
$t = 5$	0.396	0.236	0.487	0.19	0.174	0.628	0.486	0.734	0.497	0.292
$t = 6$	0.448	0.324	0.419	0.215	0.554	0.566	0.696	0.724	0.536	0.194
$t = 7$	0.454	0.498	0.457	0.347	0.323	0.169	0.479	0.39	0.543	0.546
$t = 8$	0.678	0.547	0.674	0.396	0.188	0.278	0.326	0.704	0.406	0.2
$t = 9$	0.17	0.379	0.711	0.209	0.358	0.392	0.582	0.472	0.226	0.536
$t = 10$	0.659	0.692	0.474	0.178	0.434	0.369	0.604	0.78	0.412	0.784

Table 5: Imprecise consequence intervals medians for at time t, for each a_k.

Time (t)	**Imprecise Consequence Weighted Values** ($\hat{s_{q_k}}$) **for** a_k **at Time** t										$\sum_{k=1}^{m} \hat{s_{q_k}}$
	$\hat{s_{q_1}}$	$\hat{s_{q_2}}$	$\hat{s_{q_3}}$	$\hat{s_{q_4}}$	$\hat{s_{q_5}}$	$\hat{s_{q_6}}$	$\hat{s_{q_7}}$	$\hat{s_{q_8}}$	$\hat{s_{q_9}}$	$\hat{s_{q_{10}}}$	
$t = 0$	0.106	0.135	0.027	0.047	0.177	0.094	0.103	0.026	0.169	0.116	1.00
$t = 1$	0.148	0.113	0.111	0.08	0.096	0.158	0.092	0.033	0.076	0.093	1.00
$t = 2$	0.109	0.138	0.104	0.074	0.067	0.129	0.106	0.071	0.047	0.156	1.00
$t = 3$	0.062	0.067	0.035	0.151	0.036	0.163	0.145	0.149	0.125	0.067	1.00
$t = 4$	0.068	0.108	0.072	0.16	0.052	0.126	0.075	0.123	0.165	0.05	1.00
$t = 5$	0.096	0.057	0.118	0.046	0.042	0.152	0.118	0.178	0.121	0.071	1.00
$t = 6$	0.096	0.069	0.09	0.046	0.118	0.121	0.149	0.155	0.115	0.041	1.00
$t = 7$	0.108	0.118	0.109	0.083	0.077	0.04	0.114	0.093	0.129	0.13	1.00
$t = 8$	0.154	0.124	0.153	0.09	0.043	0.063	0.074	0.16	0.092	0.045	1.00
$t = 9$	0.042	0.094	0.176	0.052	0.089	0.097	0.144	0.117	0.056	0.133	1.00
$t = 10$	0.122	0.128	0.088	0.033	0.081	0.069	0.112	0.145	0.076	0.146	1.00

Table 6: Imprecise consequence weight values at time t for each a_k, obtained via the normalization of the imprecise consequence values of a_k for each t.

3.3. Interpretation of results and sensitivity analysis

By corresponding each weight value from Table 6 to its relative attribute, the decision maker can choose to prioritize directly based on these weight values. Additionally, these values can also be used to perform discounting on each attribute with the end goal being the prioritization of attributes at time t. Using discounting, however, is an extra measure for comparing attributes, for the purpose of this paper; it has been left to the decision maker's discretion.

At first glance, it is easy to see that at time $t = 0$, attribute a_5 would be ranked first, followed closely by attribute a_9, then by a_2, a_{10}, a_1, a_7, a_6, a_4, a_3, and a_8, respectively. Doing the same for $t = 1$, the attribute ranking is a_6, a_1, a_2, a_3, a_5, a_{10}, a_7, a_4, a_9, and a_8, respectively. For $t = 2$, we have a_{10}, a_2, a_6, a_1, a_7, a_3, a_4, a_8, a_5, and a_9, in that order. The same process is used to determine the ranking or priority for the same attributes at times $t = 3, \ldots, 10$.

