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Multiple-Valued Neutrosophic Uncertain Linguistic Sets With Dombi Normalized Weighted Bonferroni Mean Operator and Their Applications in Multiple Attribute Decision Making Problem

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ABSTRACT In order to take into account quantitative and qualitative information in real complex decision making issue, a multiple-valued neutrosophic uncertain linguistic set (MVNULS) is initially proposed, which includes the uncertain linguistic part and the multiple-valued neutrosophic set (MVNS). Consequently, it has the advantages of them in expressing evaluation information. In the article, we initially define some concepts containing a multiple-valued neutrosophic uncertain linguistic set (MVNULS) and a multiple-valued neutrosophic uncertain linguistic element (MVNULE). Next, we present some basic operational rules regarding Dombi t-conorm and t-norm, which is more flexibility with a general parameter, and introduce a comparison method for multiple-valued neutrosophic uncertain linguistic numbers (MVNULNs). Since the aggregating operator of normalized weighted Bonferroni mean (NWBM) cannot only capture the interrelationship among multiple arguments, but also consider weight vector on each criterion and has the reducibility property, the NWBM operator has gained many attentions of scholars. Hence, in order to deal with the aggregation of MVNULEs, we put forward the multiple-valued neutrosophic uncertain linguistic Dombi normalized weighted Bonferroni mean (MVNULDNWBM) operator on account of Dombi t-conorm and t-norm, and discuss the desirable properties of the MVNULDNWBM operator. Furthermore, we investigate a multiple attribute decision making (MADM) approach in view of the aggregating operator to handle the MADM problem with multiple-valued neutrosophic uncertain linguistic information. Eventually, we conduct a concrete instance to validate the effectiveness and application of the developed approach.

INDEX TERMS Multiple-valued neutrosophic uncertain linguistic, Dombi, NWBM, multiple attribute decision making.

I. INTRODUCTION

Neutrosophic Sets (NSs) as a generalization of intuitionistic fuzzy sets (IFSs) is initially proposed by Smarandache [1]. However, NSs without being specified are hard to apply in real life. Therefore, many researchers have defined different particular NSs, for instance, single-valued neutrosophic sets (SVNs) [2], simplified neutrosopic sets (SNSs) [3], interval neutrosopic sets (INSs) [4] and multiple-valued neutrosophic sets (MVNSs) [5]. A MVNSs is also called as single-valued neutrosophic hesitant fuzzy sets (SVNHFSs) [6], which is the merging of SVNs and hesitant fuzzy sets (HFSs) [7], and have several values from zero to one for the truth-membership, indeterminacy-membership, and falsity-membership function, respectively, to better depict more complexity decision making information in real life.

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Regarding the characteristics of particular NSs, which have gained many attentions and wildly applied in solving the problem of multiple attribute decision making (MADM). Although different particular NSs have been proposed and developed, they still cannot deal with various kinds of fuzziness in real situation, for example qualitative information can't be depicted by quantitative values. Linguistic variables (LVs) proposed by Zadeh [8] are useful in expressing qualitative information, and a series of works on it have been studied and investigated in recently years [9]–[12]. Some neutrosophic linguistic sets have also been developed and carried out for dealing with MADM problem, containing single-valued neutrosophic linguistic sets (SVNLSs) [13], simplified neutrosophic linguistic sets (SNLSs) [14], interval neutrosophic linguistic sets (INLSs) [15], and multiple-valued neutrosophic linguistic sets (MVNLSs) [16]. In above neutrosophic linguistic sets, the linguistic part is the certain linguistic not uncertain linguistic, which can be better to address qualitative information. Subsequently, some researchers defined single-valued neutrosophic uncertain linguistic sets (SVNULSs) [17], simplified neutrosophic uncertain linguistic sets (SNULSs) [18], interval neutrosophic uncertain linguistic sets (INULSs) [19], which is as an extension of the SVNLSs, SNLSs and INLSs respectively. To be better express people's hesitant, MVNLSs defined by Li contain a linguistic variable and a multiple-valued neutrosophic part, the linguistic part of the multiple-valued neutrosophic linguistic number (MVNLN) is represented by certain linguistic variable, and the multiple-valued neutrosophic part for the MVNLN is the evaluation values regarding the given linguistic variable, while the MVNLSs can't better represent the ambiguity of people's judgment to the evaluated object. To overcome this limitation, we can utilize an uncertain linguistic variable instead of the linguistic variable for MVNLSs. Therefore, in this paper, the concepts of the MVNULSs and the multiple-valued neutrosophic uncertain linguistic numbers (MVNULNs) are proposed, which contains two part of uncertain linguistic and multiple-valued neutrosophic. Thus, the MVNULSs will better depict the complexity of the evaluated object than MVNLSs.

Aggregation operator merging many input values into a synthesis value is an effective way and has become the research hotspot in the filed MADM. Some agile aggregation operators have been defined and conducted by scholars, including the weighting arithmetic average (WAA) aggregating operator [20], the weighting geometric average (WGA) aggregating operator [21], power average (PA) aggregating operator [22], Heronian mean (HM) aggregating operator [23], Maclaurin symmetric mean (MSM) aggregating operator [24], Muirhead mean (MM) operator [25], Bonferroni mean (BM) operator [26], Hamy mean (HM) operator [53] and so on. As we known, different aggregation operator has different function form. For example, BM is initially proposed by Bonferroni, which can take into account the interrelationship of massive input arguments. This desirable characteristic of BM operator has attracted many researchers extend it to fuzzy sets, and different forms of BM operators have also been developed and applied in solving MADM and MAGDM problems, such as the generalized Bonferroni mean(GBM) [27], generalized weighted Bonferroni mean (GWBM) [28], the normalized weighted BM (NWBM) [29]. The NWBM operator proposed by zhou has the advantages of reflecting the weight and interrelationship of the input arguments and satisfying desirable properties of the aggregation operation, which has been applied to merge multiple values for intuitionistic fuzzy sets [29], Pythagorean fuzzy sets [30] ,single-valued neutrosophic sets [31], [32], simplified neutrosophic linguistic sets [14], multiple-valued neutrosophic linguistic sets [16], but it fail to handle the MVNULSs.

In fact, different aggregation operators are based on different t-conorm and t-norm, for instance, algebraic operations [33], [34], einstein operations [35]–[37], hamacher operations [16] [38]–[40], and dombi operations [41]. Dombi operations are more flexibility in the aggregating process because of a general parameter, some scholars presented a few aggregating operators in view of dombi operation, for example Zhang [42] proposed Heronian mean aggregating operator on the basis of dombi operational rules to manage picture fuzzy numbers, Liu [43] developed dombi Bonferroni mean operators to fuse intuitionistic fuzzy elements in MAGDM problem, He [44] defined dombi aggregating operators with hesitant information, He [45] established a group decision making approach utilizing dombi operations to aggregate interval-valued intuitionistic fuzzy elements and applied it to personnel evaluation, Li [46] investigated dombi prioritized weighted aggregation operators for MAGDM problem under intuitionistic fuzzy environment, Chen [47] studied dombi weighted aggregating operator under single-valued neutrosophic environment. Wei [48] explored dombi prioritized weighted aggregating operator with single-valued neutrosophic information, Khan [49] utilized dombi power bonferroni mean operators for solving MADM with interval neutrosophic information. Shi [50] extended the Dombi operations to neutrosophic cubic sets and introduced dombi weighted arithmetic average operator and Dombi weighted geometric average operator under neutrosophic cubic Sets.

From the existing achievements, we know these proposed aggregating operators based on dombi opertaions have been applied to fuse intuitionistic elements, hesitant elements, single-valued neutrosophic elements, interval neutrosophic elements, and neutrosophic cubic elements. However, it has not been utilized to aggregate multiple-valued neutrosophic information and multiple-valued neutrosophic uncertain linguistic information. There has been no research on the utilization of combing NWBM operator with Dombi operations to aggregate MVNULNs information. Thus, it is necessary to extend NWBM based on Dombi operations to MVNULNs.

In sum, the Dombi NWBM operator has the following characteristics. Firstly, it is more flexibility with general parameters. Meanwhile, it takes into account both the relationship and the weight of multiple arguments. In addition, MVNULSs are more suitable to handle quantitative and qualitative information in solving MADM and MAGDM problems. Considering these advantages, therefore, the goal of the paper is as follows.

(1) The MVNS and ULV are combined to easily express people ambiguous in real decision making, and we define the concepts of MVNULS and MVNULN. Moreover, three functions comparing MVNULNs are also given.

(2) Since the flexibility regarding dombi operations, we develop novel operational rules for MVNULNs on the basis of dombi operations.

(3) The NWBM aggregating operator regarding the interactive relationships among different arguments has attracted many attentions, we expand NWBM aggregating operator based on dombi operations to MVNULN environment, and the DNWBM operator is defined. In addition, some desirable characteristics of the proposed aggregation operator are as well discussed.

(4) To demonstrate the practicality, an illustrative instance of MADM using DNWBM aggregating operator under multiple-valued neutrosophic uncertain linguistic environment is shown. In addition, the comparative analysis and sensitivity analysis are also conducted. The instance demonstrates the generalization and flexibility of the proposed operator.

The remainder of the paper is arranged as follows. We briefly introduce some concepts relating to MVNS, ULV, NWBM and Dombi operations in Section 2. Section 3 develops the concepts for MVNULS and MVNULN, and investigates operational rules for MVNULNs based on Dombi operations. In addition, the comparison approach of MVNULNs is as well given. We explore the MVNULDNWBM operator and discuss some desirable properties in Section 4. In Section 5, we present the MADM method based on the novel aggregating operator under MVNUL environment. In Section 6, we demonstrate an illustrative example to verify the generalization of the proposed approach, as well as comparative analysis and sensitivity analysis. Eventually, conclusions are shown and future work is given in Section 7.

II. PRELIMINARIES

A. MULTIPLE-VALUED NEUTROSOPHIC SETS

Definition 1. [5] Suppose X be a collection of objects, MVNSs A on X is denoted as

$$A=\left\{\left\langle x,\tilde{T}_A(x),\tilde{I}_A(x),\tilde{F}_A(x)\right\rangle \middle| x\in X\right\},$$

where $\tilde{T}_A(x)=\left\{\gamma\,\middle|\gamma\in\tilde{T}_A(x)\right\}$, $\tilde{I}_A(x)=\left\{\delta\,\middle|\delta\in\tilde{I}_A(x)\right\}$, $\tilde{F}_A(x)=\left\{\eta\,\middle|\eta\in\tilde{F}_A(x)\right\}$,

$\tilde{T}_A(x)$, $\tilde{I}_A(x)$, and $\tilde{F}_A(x)$ are three collections of real values between 0 and 1, indicating the degrees of truth-membership, indeterminacy-membership and falsity-membership, respectively, where x in X belonging to A, meeting the following conditions $0\le\gamma,\delta,\eta\le 1$, and $0\le\sup\tilde{T}_A(x)+\sup\tilde{I}_A(x)+\tilde{F}_A(x)\le 3$. If it contains only one value in X, A is represented by the tuple $A=\left\langle\tilde{T}_A(x),\tilde{I}_A(x),\tilde{F}_A(x)\right\rangle$ which is called as a multiple-valued neutrosophic number (MVNN). Commonly, MVNSs is a generalization of FSs, IFSs, HFSs, and SVNSs.

B. UNCERTAIN LINGUISTIC TERM SETS

Let $S=\{s_1,s_2,\cdots,s_\iota\}$ be a finite and ordered linguistic term set, where s_j indicates a semantic variable and ι represents an odd element number. Supposing ι is equal to seven, then the relating linguistic set is given in the following:

$$\begin{aligned}S&=\{s_1,s_2,s_3,s_4,s_5,s_6,s_7\}\\&=\{\text{extremely poor, very poor, poor, medium,}\\&\quad\ \text{good, very good, extremely good}\}.\end{aligned}$$

In order to avoid the loss of relating information in the calculating procedure, the above discrete set can be extended to a contiguous set, $\bar{S}=\{s_\alpha\,|\alpha\in R\}$.

Definition 2 [51]: Suppose $\tilde{s}=[s_a,s_b]$, where $a\le b$ and $s_a,s_b\in\bar{S}$, s_a is the lower boundary, s_b is the upper boundary of $\tilde{s}$, and $\tilde{s}$ is called as uncertain linguistic variable.

