Unification of Fusion Theories (UFT)

Florentin Smarandache
The University of New Mexico
200 College Road
Gallup, NM 87301, USA
smarand@unm.edu

Abstract. We propose a Unification of Fusion Theories and Fusion Rules in solving problems/applications. For each particular application, check the reliability of sources, and selects the most appropriate model, rule(s), fusion theories, and algorithm of implementation.
The unification scenario presented herein, which is in an incipient form, should periodically be updated incorporating new discoveries from the fusion and engineering research.

Keywords. Fusion theories, fusion rules, lattice, Boolean algebra, Lindenbaum algebra, frame of discernment, model, static / dynamic fusion, incomplete / paraconsistent / imprecise information, specificity chains, specialization

Introduction

Each theory works better for some applications, and less for others. This unification, which is a fusion overview attempt, might look like a cooking recipe, or better saying like a logical chart or a computer program, yet we don't see another method to comprise/unify all things.
We extend the power set and hyper-power set from previous theories to a Boolean algebra that we construct by closing the frame of discernment under union, intersection, and complement of sets. All basic belief assignments (bba) and rules are extended on this Boolean algebra.
A similar generalization has been previously used by Guan-Bell (1993) for the Dempster-Shafer rule using propositions in sequential logic, herein we reconsider the Boolean algebra for all fusion rules and theories but using sets instead of propositions, because it is generally harder to work in sequential logic with summations and inclusions than in the set theory.
We present the definition of a model, some classifications of frames of discernment and their elements, the types of information, what specificity chains and specialization mean, also the definition of static and dynamic fusions, and the algebraic properties of rules. We list the fusion rules and theories but we are not able to present them due to space limitation, also we propose a partial Unification of Fusion Rules (UFR) too.

1. Fusion Space

For $n \geq 2$ let $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ be the frame of discernment of the fusion problem/application under consideration. Then $(\Theta, \cup, \cap, \mathcal{C})$, Θ closed under these three operations: union, intersection, and complementation of sets respectively, forms a Boolean algebra. With respect to the partial ordering relation, the inclusion $\subseteq$, the minimum element is the empty set ϕ, and the maximal element is the total ignorance $I = \bigcup_{i=1}^{n} \theta i$.

Similarly one can define: $(\Theta, \cup, \cap, \setminus)$ for sets, Θ closed with respect to each of these operations: union, intersection, and difference of sets respectively.

$(\Theta, \cup, \cap, \mathcal{C})$ and $(\Theta, \cup, \cap, \setminus)$ generate the same super-power set S^Θ closed under $\cup$, $\cap$, $\mathcal{C}$, and $\setminus$ because for any $A, B \in S^\Theta$ one has $\mathcal{C}(A) = I \setminus A$ and reciprocally $A \setminus B = A \cap \mathcal{C}(B)$.

If one considers propositions, then $(\Theta, \vee, \wedge, \neg)$ forms a Lindenbaum algebra in sequential logic, which is isomorphic with the above $(\Theta, \cup, \cap, \mathcal{C})$ Boolean algebra.

By choosing the frame of discernment Θ with exclusive elements, closed under $\cup$ only , one gets Dempster-Shafer's, Yager's, Transferable Belief Model, Dubois-Prade's power set . Then making Θ closed under both $\cup$, $\cap$ one gets Dezert-Smarandache's hyper-power set. While, extending Θ for closure under $\cup$, $\cap$, and $\mathcal{C}$ one also includes the complement of set (or negation of proposition if working in sequential logic); in the case of non-exclusive vague elements in the frame of discernment one considers the complement is involutive, i.e. $C(C(A))=A$ for any set A in order to avoid an infinite loop in the closure under complement process. Therefore the super-power set $(\Theta, \cup, \cap, \mathcal{C})$ includes all the previous fusion spaces.

The power set 2^Θ, used in DST, Yager's, TBM, DP, which is the set of all subsets of Θ, is also a Boolean algebra, closed under $\cup$, $\cap$, and $\mathcal{C}$, but does not contain intersections of elements from Θ since the elements are supposed exclusive.