The approximation-based approach using the medians of the imprecise consequence intervals in Subsection 3.2 gives us an outlet in assigning a weighted value to each attribute because it allows us to bypass the need to compare each imprecise consequence interval with another (See Remark 2.3). That approach, however, does not take into account whether attribute a_k contains a favorable or unfavorable indeterminacy at time t. This stems from the fact that the median is the closest to the halfway point between $s_k^L(t)$ and $s_k^U(t)$, therefore that approach does not take into account whether $i_{k_t} = \min(i_{k_t}, \bar{i_{k_t}})$ (favorable indeterminacy) or $i_{k_t} = \max(i_{k_t}, \bar{i_{k_t}})$ (unfavorable indeterminacy). We know that each $[s_k^L(t), s_k^U(t)]$ is a continuous distribution, so we define $\epsilon_{k_t} = \mathrm{d}s_{q_k(t)}$ as an arbitrary infinitesimal variation from the median $\bar{s_{q_k}}(t)$ such that $\bar{s_{q_k}}(t) - \epsilon_{k_t} \leq \bar{s_{q_k}}(t) \leq \bar{s_{q_k}}(t) + \epsilon_{k_t}$, at time t. Thus, we recompute the weighted values ($\hat{s_{q_k}}$) using

$$\hat{s_{q_k}} = \begin{cases} \dfrac{\bar{s_{q_k}} + \epsilon_{k_t}}{\sum_{k=1}^{m} \bar{s_{q_k}}}, & \text{if } i_{k_t} = \min(i_{k_t}, \bar{i_{k_t}}); \\ \\ \dfrac{\bar{s_{q_k}} - \epsilon_{k_t}}{\sum_{k=1}^{m} \bar{s_{q_k}}}, & \text{otherwise.} \end{cases} \tag{50}$$

For simplicity, we use the same ϵ_{k_t} throughout $t = 1, \ldots, 10$. However, it is normal to envision a case where the decision maker would choose a different ϵ_{k_t} value as t progresses. Moreover, since at $t = 0$, there is no imprecise consequence interval for any of the attributes (See Table 4), the newly computed values only affect the previous weighted values from Table 6 for $t > 0$. Those newly computed weighted values are then reflected in Table 7 using $\epsilon_{k_t} = 0.1$.

Time ($t > 0$)	**Imprecise Consequence Weighted Values** ($\hat{s_{q_k}}$) **for** a_k **at Future** t										$\sum_{k=1}^{m} \hat{s_{q_k}}$
	$\hat{s_{q_1}}$	$\hat{s_{q_2}}$	$\hat{s_{q_3}}$	$\hat{s_{q_4}}$	$\hat{s_{q_5}}$	$\hat{s_{q_6}}$	$\hat{s_{q_7}}$	$\hat{s_{q_8}}$	$\hat{s_{q_9}}$	$\hat{s_{q_{10}}}$	
$t = 1$	0.167	0.094	0.092	0.06	0.115	0.177	0.073	0.014	0.057	0.074	0.923
$t = 2$	0.089	0.158	0.084	0.053	0.047	0.149	0.086	0.051	0.067	0.176	0.96
$t = 3$	0.042	0.047	0.015	0.171	0.056	0.183	0.164	0.169	0.145	0.047	1.039
$t = 4$	0.09	0.086	0.093	0.182	0.031	0.148	0.054	0.145	0.186	0.028	1.043
$t = 5$	0.12	0.033	0.142	0.07	0.067	0.177	0.094	0.202	0.145	0.095	1.145
$t = 6$	0.117	0.048	0.068	0.025	0.097	0.1	0.17	0.176	0.136	0.02	0.957
$t = 7$	0.132	0.142	0.132	0.059	0.101	0.064	0.09	0.069	0.153	0.154	1.096
$t = 8$	0.177	0.147	0.176	0.067	0.065	0.04	0.051	0.183	0.07	0.023	0.999
$t = 9$	0.067	0.069	0.201	0.027	0.064	0.122	0.119	0.142	0.081	0.158	1.05
$t = 10$	0.141	0.147	0.069	0.014	0.062	0.087	0.131	0.163	0.058	0.164	1.036

Table 7: Imprecise consequence weight values at future time t for each a_k, obtained via adding or subtracting an arbitrary $\epsilon_{k_t} = 0.1$ from the imprecise consequence interval median for each attribute a_k, depending on whether the value of i_{k_t} is less than 0.5 or not, at time $t > 0$.