Let $\tilde{s}_i=[s_{i1},s_{i2}]$ and $\tilde{s}_j=[s_{j1},s_{j2}]$ be two uncertain linguistic variables, the corresponding operations are presented as below:

$$(1)\lambda\tilde{s}_i=[s_{\lambda\times i1},s_{\lambda\times i2}];\ \lambda\ge 0;$$
$$(2)\tilde{s}_i\oplus\tilde{s}_j=[s_{i1+j1},s_{i2+j2}];$$
$$(3)\tilde{s}_i\otimes\tilde{s}_j=[s_{i1\times j1},s_{i2\times j2}];$$
$$(4)\ (\tilde{s}_i)^\lambda=[s_{(i1)^\lambda},s_{(i2)^\lambda}];\ \lambda\ge 0;$$
$$(5)\tilde{s}_i/\tilde{s}_j=[s_{i1/j1},s_{i2/j2}];\ j_1\ne 0,j_2\ne 0.$$

C. DOMBI OPERATIONS

It is worthy to notice that different aggregating operators are in the light of different t-conorm and t-norms operations.

Definition 3 [41]: Let g and h be any two real numbers. Then, the Dombi t-norms and t-conorms between g and h can be explained as follows:

$$D(g,h)=\frac{1}{1+\left\{\left(\frac{1-g}{g}\right)^\rho+\left(\frac{1-h}{h}\right)^\rho\right\}^{1/\rho}};$$

$$D(g,h)=1-\frac{1}{1+\left\{\left(\frac{g}{1-g}\right)^\rho+\left(\frac{h}{1-h}\right)^\rho\right\}^{1/\rho}},$$

where $\rho\ge 0$ and $(g,h)\in[0,1]\times[0,1]$.

D. NORMALIZED WEIGHTED BONFERRONI MEAN

Definition 4 [29]: Let $p,q\ge 0$, and $a_i(i=1,2,\cdots n)$ be a set of nonnegative values, and the corresponding NWBM can be expressed as below:

$$NWBM^{p,q}(a_1,a_2,\cdots,a_n)=\left(\sum_{\substack{i,j=1,\\ i\ne j}}^{n}\frac{w_iw_j}{1-w_i}(a_i^pa_j^q)\right)^{\frac{1}{p+q}}$$

where $w = (w_1, w_2, \cdots, w_n)$ represents the corresponding weighted value for a_i $(i = 1, 2, \cdots, n)$, meeting $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$. The weight vector can be given according to the preference of decision-makers in actual problem.

Apparently, the NWBM aggregating operator has some characteristics, for instance idempotency, commutativity, boundedness, monotonicity, and reducibility.

III. MULTIPLE-VALUED NEUTROSOPHIC UNCERTAIN LINGUISTIC SET

A. MVNULS AND ITS DOMBI OPERATIONS

Definition 5: Let X be a collection of objects, the MVNULS A in X is presented as below:

$$A = \left\{ \left\langle x, \left[s_{\theta(x)}, s_{\sigma(x)} \right], \left(\tilde{T}_A(x), \tilde{I}_A(x), \tilde{F}_A(x) \right) \right\rangle | x \in X \right\},$$

where $s_{\theta(x)}, s_{\sigma(x)} \in S$, $\tilde{T}_A(x) = \left\{ \gamma \left| \gamma \in \tilde{T}_A(x) \right. \right\}$, $\tilde{I}_A(x) = \left\{ \delta \left| \delta \in \tilde{I}_A(x) \right. \right\}$, $\tilde{F}_A(x) = \left\{ \eta \left| \eta \in \tilde{F}_A(x) \right. \right\}$, $\tilde{T}_A(x), \tilde{I}_A(x)$, and $\tilde{F}_A(x)$ are three sets with numbers between 0 and 1, indicating degrees of the true, indeterminacy and false-membership for x in X belonging to the uncertain linguistic variable $\left[s_{\theta(x)}, s_{\sigma(x)} \right]$, meeting the following conditions $0 \leq \gamma, \delta, \eta \leq 1$, and $0 \leq \sup \tilde{T}_A(x) + \sup \tilde{I}_A(x) + \sup \tilde{F}_A(x) \leq 3$.

Definition 6: Suppose $A = \left\{ \left\langle x, \left[s_{\theta(x)}, s_{\sigma(x)} \right], \left(\tilde{T}_A(x), \tilde{I}_A(x), \tilde{F}_A(x) \right) \right\rangle | x \in X \right\}$ be an MVNULS, and X contains only one element, then tuple $\left\langle \left[s_{\theta(x)}, s_{\sigma(x)} \right], \left(\tilde{T}_A(x), \tilde{I}_A(x), \tilde{F}_A(x) \right) \right\rangle$ can be denoted by a multiple-valued neutrosophic uncertain linguistic number (MVNULN). Thus, the MVNULN can be presented as follows:

$$A = \left\{ \left\langle \left[s_{\theta(x)}, s_{\sigma(x)} \right], \left(\tilde{T}_A(x), \tilde{I}_A(x), \tilde{F}_A(x) \right) \right\rangle | x \in X \right\}$$

Definition 7: Let $a_1 = \left\langle \left[s_{\theta(a_1)}, s_{\sigma(a_1)} \right], \left(\tilde{T}(a_1), \tilde{I}(a_1), \tilde{F}(a_1) \right) \right\rangle$ and $a_2 = \left\langle \left[s_{\theta(a_2)}, s_{\sigma(a_2)} \right], \left(\tilde{T}(a_2), \tilde{I}(a_2), \tilde{F}(a_2) \right) \right\rangle$ be two MVNULNs, and $\lambda > 0$, then the operations for MVNULNs are defined on the basis of Dombi operations.

$$(1)\ a_1 \oplus a_2 = \Bigg\langle \left[s_{\theta(a_1)+\theta(a_2)}, s_{\sigma(a_1)+\sigma(a_2)} \right], \Bigg(\cup_{\gamma_1 \in \tilde{T}(a_1), \gamma_2 \in \tilde{T}(a_2)} \left\{ 1 - \frac{1}{1 + \left\{ \left(\frac{\gamma_1}{1-\gamma_1} \right)^{\rho} + \left(\frac{\gamma_2}{1-\gamma_2} \right)^{\rho} \right\}^{1/\rho}} \right\},$$
$$\cup_{\delta_1 \in \tilde{I}(a_1), \delta_2 \in \tilde{I}(a_2)} \left\{ \frac{1}{1 + \left\{ \left(\frac{1-\delta_1}{\delta_1} \right)^{\rho} + \left(\frac{1-\delta_2}{\delta_2} \right)^{\rho} \right\}^{1/\rho}} \right\},$$
$$\cup_{\eta_1 \in \tilde{F}(a_1), \eta_2 \in \tilde{F}(a_2)} \left\{ \frac{1}{1 + \left\{ \left(\frac{1-\eta_1}{\eta_1} \right)^{\rho} + \left(\frac{1-\eta_2}{\eta_2} \right)^{\rho} \right\}^{1/\rho}} \right\} \Bigg) \Bigg\rangle;$$

$$(2)\ a_1 \otimes a_2 = \Bigg\langle \left[s_{\theta(a_1)\times\theta(a_2)}, s_{\sigma(a_1)\times\sigma(a_2)} \right], \Bigg(\cup_{\gamma_1 \in \tilde{T}(a_1), \gamma_2 \in \tilde{T}(a_2)} \left\{ \frac{1}{1 + \left\{ \left(\frac{1-\gamma_1}{\gamma_1} \right)^{\rho} + \left(\frac{1-\gamma_2}{\gamma_2} \right)^{\rho} \right\}^{1/\rho}} \right\},$$
$$\cup_{\delta_1 \in \tilde{I}(a_1), \delta_2 \in \tilde{I}(a_2)} \left\{ 1 - \frac{1}{1 + \left\{ \left(\frac{\delta_1}{1-\delta_1} \right)^{\rho} + \left(\frac{\delta_2}{1-\delta_2} \right)^{\rho} \right\}^{1/\rho}} \right\},$$
$$\cup_{\eta_1 \in \tilde{F}(a_1), \eta_2 \in \tilde{F}(a_2)} \left\{ 1 - \frac{1}{1 + \left\{ \left(\frac{\eta_1}{1-\eta_1} \right)^{\rho} + \left(\frac{\eta_2}{1-\eta_2} \right)^{\rho} \right\}^{1/\rho}} \right\} \Bigg) \Bigg\rangle;$$

$$(3)\ \lambda a_1 = \Bigg\langle \left[s_{\lambda\theta(a_1)}, s_{\lambda\sigma(a_1)} \right], \Bigg(\cup_{\gamma_1 \in \tilde{T}(a_1)} \left\{ 1 - \frac{1}{1 + \left\{ \lambda \left(\frac{\gamma_1}{1-\gamma_1} \right)^{\rho} \right\}^{1/\rho}} \right\},$$
$$\cup_{\delta_1 \in \tilde{I}(a_1)} \left\{ \frac{1}{1 + \left\{ \lambda \left(\frac{1-\delta_1}{\delta_1} \right)^{\rho} \right\}^{1/\rho}} \right\},$$
$$\cup_{\eta_1 \in \tilde{F}(a_1)} \left\{ \frac{1}{1 + \left\{ \lambda \left(\frac{1-\eta_1}{\eta_1} \right)^{\rho} \right\}^{1/\rho}} \right\} \Bigg) \Bigg\rangle;$$

$$(4)\ a_1^{\lambda} = \Bigg\langle \left[s_{\theta^{\lambda}(a_1)}, s_{\sigma^{\lambda}(a_1)} \right], \Bigg(\cup_{\gamma_1 \in \tilde{T}(a_1)} \left\{ \frac{1}{1 + \left\{ \lambda \left(\frac{1-\gamma_1}{\gamma_1} \right)^{\rho} \right\}^{1/\rho}} \right\},$$
$$\cup_{\delta_1 \in \tilde{I}(a_1)} \left\{ 1 - \frac{1}{1 + \left\{ \lambda \left(\frac{\delta_1}{1-\delta_1} \right)^{\rho} \right\}^{1/\rho}} \right\},$$
$$\cup_{\eta_1 \in \tilde{F}(a_1)} \left\{ 1 - \frac{1}{1 + \left\{ \lambda \left(\frac{\eta_1}{1-\eta_1} \right)^{\rho} \right\}^{1/\rho}} \right\} \Bigg) \Bigg\rangle.$$