The Dedekind distributive lattice D^Θ, used in DSmT, is closed under $\cup$, $\cap$, and if negations/complements arise they are directly introduced in the frame of discernment, say Θ', which is then closed under $\cup$, $\cap$. Unlike others, DSmT allows intersections, generalizing the previous theories.

The Unifying Theory contains intersections and complements as well.

Model means to know the empty intersections in the super-power set, whose conflicting masses should be transferred to non-empty sets.

Comments on Frames and their extensions.

F.1.1. **Open World** is a frame which misses some hypotheses (*non-exhaustive*) [Smets].
Ex. Ω = {John, George}, but later we find another suspect: David.
An open world becomes closed if one adds in the frame of discernment another hypothesis θ_c which includes all missing hypotheses.

F.1.2. **Closed World** is a frame which includes all hypotheses (*exhaustive*).

F.2.1. **Homogeneous frame**: all its elements are of the same nature.

F.2.2. **Heterogeneous frame**: at least two of its elements are of different nature.
Ex. Ω = {White, Bird, Long}.
It is split into homogeneous sub-frames; the *complement* is computed with respect to element's sub-frame; construct super-power sets for each sub-frame.

F.3.1. **Finite frame**.

F.3.2. **Infinite frame**.

E.1.1. **Exclusive elements**: their intersection is empty.

E.1.1. **Non-exclusive elements**: their intersection is not empty.

Ex. Ω = {A, B} of target cross-sections, A={x| 1.5<x<2.5}, B={x| 2<x<3}.

E.2.1. **Classical elements**: their boundaries are well defined.

E.2.2. **Vague elements**: their boundaries are not well defined.

Ex. Ω = {Red, Orange}.

Let's consider a frame of discernment Θ with exclusive or non-exclusive hypotheses, exhaustive or non-exhaustive, closed or open world (all possible cases).

We need to make the remark that in case when these $n \geq 2$ elementary hypotheses $\theta_1, \theta_2, \ldots, \theta_n$ are *exhaustive and exclusive* one gets the Dempster-Shafer Theory, Yager's, Dubois-Prade Theory, Dezert-Smarandache Theory, but for the case when the hypotheses are *non-exclusive* one gets Dezert-Smarandache Theory, while for *non-exhaustivity* one gets TBM.

An exhaustive frame of discernment is called *close world*, and a non-exhaustive frame of discernment is called *open world* (meaning that new hypotheses might exist in the frame of discernment that we are not aware of). Θ may be finite or infinite.

Let $m_j: S^\Theta \rightarrow [0.\ 1]$, $1 \leq j \leq s$, be $s \geq 2$ basic belief assignments, (when bbas are working with crisp numbers), or with subunitary subsets, $m_j: S^\Theta \rightarrow \wp([0.\ 1])$, where $\wp([0.\ 1])$ is the set of all subsets of the interval [0,1] (when dealing with very imprecise information).

2. Types of Information

2.1. Complete information: normally the sum of crisp masses of a bba, m(.), is 1,

i.e. $\sum_{X \in S^\wedge\Theta} m(X) = 1$.

2.2. Incomplete information: not enough knowledge; the sum of scalar mass components is < 1;

2.3. Paraconsistent information: conflicting/paradoxist information coming from opposite view points; the sum of scalar mass components > 1;

Some prefer to normalize incomplete and paraconsistent information; others don't (wanting to learn the type of information after fusion).

2.4. Imprecise information (Dezert-Smarandache): mass components are subsets (not necessarily intervals) of [0, 1].

2.4.1. Admissibly condition for completeness: $\exists x \in A$: m(A): $\sum_{A \in S^\Theta} m(A) = 1$;

otherwise it is imprecise incomplete or paraconsistent information.

Similarly, for a bba m(.) valued on subunitary subsets dealing with paraconsistent and incomplete information respectively:

