ISSN 1068-3623, Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences), 2022, Vol. 57, No. 2, pp. 90–101. © Allerton Press, Inc., 2022.
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REAL AND COMPLEX ANALYSIS

Fekete–Szegő Inequalities for Certain Subclasses of Analytic Functions Related with Nephroid Domain

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Received May 20, 2020; revised June 15, 2021; accepted October 15, 2021

Abstract—The purpose of this paper is to consider coefficient estimates in a class of functions $\mathcal{M}_{\alpha,\lambda}(q)$ consisting of analytic functions f normalized by $f(0) = f'(0) - 1 = 0$ in the open unit disk $\Delta = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$ subordinating with nephroid domain, to derive certain coefficient estimates a_2, a_3 and Fekete–Szegő inequality for $f \in \mathcal{M}_{\alpha,\lambda}(q)$. A similar result have been done for the function f^{-1}. Further application of our results to certain functions defined by convolution products with a normalized analytic function is given, and in particular we obtain Fekete–Szegő inequalities for certain subclasses of functions defined through neutrosophic Poisson distribution.

MSC2010 numbers: 30C80, 30C45

DOI: 10.3103/S1068362322020066

Keywords: *analytic function, starlike function, convex function, subordination, Fekete–Szegő inequality, neutrosophic Poisson distribution, Hadamard product*

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\Delta := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. A function $f \in \mathcal{S}$ is said to be *starlike* in Δ if and only if

$$\Re\Big(\frac{zf'(z)}{f(z)}\Big) > 0, \qquad (z \in \Delta),$$

and, on the other hand, a function $f \in \mathcal{S}$ is said to be *convex* in Δ if and only if

$$\Re\Big(1 + \frac{zf''(z)}{f'(z)}\Big) > 0, \quad (z \in \Delta)$$

denoted by $\mathcal{S}^*$ and $\mathcal{C}$, respectively.

Let f_1 and f_2 be functions analytic in Δ. Then we say that the function f_1 is subordinate to f_2 if there exists a Schwarz function $w(z)$, analytic in Δ with $w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \Delta)$, such that $f_1(z) = f_2(w(z)) \quad (z \in \Delta)$. We denote this subordination by

$$f_1 \prec f_2 \quad \text{or} \quad f_1(z) \prec f_2(z) \quad (z \in \Delta).$$

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In particular, if the function f_2 is univalent in Δ, the above subordination is equivalent to $f_1(0) = f_2(0)$ and $f(\Delta) \subset f_2(\Delta)$. The function $q(z) = 1 + z - \frac{z^3}{3}$ maps Δ onto the region bounded by the nephroid

$$\left((u-1)^2 + v^2 - \frac{4}{9}\right)^3 - \frac{4v^2}{3} = 0,$$

which is symmetric about the real axis and lies completely inside the right-half plane $u > 0$. Geometrically, a nephroid is the locus of a point on the circumference of a circle of radius ρ traversing positively the outside of a fixed circle of radius 2ρ. It is an algebraic curve of degree six and is an epicycloid having two cusps. The plane curve nephroid was studied by Huygens and Tschirnhausen around 1679 in connection with the theory of caustics, a method of deriving a new curve based on a given curve and a point. In 1692, J. Bernoulli showed that the nephroid is the catacaustic (envelope of rays emanating from a specified point) of a cardioid for a luminous cusp. However, the name nephroid, which means kidney shaped, was first used by the English mathematician Richard A. Proctor in 1878 in his book *Geometry of Cycloids* (for more details, see [20] and references cited therein).

Definition 1.1 [20]. Let $\mathcal{S}^*(q)$ denote the class of analytic functions f in the unit disc Δ normalized by $f(0) = f'(0) - 1 = 0$ and satisfying the condition that

$$\frac{zf'(z)}{f(z)} \prec 1 + z - \frac{z^3}{3} =: q(z), \quad z \in \Delta. \tag{1.2}$$

and $\mathcal{C}(q)$ if

$$\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec 1 + z - \frac{z^3}{3} =: q(z), \quad z \in \Delta. \tag{1.3}$$

Further, they proved by considering, $q(z)$ as a holomorphic solution of the differential equation

$$\frac{zq'(z)}{q(z)} = 1 + z - \frac{z^3}{3}, \quad z \in \Delta,\ q(0) = 0,\ q'(0) = 1,$$

i.e.,

$$\Omega_n(z) = z\exp\left(\int_0^z \frac{q(\zeta^{n-1}) - 1}{\zeta} d\zeta\right) = z + \frac{z^n}{n-1} + \frac{z^{2n-1}}{2(n-1)^2} + \cdots, \quad z \in \Delta \tag{1.4}$$

plays the extremal role of the class $\mathcal{S}_q^*$ as noted by Wani and Swaminathan [20]. Also

$$\Upsilon_n(z) = \exp\left(\int_0^z \frac{q(\zeta^{n-1}) - 1}{\zeta} d\zeta\right) = z + \frac{z^n}{n(n-1)} + \frac{z^{2n-1}}{2(2n-1)(n-1)^2} + \cdots, \tag{1.5}$$

$z \in \Delta$, plays the extremal role of the class $\mathcal{C}_q$ as noted by Wani and Swaminathan [20]. It may be noted from (1.3) of Definition 1.1 that the set $q(\Delta)$ lies in the right half-plane and it is not a starlike domain with respect to the origin.