If there is only one value in $\tilde{T}(a_1), \tilde{I}(a_1), \tilde{F}(a_1), \tilde{T}(a_2), \tilde{I}(a_2)$, and $\tilde{F}(a_2)$, then the operational laws on the basis of Dombi operations in Definition 7 are reduced to the operational laws of single-valued neutrosophic uncertain linguistic numbers (SVNULNs) on the basis of Dombi t-connorm and t-norm in the following:

$$(5)\ a_1 \oplus a_2 = \left\langle \left[s_{\theta(a_1)+\theta(a_2)}, s_{\sigma(a_1)+\sigma(a_2)}\right], \left(1-\frac{1}{1+\left\{\left(\frac{\gamma_1}{1-\gamma_1}\right)^{\rho}+\left(\frac{\gamma_2}{1-\gamma_2}\right)^{\rho}\right\}^{1/\rho}}, \frac{1}{1+\left\{\left(\frac{1-\delta_1}{\delta_1}\right)^{\rho}+\left(\frac{1-\delta_2}{\delta_2}\right)^{\rho}\right\}^{1/\rho}}, \frac{1}{1+\left\{\left(\frac{1-\eta_1}{\eta_1}\right)^{\rho}+\left(\frac{1-\eta_2}{\eta_2}\right)^{\rho}\right\}^{1/\rho}} \right) \right\rangle;$$

$$(6)\ a_1 \otimes a_2 = \left\langle \left[s_{\theta(a_1)\times\theta(a_2)}, s_{\sigma(a_1)\times\sigma(a_2)}\right], \left(\frac{1}{1+\left\{\left(\frac{1-\gamma_1}{\gamma_1}\right)^{\rho}+\left(\frac{1-\gamma_2}{\gamma_2}\right)^{\rho}\right\}^{1/\rho}}, 1-\frac{1}{1+\left\{\left(\frac{\delta_1}{1-\delta_1}\right)^{\rho}+\left(\frac{\delta_2}{1-\delta_2}\right)^{\rho}\right\}^{1/\rho}}, 1-\frac{1}{1+\left\{\left(\frac{\eta_1}{1-\eta_1}\right)^{\rho}+\left(\frac{\eta_2}{1-\eta_2}\right)^{\rho}\right\}^{1/\rho}} \right) \right\rangle;$$

$$(7)\ \lambda a_1 = \left\langle \left[s_{\lambda\theta(a_1)}, s_{\lambda\sigma(a_1)}\right], \left(1-\frac{1}{1+\left\{\lambda\left(\frac{\gamma_1}{1-\gamma_1}\right)^{\rho}\right\}^{1/\rho}}, \frac{1}{1+\left\{\lambda\left(\frac{1-\delta_1}{\delta_1}\right)^{\rho}\right\}^{1/\rho}}, \frac{1}{1+\left\{\lambda\left(\frac{1-\eta_1}{\eta_1}\right)^{\rho}\right\}^{1/\rho}} \right) \right\rangle;$$

$$(8)\ a_1^{\lambda} = \left\langle \left[s_{\theta^{\lambda}(a_1)}, s_{\sigma^{\lambda}(a_1)}\right], \left(\frac{1}{1+\left\{\lambda\left(\frac{1-\gamma_1}{\gamma_1}\right)^{\rho}\right\}^{1/\rho}}, 1-\frac{1}{1+\left\{\lambda\left(\frac{\delta_1}{1-\delta_1}\right)^{\rho}\right\}^{1/\rho}}, 1-\frac{1}{1+\left\{\lambda\left(\frac{\eta_1}{1-\eta_1}\right)^{\rho}\right\}^{1/\rho}} \right) \right\rangle.$$

Theorem 1: Let $a_1 = \left\langle \left[s_{\theta(a_1)}, s_{\sigma(a_1)}\right], \left(\tilde{T}(a_1), \tilde{I}(a_1), \tilde{F}(a_1)\right) \right\rangle a_2 = \left\langle \left[s_{\theta(a_2)}, s_{\sigma(a_2)}\right], \left(\tilde{T}(a_2), \tilde{I}(a_2), \tilde{F}(a_2)\right) \right\rangle$ and $a_3 = \left\langle \left[s_{\theta(a_3)}, s_{\sigma(a_3)}\right], \left(\tilde{T}(a_3), \tilde{I}(a_3), \tilde{F}(a_3)\right) \right\rangle$ be any three MVNULNs, and $\lambda, \lambda_1, \lambda_2 > 0$, then the equations below are true.

(1) $a_1 \oplus a_2 = a_2 \oplus a_1$;
(2) $a_1 \otimes a_2 = a_2 \otimes a_1$;
(3) $\lambda (a_1 \oplus a_2) = \lambda a_1 \oplus \lambda a_2$;
(4) $\lambda_1 a_1 \oplus \lambda_2 a_1 = (\lambda_1 + \lambda_2) a_1$;
(5) $a_1^{\lambda_1} \otimes a_1^{\lambda_2} = a_1^{\lambda_1+\lambda_2}$;
(6) $a_1^{\lambda} \otimes a_2^{\lambda} = (a_1 \otimes a_2)^{\lambda}$;
(7) $(a_1 \oplus a_2) \oplus a_3 = a_1 \oplus (a_2 \oplus a_3)$;
(8) $(a_1 \otimes a_2) \otimes a_3 = a_1 \otimes (a_2 \otimes a_3)$.

Then, equation (3) will be proved in the following:

Proof: (3) Since $\lambda > 0$, $\lambda (a_1 \oplus a_2)$, as shown at the bottom of the next page. And $\lambda a_1 \oplus \lambda a_2$, as shown at the bottom of the 7th page.

Therefore, equation (3) $\lambda (a_1 \oplus a_2) = \lambda a_1 \oplus \lambda a_2$ can be obtained.

Similarly, the other equations in Theorem 1 can be proved based on Dombi operations in Definition 7.

B. COMPARED METHOD

To rank MVNULNs, the three functions of score, accuracy, and certainty play an important criteria, and its concept is presented as below:

Definition 8: Let $a = \left\langle \left[s_{\theta(a)}, s_{\sigma(a)}\right], \left(\tilde{T}(a), \tilde{I}(a), \tilde{F}(a)\right) \right\rangle$ be an MVNULN, and the three functions of score, accuracy, and certainty can be calculated in the following.

$$(1)\ \mathrm{E}(a) = \left(\frac{1}{\iota_{\tilde{T}(a)}\iota_{\tilde{I}(a)}\iota_{\tilde{F}(a)}} \sum_{\gamma\in\tilde{T}(a),\delta\in\tilde{I}(a),\eta\in\tilde{F}(a)} \times \left(\frac{\gamma+1-\delta+1-\eta}{3}\right) \right) \times s_{\frac{\theta(a)+\rho(a)}{2}} = s_{\left(\frac{1}{\iota_{\tilde{T}(a)}\iota_{\tilde{I}(a)}\iota_{\tilde{F}(a)}} \sum_{\gamma\in\tilde{T}(a),\delta\in\tilde{I}(a),\eta\in\tilde{F}(a)} \left(\frac{\gamma+1-\delta+1-\eta}{3}\right)\right)\cdot\frac{\theta(a)+\rho(a)}{2}}$$

$$(2)\ \mathrm{H}(a) = \left(\frac{1}{\iota_{\tilde{T}(a)}\iota_{\tilde{F}(a)}} \sum_{\gamma\in\tilde{T}(a),\eta\in\tilde{F}(a)} (\gamma-\eta) \right) \times s_{\frac{\theta(a)+\rho(a)}{2}} = s_{\left(\frac{1}{\iota_{\tilde{T}(a)}\iota_{\tilde{F}(a)}} \sum_{\gamma\in\tilde{T}(a),\eta\in\tilde{F}(a)} (\gamma-\eta)\right)\cdot\frac{\theta(a)+\rho(a)}{2}}$$

$$(3)\ \mathrm{C}(a) = \left(\frac{1}{\iota_{\tilde{T}(a)}} \sum_{\gamma\in\tilde{T}(a)} \gamma \right) \times s_{\frac{\theta(a)+\rho(a)}{2}} = s_{\left(\frac{1}{\iota_{\tilde{T}(a)}} \sum_{\gamma\in\tilde{T}(a)} \gamma\right)\cdot\frac{\theta(a)+\rho(a)}{2}}$$

where $\iota_{\tilde{T}(a)}$, $\iota_{\tilde{I}(a)}$, and $\iota_{\tilde{F}(a)}$ is the number of the values belonging to $\tilde{T}_A(x)$, $\tilde{I}_A(x)$, and $\tilde{F}_A(x)$ respectively.

The compared functions given in Definition 8 are represented by the linguistic variable due to its important for an MVNULN. The higher the truth-degree of $\tilde{T}(a)$ relating to uncertain linguistic variable $\left[s_{\theta(a)}, s_{\sigma(a)}\right]$ is, meanwhile, the lower the indeterminacy-degree of $\tilde{I}(a)$ as well as the false -degree of $\tilde{F}(a)$ relating to the uncertain linguistic variable $\left[s_{\theta(a)}, s_{\sigma(a)}\right]$ are, then the greater the MVNULN is. If the higher the difference between the truth-degree of $\tilde{T}(a)$ and the false-degree of $\tilde{F}(a)$ corresponding to $\left[s_{\theta(a)}, s_{\sigma(a)}\right]$ is, the greater the affirmative statement is. The higher the truth -degree of $\tilde{T}(a)$ is, the bigger the certainty of the statement is. Thus, the bigger the functions of score, accuracy, and certainty are, the greater the MVNULN.

The compared method for MVNULNs can be concluded according to Definition 8.

Definition 9: Let a_1 and a_2 be any two MVNULNs, the comparative method is presented in the following:

Supposing that $E(a_1) > E(a_2)$, then a_1 is greater than a_2, represented as $a_1 \succ a_2$.

Supposing that $E(a_1) = E(a_2)$, and $H(a_1) > H(a_2)$, then a_1 is greater than a_2, represented as $a_1 \succ a_2$.

Supposing that $E(a_1) = E(a_2)$, $H(a_1) = H(a_2)$, and $C(a_1) > C(a_2)$, then a_1 is greater than a_2, represented as $a_1 \succ a_2$.

Supposing that $E(a_1) = E(a_2)$, $H(a_1) = H(a_2)$, and $C(a_1) = C(a_2)$, then a_1 equals a_2, represented as $a_1 \sim a_2$.

IV. THE MULTIPLE-VALUED NEUTROSOPHIC UNCERTAIN LINGUISTIC DOMBI NORMALIZED WEIGHTED BONFERRONI MEAN OPERATOR

The NWBM aggregating operator has many desirable characteristics. However, it cannot handle the multiple-valued neutrosophic uncertain linguistic information. Thus, in this section, we will expand NWBM aggregating operator to MVNULN environment.

Definition 10: Let $a_i (i = 1, 2, \cdots, n)$ be a collection of MVNULNs,

$$a_i = \left\langle \left[s_{\theta(a_i)}, s_{\sigma(a_i)}\right], \left(\tilde{T}(a_i), \tilde{I}(a_i), \tilde{F}(a_i)\right)\right\rangle, \quad p, q \geq 0,$$

$$\begin{aligned}
\lambda\left(a_1 \oplus a_2\right) = \Bigg\langle & \left[s_{\lambda(\theta(a_1)+\theta(a_2))}, s_{\lambda(\sigma(a_1)+\sigma(a_2))}\right], \\
& \left(\cup_{\gamma_1 \in \tilde{T}(a_1), \gamma_2 \in \tilde{T}(a_2)} \left\{1 - 1 \Bigg/ \left(1 + \left\{\lambda\left(\frac{1 - \frac{1}{1+\left\{\left(\frac{\gamma_1}{1-\gamma_1}\right)^{\rho}+\left(\frac{\gamma_2}{1-\gamma_2}\right)^{\rho}\right\}^{1/\rho}}}{1 - \left(1 - \frac{1}{1+\left\{\left(\frac{\gamma_1}{1-\gamma_1}\right)^{\rho}+\left(\frac{\gamma_2}{1-\gamma_2}\right)^{\rho}\right\}^{1/\rho}}\right)}\right)^{\rho}\right\}^{1/\rho}\right)\right\},\right. \\
& \cup_{\delta_1 \in \tilde{I}(a_1), \delta_2 \in \tilde{I}(a_2)} \left\{1 \Bigg/ \left(1 + \left\{\lambda\left(\frac{1 - \frac{1}{1+\left\{\left(\frac{1-\delta_1}{\delta_1}\right)^{\rho}+\left(\frac{1-\delta_2}{\delta_2}\right)^{\rho}\right\}^{1/\rho}}}{\frac{1}{1+\left\{\left(\frac{1-\delta_1}{\delta_1}\right)^{\rho}+\left(\frac{1-\delta_2}{\delta_2}\right)^{\rho}\right\}^{1/\rho}}}\right)^{\rho}\right\}^{1/\rho}\right)\right\}, \\
& \left.\cup_{\eta_1 \in \tilde{F}(a_1), \eta_2 \in \tilde{F}(a_2)} \left\{1 \Bigg/ \left(1 + \left\{\lambda\left(\frac{1 - \frac{1}{1+\left\{\left(\frac{1-\eta_1}{\eta_1}\right)^{\rho}+\left(\frac{1-\eta_2}{\eta_2}\right)^{\rho}\right\}^{1/\rho}}}{\frac{1}{1+\left\{\left(\frac{1-\eta_1}{\eta_1}\right)^{\rho}+\left(\frac{1-\eta_2}{\eta_2}\right)^{\rho}\right\}^{1/\rho}}}\right)^{\rho}\right\}^{1/\rho}\right)\right\}\right)\Bigg\rangle \\
= \Bigg\langle & \left[s_{\lambda\theta(a_1)+\lambda\theta(a_2)}, s_{\lambda\sigma(a_1)+\lambda\sigma(a_2)}\right], \\
& \left(\cup_{\gamma_1 \in \tilde{T}(a_1), \gamma_2 \in \tilde{T}(a_2)} \left\{1 - 1 \Bigg/ \left(1 + \left\{\lambda\left(\frac{\gamma_1}{1-\gamma_1}\right)^{\rho} + \lambda\left(\frac{\gamma_2}{1-\gamma_2}\right)^{\rho}\right\}^{1/\rho}\right)\right\},\right. \\
& \cup_{\delta_1 \in \tilde{I}(a_1), \delta_2 \in \tilde{I}(a_2)} \left\{1 \Bigg/ \left(1 + \left\{\lambda\left(\frac{1-\delta_1}{\delta_1}\right)^{\rho} + \lambda\left(\frac{1-\delta_2}{\delta_2}\right)^{\rho}\right\}^{1/\rho}\right)\right\}, \\
& \left.\cup_{\eta_1 \in \tilde{F}(a_1), \eta_2 \in \tilde{F}(a_2)} \left\{1 \Bigg/ \left(1 + \left\{\lambda\left(\frac{1-\eta_1}{\eta_1}\right)^{\rho} + \lambda\left(\frac{1-\eta_2}{\eta_2}\right)^{\rho}\right\}^{1/\rho}\right)\right\}\right)\Bigg\rangle
\end{aligned}$$

and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ be the weighted vector for a_i, $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \omega_i = 1$. Then the aggregated operator of MVNULDNWBM can be derived in the following, the aggregating result is still an MVNULN.

$$MVNULDNWBM(a_1, a_2, \cdots a_n) = \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} (a_i^p \otimes a_j^q) \right)^{\frac{1}{p+q}}$$

Based on the operations of Definition 7, the result is achieved in the following (1), as shown at the bottom of the next page.