2.4.2. For **incomplete imprecise information**, one has $\sum_{X \in S^\wedge T} \sup\{m(X)\} < 1$.

2.4.3. While for **paraconsistent imprecise information** one has $\sum_{X \in S^\wedge T} \inf\{m(X)\} > 1$.

3. Specificity Chains and Specialization

3.1. Specificity chains are inclusion chains that use the *min principle, i.e.* a cautious way to transfer conflicting masses to less and less specific elements.
The transfer of conflicting mass and normalization diminish the specificity.
If $A \cap B = \phi$, its mass is moved to a less specific element A (also to B) in an optimistic view on them, but if we have a pessimistic view on A and B we can move the mass $m(A \cap B)$ to $A \cup B$ (entropy increases, imprecision increases), and even more if we are very pessimistic about A and B we move the conflicting mass to total ignorance in a closed world, or to the empty set in an open world.
Examples of Specificity Chains:
a) In a closed world: $A \cap B \cap C \subset A \cap B \subset A \subset A \cup B \subset A \cup B \cup C \subset \Theta$
b) In an open world: $A \cap (B \cup C) \rightarrow \phi$.
3.2. Specialization means transfer of a set mass to its subsets (opposite of specificity chain).
Ex. $A \supset A \cap B \supset A \cap B \cap C$.

4. Static and Dynamic Fusion

According to Wu Li we have the following classification and definitions:
4.1. Static fusion means to combine all belief functions simultaneously.
4.2. Dynamic fusion means that the belief functions become available one after another sequentially, and the current belief function is updated by combining itself with a newly available belief function.

5. Summary of Library of Fusion Rules

The following 25 old and 19 new rules have been collected herein:

Conjunctive, Disjunctive, Exclusive Disjunctive, Mixed Conjunctive-Disjunctive rules, Conditional rule, Dempster's, Yager's, Smets' TBM rule, Dubois-Prade's, Dezert-Smarandache classical and hybrid rules, Murphy's average rule, Inagaki-Lefevre-Colot-Vannoorenberghe Unified Combination rules [and, as particular cases: Iganaki's parameterized rule, Weighting Average Operator (Vannoorenberghe), minC (M. Daniel), and newly Proportional Conflict Redistribution 1-5 and 3.4 rules (Smarandache-Dezert) among which PCR5 is the most exact way of redistribution of the conflicting mass to non-empty sets following the path of the conjunctive rule], Zhang's Center Combination rule, Convolutive x-Averaging, Consensus Operator (Jøsang), Cautious Rule (Smets), α-junctions rules (Smets)**,** Yen's rule, p-boxes method, Yao and Wong's Qualitative rule, Baldwin's rule, Besnard's rule, and six new T-norm & T-conorm rules (Tchamova-Smarandache) adjusted from fuzzy sets, plus six new N-norm & N-conorm rules adjusted from neutrosophic logic (Smarandache), partial Unification of Fusion Rules (Smarandache, 2005).

Introducing the degree of union, inclusion, besides that of intersection, with respect to the cardinal of sets (not from a fuzzy set point of view), many from the above fusion rules can be improved.

Due to space limitation we are unable to present these rules nor the below fusion theories. Reader can download author's NASA presentation article for more information: http://xxx.lanl.gov/ftp/cs/papers/0410/0410033.pdf.

6. Summary of Fusion Theories

Bayesian Theory
Dempster-Shafer Theory of Evidence (1976)
TBM (Transferable Belief Model) [P. Smets]
Fuzzy Theory (Zadeh, 1965)
Dezert-Smarandache Theory of Plausible, Uncertain, and Paradoxist Reasoning (2001)
Neutrosophic Theory (Smarandache, 1995) – generalization of Fuzzy Theory

7. Algebraic Properties of Fusion Rules

Let R be a fusion rule, and $m_1(.)$, $m_2(.)$, …, $m_s(.)$ bba's.

7.1. R is **commutative** if $R(m_1, m_2)=R(m_2, m_1)$.

7.2.1. R is **associative** if $R(R(m_1, m_2), m_3)=R(m_1, R(m_2, m_3))$.

7.2.2. R is **quasi-associative** if there exists an algorithm/method that transforms a non-associative rule into an associative one.

Ex. Rules, based on conjunctive rule and then the transfer of conflicting mass, are quasi-associative since one can store the conjunctive rule result for the combination with the next evidence coming in.

7.3.1. R is **idempotent** if $R(m_1, m_1)=m_1$.

7.3.2. R is **convergent towards idempotence** (Smarandache, 2004) if

$$\lim_{k\to\infty} R(\underbrace{m_1, ,,, , m_1}_{k \text{ times}})=m_1$$

7.4. R satisifies the Vacuum Belief Assignment (VBA), where VBA or **neutral element** is mVBA(total ignorance)=1, if $R(m_1, VBA)=m_1$.