Recently, Raina and Sokol [15] have studied and obtained some coefficient inequalities for the class $\mathcal{S}^\star(z + \sqrt{1+z^2})$ and these results are further improved by Sokol and Thomas [19] further the Fekete–Szegő inequality for functions in the class $\mathcal{C}(q)$ were obtained and in view of the Alexander result between the class $\mathcal{S}^*(z + \sqrt{1+z^2})$ and $\mathcal{C}(z + \sqrt{1+z^2})$, the Fekete–Szegő inequality for functions in $\mathcal{S}^*(z + \sqrt{1+z^2})$ were also obtained. For a brief history of Fekete–Szegő problem for the class of starlike, convex and various other subclasses of analytic functions, we refer the interested reader to [18]. Let $\alpha \geqslant 0$, $\lambda \geqslant 0$ and $0 \leqslant \rho < 1$ and $f \in \mathcal{A}$. We say that $f \in M(\alpha, \lambda, \rho)$ if it satisfies the condition

$$\Re\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^\alpha + \lambda\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)} - 1\right)\right]\right\} > \rho.$$

The class $M(\alpha, \lambda, \rho)$ was introduced by Guo and Liu [4].

Motivated essentially by the aforementioned works, (see [15, 17 and 1]) in this paper we define the following class $\mathcal{M}_{\alpha,\lambda}(q)$ given in Definition 1.2. First, we shall find estimations of first few coefficients of functions f of the form (1.1) belonging to $\mathcal{M}_{\alpha,\lambda}(q)$ and we prove the Fekete–Szegő inequality $f\in\mathcal{M}_{\alpha,\lambda}(q)$ and also for $f^{-1}\in\mathcal{M}_{\alpha,\lambda}(q)$. Also we give applications of our results to certain functions defined through Poisson distribution.

Now, we define the following class $\mathcal{M}_{\alpha,\lambda}(q)$:

Definition 1.2. For $\alpha\geqslant 0,\ \lambda\geqslant 0$ a function $f\in\mathcal{A}$ is in the class $\mathcal{M}_{\alpha,\lambda}(q)$ if

$$\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha}+\lambda\left[1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}+\alpha\left(\frac{zf'(z)}{f(z)}-1\right)\right]\right\}$$
$$\prec\ 1+z-\frac{z^3}{3}=q(z);\ z=re^{i\theta}\in\Delta. \tag{1.6}$$

Note that by specializing the parameter we get the following subclasses based on nephroid domain (see [20]).

- $\mathcal{M}_{0,0}(q)\equiv\mathcal{S}^*(q)=\left\{f\in\mathcal{A}:\dfrac{zf'(z)}{f(z)}\prec q(z)=1+z-\dfrac{z^3}{3},z\in\Delta\right\}$
- $\mathcal{M}_{0,1}(q)\equiv\mathcal{C}(q)=\left\{f\in\mathcal{A}:\left(1+\frac{zf''(z)}{f'(z)}\right)\prec q(z)=1+z-\frac{z^3}{3},z\in\Delta\right\}$
- $\mathcal{M}_{0,\lambda}(q)\equiv\mathcal{M}_{\lambda}(q)$
  $=\{f\in\mathcal{A}:(1-\lambda)\dfrac{zf'(z)}{f(z)}+\lambda\left(1+\dfrac{zf''(z)}{f'(z)}\right)\prec q(z)=1+z-\dfrac{z^3}{3},z\in\Delta\}$
- $\mathcal{M}_{\alpha,0}(q)\equiv\mathcal{B}^{\alpha}(q)=\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha}\prec q(z)=1+z-\frac{z^3}{3},z\in\Delta\}$ . .

2. A COEFFICIENT ESTIMATE

To prove our main result, we need the following:

Lemma 2.1 [8]. *If $p_1(z)=1+c_1z+c_2z^2+\cdots$ is a function with positive real part in Δ, then*

$$|c_2-vc_1^2|\leqslant\begin{cases}-4v+2, & if\quad v\leqslant 0,\\ 2, & if\quad 0\leqslant v\leqslant 1,\\ 4v-2, & if\quad v\geqslant 1.\end{cases}$$

When $v<0$ or $v>1$, the equality holds if and only if $p_1(z)$ is $\dfrac{1+z}{1-z}$ or one of its rotations. If $0<v<1$, then equality holds if and only if $p_1(z)$ is $\dfrac{1+z^2}{1-z^2}$ or one of its rotations. If $v=0$, the equality holds if and only if

$$p_1(z)=\left(\frac{1}{2}+\frac{1}{2}\eta\right)\frac{1+z}{1-z}+\left(\frac{1}{2}-\frac{1}{2}\eta\right)\frac{1-z}{1+z}\quad(0\leqslant\eta\leqslant 1)$$

or one of its rotations. If $v=1$, the equality holds if and only if p_1 is the reciprocal of one of the functions such that the equality holds in the case of $v=0$.