Proof: Based on the operations of MVNULNs with Dombi operations in Definition 7, we can gain the following equation.

$$a_i^p = \left\langle \left[s_{\theta^p(a_i)}, s_{\sigma^p(a_i)} \right], \left(\cup_{\gamma_i \in \tilde{T}(a_i)} \left\{ \frac{1}{1 + \left\{ p \left(\frac{1-\gamma_i}{\gamma_i} \right)^{\rho} \right\}^{1/\rho}} \right\}, \cup_{\delta_i \in \tilde{I}(a_i)} \left\{ 1 - \frac{1}{1 + \left\{ p \left(\frac{\delta_i}{1-\delta_i} \right)^{\rho} \right\}^{1/\rho}} \right\}, \cup_{\eta_i \in \tilde{F}(a_i)} \left\{ 1 - \frac{1}{1 + \left\{ p \left(\frac{\eta_i}{1-\eta_i} \right)^{\rho} \right\}^{1/\rho}} \right\} \right) \right\rangle.$$

$$\begin{aligned}
&\lambda a_1 \oplus \lambda a_2 \\
&= \left\langle \left[s_{\lambda\theta(a_1)+\lambda\theta(a_2)}, s_{\lambda\sigma(a_1)+\lambda\sigma(a_2)} \right], \right. \\
&\quad \left(\cup_{\gamma_1 \in \tilde{T}(a_1), \gamma_2 \in \tilde{T}(a_2)} \left\{ 1 - 1 \middle/ \left(1 + \left\{ \left(\frac{1 - \frac{1}{1+\left\{\lambda\left(\frac{\gamma_1}{1-\gamma_1}\right)^{\rho}\right\}^{1/\rho}}}{1 - \left(1 - \frac{1}{1+\left\{\lambda\left(\frac{\gamma_1}{1-\gamma_1}\right)^{\rho}\right\}^{1/\rho}}\right)} \right)^{\rho} + \left(\frac{1 - \frac{1}{1+\left\{\lambda\left(\frac{\gamma_2}{1-\gamma_2}\right)^{\rho}\right\}^{1/\rho}}}{1 - \left(1 - \frac{1}{1+\left\{\lambda\left(\frac{\gamma_2}{1-\gamma_2}\right)^{\rho}\right\}^{1/\rho}}\right)} \right)^{\rho} \right\}^{1/\rho} \right) \right\}, \right. \\
&\quad \cup_{\delta_1 \in \tilde{I}(a_1), \delta_2 \in \tilde{I}(a_2)} \left\{ 1 \middle/ \left(1 + \left\{ \left(\frac{1 - \frac{1}{1+\left\{\lambda\left(\frac{1-\delta_1}{\delta_1}\right)^{\rho}\right\}^{1/\rho}}}{\frac{1}{1+\left\{\lambda\left(\frac{1-\delta_1}{\delta_1}\right)^{\rho}\right\}^{1/\rho}}} \right)^{\rho} + \left(\frac{1 - \frac{1}{1+\left\{\lambda\left(\frac{1-\delta_2}{\delta_2}\right)^{\rho}\right\}^{1/\rho}}}{\frac{1}{1+\left\{\lambda\left(\frac{1-\delta_2}{\delta_2}\right)^{\rho}\right\}^{1/\rho}}} \right)^{\rho} \right\}^{1/\rho} \right) \right\}, \\
&\quad \left. \left. \cup_{\eta_1 \in \tilde{F}(a_1), \eta_2 \in \tilde{F}(a_2)} \left\{ 1 \middle/ \left(1 + \left\{ \left(\frac{1 - \frac{1}{1+\left\{\lambda\left(\frac{1-\eta_1}{\eta_1}\right)^{\rho}\right\}^{1/\rho}}}{\frac{1}{1+\left\{\lambda\left(\frac{1-\eta_1}{\eta_1}\right)^{\rho}\right\}^{1/\rho}}} \right)^{\rho} + \left(\frac{1 - \frac{1}{1+\left\{\lambda\left(\frac{1-\eta_2}{\eta_2}\right)^{\rho}\right\}^{1/\rho}}}{\frac{1}{1+\left\{\lambda\left(\frac{1-\eta_2}{\eta_2}\right)^{\rho}\right\}^{1/\rho}}} \right)^{\rho} \right\}^{1/\rho} \right) \right\} \right) \right\rangle \\
&= \left\langle \left[s_{\lambda\theta(a_1)+\lambda\theta(a_2)}, s_{\lambda\sigma(a_1)+\lambda\sigma(a_2)} \right], \right. \\
&\quad \left(\cup_{\gamma_1 \in \tilde{T}(a_1), \gamma_2 \in \tilde{T}(a_2)} \left\{ 1 - 1 \middle/ \left(1 + \left\{ \lambda \left(\frac{\gamma_1}{1-\gamma_1} \right)^{\rho} + \lambda \left(\frac{\gamma_2}{1-\gamma_2} \right)^{\rho} \right\}^{1/\rho} \right) \right\}, \right. \\
&\quad \cup_{\delta_1 \in \tilde{I}(a_1), \delta_2 \in \tilde{I}(a_2)} \left\{ 1 \middle/ \left(1 + \left\{ \lambda \left(\frac{1-\delta_1}{\delta_1} \right)^{\rho} + \lambda \left(\frac{1-\delta_2}{\delta_2} \right)^{\rho} \right\}^{1/\rho} \right) \right\}, \\
&\quad \left. \left. \cup_{\eta_1 \in \tilde{F}(a_1), \eta_2 \in \tilde{F}(a_2)} \left\{ 1 \middle/ \left(1 + \left\{ \lambda \left(\frac{1-\eta_1}{\eta_1} \right)^{\rho} + \lambda \left(\frac{1-\eta_2}{\eta_2} \right)^{\rho} \right\}^{1/\rho} \right) \right\} \right) \right\rangle \\
&= \lambda (a_1 \oplus a_2)
\end{aligned}$$

$$
\begin{aligned}
&MVNULDNWBM(a_1, a_2, \cdots a_n) = \left\langle \left[s_{\left(\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\frac{\omega_i \omega_j}{1-\omega_i} \cdot \theta^p(a_i) \cdot \theta^q(a_j) \right) \right)^{\frac{1}{p+q}}}, s_{\left(\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\frac{\omega_i \omega_j}{1-\omega_i} \cdot \sigma^p(a_i) \cdot \sigma^q(a_j) \right) \right)^{\frac{1}{p+q}}} \right], \right. \\
&\left(\bigcup_{\gamma_i \in \tilde{T}(a_i), \gamma_j \in \tilde{T}(a_j)} \left\{ 1 \Big/ \left(1 + \left(\frac{1}{p+q} \cdot \frac{1}{\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\frac{\omega_i \omega_j}{1-\omega_i} \cdot \frac{1}{p\left(\frac{1-\gamma_i}{\gamma_i}\right)^{\rho} + q\left(\frac{1-\gamma_j}{\gamma_j}\right)^{\rho}} \right)} \right)^{\frac{1}{\rho}} \right) \right\}, \right. \\
&\bigcup_{\delta_i \in \tilde{I}(a_i), \delta_j \in \tilde{I}(a_j)} \left\{ 1 - 1 \Big/ \left(1 + \left(\frac{1}{p+q} \cdot \frac{1}{\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\frac{\omega_i \omega_j}{1-\omega_i} \cdot \frac{1}{p\left(\frac{\delta_i}{1-\delta_i}\right)^{\rho} + q\left(\frac{\delta_j}{1-\delta_j}\right)^{\rho}} \right)} \right)^{\frac{1}{\rho}} \right) \right\}, \\
&\left. \left. \bigcup_{\eta_i \in \tilde{F}(a_i), \eta_j \in \tilde{F}(a_j)} \left\{ 1 - 1 \Big/ \left(1 + \left(\frac{1}{p+q} \cdot \frac{1}{\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\frac{\omega_i \omega_j}{1-\omega_i} \cdot \frac{1}{p\left(\frac{\eta_i}{1-\eta_i}\right)^{\rho} + q\left(\frac{\eta_j}{1-\eta_j}\right)^{\rho}} \right)} \right)^{\frac{1}{\rho}} \right) \right\} \right) \right\rangle \\
&= \left\langle \left[s_{\left(\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\frac{\omega_i \omega_j}{1-\omega_i} \cdot \theta^p(a_i) \cdot \theta^q(a_j) \right) \right)^{\frac{1}{p+q}}}, s_{\left(\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\frac{\omega_i \omega_j}{1-\omega_i} \cdot \sigma^p(a_i) \cdot \sigma^q(a_j) \right) \right)^{\frac{1}{p+q}}} \right], \right. \\
&\left(\bigcup_{\gamma_i \in \tilde{T}(a_i), \gamma_j \in \tilde{T}(a_j)} \left\{ \frac{\left((p+q) \cdot \sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\frac{\omega_i \omega_j}{1-\omega_i} \cdot \frac{1}{p\left(\frac{1-\gamma_i}{\gamma_i}\right)^{\rho} + q\left(\frac{1-\gamma_j}{\gamma_j}\right)^{\rho}} \right) \right)^{\frac{1}{\rho}}}{1 + \left((p+q) \cdot \sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\frac{\omega_i \omega_j}{1-\omega_i} \cdot \frac{1}{p\left(\frac{1-\gamma_i}{\gamma_i}\right)^{\rho} + q\left(\frac{1-\gamma_j}{\gamma_j}\right)^{\rho}} \right) \right)^{\frac{1}{\rho}}} \right\}, \quad \bigcup_{\delta_i \in \tilde{I}(a_i), \delta_j \in \tilde{I}(a_j)} \right. \\
&\left. \left. \left\{ \frac{1}{1 + \left((p+q) \cdot \sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\frac{\omega_i \omega_j}{1-\omega_i} \cdot \frac{1}{p\left(\frac{\delta_i}{1-\delta_i}\right)^{\rho} + q\left(\frac{\delta_j}{1-\delta_j}\right)^{\rho}} \right) \right)^{\frac{1}{\rho}}} \right\}, \bigcup_{\eta_i \in \tilde{F}(a_i), \eta_j \in \tilde{F}(a_j)} \left\{ \frac{1}{1 + \left((p+q) \cdot \sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\frac{\omega_i \omega_j}{1-\omega_i} \cdot \frac{1}{p\left(\frac{\eta_i}{1-\eta_i}\right)^{\rho} + q\left(\frac{\eta_j}{1-\eta_j}\right)^{\rho}} \right) \right)^{\frac{1}{\rho}}} \right\} \right) \right\rangle
\end{aligned}
\tag{1}
$$

$$a_j^q=\left\langle\left[s_{\theta^q(a_j)},s_{\sigma^q(a_j)}\right],\left(\cup_{\gamma_j\in\tilde{T}(a_j)}\left\{\frac{1}{1+\left\{q\left(\frac{1-\gamma_j}{\gamma_j}\right)^{\rho}\right\}^{1/\rho}}\right\},\cup_{\delta_j\in\tilde{I}(a_j)}\left\{1-\frac{1}{1+\left\{q\left(\frac{\delta_j}{1-\delta_j}\right)^{\rho}\right\}^{1/\rho}}\right\},\cup_{\eta_j\in\tilde{F}(a_j)}\left\{1-\frac{1}{1+\left\{q\left(\frac{\eta_j}{1-\eta_j}\right)^{\rho}\right\}^{1/\rho}}\right\}\right)\right\rangle.$$