7.5. R satisfies the **Markovian requirement** (Smets) if $R(m_1, m_2, \ldots, m_s)=R(R(m_1, m_2, \ldots, m_{s-1}),m_s))$.

7.6.1. R satisfies the **majority opinion** (Wu Li, 2004) if $R(m_1, m_2, \ldots, m_2) \approx m_2$.

7.6.2. R is **convergent towards the majority opinion** (Smarandache, 2004) if

$$\lim_{k\to\infty} R(m_1, \underbrace{m_2, ,,, , m_2}_{k \text{ times}})= m_2.$$

7.7. R **discounts the old sources** if $d[R(m_1, m_2, \ldots, m_s), m_1] > d[R(m_1, m_2, \ldots, m_{s-1}), m_1]$ for $m_s \neq m_1$, where the *distance* $d(m_1, m_2) = \sum|m_1(X)-m_2(X)|$ for all $X \in S^{\Theta}$.

7.8. **Continuity** of rule R (Smarandache, 2004):

$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon)$, such that if $|m_1-m_2| < \varepsilon$ then $|R(m_1,m_3) - R(m_2,m_3)| < \delta$.

This property means *smooth behavior* of the rule.

7.9. **Coherence** of a rule: it has some justification for its construction and able to provide fusion performances close to what human experts would expect "rationale“ (this is relatively subjective) [J. Dezert].

8. Unification of Fusion Rules (UFR)

If variable y is *directly proportional* with variable p, then $z=k\cdot p$, where k is a constant.
If variable y is *inversely proportional* with variable q, then $z=k\cdot(1/q)$; we can also say that z is directly proportional with variable 1/q.
In a general way, we say that y is directly proportional with variables $p_1, p_2, \ldots, p_m$ and inversely proportionally with variables $q_1, q_2, \ldots, q_n$, then

$y = k\cdot(p_1\cdot p_2\cdot\ldots\cdot p_m)/(q_1\cdot q_2\cdot\ldots\cdot q_n) = kP/Q$, where $P=\prod_{i=1}^{m} p_i$, $Q=\prod_{j=1}^{n} q_j$.

In a general definition UFR is:

$m_{UFR}(\phi) = 0$, and $\forall A \in S^{\Theta} \setminus \phi$ one has $m_{UFR}(A) = \sum_{\substack{X1, X2 \in S^{\wedge}\Theta \\ X1 * X2 = A}} d(X1 * X2) T(X1, X2)$

$$+ \frac{P(A)}{Q(A)} \sum_{\substack{X \in S^{\wedge}\Theta \setminus A \\ X * A \in Tr}} d(X * A) \frac{T(A, X)}{P(A)/Q(A) + P(X)/Q(X)}$$

where * is intersection or union of sets, d(X*Y) is the degree of intersection or union, T(X,Y) is a T-norm fusion combination rule (extension of conjunctive or disjunctive rules), Tr is the ensemble of sets (in majority cases they are empty sets) whose masses must be transferred, P(A) is the product of all parameters directly proportional with A, while Q(A) the product of all parameters inversely proportional with A.

9. Scenario of Unification of Fusion Theories

A. CHECK THE RELIABILITY OF SOURCES
B. FIND THE MODEL
C. CHOOSE THE FUSION RULES AND THEORIES
D. MORE CHECKINGS

Since everything depends on the application/problem to solve, this scenario looks like a logical chart designed by the programmer in order to write and implement a computer program, or like a cooking recipe.

Here it is the attempting **scenario for a unification and reconciliation of the fusion theories and rules**:

9.1) If all sources of information are reliable, then apply the conjunctive rule, which means consensus between them (or their common part):

9.2) If some sources are reliable and others are not, but we don't know which ones are unreliable, apply the disjunctive rule as a cautious method (and no transfer or normalization is needed).

9.3) If only one source of information is reliable, but we don't know which one, then use the exclusive disjunctive rule based on the fact that $X_1 \veebar X_2 \veebar \ldots \veebar X_n$ means either X_1 is reliable, or X_2, or and so on or X_n, but not two or more in the same time.