Although the above upper bound is sharp, when $0<v<1$, it can be improved as follows:

$$|c_2-vc_1^2|+v|c_1|^2\leqslant 2\quad(0<v\leqslant 1/2)$$

and

$$|c_2-vc_1^2|+(1-v)|c_1|^2\leqslant 2\quad(1/2<v\leqslant 1).$$

We also need the following:

Lemma 2.2 [3]. *If $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part in Δ, then*

$$|\, c_n \,| \leqslant 2 \ \ \textit{for all } n \geqslant 1 \qquad \textit{and} \qquad |c_2 - \frac{c_1^2}{2}| \leqslant 2 - \frac{|c_1|^2}{2}.$$

The class of all such functions with positive real part are denoted by $\mathcal{P}$.

Lemma 2.3 [7]. *If $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part in Δ, then*

$$|c_2 - vc_1^2| \leqslant 2\max(1, |2v-1|).$$

The result is sharp for the functions

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z}.$$

Lemma 2.4 [6]. *Let $P(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ...$ be in $\mathcal{P}$ then for any complex number μ,*

$$\left|c_2 - \mu\frac{c_1^2}{2}\right| \leqslant \ \max\{2, 2|\mu-1|\} = \begin{cases} 2, & 0 \leqslant \mu \leqslant 2; \\ 2|\mu - 1|, & \textit{elsewhere}. \end{cases}$$

The result is sharp for the functions defined by $P(z) = \frac{1+z^2}{1-z^2}$ or $P(z) = \frac{1+z}{1-z}$.

Theorem 2.1. *Let $\alpha \geqslant 0$ and $\lambda \geqslant 0$. If $f(z)$ given by (1.1) belongs to $\mathcal{M}_{\alpha,\lambda}(q)$, then*

$$|a_2| \ \leqslant \ \frac{1}{(1+\alpha)(1+\lambda)},$$
$$|a_3| \ \leqslant \ \frac{1}{(\alpha+2)(1+2\lambda)} \max\{1, \left| \left(\frac{\alpha^2 + \alpha - 2(\alpha+3)\lambda - 2}{2((1+\alpha)(1+\lambda))^2} \right) \right| \}.$$

Proof. If $f \in \mathcal{M}_{\alpha,\lambda}(q)$, then there is a Schwarz function $w(z)$, analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ in Δ such that

$$\begin{aligned} &\left\{ \frac{zf'(z)}{f(z)} \left(\frac{f(z)}{z} \right)^{\alpha} + \lambda \left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha \left(\frac{zf'(z)}{f(z)} - 1 \right) \right] \right\} \\ &= q(w(z)) = 1 + w(z) - \frac{(w(z))^3}{3}. \end{aligned} \tag{2.1}$$

Define the function $P(z)$ by

$$P(z) := \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots$$

it is easy to see that

$$w(z) = \frac{P(z)-1}{P(z)+1} = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2} \right) z^2 + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) z^3 + \cdots \right]. \tag{2.2}$$

Since $w(z)$ is a Schwarz function, we see that $\Re(p_1(z)) > 0$ and $p_1(0) = 1$. Let us define the function $p(z)$ by

$$\begin{aligned} p(z) :\ &= \ \left\{ \frac{zf'(z)}{f(z)} \left(\frac{f(z)}{z} \right)^{\alpha} + \lambda \left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha \left(\frac{zf'(z)}{f(z)} - 1 \right) \right] \right\} \\ &= \ 1 + b_1 z + b_2 z^2 + \cdots . \end{aligned} \tag{2.3}$$

In view of the Eqs. (2.1)–(2.3), we have

$$p(z) = q\left(\frac{P(z)-1}{P(z)+1} \right). \tag{2.4}$$

Hence,

$$\begin{aligned} 1 + w(z) - \frac{(w(z))^3}{3} &= 1 + \frac{c_1}{2}z + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right) z^2 + \left(\frac{c_3}{2} - \frac{c_1 c_2}{2} + \frac{c_1^3}{8}\right) z^3 - \frac{{c_1}^3}{24} z^3 + \cdots \\ &= 1 + \frac{c_1}{2}z + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right) z^2 + \left(\frac{c_3}{2} - \frac{c_1 c_2}{2} + \frac{{c_1}^3}{12}\right) z^3 + \cdots, \quad z \in \mathbb{D}. \end{aligned} \tag{2.5}$$

Using (2.2) in (2.4), we get

$$b_1 = \frac{c_1}{2} \qquad \text{and} \qquad b_2 = \frac{c_2}{2} - \frac{c_1^2}{4}.$$

A computation shows that

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + (2a_3 - a_2^2)z^2 + (3a_4 + a_2^3 - 3a_3 a_2)z^3 + \cdots .$$

Similarly we have

$$1 + \frac{zf''(z)}{f'(z)} = 1 + 2a_2 z + (6a_3 - 4a_2^2)z^2 + \cdots .$$

An easy computation shows that

$$\begin{aligned} &\left\{\frac{zf'(z)}{f(z)} \left(\frac{f(z)}{z}\right)^{\alpha} + \lambda \left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha \left(\frac{zf'(z)}{f(z)} - 1\right)\right]\right\} \\ &\qquad = 1 + (1+\alpha)(1+\lambda)a_2 z + (\alpha+2)(1+2\lambda)a_3 z^2 \\ &\qquad\quad + \left(\frac{\alpha^2+\alpha}{2} - (\alpha+3)\lambda - 1\right) a_2^2 z^2 + \cdots . \end{aligned}$$