And

$$a_i^p\otimes a_j^q=\left\langle\left[s_{\theta^p(a_i)\times\theta^q(a_j)},s_{\sigma^p(a_i)\times\sigma^q(a_j)}\right],\right.$$

$$\left(\cup_{\gamma_i\in\tilde{T}(a_i),\gamma_j\in\tilde{T}(a_j)}\left\{1\Bigg/\left(1+\left\{\left(\frac{1-\frac{1}{1+\left\{p\left(\frac{1-\gamma_i}{\gamma_i}\right)^{\rho}\right\}^{1/\rho}}}{\frac{1}{1+\left\{p\left(\frac{1-\gamma_i}{\gamma_i}\right)^{\rho}\right\}^{1/\rho}}}\right)^{\rho}+\left(\frac{1-\frac{1}{1+\left\{q\left(\frac{1-\gamma_j}{\gamma_j}\right)^{\rho}\right\}^{1/\rho}}}{\frac{1}{1+\left\{q\left(\frac{1-\gamma_j}{\gamma_j}\right)^{\rho}\right\}^{1/\rho}}}\right)^{\rho}\right\}^{1/\rho}\right)\right\},\right.$$

$$\cup_{\delta_i\in\tilde{I}(a_i),\delta_j\in\tilde{I}(a_j)}\left\{1-1\Bigg/\left(1+\left\{\left(\frac{1-\frac{1}{1+\left\{p\left(\frac{\delta_i}{1-\delta_i}\right)^{\rho}\right\}^{1/\rho}}}{1-\left(1-\frac{1}{1+\left\{p\left(\frac{\delta_i}{1-\delta_i}\right)^{\rho}\right\}^{1/\rho}}\right)}\right)^{\rho}+\left(\frac{1-\frac{1}{1+\left\{q\left(\frac{\delta_j}{1-\delta_j}\right)^{\rho}\right\}^{1/\rho}}}{1-\left(1-\frac{1}{1+\left\{q\left(\frac{\delta_j}{1-\delta_j}\right)^{\rho}\right\}^{1/\rho}}\right)}\right)^{\rho}\right\}^{1/\rho}\right)\right\},$$

$$\left.\left.\cup_{\eta_i\in\tilde{F}(a_i),\eta_j\in\tilde{F}(a_j)}\left\{1-1\Bigg/\left(1+\left\{\left(\frac{1-\frac{1}{1+\left\{p\left(\frac{\eta_i}{1-\eta_i}\right)^{\rho}\right\}^{1/\rho}}}{1-\left(1-\frac{1}{1+\left\{p\left(\frac{\eta_i}{1-\eta_i}\right)^{\rho}\right\}^{1/\rho}}\right)}\right)^{\rho}+\left(\frac{1-\frac{1}{1+\left\{q\left(\frac{\eta_j}{1-\eta_j}\right)^{\rho}\right\}^{1/\rho}}}{1-\left(1-\frac{1}{1+\left\{q\left(\frac{\eta_j}{1-\eta_j}\right)^{\rho}\right\}^{1/\rho}}\right)}\right)^{\rho}\right\}^{1/\rho}\right)\right\}\right)\right\rangle;$$

$$=\left\langle\left[s_{\theta^p(a_i)\times\theta^q(a_j)},s_{\sigma^p(a_i)\times\sigma^q(a_j)}\right],\left(\cup_{\gamma_i\in\tilde{T}(a_i),\gamma_j\in\tilde{T}(a_j)}\left\{\frac{1}{1+\left(p\left(\frac{1-\gamma_i}{\gamma_i}\right)^{\rho}+q\left(\frac{1-\gamma_j}{\gamma_j}\right)^{\rho}\right)^{1/\rho}}\right\},\cup_{\delta_i\in\tilde{I}(a_i),\delta_j\in\tilde{I}(a_j)}\left\{1-\frac{1}{1+\left(p\left(\frac{\delta_i}{1-\delta_i}\right)^{\rho}+q\left(\frac{\delta_j}{1-\delta_j}\right)^{\rho}\right)^{1/\rho}}\right\},\cup_{\eta_i\in\tilde{F}(a_i),\eta_j\in\tilde{F}(a_j)}\left\{1-\frac{1}{1+\left(p\left(\frac{\eta_i}{1-\eta_i}\right)^{\rho}+q\left(\frac{\eta_j}{1-\eta_j}\right)^{\rho}\right)^{1/\rho}}\right\}\right)\right\rangle$$

(1) At first, the following mathematical equation (2) should be proved.

$$\bigoplus_{\substack{i,j=1\\i\neq j}}^{n}\frac{\omega_i\omega_j}{1-\omega_i}\left(a_i^p\otimes a_j^q\right)=\left\langle\left[s_{\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\theta^p(a_i)\cdot\theta^q(a_j)\right)},s_{\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\sigma^p(a_i)\cdot\sigma^q(a_j)\right)}\right],\right.$$

$$\left(\cup_{\gamma_i\in\tilde{T}(a_i),\gamma_j\in\tilde{T}(a_j)}\left\{1-\frac{1}{1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{1-\gamma_i}{\gamma_i}\right)^{\rho}+q\left(\frac{1-\gamma_j}{\gamma_j}\right)^{\rho}}\right)\right)^{1/\rho}}\right\},\right.$$

$$\cup_{\delta_i\in\tilde{I}(a_i),\delta_j\in\tilde{I}(a_j)}\left\{\frac{1}{1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{\delta_i}{1-\delta_i}\right)^{\rho}+q\left(\frac{\delta_j}{1-\delta_j}\right)^{\rho}}\right)\right)^{1/\rho}}\right\},$$

$$\left.\left.\cup_{\eta_i\in\tilde{F}(a_i),\eta_j\in\tilde{F}(a_j)}\left\{\frac{1}{1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{\eta_i}{1-\eta_i}\right)^{\rho}+q\left(\frac{\eta_j}{1-\eta_j}\right)^{\rho}}\right)\right)^{1/\rho}}\right\}\right)\right\rangle \tag{2}$$

In order to prove Eq.(2), the mathematical induction on n is adopt as below.

When $n = 2$, we can gain the equation below. $\underset{\substack{i,j=1\\i\neq j}}{\overset{2}{\oplus}}$, as shown at the bottom of the next page. Then, If $n = 2$, Eq. (2) is right.

Supposing if $n = k$, Eq. (2) is right, then

$$\underset{\substack{i,j=1\\i\neq j}}{\overset{k}{\oplus}}\frac{\omega_i\omega_j}{1-\omega_i}(a_i^p\otimes a_j^q) = \left\langle \left[s_{\sum\limits_{\substack{i,j=1\\i\neq j}}^{k}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\theta^p(a_i)\cdot\theta^q(a_j)\right)}, s_{\sum\limits_{\substack{i,j=1\\i\neq j}}^{k}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\sigma^p(a_i)\cdot\sigma^q(a_j)\right)}\right], \left(\bigcup_{\gamma_i\in\tilde{T}(a_i),\gamma_j\in\tilde{T}(a_j)}\left\{1-\frac{1}{1+\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{k}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{1-\gamma_i}{\gamma_i}\right)^\rho+q\left(\frac{1-\gamma_j}{\gamma_j}\right)^\rho}\right)\right)^{1/\rho}}\right\}, \bigcup_{\delta_i\in\tilde{I}(a_i),\delta_j\in\tilde{I}(a_j)}\left\{\frac{1}{1+\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{k}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{\delta_i}{1-\delta_i}\right)^\rho+q\left(\frac{\delta_j}{1-\delta_j}\right)^\rho}\right)\right)^{1/\rho}}\right\}, \bigcup_{\eta_i\in\tilde{F}(a_i),\eta_j\in\tilde{F}(a_j)}\left\{\frac{1}{1+\left(\sum\limits_{\substack{i,j=1\\i\neq j}}^{k}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{\eta_i}{1-\eta_i}\right)^\rho+q\left(\frac{\eta_j}{1-\eta_j}\right)^\rho}\right)\right)^{1/\rho}}\right\}\right)\right\rangle. \tag{3}$$

Then, when $n = k + 1$, the following result should be proved.

$$\begin{aligned}\underset{\substack{i,j=1\\i\neq j}}{\overset{k+1}{\oplus}}\frac{\omega_i\omega_j}{1-\omega_i}(a_i^p\otimes a_j^q) &= \underset{\substack{i,j=1\\i\neq j}}{\overset{k}{\oplus}}\frac{\omega_i\omega_j}{1-\omega_i}(a_i^p\otimes a_j^q)\\ &\oplus \underset{i=1}{\overset{k}{\oplus}}\frac{\omega_i\omega_{k+1}}{1-\omega_i}(a_i^p\otimes a_{k+1}^q)\\ &\oplus \underset{j=1}{\overset{k}{\oplus}}\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}(a_{k+1}^p\otimes a_j^q)\end{aligned} \tag{4}$$

Thus, the following equations need to be proved.

$$\underset{i=1}{\overset{k}{\oplus}}\frac{\omega_i\omega_{k+1}}{1-\omega_i}(a_i^p\otimes a_{k+1}^q) = \left\langle \left[s_{\sum\limits_{i=1}^{k}\left(\frac{\omega_i\omega_{k+1}}{1-\omega_i}\cdot\theta^p(a_i)\cdot\theta^q(a_{k+1})\right)}, s_{\sum\limits_{i=1}^{k}\left(\frac{\omega_i\omega_{k+1}}{1-\omega_i}\cdot\sigma^p(a_i)\cdot\sigma^q(a_{k+1})\right)}\right], \left(\bigcup_{\gamma_i\in\tilde{T}(a_i),\gamma_{k+1}\in\tilde{T}(a_{k+1})}\left\{1-\frac{1}{1+\left(\sum\limits_{i=1}^{k}\left(\frac{\omega_i\omega_{k+1}}{1-\omega_i}\cdot\frac{1}{p\left(\frac{1-\gamma_i}{\gamma_i}\right)^\rho+q\left(\frac{1-\gamma_{k+1}}{\gamma_{k+1}}\right)^\rho}\right)\right)^{1/\rho}}\right\}, \bigcup_{\delta_i\in\tilde{I}(a_i),\delta_{k+1}\in\tilde{I}(a_{k+1})}\left\{\frac{1}{1+\left(\sum\limits_{i=1}^{k}\left(\frac{\omega_i\omega_{k+1}}{1-\omega_i}\cdot\frac{1}{p\left(\frac{\delta_i}{1-\delta_i}\right)^\rho+q\left(\frac{\delta_{k+1}}{1-\delta_{k+1}}\right)^\rho}\right)\right)^{1/\rho}}\right\}, \bigcup_{\eta_i\in\tilde{F}(a_i),\eta_{k+1}\in\tilde{F}(a_{k+1})}\left\{\frac{1}{1+\left(\sum\limits_{i=1}^{k}\left(\frac{\omega_i\omega_{k+1}}{1-\omega_i}\cdot\frac{1}{p\left(\frac{\eta_i}{1-\eta_i}\right)^\rho+q\left(\frac{\eta_{k+1}}{1-\eta_{k+1}}\right)^\rho}\right)\right)^{1/\rho}}\right\}\right)\right\rangle. \tag{5}$$