9.4) If a mixture of the previous three cases, in any possible way, use the mixed conjunctive-disjunctive rule.

As an example, suppose we have four sources of information and we know that: either the first two are telling the truth or the third, or the fourth is telling the truth.

The mixed formula becomes:

$m_{\cap\cup}(\phi) = 0$, and $\forall\, A \in S^{\Theta} \setminus \phi$, one has $m_{\cap\cup}(A) = \sum_{\substack{X_1, X_2, X_3, X_4 \in S^{\Theta} \\ ((X_1 \cap X_2) \cup X_3) \veebar X_4 = A}} m_1(X_1) m_2(X_2) m_3(X_3) m_4(X_4)$.

9.5) If we know the sources which are unreliable, we discount them. But if all sources are fully unreliable (100%), then the fusion result becomes vacuum bba (i.e. $m(\Theta) = 1$, and the problem is indeterminate. We need to get new sources which are reliable or at least they are not fully unreliable.

9.6) If all sources are reliable, or the unreliable sources have been discounted (in the default case), then use the DSm classic rule (which is commutative, associative, Markovian) on Boolean algebra $(\Theta, \cup, \cap, \mathcal{C})$, no matter what contradictions (or model) the problem has. I emphasize that the super-power set S^{Θ} generated by this Boolean algebra contains singletons, unions, intersections, and complements of sets.

9.7) If the sources are considered from a statistical point of view, use *Murphy's average rule* (and no transfer or normalization is needed).

9.8) In the case the model is not known (the default case), it is prudent/cautious to use the free model (i.e. all intersections between the elements of the frame of discernment are non-empty) and DSm classic rule on S^{Θ}, and later if the model is found out (i.e. the constraints of empty intersections become known), one can adjust the conflicting mass at any time/moment using the DSm hybrid rule.

9.9) Now suppose the model becomes known [i.e. we find out about the contradictions (= empty intersections) or consensus (= non-empty intersections) of the problem/application]. Then :

9.9.1 If an intersection $A \cap B$ is not empty, we keep the mass $m(A \cap B)$ on $A \cap B$, which means consensus (common part) between the two hypotheses A and B (i.e. both hypotheses A and B are right) [here one gets *DSmT*].

9.9.2. If the intersection $A \cap B = \phi$ is empty, meaning contradiction, we do the following :

9.9.2.1) if one knows that between these two hypotheses A and B one is right and the other is false, but we don't know which one, then one transfers the mass m(A∩B) to m(A∪B), since A∪B means at least one is right [here one gets *Yager's* if n=2, or *Dubois-Prade,* or *DSmT*];

9.9.2.2) if one knows that between these two hypotheses A and B one is right and the other is false, and we know which one is right, say hypothesis A is right and B is false, then one transfers the whole mass m(A∩B) to hypothesis A (nothing is transferred to B);

9.9.2.3) if we don't know much about them, but one has an optimistic view on hypotheses A and B, then one transfers the conflicting mass m(A∩B) to A and B (the nearest specific sets in the Specificity Chains) [using *Dempster's*, *PCR2-5*]

9.9.2.4) if we don't know much about them, but one has a pessimistic view on hypotheses A and B, then one transfers the conflicting mass m(A∩B) to A∪B (the more pessimistic the further one gets in the Specificity Chains: (A∩B) ⊂ A ⊂ (A∪B) ⊂ I); this is also the default case [using *DP's*, *DSm hybrid rule*, *Yager's*];

if one has a very pessimistic view on hypotheses A and B then one transfers the conflicting mass m(A∩B) to the total ignorance in a closed world [*Yager's*, *DSmT*], or to the empty set in an open world [*TBM*];

9.9.2.5.1) if one considers that no hypothesis between A and B is right, then one transfers the mass m(A∩B) to other non-empty sets (in the case more hypotheses do exist in the frame of discernment) - different from A, B, A∪B - for the reason that: if A and B are not right then there is a bigger chance that other hypotheses in the frame of discernment have a higher subjective probability to occur; we do this transfer in a **closed world** [DSm hybrid rule]; but, if it is an **open world**, we can transfer the mass m(A∩B) to the empty set leaving room for new possible hypotheses [here one gets *TBM*];

9.9.2.5.2) if one considers that none of the hypotheses A, B is right and no other hypothesis exists in the frame of discernment (i.e. n = 2 is the size of the frame of discernment), then one considers the **open world** and one transfers the mass to the empty set [here *DSmT* and *TBM* converge to each other].