In view of the Eq. (2.3), we see that

$$b_1 = (1+\alpha)(1+\lambda)a_2, \tag{2.6}$$

$$b_2 = (\alpha+2)(1+2\lambda)a_3 + \left(\frac{\alpha^2+\alpha}{2} - (\alpha+3)\lambda - 1\right) a_2^2 \tag{2.7}$$

or, equivalently, we have

$$a_2 = \frac{c_1}{2(1+\alpha)(1+\lambda)}, \tag{2.8}$$

$$\begin{aligned} a_3 &= \frac{1}{(\alpha+2)(1+2\lambda)} \left(\frac{c_2}{2} - \frac{c_1^2}{4}\left[1 + \frac{\alpha^2+\alpha-2(\alpha+3)\lambda-2}{2\left((1+\alpha)(1+\lambda)\right)^2}\right]\right) \\ &= \frac{1}{2(\alpha+2)(1+2\lambda)} \left(c_2 - \frac{c_1^2}{2}\left[1 + \frac{\alpha^2+\alpha-2(\alpha+3)\lambda-2}{2\left((1+\alpha)(1+\lambda)\right)^2}\right]\right) \\ &= \frac{1}{2(\alpha+2)(1+2\lambda)} \left(c_2 - v c_1^2\right), \end{aligned} \tag{2.9}$$

where

$$v = \frac{1}{2}\left(1 + \frac{\alpha^2+\alpha-2(\alpha+3)\lambda-2}{2\left((1+\alpha)(1+\lambda)\right)^2}\right). \tag{2.10}$$

Therefore, we have

$$|a_2| \leqslant \frac{1}{(1+\alpha)(1+\lambda)}$$

and, by using the estimate

$$|c_2 - vc_1^2| \leqslant 2\max(1, |2v-1|)$$

given in Lemma 2.3, we have

$$\begin{aligned} |a_3| &\leqslant \frac{1}{(\alpha+2)(1+2\lambda)} \max\{1, \left|2\times\frac{1}{2}\left(1+\frac{\alpha^2+\alpha-2(\alpha+3)\lambda-2}{2\left((1+\alpha)(1+\lambda)\right)^2}\right)-1\right|\} \\ &= \frac{1}{(\alpha+2)(1+2\lambda)} \max\{1, \left|\left(\frac{\alpha^2+\alpha-2(\alpha+3)\lambda-2}{2\left((1+\alpha)(1+\lambda)\right)^2}\right)\right|\}. \end{aligned}$$

Remark 2.1. Let $\alpha = 0$ and $\lambda \geqslant 0$. If $f(z)$ given by (1.1) belongs to $\mathcal{M}_\lambda(q)$, then

$$\begin{aligned} |a_2| &\leqslant \frac{1}{1+\lambda}, \\ |a_3| &\leqslant \frac{1}{2(1+2\lambda)} \max\{1, \left|\frac{3\lambda+1}{2(1+\lambda)^2}\right|\} = \frac{3\lambda+1}{4(1+2\lambda)(1+\lambda)^2}. \end{aligned}$$

Remark 2.2. Let $\lambda = 0$. If $f(z)$ given by (1.1) belongs to $\mathcal{B}^\alpha(q)$, then

$$|a_2| \leqslant \frac{1}{1+\alpha}, \qquad \text{and} \qquad |a_3| \leqslant \frac{1}{\alpha+2} \max\{1, \left|\left(\frac{\alpha^2+\alpha--2}{2\left(1+\alpha\right)^2}\right)\right|\}.$$

Remark 2.3 (see [20])**.** Let $\alpha = 0$ and $\lambda = 0$. If $f(z)$ given by (1.1) belongs to $\mathcal{S}^*(q)$, then

$$|a_2| \leqslant 1, \qquad \text{and} \qquad |a_3| \leqslant \frac{1}{2} \max\{1, \left|\frac{1}{2}\right|\} = \frac{1}{2}.$$

Remark 2.4 (see [20])**.** Let $\alpha = 0$ and $\lambda = 1$. If $f(z)$ given by (1.1) belongs to $\mathcal{C}(q)$, then

$$|a_2| \leqslant \frac{1}{2}, \qquad \text{and} \qquad |a_3| \leqslant \frac{1}{6} \max\{1, \left|\frac{1}{2}\right|\} = \frac{1}{6}.$$

Theorem 2.2. *Let $0 \leqslant \mu \leqslant 1$, $\alpha \geqslant 0$ and $\lambda \geqslant 0$. If $f(z)$ given by (1.1) belongs to $\mathcal{M}_{\alpha,\lambda}(q)$, then*

$$|a_3 - \mu a_2^2| \leqslant \begin{cases} \frac{1}{2\xi}\left(-\frac{\gamma}{\tau^2}\right), & \text{if} \quad \mu \leqslant \sigma_1, \\ \frac{1}{\xi}, & \text{if} \quad \sigma_1 \leqslant \mu \leqslant \sigma_2, \\ \frac{1}{2\xi}\left(\frac{\gamma}{\tau^2}\right), & \text{if} \quad \mu \geqslant \sigma_2, \end{cases}$$

where, for convenience,

$$\sigma_1 = \frac{-2\tau^2 + 2(\alpha+3)\lambda - \rho}{2\xi}; \sigma_2 = \frac{2\tau^2 + 2(\alpha+3)\lambda - \rho}{2\xi}; \sigma_3 = \frac{2(\alpha+3)\lambda - \rho}{2\xi},$$

$$\gamma := \rho - 2(\alpha+3)\lambda + 2\mu\xi, \tag{2.11}$$

$$\rho := \alpha^2 + \alpha - 2, \tag{2.12}$$

$$\xi := (\alpha+2)(1+2\lambda), \tag{2.13}$$

and

$$\tau := (1+\alpha)(1+\lambda). \tag{2.14}$$

Further, if $\sigma_1 \leqslant \mu \leqslant \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{\tau^2}{\xi}\left(1 + \frac{\gamma}{2\tau^2}\right)|a_2|^2 \leqslant \frac{1}{\xi}.$$