And

$$\underset{j=1}{\overset{k}{\oplus}}\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}(a_{k+1}^p\otimes a_j^q) = \left\langle \left[s_{\sum\limits_{j=1}^{k}\left(\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}\cdot\theta^p(a_{k+1})\cdot\theta^q(a_j)\right)}, s_{\sum\limits_{j=1}^{k}\left(\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}\cdot\sigma^p(a_{k+1})\cdot\sigma^q(a_j)\right)}\right], \left(\bigcup_{\gamma_{k+1}\in\tilde{T}(a_{k+1}),\gamma_j\in\tilde{T}(a_j)}\left\{1-\frac{1}{1+\left(\sum\limits_{j=1}^{k}\left(\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}\cdot\frac{1}{p\left(\frac{1-\gamma_{k+1}}{\gamma_{k+1}}\right)^\rho+q\left(\frac{1-\gamma_j}{\gamma_j}\right)^\rho}\right)\right)^{1/\rho}}\right\}, \bigcup_{\delta_{k+1}\in\tilde{I}(a_{k+1}),\delta_j\in\tilde{I}(a_j)}\left\{\frac{1}{1+\left(\sum\limits_{j=1}^{k}\left(\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}\cdot\frac{1}{p\left(\frac{\delta_{k+1}}{1-\delta_{k+1}}\right)^\rho+q\left(\frac{\delta_j}{1-\delta_j}\right)^\rho}\right)\right)^{1/\rho}}\right\}, \bigcup_{\eta_{k+1}\in\tilde{F}(a_{k+1}),\eta_j\in\tilde{F}(a_j)}\left\{\frac{1}{1+\left(\sum\limits_{j=1}^{k}\left(\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}\cdot\frac{1}{p\left(\frac{\eta_{k+1}}{1-\eta_{k+1}}\right)^\rho+q\left(\frac{\eta_j}{1-\eta_j}\right)^\rho}\right)\right)^{1/\rho}}\right\}\right)\right\rangle \tag{6}$$

Eq.(5) and Eq.(6) can be gained utilizing the mathematical induction on k.

$$
\begin{aligned}
&\mathop{\oplus}_{\substack{i,j=1\\ i\neq j}}^{2}\frac{\omega_i\omega_j}{1-\omega_i}(a_i^p\otimes a_j^q)=\frac{\omega_1\omega_2}{1-\omega_1}(a_1^p\otimes a_2^q)\oplus\frac{\omega_2\omega_1}{1-\omega_2}(a_2^p\otimes a_1^q)\\
&=\left\langle\left[s_{\frac{\omega_1\omega_2}{1-\omega_1}\cdot\theta^p(a_1)\cdot\theta^q(a_2)+\frac{\omega_2\omega_1}{1-\omega_2}\cdot\theta^p(a_2)\cdot\theta^q(a_1)},s_{\frac{\omega_1\omega_2}{1-\omega_1}\cdot\sigma^p(a_1)\cdot\sigma^q(a_2)+\frac{\omega_2\omega_1}{1-\omega_2}\cdot\sigma^p(a_2)\cdot\sigma^q(a_1)}\right],\right.\\
&\left(\bigcup_{\gamma_1\in\tilde{T}(a_1),\gamma_2\in\tilde{T}(a_2)}\left\{1-\frac{1}{1+\left(\frac{\omega_1\omega_2}{1-\omega_1}\cdot\frac{1}{p\left(\frac{1-\gamma_1}{\gamma_1}\right)^\rho+q\left(\frac{1-\gamma_2}{\gamma_2}\right)^\rho}+\frac{\omega_2\omega_1}{1-\omega_2}\cdot\frac{1}{p\left(\frac{1-\gamma_2}{\gamma_2}\right)^\rho+q\left(\frac{1-\gamma_1}{\gamma_1}\right)^\rho}\right)^{1/\rho}}\right\},\right.\\
&\bigcup_{\delta_1\in\tilde{I}(a_2),\delta_2\in\tilde{I}(a_2)}\left\{\frac{1}{1+\left(\frac{\omega_1\omega_2}{1-\omega_1}\cdot\frac{1}{p\left(\frac{\delta_1}{1-\delta_1}\right)^\rho+q\left(\frac{\delta_2}{1-\delta_2}\right)^\rho}+\frac{\omega_2\omega_1}{1-\omega_2}\cdot\frac{1}{p\left(\frac{\delta_2}{1-\delta_2}\right)^\rho+q\left(\frac{\delta_1}{1-\delta_1}\right)^\rho}\right)^{1/\rho}}\right\},\\
&\left.\left.\bigcup_{\eta_1\in\tilde{F}(a_1),\eta_2\in\tilde{F}(a_2)}\left\{\frac{1}{1+\left(\frac{\omega_1\omega_2}{1-\omega_1}\cdot\frac{1}{p\left(\frac{\eta_1}{1-\eta_1}\right)^\rho+q\left(\frac{\eta_2}{1-\eta_2}\right)^\rho}+\frac{\omega_2\omega_1}{1-\omega_2}\cdot\frac{1}{p\left(\frac{\eta_2}{1-\eta_2}\right)^\rho+q\left(\frac{\eta_1}{1-\eta_1}\right)^\rho}\right)^{1/\rho}}\right\}\right)\right\rangle\\
&=\left\langle\left[s_{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{2}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\theta^p(a_i)\cdot\theta^q(a_j)\right)},s_{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{2}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\sigma^p(a_i)\cdot\sigma^q(a_j)\right)}\right],\right.\\
&\left(\bigcup_{\gamma_i\in\tilde{T}(a_i),\gamma_j\in\tilde{T}(a_j)}\left\{1-\frac{1}{1+\left(\sum\limits_{\substack{i,j=1\\ i\neq j}}^{2}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{1-\gamma_i}{\gamma_i}\right)^\rho+q\left(\frac{1-\gamma_j}{\gamma_j}\right)^\rho}\right)\right)^{1/\rho}}\right\},\right.\\
&\bigcup_{\delta_i\in\tilde{I}(a_i),\delta_j\in\tilde{I}(a_j)}\left\{\frac{1}{1+\left(\sum\limits_{\substack{i,j=1\\ i\neq j}}^{2}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{\delta_i}{1-\delta_i}\right)^\rho+q\left(\frac{\delta_j}{1-\delta_j}\right)^\rho}\right)\right)^{1/\rho}}\right\},\\
&\left.\left.\bigcup_{\eta_i\in\tilde{F}(a_i),\eta_j\in\tilde{F}(a_j)}\left\{\frac{1}{1+\left(\sum\limits_{\substack{i,j=1\\ i\neq j}}^{2}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{\eta_i}{1-\eta_i}\right)^\rho+q\left(\frac{\eta_j}{1-\eta_j}\right)^\rho}\right)\right)^{1/\rho}}\right\}\right)\right\rangle.
\end{aligned}
$$

Therefore, based on the Eq.(3), Eq.(5) and Eq.(6), Eq.(4) can be represented as

$$\overset{k+1}{\underset{\substack{i,j=1\\ i\neq j}}{\oplus}} \frac{\omega_i\omega_j}{1-\omega_i}(a_i^p \otimes a_j^q)$$

$$= \overset{k}{\underset{\substack{i,j=1\\ i\neq j}}{\oplus}} \frac{\omega_i\omega_j}{1-\omega_i}(a_i^p \otimes a_j^q)$$

$$\oplus \overset{k}{\underset{i=1}{\oplus}} \frac{\omega_i\omega_{k+1}}{1-\omega_i}(a_i^p \otimes a_{k+1}^q) \oplus \overset{k}{\underset{j=1}{\oplus}} \frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}(a_{k+1}^p \otimes a_j^q)$$

$$= \left\langle \left[s_{\sum_{\substack{i,j=1\\ i\neq j}}^{k+1}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\theta^p(a_i)\cdot\theta^q(a_j)\right)}, s_{\sum_{\substack{i,j=1\\ i\neq j}}^{k+1}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\sigma^p(a_i)\cdot\sigma^q(a_j)\right)} \right], \right.$$

$$\left(\bigcup_{\gamma_i\in\tilde{T}(a_i),\gamma_j\in\tilde{T}(a_j)} \left\{ 1 - \frac{1}{1+\left(\sum_{\substack{i,j=1\\ i\neq j}}^{k+1}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{1-\gamma_i}{\gamma_i}\right)^\rho + q\left(\frac{1-\gamma_j}{\gamma_j}\right)^\rho}\right)\right)^{1/\rho}} \right\}, \right.$$

$$\bigcup_{\delta_i\in\tilde{I}(a_i),\delta_j\in\tilde{I}(a_j)} \left\{ \frac{1}{1+\left(\sum_{\substack{i,j=1\\ i\neq j}}^{k+1}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{\delta_i}{1-\delta_i}\right)^\rho + q\left(\frac{\delta_j}{1-\delta_j}\right)^\rho}\right)\right)^{1/\rho}} \right\},$$

$$\left.\left. \bigcup_{\eta_i\in\tilde{F}(a_i),\eta_j\in\tilde{F}(a_j)} \left\{ \frac{1}{1+\left(\sum_{\substack{i,j=1\\ i\neq j}}^{k+1}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{\eta_i}{1-\eta_i}\right)^\rho + q\left(\frac{\eta_j}{1-\eta_j}\right)^\rho}\right)\right)^{1/\rho}} \right\} \right) \right\rangle.$$

That is, when $n = k+1$, Eq. (2) is right. So, Eq. (2) is true for all n.

(2) Since Eq. (2) is true, Eq. (1) can be proved true. That is $MVNULDNWBM(a_1, a_2, \cdots a_n)$, as shown at the bottom of the next page.

Similar to the NWBM operator, the MVNULDNWBM operator contains the following properties of commutativity, reducibility, boundedness, monotonicity, and idempotency.

Next, the desirable properties for MVNULDNWBM operator are proved.

(1) Reducibility. Let a_i $(i = 1, 2, \cdots, n)$ be a collection of MVNULNs and $\omega = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)$. Then

$$MVNULDNWBM(a_1, a_2, \cdots a_n) = MVNULDBM(a_1, a_2, \cdots a_n).$$

Proof: Since $\omega = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)$, then according to the equation in Definition 10, we can obtain the following equation.

$$MVNULDNWBM(a_1, a_2, \cdots a_n)$$

$$= \left(\overset{n}{\underset{\substack{i,j=1\\ i\neq j}}{\oplus}} \frac{\omega_i\omega_j}{1-\omega_i}(a_i^p \otimes a_j^q)\right)^{\frac{1}{p+q}}$$

$$= \left(\overset{n}{\underset{\substack{i,j=1\\ i\neq j}}{\oplus}} \frac{\frac{1}{n^2}}{1-\frac{1}{n}}(a_i^p \otimes a_j^q)\right)^{\frac{1}{p+q}}$$

$$= \left(\overset{n}{\underset{\substack{i,j=1\\ i\neq j}}{\oplus}} \frac{1}{n(n-1)}(a_i^p \otimes a_j^q)\right)^{\frac{1}{p+q}}$$

$$= \left(\frac{1}{n(n-1)} \overset{n}{\underset{\substack{i,j=1\\ i\neq j}}{\oplus}} (a_i^p \otimes a_j^q)\right)^{\frac{1}{p+q}}$$

$$= MVNULDBM(a_1, a_2, \cdots a_n).$$

(2) Idempotency. Suppose a_i $(i = 1, 2, \cdots, n)$ be a space of MVNULNs and $a_i = a$ $(i = 1, 2, \cdots, n)$. Then

$$MVNULDNWBM(a_1, a_2, \cdots a_n) = a.$$

Proof: For each i, owing to $a_i = a$, the following formula can be obtained on the basis of Eq. (5) in Theorem 1.