Of course, this procedure is extended for any intersections of two or more sets: A∩B∩C, etc. and even for mixed sets: A∩ (B∪C), etc.

If it is a dynamic fusion in a real time and associativity and/or Markovian process are needed, use an algorithm which transforms a rule (which is based on the conjunctive rule and the transfer of the conflicting mass) into an associative and Markovian rule by storing the previous result of the conjunctive rule and, depending of the rule, other data. Such rules are called quasi-associative and quasi-Markovian.

Some applications require the necessity of **decaying the old sources** because their information is considered to be worn out.

If some scalar bba is not normalized (i.e. the sum of its components is < 1 as in incomplete information, or > 1 as in paraconsistent information) we can easily divide each component by the sum of the components and normalize it. But also it is possible to fusion incomplete and paraconsistent masses, and then normalize them after fusion. Or leave them unnormalized since they are incomplete or paraconsistent.

PCR5 (Smarandache-dezert) does the most mathematically exact (in the fusion literature) redistribution of the conflicting mass to the elements involved in the conflict, redistribution which exactly follows the tracks of the conjunctive rule. Here it is its formula:

$$\forall\, X \in S^{\Theta} \setminus \phi,\; m_{PCR5}(X) = m_{12}(X) + \sum_{\substack{Y \in (S^{\wedge}\Theta)\setminus X \\ c(X\cap Y)=\phi}} \left\{ \frac{[m_1(X)^{\wedge}2]m_2(Y)}{m_1(X)+m_2(Y)} + \frac{[m_2(X)^{\wedge}2]m_1(Y)}{m_2(X)+m_1(Y)} \right\}$$

where c(X) is the conjunctive normal form of X, $m_{12}(X)$ is the conjunctive rule result of X, and all denominators are non-null; if one is null, its fraction is discarded.

10. Example

Let $\Theta = \{A, B, C, D, E\}$ be the frame of discernment. We present an example that passes through many possibilities.

Suppose $m_1(A) = 0.2$, $m_1(B) = 0$, $m_1(C) = 0.3$, $m_1(D) = 0.4$, $m_1(E) = 0.1$, and $m_2(A) = 0.5$, $m_2(B) = 0.2$, $m_2(C) = 0.1$, $m_2(D) = 0$, $m_2(E) = 0.2$. Suppose both sources are reliable, then we use the conjunctive rule and we get: $m_{12}(A) = 0.10$, $m_{12}(B) = 0$, $m_{12}(C) = 0.03$, $m_{12}(D) = 0$, $m_{12}(E) = 0.02$, and also: $m_{12}(A\cap B) = 0.04$, $m_{12}(A\cap C) = 0.17$, $m_{12}(A\cap D) = 0.20$, $m_{12}(A\cap E) = 0.09$ $m_{12}(B\cap C) = 006$. $m_{12}(B\cap D) = 0.08$, $m_{12}(B\cap E) = 0.02$, $m_{12}(C\cap D) = 0.04$, $m_{12}(C\cap E) = 0.07$, $m_{12}(D\cap E) = 0.08$.

For the redistribution of the intersection masses, let's suppose we know that:

a) $A\cap B\neq\phi$, i.e. consensus (common part) between A and B, hence the mass $m_{12}(A\cap B) = 0.04$ remains on intersection: $m_{UFT}(A\cap B) = 0.04$.

b) $A\cap C=\phi$, i.e. contradiction between A and C, but we are optimistic in both of them, then we can transfer the mass 0.17 to A and C (using PCR5, but other rules can also be used such as Dempster's, PCR3, etc.) the redistribution mass $m_r(A)=0.107$, $m_r(C)=0.107$, where $m_r(X)$ means the redistributed mass gained by set X at a respective step.

c) $A\cap D=\phi$, and suppose we know that one hypothesis is right, one wrong, but we don't know which one, then the mass 0.20 is transferred to $m_r(A\cup D)$.