If $\sigma_3 \leqslant \mu \leqslant \sigma_2$, *then*

$$|a_3 - \mu a_2^2| + \frac{\tau^2}{\xi}\left(1 - \frac{\gamma}{2\tau^2}\right)|a_2|^2 \leqslant \frac{1}{\xi}.$$

These results are sharp.

Proof. Now, by making use of (2.8) and (2.9) we get

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{1}{(\alpha+2)(1+2\lambda)}\left(\frac{c_2}{2} - \frac{c_1^2}{4} - \left(\frac{\alpha^2+\alpha-2-2(\alpha+3)\lambda+2\mu(\alpha+2)(1+2\lambda)}{8\left((1+\alpha)(1+\lambda)\right)^2}\right)c_1^2\right) \\ &= \frac{1}{2(\alpha+2)(1+2\lambda)}\left(c_2 - \frac{c_1^2}{2}\left(1 + \frac{\alpha^2+\alpha-2-2(\alpha+3)\lambda+2\mu(\alpha+2)(1+2\lambda)}{2\left((1+\alpha)(1+\lambda)\right)^2}\right)\right), \end{aligned}$$

where

$$v := \frac{1}{2}\left(1 + \frac{\alpha^2+\alpha-2-2(\alpha+3)\lambda+2\mu(\alpha+2)(1+2\lambda)}{2\left((1+\alpha)(1+\lambda)\right)^2}\right).$$

That is, simply

$$v := \frac{1}{2}\left(1 + \frac{\rho - 2(\alpha+3)\lambda + 2\mu\xi}{2\tau^2}\right) = \frac{1}{2}\left(1 + \frac{\gamma}{2\tau^2}\right).$$

The assertion of Theorem 2.2 now follows by an application of Lemma 2.1.

To show that the bounds are sharp, we define the functions F_η and G_η $(0 \leqslant \eta \leqslant 1)$, respectively, with $F_\eta(0) = 0 = F'_\eta(0) - 1$ and $G_\eta(0) = 0 = G'_\eta(0) - 1$ by

$$\frac{z(F_\eta)'(z)}{F_\eta(z)}\left(\frac{F_\eta(z)}{z}\right)^\alpha + \lambda\left[1 + \frac{z(F_\eta)''(z)}{(F_\eta)'(z)} - \frac{z(F_\eta)'(z)}{F_\eta(z)} + \alpha\left(\frac{z(F_\eta)'(z)}{F_\eta(z)} - 1\right)\right] = q\left(\frac{z(z+\eta)}{1+\eta z}\right),$$

and

$$\frac{z(G_\eta)'(z)}{G_\eta(z)}\left(\frac{G_\eta(z)}{z}\right)^\alpha + \lambda\left[1 + \frac{z(G_\eta)''(z)}{(G_\eta)'(z)} - \frac{z(G_\eta)'(z)}{G_\eta(z)} + \alpha\left(\frac{z(G_\eta)'(z)}{G_\eta(z)} - 1\right)\right] = q\left(-\frac{z(z+\eta)}{1+\eta z}\right),$$

respectively. Clearly, the functions $K_q := q(z)$, F_η, and G_η are members of $\mathcal{M}_{\alpha,\lambda}(q)$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is K_q or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if and only if f is $K_q = q(z^2)$ or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if f is F_η or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if f is G_η or one of its rotations.
□

By making use of Lemma 2.3, we immediately obtain the following:

Theorem 2.3. *Let* $0 \leqslant \alpha \leqslant 1$, *and* $0 \leqslant \lambda \leqslant 1$. *If* $f \in \mathcal{M}_{\alpha,\lambda}(q)$, *then for complex* μ, *we have*

$$\begin{aligned} &|a_3 - \mu a_2^2| \\ &\leqslant \frac{1}{(\alpha+2)(1+2\lambda)}\max\left\{1, \left|\frac{\alpha^2+\alpha-2-2(\alpha+3)\lambda+2\mu(\alpha+2)(1+2\lambda)}{2((1+\alpha)(1+\lambda))^2}\right|\right\} \\ &= \frac{1}{\xi}\max\left\{1, \left|\frac{\rho-2(\alpha+3)\lambda+2\mu\xi}{2\tau^2}\right|\right\}, \end{aligned}$$

where ρ, ξ, τ *are as defined in (2.12), (2.13), and (2.14). The result is sharp.*

Remark 2.5.