$$MVNULDNWBM(a_1, a_2, \cdots a_n)$$

$$= \left(\overset{n}{\underset{\substack{i,j=1\\ i\neq j}}{\oplus}} \frac{\omega_i\omega_j}{1-\omega_i}(a_i^p \otimes a_j^q)\right)^{\frac{1}{p+q}}$$

$$= \left(\overset{n}{\underset{\substack{i,j=1\\ i\neq j}}{\oplus}} \frac{\omega_i\omega_j}{1-\omega_i}(a^p \otimes a^q)\right)^{\frac{1}{p+q}}$$

$$= \left(\overset{n}{\underset{\substack{i,j=1\\ i\neq j}}{\oplus}} \frac{\omega_i\omega_j}{1-\omega_i} a^{p+q}\right)^{\frac{1}{p+q}}$$

$$= a\left(\overset{n}{\underset{\substack{i,j=1\\ i\neq j}}{\oplus}} \frac{\omega_i\omega_j}{1-\omega_i}\right)^{\frac{1}{p+q}}$$

$$= a\left(\overset{n}{\underset{i=1}{\oplus}} \omega_i \overset{n}{\underset{\substack{j=1\\ j\neq i}}{\oplus}} \frac{\omega_j}{1-\omega_i}\right)^{\frac{1}{p+q}}$$

$$= a.$$

(3) Commutativity. Let $(\tilde{a}_1, \tilde{a}_2, \cdots \tilde{a}_n)$ be any permutation of $(a_1, a_2, \cdots a_n)$. Then

$$MVNULDNWBM\ (\tilde{a}_1, \tilde{a}_2, \cdots \tilde{a}_n) = MVNULDNWBM(a_1, a_2, \cdots a_n).$$

$$
\begin{aligned}
& MVNULDNWBM(a_1, a_2, \cdots a_n) \\
& = \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} \left(a_i^p \otimes a_j^q \right) \right)^{\frac{1}{p+q}} \\
& = \left\langle \left[s_{\left(\sum_{\substack{i,j=1 \\ i\neq j}}^{n} \left(\frac{\omega_i\omega_j}{1-\omega_i} \cdot \theta^p(a_i) \cdot \theta^q(a_j) \right) \right)^{\frac{1}{p+q}}}, s_{\left(\sum_{\substack{i,j=1 \\ i\neq j}}^{n} \left(\frac{\omega_i\omega_j}{1-\omega_i} \cdot \sigma^p(a_i) \cdot \sigma^q(a_j) \right) \right)^{\frac{1}{p+q}}} \right], \right. \\
& \left(\cup_{\gamma_i \in \tilde{T}(a_i), \gamma_j \in \tilde{T}(a_j)} \left\{ 1 \middle/ \left(1 + \left\{ \frac{1}{\text{p+q}} \left(\frac{1 - \left(1 - \frac{1}{1 + \left(\sum_{\substack{i,j=1 \\ i\neq j}}^{n} \left(\frac{\omega_i\omega_j}{1-\omega_i} \cdot \frac{1}{p\left(\frac{1-\gamma_i}{\gamma_i}\right)^{\rho} + q\left(\frac{1-\gamma_j}{\gamma_j}\right)^{\rho}} \right) \right)^{1/\rho}} \right)}{1 - \frac{1}{1 + \left(\sum_{\substack{i,j=1 \\ i\neq j}}^{n} \left(\frac{\omega_i\omega_j}{1-\omega_i} \cdot \frac{1}{p\left(\frac{1-\gamma_i}{\gamma_i}\right)^{\rho} + q\left(\frac{1-\gamma_j}{\gamma_j}\right)^{\rho}} \right) \right)^{1/\rho}}} \right)^{\rho} \right\}^{1/\rho} \right) \right\} \right), \\
& \cup_{\delta_i \in \tilde{I}(a_i), \delta_j \in \tilde{I}(a_j)} \left\{ 1 - 1 \middle/ \left(1 + \left\{ \frac{1}{\text{p+q}} \left(\frac{\frac{1}{1 + \left(\sum_{\substack{i,j=1 \\ i\neq j}}^{n} \left(\frac{\omega_i\omega_j}{1-\omega_i} \cdot \frac{1}{p\left(\frac{\delta_i}{1-\delta_i}\right)^{\rho} + q\left(\frac{\delta_j}{1-\delta_j}\right)^{\rho}} \right) \right)^{1/\rho}}}{1 - \frac{1}{1 + \left(\sum_{\substack{i,j=1 \\ i\neq j}}^{n} \left(\frac{\omega_i\omega_j}{1-\omega_i} \cdot \frac{1}{p\left(\frac{\delta_i}{1-\delta_i}\right)^{\rho} + q\left(\frac{\delta_j}{1-\delta_j}\right)^{\rho}} \right) \right)^{1/\rho}}} \right)^{\rho} \right\}^{1/\rho} \right) \right\}, \\
& \left. \cup_{\eta_i \in \tilde{F}(a_i), \eta_j \in \tilde{F}(a_j)} \left\{ 1 - 1 \middle/ \left(1 + \left\{ \frac{1}{\text{p+q}} \left(\frac{\frac{1}{1 + \left(\sum_{\substack{i,j=1 \\ i\neq j}}^{n} \left(\frac{\omega_i\omega_j}{1-\omega_i} \cdot \frac{1}{p\left(\frac{\eta_i}{1-\eta_i}\right)^{\rho} + q\left(\frac{\eta_j}{1-\eta_j}\right)^{\rho}} \right) \right)^{1/\rho}}}{1 - \frac{1}{1 + \left(\sum_{\substack{i,j=1 \\ i\neq j}}^{n} \left(\frac{\omega_i\omega_j}{1-\omega_i} \cdot \frac{1}{p\left(\frac{\eta_i}{1-\eta_i}\right)^{\rho} + q\left(\frac{\eta_j}{1-\eta_j}\right)^{\rho}} \right) \right)^{1/\rho}}} \right)^{\rho} \right\}^{1/\rho} \right) \right\} \right) \right\rangle
\end{aligned}
$$

Proof: Owing to $(\tilde{a}_1, \tilde{a}_2, \cdots \tilde{a}_n)$ is permutation of $(a_1, a_2, \cdots a_n)$, then the following equation is obtained.

$$MVNULDNWBM(a_1, a_2, \cdots a_n) = \left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}(a_i^p \otimes a_j^q)\right)^{\frac{1}{p+q}} = \left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}(\tilde{a}_i^p \otimes \tilde{a}_j^q)\right)^{\frac{1}{p+q}}$$

(4) Monotonicity. Suppose $a_i = \left\langle\left[s_{\theta(a_i)}, s_{\sigma(a_i)}\right], \left(\tilde{T}(a_i), \tilde{I}(a_i), \tilde{F}(a_i)\right)\right\rangle (i = 1, 2, \cdots, n)$ and $b_i = \left\langle\left[s_{\theta(b_i)}, s_{\sigma(b_i)}\right], \left(\tilde{T}(b_i), \tilde{I}(b_i), \tilde{F}(b_i)\right)\right\rangle (i = 1, 2, \cdots, n)$ are two collections of MVNULNs, when $s_{\theta(a_i)} \geq s_{\theta(b_i)}, s_{\sigma(a_i)} \geq s_{\sigma(b_i)}, \tilde{T}(a_i) \geq \tilde{T}(b_i), \tilde{I}(a_i) \leq \tilde{I}(b_i)$ and $\tilde{F}(a_i) \leq \tilde{F}(b_i)$ for any. Then

$$MVNULDNWBM(a_1, a_2, \cdots a_n) \geq MVNULDNWBM(b_1, b_2, \cdots b_n).$$

Proof: (I) For uncertain linguistic term part

Owing to $p, q \geq 0$, and $s_{\theta(a_i)} \geq s_{\theta(b_i)}, s_{\sigma(a_i)} \geq s_{\sigma(b_i)}$, for any i, the result is gained as below.

$$\theta(a_i) \geq \theta(b_i) \geq 0, \quad \sigma(a_i) \geq \sigma(b_i) \geq 0,$$

And

$$\theta^p(a_i) \geq \theta^p(b_i), \quad \theta^q(a_j) \geq \theta^q(b_j), \ \sigma^p(a_i) \geq \sigma^p(b_i),$$

and

$$\sigma^q(a_i) \geq \sigma^q(b_i) \Rightarrow \theta^p(a_i)\theta^q(a_j) \geq \theta^p(b_i)\theta^q(b_j) \Rightarrow \frac{\omega_i\omega_j}{1-\omega_i}\theta^p(a_i)\theta^q(a_j) \geq \frac{\omega_i\omega_j}{1-\omega_i}\theta^p(b_i)\theta^q(b_j)$$

$$= \left\langle \left[s_{\left(\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\theta^p(a_i)\cdot\theta^q(a_j)\right)\right)^{\frac{1}{p+q}}}, s_{\left(\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\sigma^p(a_i)\cdot\sigma^q(a_j)\right)\right)^{\frac{1}{p+q}}} \right], \right.$$

$$\left(\bigcup_{\gamma_i\in\tilde{T}(a_i),\gamma_j\in\tilde{T}(a_j)} \left\{ \frac{\left((p+q)\cdot\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{1-\gamma_i}{\gamma_i}\right)^{\rho}+q\left(\frac{1-\gamma_j}{\gamma_j}\right)^{\rho}}\right)\right)^{\frac{1}{\rho}}}{1+\left((p+q)\cdot\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{1-\gamma_i}{\gamma_i}\right)^{\rho}+q\left(\frac{1-\gamma_j}{\gamma_j}\right)^{\rho}}\right)\right)^{\frac{1}{\rho}}} \right\},\right.$$

$$\bigcup_{\delta_i\in\tilde{I}(a_i),\delta_j\in\tilde{I}(a_j)} \left\{ \frac{1}{1+\left((p+q)\cdot\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{\delta_i}{1-\delta_i}\right)^{\rho}+q\left(\frac{\delta_j}{1-\delta_j}\right)^{\rho}}\right)\right)^{\frac{1}{\rho}}} \right\},$$

$$\left.\left.\bigcup_{\eta_i\in\tilde{F}(a_i),\eta_j\in\tilde{F}(a_j)} \left\{ \frac{1}{1+\left((p+q)\cdot\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{\eta_i}{1-\eta_i}\right)^{\rho}+q\left(\frac{\eta_j}{1-\eta_j}\right)^{\rho}}\right)\right)^{\frac{1}{\rho}}} \right\}\right)\right\rangle$$

$$\Rightarrow \sum_{\substack{i,j=1\\i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}\theta^p(a_i)\theta^q(a_j) \geq \sum_{\substack{i,j=1\\i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}\theta^p(b_i)\theta^q(b_j)$$

$$\Rightarrow \left(\sum_{\substack{i,j=1\\i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}\theta^p(a_i)\theta^q(a_j)\right)^{\frac{1}{p+q}}$$

$$\geq \left(\sum_{\substack{i,j=1\\i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}\theta^p(b_i)\theta^q(b_j)\right)^{\frac{1}{p+q}}$$

Similarly,

$$\left(\sum_{\substack{i,j=1\\i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}\sigma^p(a_i)\sigma^q(a_j)\right)^{\frac{1}{p+q}} \geq \left(\sum_{\substack{i,j=1\\i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}\sigma^p(b_i)\sigma^q(b_j)\right)^{\frac{1}{p+q}}$$

can also be proved. That is,

$$s\left(\sum_{\substack{i,j=1\\i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}\theta^p(a_i)\theta^q(a_j)\right)^{\frac{1}{p+q}} \geq s\left(\sum_{\substack{i,j=1\\i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}\theta^p(b_i)\theta^q(b_j)\right)^{\frac{1}{p+q}}$$

and

$$s\left(\sum_{\substack{i,j=1\\i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}\sigma^p(a_i)\sigma^q(a_j)\right)^{\frac{1}{p+q}} \geq s\left(\sum_{\substack{i,j=1\\i\neq j}}^{n} \frac{\omega_i\omega_j}{1-\omega_i}\sigma^p(b_i)\sigma^q(b_j)\right)^{\frac{1}{p+q}}.$$

(II) For the parts of true, indeterminacy and falsity membership

We need to prove the following equation.