As an interpretation of the results in Table 7, we can see that there are no new attribute weighting or ranking for $t = 0$, as expected. For $t = 1$, we have the attributes in which $i_{k_t} = \min(i_{k_t}, \bar{i_{k_t}})$ (i.e., a_6, a_1, and a_5, respectively), then followed by those in which $i_{k_t} \neq \min(i_{k_t}, \bar{i_{k_t}})$, such as a_2, a_3, a_{10}, a_7, a_4, a_9, and a_8, in that order. For $t = 2$, we have a_{10}, a_2, a_6, a_1, a_7, a_3, a_9, a_4, a_8, and a_5, in that order. The same process is repeated to determine the rankings at times $t = 3, \ldots, 10$.

As expected, our first observation is that when $i_{k_t} = \min(i_{k_t}, \bar{i_{k_t}})$, attributes with the largest τ_{k_t} and smallest i_{k_t} values yield a larger $\hat{s_{q_k}}$ than those that do not. Moreover, we can see that when two or more attributes contain $i_{k_t} \neq \min(i_{k_t}, \bar{i_{k_t}})$, priority is given to the one(s) with the largest τ_{k_t}. This new ranking aligns more with Remark 2.3 than the approximation-based approach of just using the normalized weights of the median values of the imprecise consequence intervals reflected in Table 6. For instance, at $t = 4$, a_4 has $i_{k_t} = 0.39$ while i_{k_t} for a_9 is 0.198; we also can see that $[0.711, 0.785] \subseteq [0.669, 0.87]$, and according to both Remark 2.3 and the $\hat{s_{q_k}}$ values obtained in Table 7, we observe that a_9 would be ranked just ahead of a_4. As a result, where applicable, either Remark 2.3 or the approach in equation (50) can be used to determine with attribute has the most impactful strategy between any given set of attributes, at a specific time in the future. The challenge, in applying the approach in equation (50), is how to determine which ϵ_{k_t} is best to facilitate prioritizing attributes with favorable indeterminacy over those with unfavorable indeterminacy. We can see that the standard deviations from the imprecise consequence interval medians, at each $t > 0$, are 0.178, 0.163, 0.247, 0.189, 0.179, 0.174, 0.11, 0.189, 0.166, and 0.187, respectively. Thus, choosing $\epsilon_{k_t} = 0.1$ is a sensible pick. Any pick too small (i.e., $\epsilon_{k_t} \to 0$) would get us right around the median value, which defeats the purpose of establish some bias towards favorable indeterminacy. Any pick too large creates a significant gap between attributes with favorable indeterminacy and those with unfavorable indeterminacy. Ultimately, having the intervals available to the decision maker empowers him/her in choosing any approximation approach that suits the end-goal of the scenario at hand.

4. Future work and discussion

We have developed a DSS computation model for decision-support scenarios where we have N objectives with m attributes spanning over n time periods. We present each attribute in the form of a single-valued neutrosophic set and performed necessary operations to determine a specific set of strategies in which each attribute's imprecise consequence can be presented as a continuous distribution interval. We proceed with defining the model to detect when indeterminacy is favorable or unfavorable, and present approaches that help achieve that bias. We provide a computation example in which the model is used, and conduct a sensitivity analysis of the results. The example seems to have provided a clear application of the model but we are aware that certain areas still need further discussion. These areas, which can be addressed in future discussions regarding this computation model, so far include, but are not limited to:

1. Since each imprecise consequence is given in the form of a continuous distribution interval, assuming that the distribution is normal, would the normal density of each interval help establish a trend about each attribute?

2. Can the neutrosophic components of an attribute be linked with any property other than how impactful the attribute would be in the future?

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