1. For the choice $\alpha = 0$ and $\lambda = 1$, Theorem 2.3 coincides with the result obtained for the class $f \in \mathcal{C}(q)$ as

$$|a_3 - \mu a_2^2| \leqslant \frac{1}{6}\max\left\{1, \left|\frac{3\mu}{2} - 1\right|\right\}.$$

2. For the choices $\alpha = 0$ and $\lambda = 0$, Theorem 2.3 reduces to the result for the class $f \in \mathcal{S}^*(q)$ (see [20]) as

$$|a_3 - \mu a_2^2| \leqslant \frac{1}{2} \max\{1, |2\mu - 1|\}.$$

3. For the choice of $\alpha = 0$, Theorem 2.3, reduces the result for the class $f \in \mathcal{M}_\lambda(q)$ as

$$|a_3 - \mu a_2^2| \leqslant \frac{1}{1+2\lambda} \max\left\{1, \left|\frac{-2-6\lambda+4\mu(1+2\lambda)}{2(1+\lambda)^2}\right|\right\}.$$

4. For the choice of $\lambda = 0$, Theorem 2.3 reduces the result for $f \in \mathcal{B}^\alpha(q)$

$$|a_3 - \mu a_2^2| \leqslant \frac{2}{\alpha+2} \max\left\{1, \left|\frac{\alpha^2+\alpha-2+2\mu(\alpha+2)}{2(1+\alpha)^2}\right|\right\}.$$

3. COEFFICIENT INEQUALITIES FOR THE FUNCTION f^{-1}

Theorem 3.1. *If $f \in \mathcal{M}_{\alpha,\lambda}(q)$ and $f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n$ is the inverse function of f with $|w| < r_0$, where r_0 is greater than the radius of the Koebe domain of the class $f \in \mathcal{M}_{\alpha,\lambda}(q)$, then for any complex number μ, we have*

$$|\, d_3 - \mu d_2^2 \,| \leqslant \frac{1}{\xi} \max\left\{1, \left|\, \frac{2\tau^2+\rho-2(\alpha+3)\lambda+(4+2\mu)\xi}{\tau^2} - 1 \,\right|\right\} \tag{3.1}$$

where ρ, ξ, and τ are as defined in (2.12), (2.13), and (2.14).

Proof. As

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n \tag{3.2}$$

is the inverse function of f, it can be seen that

$$f^{-1}(f(z)) = f\{f^{-1}(z)\} = z. \tag{3.3}$$

From Eqs. (1.1) and (3.3), it can be reduced to

$$f^{-1}(z + \sum_{n=2}^{\infty} a_n z^n) = z. \tag{3.4}$$

From (3.3) and (3.4), one can obtain

$$z + (a_2 + d_2)z^2 + (a_3 + 2a_2d_2 + d_3)z^3 + ... = z. \tag{3.5}$$

By comparing the coefficients of z and z^2 from relation (3.5), it can be seen that

$$d_2 = -a_2, \qquad d_3 = 2a_2^2 - a_3. \tag{3.6}$$

From relations (2.8), (2.9), and (3.6)

$$d_2 = -\frac{c_1}{2(1+\alpha)(1+\lambda)}; \tag{3.7}$$

$$\begin{aligned} d_3 &= \frac{1}{2(\alpha+2)(1+2\lambda)} \\ &\times \left(c_2 - \frac{2\left((1+\alpha)(1+\lambda)\right)^2 + 4(\alpha+2)(1+2\lambda) + \alpha^2 + \alpha - 2(\alpha+3)\lambda - 2}{4\left((1+\alpha)(1+\lambda)\right)^2} c_1^2\right) \\ &= \frac{1}{2\xi}\left(c_2 - \frac{2\tau^2 + 4\xi + \rho - 2(\alpha+3)\lambda}{2\tau^2} c_1^2\right); \end{aligned} \tag{3.8}$$

and ρ, ξ, and τ are as defined in (2.12)–(2.14). For any complex number μ, consider

$$d_3 - \mu d_2^2 = \frac{1}{2\xi}\Big(c_2 - \frac{2\tau^2 + \rho - 2(\alpha+3)\lambda + (4+2\mu)\xi}{2\tau^2}c_1^2\Big). \tag{3.9}$$

Taking modulus on both sides and by applying Lemma 2.3 on the right-hand side of (3.9), one can obtain the result as in (3.1). Hence this completes the proof. □

Remark 3.1. Suitably specializing the parameters in Theorem 3.1 one can easily state above result for the function classes $\mathcal{M}_{0,\lambda}(q) \equiv \mathcal{M}_\lambda(q)$, $\mathcal{M}_{\alpha,0}(q) \equiv \mathcal{B}^\alpha(q)$, $\mathcal{M}_{0,0}(q) \equiv \mathcal{S}^*(q)$, and $\mathcal{M}_{0,1}(q) \equiv \mathcal{C}(q)$.