d) $A\cap E=\phi$, and suppose we know that A is right, E is wrong, then whole mass 0.09 is transferred to A only, i.e. $m_r(A)=0.09$.

e) we don't know if $B\cap C = \phi$ or $\neq \phi$, therefore the model is unknown, hence we keep the mass 0.06 on $B\cap C$ just in case we might find out more information on the model (this is considered the default model).

f) $B\cap D=\phi$, but we don't know any relationship between B and D, hence in a prudent way we transfer the mass 0.08 to the uncertainty: $m_r(B\cup D)=0.08$.

g) $B\cap E\neq\phi$, the intersection is not empty, but suppose neither $B\cap E$ nor $B\cup E$ interest us, then we can transfer the mass 0.02 to B and E (using PCR5), hence $m_r(B)=0.013$, $m_r(E)=0.007$.

h) $C\cap D=\phi$, and suppose we are pessimistic in both C and D, then the mass 0.04 is transferred to $C\cup D$, i.e. $m_r(C\cup D)=0.04$.

i) $C\cap E=\phi$, and suppose we are very pessimistic in both C and E, then we cautiously transfer the mass of this intersection, 0.07, to the total ignorance: $m_r(A\cup B\cup C\cup D\cup E)=0.07$.

j) $D \cap E = \phi$, and suppose we know that both D and E are wrong, then its mass 0.08 is redistributed among A, B, C equally, $m_r(A) = m_r(B) = m_r(C) = 0.027$.

Then one sums the masses of the conjunctive rule $\mathbf{m_{12}}$ and the redistribution of conflicting masses $\mathbf{m_r}$ (according to the information we supposedly have on each intersection, model, and relationship between conflicting hypotheses) in order to get the mass of the Unification of Fusion Theories $\mathbf{m_{UFT}}$, and we get:
$m_{UFT}(A) = 0.324$, $m_{UFT}(B) = 0.040$, $m_{UFT}(C) = 0.119$, $m_{UFT}(D) = 0$, $m_{UFT}(E) = 0.027$, $m_{UFT}(A \cap B) = 0.04$, $m_{UFT}(B \cap C) = 0.06$, $m_{UFT}(A \cup D) = 0.20$, $m_{UFT}(B \cup D) = 0.08$, $m_{UFT}(C \cup D) = 0.04$, $m_{UFT}(A \cup B \cup C \cup D \cup E) = 0.07$, $m_{UFT}(\phi) = 0$.

$\mathbf{m_{UFT}}$, the Unification of Fusion Theories rules, are combinations of many rules and give the optimal redistribution of the conflicting mass for each particular problem, following the given model and relationships between hypotheses; this extra-information allows the choice of the combination rule to be used for each intersection. The algorithm is presented above.

$\mathbf{m_{lower}}$, the lower bound believe assignment, the most pessimistic/prudent belief, is obtained by transferring the whole conflicting mass to the total ignorance (Yager's rule) in a closed world, or to the empty set (Smets' TBM) in an open world herein meaning that other hypotheses might belong to the frame of discernment.
For the previous example we have: $m_{lower}(A)=0.10$, $m_{lower}(B)=0$, $m_{lower}(C)=0.03$, $m_{lower}(D)=0$, $m_{lower}(E)=0.02$, and $m_{lower}(A \cup B \cup C \cup D \cup E)= 0.85$ in a closed world or $m_{lower}(\phi)= 0.85$.

$\mathbf{m_{middle}}$, or the default case, the middle believe assignment, half optimistic and half pessimistic, is obtained by transferring the partial conflicting masses $m_{12}(X \cap Y)$ to the partial ignorance $X \cup Y$ (as in Dubois-Prade's rule or more general as in Dezert-Smarandache theory).
For the previous example we have: $m_{middle}(A)=0.10$, $m_{middle}(B)=0$, $m_{middle}(C)=0.03$, $m_{middle}(D)=0$, $m_{middle}(E)=0.02$, $m_{middle}(A \cup B)=0.04$, $m_{middle}(A \cup C)=0.17$, $m_{middle}(A \cup D)=0.20$, $m_{middle}(A \cup E)=0.09$, $m_{middle}(B \cup C)=0.06$, $m_{middle}(B \cup D)=0.08$, $m_{middle}(B \cup E)=0.02$, $m_{middle}(C \cup D)=0.04$, $m_{middle}(C \cup E)=0.07$, $m_{middle}(D \cup E)=0.08$.

$\mathbf{m_{upper}}$, the upper bound believe assignment, the most optimistic (less prudent) belief, is obtained by transferring the masses of intersections (empty or non-empty) to the elements in the frame of discernment using the PCR5 rule of combination, i.e. $m_{12}(X \cap Y)$ is split to the elements X, Y (see Table 2). We use PCR5 because it is more exact mathematically (following backwards the tracks of the conjunctive rule) than Dempster's rule, minC, and PCR1-4.
For the previous example we have: $m_{upper}(A)= 0.400$, $m_{upper}(B)= 0.084$, $m_{upper}(C)= 0.178$, $m_{upper}(D)= 0.227$, $m_{upper}(E)= 0.111$.

11. Conclusion

The Unification of Fusion Theories (UFT) and partial Unification of Fusion Rules (UFR) are presented in this short article. They combine existing and new fusion rules and theories in an attempt to provide an optimal fusion for practical applications. The partial or total conflicting masses are better redistributed if we have more information about the sources and the relationship between hypotheses in conflict. It is possible to do a prudent/pessimistic (low belief) transfer, or average optimistic (middle belief) transfer, or most optimistic (less prudent) transfer.

References

[1] D. Dubois., H. Prade, *On the combination of evidence in various mathematical frameworks*, Reliability Data Collection and Analysis, J. Flamm and T. Luisi, Brussels, ECSC, EEC, EAFC: pp. 213-241, 1992.

[2] J. W. Guan, D. A. Bell, *Generalizing the Dempster-Shaffer Rule of Combination to Boolean Algebras*, IEEE, 229-236, 1993.

[3] T. Inagaki, *Interdependence between safety-control policy and multiple-senor schemes via Dempster-Shafer theory*, IEEE Trans. on reliability, Vol. 40, no. 2, pp. 182-188, 1991.

[4] E. Lefevre, O. Colot, P. Vannoorenberghe, *Belief functions combination and conflict management*, Information Fusion Journal, Elsevier Publisher, Vol. 3, No. 2, pp. 149-162, 2002.

[5] C. K. Murphy, *Combining belief functions when evidence conflicts*, Decision Support Systems, Elsevier Publisher, Vol. 29, pp. 1-9, 2000.

[6] K. Sentz, S. Ferson, *Combination of evidence in Dempster-Shafer Theory*, SANDIA Tech. Report, SAND2002-0835, 96 pages, April 2002, www.sandia.gov/epistemic/Reports/SAND2002-0835.pdf.

[7] G. Shafer, *A Mathematical Theory of Evidence*, Princeton Univ. Press, Princeton, NJ, 1976.

[8] F. Smarandache, J. Dezert (Editors), *Applications and Advances of DSmT for Information Fusion*, Am. Res. Press, Rehoboth, 2004, http://www.gallup.unm.edu/~smarandache/DSmT-book1.pdf.

[9] F. Smarandache, J. Dezert, *Proportional Conflict Redistribution Rules*, arXiv Archives, http://xxx.lanl.gov/PS_cache/cs/pdf/0408/0408064.pdf.

[10] P. Smets, Quantified Epistemic Possibility Theory seen as an Hyper Cautious Transferable Belief Model, http://iridia.ulb.ac.be/~psmets.

[11] R. R. Yager, Hedging in the combination of evidence, Journal of Information & Optimization Sciences, Analytic Publishing Co., Vol. 4, No. 1, pp. 73-81, 1983.

[12] F. Voorbraak, On the justification of Dempster's rule of combination, Artificial Intelligence, 48, pp. 171-197, 1991.

[13] R.R. Yager, On the relationships of methods of aggregation of evidence in expert systems, Cybernetics and Systems, Vol. 16, pp. 1-21, 1985.

[14] L. Zadeh, Review of Mathematical theory of evidence, by Glenn Shafer, AI Magazine, Vol. 5, No. 3, pp. 81-83, 1984.

[15] L. Zadeh, A simple view of the Dempster-Shafer theory of evidence and its implication for the rule of combination, AI Magazine 7, No.2, pp. 85-90, 1986.