4. APPLICATION TO FUNCTIONS DEFINED BY NEUTROSOPHIC POISSON DISTRIBUTION

By letting $\wp_N(z)$ as the neutrosophic Poisson distribution series, we study the following results (for details, see [12, 14]). As is well known that the classical probability distributions only deals with specified data and specified parameter values, while neutrosophic probability distribution gives a more general and clear ones. In fact, neutrosophic Poisson distribution of a discrete variable X is a classical Poisson distribution of x with the imprecise parameter value. A variable X is said to have neutrosophic Poisson distribution if its probability with the value $k \in \mathbb{N}^* = \mathbb{N} \cup \{0\}$ is

$$NP(x=k) = \frac{(m_N)^k}{k!}e^{-m_N}, k = 0.1, 2, 3 \dots .$$

where the distribution parameter m_N is the expected value and the variance, that is to say, $NE(x) = NV(x) = m_N$ for the neutrosophic statistical number $N = d + I$ (refer to [5] and also see [14] and the references cited). Define a power series whose coefficients are probabilities of neutrosophic Poisson distribution by

$$\Phi(m_N, z) = z + \sum_{n=2}^{\infty} \frac{(m_N)^{n-1}}{(n-1)!}e^{-m_N}z^n, \quad z \in \mathbb{D}.$$

For $f \in \mathcal{A}$, we take the convolution operator $*$ and introduce the linear operator $\Lambda : \mathcal{A} \to \mathcal{A}$ defined by

$$\begin{aligned}\Lambda f(z) &= \Phi(m_N, z) * f(z) = z + \sum_{n=2}^{\infty} \frac{(m_N)^{n-1}}{(n-1)!}e^{-m_N}a_n z^n \\ &= z + \sum_{n=m+1}^{\infty} \Psi(m_N, n)a_n z^n, \end{aligned} \tag{4.1}$$

where

$$\Psi_n := \Psi(m_N, n) = \frac{(m_N)^{n-1}}{(n-1)!}e^{-m_N}.$$

Specially,

$$\Psi_2 := m_N e^{-m_N}, \quad \Psi_3 := \frac{(m_N)^2}{2}e^{-m_N}. \tag{4.2}$$

For the application of the results given in the previous section, we define the class $\mathcal{M}^\varphi_{\alpha,\lambda}(q)$ in the following way:

$$\mathcal{M}^\varphi_{\alpha,\lambda}(q) := \{f \in \mathcal{A} \quad \text{and} \quad (f * \varphi) \in \mathcal{M}_{\alpha,\lambda}(q)\},$$

where

$$\varphi(z) = z + \sum_{n=2}^{\infty} \varphi_n z^n, \quad (\varphi_n > 0); \quad (f * \varphi) = z + \sum_{n=2}^{\infty} \varphi_n a_n z^n$$

and $\mathcal{M}_{\alpha,\lambda}(q)$ is given by Definition 1.2 and $*$ denotes the convolution or Hadamard product of two series. We define the class $\mathcal{M}^{m}_{\alpha,\lambda}(q)$ in the following way:

$$\mathcal{M}^{m}_{\alpha,\lambda}(q) := \{f \in \mathcal{A} \quad \text{and} \quad \Lambda f \in \mathcal{M}_{\alpha,\lambda}(q)\},$$

where $\mathcal{M}_{\alpha,\lambda}(q)$ is given by Definition 1.2.

In following theorem, we obtain the coefficient estimate for functions in the class $\mathcal{M}^{\varphi}_{\alpha,\lambda}(q)$, from the corresponding estimate for functions in the class $\mathcal{M}_{\alpha,\lambda}(q)$. Applying Theorem 2.2 for the function $(f*\varphi)(z) = z + \varphi_2 a_2 z^2 + \varphi_3 a_3 z^3 + \cdots$, we get the next Theorems 4.1 and 4.2 after an obvious change of parameter μ.

Theorem 4.1. *Let* $0 \leqslant \alpha \leqslant 1$, *and* $0 \leqslant \lambda \leqslant 1$. *If* $f \in \mathcal{M}^{\varphi}_{\alpha,\lambda}(q)$, *then for complex* μ, *we have*

$$|a_3 - \mu a_2^2| = \frac{1}{(\alpha+2)(1+2\lambda)\varphi_3}\max\left\{1, \left|\frac{\alpha^2+\alpha-2-2(\alpha+3)\lambda}{2((1+\alpha)(1+\lambda))^2} + \frac{\mu(\alpha+2)(1+2\lambda)\varphi_3}{((1+\alpha)(1+\lambda)\varphi_2)^2}\right|\right\}.$$

Theorem 4.2. *Let* $0 \leqslant \mu \leqslant 1$, $\alpha \geqslant 0$, $\lambda \geqslant 0$, *and* $\varphi_n > 0$. *If* $f(z)$ *given by (1.1) belongs to* $\mathcal{M}^{\varphi}_{\alpha,\lambda}(q)$, *then*

$$|a_3 - \mu a_2^2| \leqslant \begin{cases} \dfrac{1}{2\xi\varphi_3}\left(-\dfrac{\gamma_2}{\tau^2}\right), & \text{if} \quad \mu \leqslant \sigma_1, \\ \dfrac{1}{\xi\varphi_3}, & \text{if} \quad \sigma_1 \leqslant \mu \leqslant \sigma_2, \\ \dfrac{1}{2\xi\varphi_3}\left(\dfrac{\gamma_2}{\tau^2}\right), & \text{if} \quad \mu \geqslant \sigma_2, \end{cases}$$

where, for convenience, $\gamma_2 := \rho - 2(\alpha+3)\lambda + 2\mu\xi\frac{\varphi_3}{\varphi_2^2}$,

$$\sigma_1 := \frac{\varphi_2^2}{\varphi_3}\left[\frac{2(\alpha+3)\lambda - \rho - 2\tau^2}{2\xi}\right], \sigma_2 = \frac{\varphi_2^2}{\varphi_3}\left[\frac{2\tau^2 + 2(\alpha+3)\lambda - \rho}{2\xi}\right],$$

and ρ, ξ, *and* τ *are as defined in (2.12), (2.13), and (2.14).*

Now, we obtain the coefficient estimate for $f \in \mathcal{M}^{m}_{\alpha,\lambda}(q)$ from the corresponding estimate for $f \in \mathcal{M}_{\alpha,\lambda}(q)$. Applying Theorem 2.2 for the function $\Lambda f = z + \Psi_2 a_2 z^2 + \Psi_3 a_3 z^3 + \cdots$, we get the following Theorems 4.3 and 4.4 after an obvious change of the parameter μ as in above theorems.

For Ψ_2 and Ψ_3 given by (4.2) Theorems 4.1 and 4.2 reduce to the following:

Theorem 4.3. *Let* $0 \leqslant \alpha \leqslant 1$, *and* $0 \leqslant \lambda \leqslant 1$. *If* $f \in \mathcal{M}^{m}_{\alpha,\lambda}(q)$, *then for complex* μ, *we have*

$$|a_3 - \mu a_2^2| = \frac{2}{(\alpha+2)(1+2\lambda)m_N^2 e^{-m_N}}\max\left\{1, \left|\frac{\alpha^2+\alpha-2-2(\alpha+3)\lambda}{2((1+\alpha)(1+\lambda))^2} + \frac{\mu(\alpha+2)(1+2\lambda)}{2((1+\alpha)(1+\lambda))^2 e^{-m_N}}\right|\right\}.$$

Theorem 4.4. *Let* $0 \leqslant \mu \leqslant 1$, $\alpha \geqslant 0$, $\lambda \geqslant 0$, *and* $\psi_n > 0$. *If* $f(z)$ *given by (1.1) belongs to* $\mathcal{M}^{m}_{\alpha,\lambda}(q)$, *then*

$$|a_3 - \mu a_2^2| \leqslant \begin{cases} \dfrac{1}{\xi m_N^2 e^{-m_N}}\left(-\dfrac{\gamma_2}{\tau^2}\right), & \text{if} \quad \mu \leqslant \sigma_1, \\ \dfrac{2}{\xi m_N^2 e^{-m_N}}, & \text{if} \quad \sigma_1 \leqslant \mu \leqslant \sigma_2, \\ \dfrac{1}{\xi m_N^2 e^{-m_N}}\left(\dfrac{\gamma_2}{\tau^2}\right), & \text{if} \quad \mu \geqslant \sigma_2, \end{cases}$$

where $\gamma_2 := \rho - 2(\alpha+3)\lambda + \frac{\mu\xi}{e^{-m_N}}$. *For convenience we write*

$$\sigma_1 := e^{-m_N}\left[\frac{2\tau^2 + 2(\alpha+3)\lambda - \rho}{2\xi}\right], \sigma_2 = e^{-m_N}\left[\frac{2\tau^2 + 2(\alpha+3)\lambda - \rho}{2\xi}\right]$$

and ρ, ξ, and τ are as defined in (2.12), (2.13), and (2.14).

A variable $\mathcal{X}$ is said to be Poisson distributed if it takes the values $0, 1, 2, 3, \cdots$ with probabilities e^{-m}, $m\frac{e^{-m}}{1!}$, $m^2\frac{e^{-m}}{2!}$, $m^3\frac{e^{-m}}{3!}$, ..., respectively, where m is called the parameter. Thus,

$$P(\mathcal{X} = r) = \frac{m^r e^{-m}}{r!},\ r = 0, 1, 2, 3, \ldots.$$

In [13], Porwal introduced a power series whose coefficients are probabilities of Poisson distribution

$$\mathcal{K}(m, z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n, \qquad z \in \Delta,$$

where $m > 0$. By ratio test the radius of convergence of above series is infinity. Using the Hadamard product, Porwal [13] (see also, [1, 9, 10] introduced a new linear operator $\mathcal{I}^m(z) : \mathcal{A} \to \mathcal{A}$ defined by

$$\mathcal{I}^m f = \mathcal{K}(m, z) * f(z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} a_n z^n, = z + \sum_{n=2}^{\infty} \psi_n(m) a_n z^n, \qquad z \in \Delta.$$

Since $\mathcal{I}^m f = z + \sum_{n=2}^{\infty} \psi_m a_n z^n$, where $\psi_n = \frac{m^{n-1}}{(n-1)!} e^{-m}$, we have

$$\psi_2 = me^{-m} \quad \text{and} \quad \psi_3 = \frac{m^2}{2} e^{-m}. \tag{4.3}$$

Remark 4.1. Suitably specializing the parameters in Theorems 4.3 and 4.4, one can easily state the results for the function classes associated with neutrosophic Poisson distribution and Poisson distribution as listed below:

1. $\mathcal{M}^m_{0,\lambda}(q) \equiv \mathcal{M}^m_{\lambda}(q)$,
2. $\mathcal{M}^m_{0,0}(q) \equiv \mathcal{S}^*_m(q)$,
3. $\mathcal{M}_{\alpha,0}(q) \equiv \mathcal{B}^{\alpha}(q)$ and
4. $\mathcal{M}^m_{0,1}(q) \equiv \mathcal{C}^m(q)$,

which are new and not been studied sofar.

ACKNOWLEDGMENTS

I am grateful to the reviewer of this article who gave valuable suggestions in order to improve and revise the paper in present form.

CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.

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