$$\frac{\left((p+q)\cdot\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{1-\gamma_{a_i}}{\gamma_{a_i}}\right)^\rho+q\left(\frac{1-\gamma_{a_j}}{\gamma_{a_j}}\right)^\rho}\right)\right)^{\frac{1}{\rho}}}{1+\left((p+q)\cdot\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{1-\gamma_{a_i}}{\gamma_{a_i}}\right)^\rho+q\left(\frac{1-\gamma_{a_j}}{\gamma_{a_j}}\right)^\rho}\right)\right)^{\frac{1}{\rho}}}$$

$$\geq \frac{\left((p+q)\cdot\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{1-\gamma_{b_i}}{\gamma_{b_i}}\right)^\rho+q\left(\frac{1-\gamma_{b_j}}{\gamma_{b_j}}\right)^\rho}\right)\right)^{\frac{1}{\rho}}}{1+\left((p+q)\cdot\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{1-\gamma_{b_i}}{\gamma_{b_i}}\right)^\rho+q\left(\frac{1-\gamma_{b_j}}{\gamma_{b_j}}\right)^\rho}\right)\right)^{\frac{1}{\rho}}};$$

$$\frac{1}{1+\left((p+q)\cdot\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{\delta_{a_i}}{1-\delta_{a_i}}\right)^\rho+q\left(\frac{\delta_{a_j}}{1-\delta_{a_j}}\right)^\rho}\right)\right)^{\frac{1}{\rho}}}$$

$$\leq \frac{1}{1+\left((p+q)\cdot\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{\delta_{b_i}}{1-\delta_{b_i}}\right)^\rho+q\left(\frac{\delta_{b_j}}{1-\delta_{b_j}}\right)^\rho}\right)\right)^{\frac{1}{\rho}}};$$

$$\frac{1}{1+\left((p+q)\cdot\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{\eta_{a_i}}{1-\eta_{a_i}}\right)^\rho+q\left(\frac{\eta_{a_j}}{1-\eta_{a_j}}\right)^\rho}\right)\right)^{\frac{1}{\rho}}}$$

$$\leq \frac{1}{1+\left((p+q)\cdot\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{\eta_{b_i}}{1-\eta_{b_i}}\right)^\rho+q\left(\frac{\eta_{b_j}}{1-\eta_{b_j}}\right)^\rho}\right)\right)^{\frac{1}{\rho}}}$$

Proof: Owing to $\tilde{T}(a_i) \geq \tilde{T}(b_i), \tilde{I}(a_i) \leq \tilde{I}(b_i)$ and $\tilde{F}(a_i) \leq \tilde{F}(b_i)$ for any $i, p, q \geq 0$, then

$$\gamma_{a_i} \geq \gamma_{b_i}, \quad \gamma_{a_j} \geq \gamma_{b_j},$$
$$\delta_{a_i} \leq \delta_{b_i}, \quad \delta_{a_j} \leq \delta_{b_j},$$
$$\eta_{a_i} \leq \eta_{b_i}, \quad \eta_{a_j} \leq \eta_{b_j},$$
$$\frac{1-\gamma_{a_i}}{\gamma_{a_i}} \leq \frac{1-\gamma_{b_i}}{\gamma_{b_i}},$$
$$\frac{1-\gamma_{a_j}}{\gamma_{a_j}} \leq \frac{1-\gamma_{b_j}}{\gamma_{b_j}},$$
$$\frac{\delta_{a_i}}{1-\delta_{a_i}} \leq \frac{\delta_{b_i}}{1-\delta_{b_i}},$$
$$\frac{\delta_{a_j}}{1-\delta_{a_j}} \leq \frac{\delta_{b_j}}{1-\delta_{b_j}},$$
$$\frac{\eta_{a_i}}{1-\eta_{a_i}} \leq \frac{\eta_{b_i}}{1-\eta_{b_i}},$$
$$\frac{\eta_{a_j}}{1-\eta_{a_j}} \leq \frac{\eta_{b_j}}{1-\eta_{b_j}},$$

Further, the following results can be proved.

$$p\left(\frac{1-\gamma_{a_i}}{\gamma_{a_i}}\right)^\rho+q\left(\frac{1-\gamma_{a_j}}{\gamma_{a_j}}\right)^\rho$$
$$\leq p\left(\frac{1-\gamma_{b_i}}{\gamma_{b_i}}\right)^\rho+q\left(\frac{1-\gamma_{b_j}}{\gamma_{b_j}}\right)^\rho$$
$$\Rightarrow \frac{1}{p\left(\frac{1-\gamma_{a_i}}{\gamma_{a_i}}\right)^\rho+q\left(\frac{1-\gamma_{a_j}}{\gamma_{a_j}}\right)^\rho} \geq \frac{1}{p\left(\frac{1-\gamma_{b_i}}{\gamma_{b_i}}\right)^\rho+q\left(\frac{1-\gamma_{b_j}}{\gamma_{b_j}}\right)^\rho}$$
$$\Rightarrow \left((p+q)\cdot\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{1-\gamma_{a_i}}{\gamma_{a_i}}\right)^\rho+q\left(\frac{1-\gamma_{a_j}}{\gamma_{a_j}}\right)^\rho}\right)\right)^{\frac{1}{\rho}}$$

$$\geq\left((p+q)\cdot\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{1-\gamma_{b_i}}{\gamma_{b_i}}\right)^{\rho}+q\left(\frac{1-\gamma_{b_j}}{\gamma_{b_j}}\right)^{\rho}}\right)\right)^{\frac{1}{\rho}}$$

$$\Rightarrow\frac{1}{\left((p+q)\cdot\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{1-\gamma_{a_i}}{\gamma_{a_i}}\right)^{\rho}+q\left(\frac{1-\gamma_{a_j}}{\gamma_{a_j}}\right)^{\rho}}\right)\right)^{\frac{1}{\rho}}}$$

$$\leq\frac{1}{\left((p+q)\cdot\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{1-\gamma_{b_i}}{\gamma_{b_i}}\right)^{\rho}+q\left(\frac{1-\gamma_{b_j}}{\gamma_{b_j}}\right)^{\rho}}\right)\right)^{\frac{1}{\rho}}}$$

$$\Rightarrow 1+\frac{1}{\left((p+q)\cdot\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{1-\gamma_{a_i}}{\gamma_{a_i}}\right)^{\rho}+q\left(\frac{1-\gamma_{a_j}}{\gamma_{a_j}}\right)^{\rho}}\right)\right)^{\frac{1}{\rho}}}$$

$$\leq 1+\frac{1}{\left((p+q)\cdot\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{1-\gamma_{b_i}}{\gamma_{b_i}}\right)^{\rho}+q\left(\frac{1-\gamma_{b_j}}{\gamma_{b_j}}\right)^{\rho}}\right)\right)^{\frac{1}{\rho}}}$$

$$\Rightarrow\frac{1}{1+\frac{1}{\left((p+q)\cdot\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{1-\gamma_{a_i}}{\gamma_{a_i}}\right)^{\rho}+q\left(\frac{1-\gamma_{a_j}}{\gamma_{a_j}}\right)^{\rho}}\right)\right)^{\frac{1}{\rho}}}}$$

$$\geq\frac{1}{1+\frac{1}{\left((p+q)\cdot\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{1-\gamma_{b_i}}{\gamma_{b_i}}\right)^{\rho}+q\left(\frac{1-\gamma_{b_j}}{\gamma_{b_j}}\right)^{\rho}}\right)\right)^{\frac{1}{\rho}}}}$$

$$\Rightarrow\frac{\left((p+q)\cdot\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{1-\gamma_{a_i}}{\gamma_{a_i}}\right)^{\rho}+q\left(\frac{1-\gamma_{a_j}}{\gamma_{a_j}}\right)^{\rho}}\right)\right)^{\frac{1}{\rho}}}{1+\left((p+q)\cdot\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{1-\gamma_{a_i}}{\gamma_{a_i}}\right)^{\rho}+q\left(\frac{1-\gamma_{a_j}}{\gamma_{a_j}}\right)^{\rho}}\right)\right)^{\frac{1}{\rho}}}$$

$$\geq\frac{\left((p+q)\cdot\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{1-\gamma_{b_i}}{\gamma_{b_i}}\right)^{\rho}+q\left(\frac{1-\gamma_{b_j}}{\gamma_{b_j}}\right)^{\rho}}\right)\right)^{\frac{1}{\rho}}}{1+\left((p+q)\cdot\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{1-\gamma_{b_i}}{\gamma_{b_i}}\right)^{\rho}+q\left(\frac{1-\gamma_{b_j}}{\gamma_{b_j}}\right)^{\rho}}\right)\right)^{\frac{1}{\rho}}}$$

Thus, the true membership part is right.

Then, we can get

$$p\left(\frac{\delta_{a_i}}{1-\delta_{a_i}}\right)^{\rho}+q\left(\frac{\delta_{a_j}}{1-\delta_{a_j}}\right)^{\rho}$$

$$\leq p\left(\frac{\delta_{b_i}}{1-\delta_{b_i}}\right)^{\rho}+q\left(\frac{\delta_{b_j}}{1-\delta_{b_j}}\right)^{\rho}$$

$$\Rightarrow\frac{1}{p\left(\frac{\delta_{a_i}}{1-\delta_{a_i}}\right)^{\rho}+q\left(\frac{\delta_{a_j}}{1-\delta_{a_j}}\right)^{\rho}}\geq\frac{1}{p\left(\frac{\delta_{b_i}}{1-\delta_{b_i}}\right)^{\rho}+q\left(\frac{\delta_{b_j}}{1-\delta_{b_j}}\right)^{\rho}}$$

$$\Rightarrow 1+\left((p+q)\cdot\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{\delta_{a_i}}{1-\delta_{a_i}}\right)^{\rho}+q\left(\frac{\delta_{a_j}}{1-\delta_{a_j}}\right)^{\rho}}\right)\right)^{\frac{1}{\rho}}$$

$$\geq 1+\left((p+q)\cdot\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{\delta_{b_i}}{1-\delta_{b_i}}\right)^{\rho}+q\left(\frac{\delta_{b_j}}{1-\delta_{b_j}}\right)^{\rho}}\right)\right)^{\frac{1}{\rho}}$$

$$\Rightarrow\frac{1}{1+\left((p+q)\cdot\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{\delta_{a_i}}{1-\delta_{a_i}}\right)^{\rho}+q\left(\frac{\delta_{a_j}}{1-\delta_{a_j}}\right)^{\rho}}\right)\right)^{\frac{1}{\rho}}}$$

$$\leq\frac{1}{1+\left((p+q)\cdot\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{\delta_{b_i}}{1-\delta_{b_i}}\right)^{\rho}+q\left(\frac{\delta_{b_j}}{1-\delta_{b_j}}\right)^{\rho}}\right)\right)^{\frac{1}{\rho}}}$$

Thus, the indeterminacy membership part is true.

Similarly, the falsity membership part can be gained.

That is, the following equation is also right.

$$\frac{1}{1+\left((p+q)\cdot\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{\eta_{a_i}}{1-\eta_{a_i}}\right)^{\rho}+q\left(\frac{\eta_{a_j}}{1-\eta_{a_j}}\right)^{\rho}}\right)\right)^{\frac{1}{\rho}}}$$

$$\leq\frac{1}{1+\left((p+q)\cdot\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{\eta_{b_i}}{1-\eta_{b_i}}\right)^{\rho}+q\left(\frac{\eta_{b_j}}{1-\eta_{b_j}}\right)^{\rho}}\right)\right)^{\frac{1}{\rho}}}$$

(III) Comparing $MVNULDNWBM(a_1,a_2,\cdots a_n)$ with $MVNULDNWBM(b_1,b_2,\cdots b_n)$

Suppose

$$a=\left\langle\left[s_{\theta(a)},s_{\sigma(a)}\right],\left(\tilde{T}(a),\tilde{I}(a),\tilde{F}(a)\right)\right\rangle$$
$$=MVNULDNWBM(a_1,a_2,\cdots a_n)$$

and

$$b=\left\langle\left[s_{\theta(b)},s_{\sigma(b)}\right],\left(\tilde{T}(b),\tilde{I}(b),\tilde{F}(b)\right)\right\rangle$$
$$=MVNULDNWBM(b_1,b_2,\cdots b_n).$$

Since $s_{\theta(a)} \geq s_{\theta(b)}, s_{\sigma(a)} \geq s_{\sigma(b)}, \tilde{T}(a) \geq \tilde{T}(b), \tilde{I}(a) \leq \tilde{I}(b)$ and $\tilde{F}(a) \leq \tilde{F}(b)$, thus $a \geq b$. Therefore, we can get

$$MVNULDNWBM(a_1, a_2, \cdots a_n) \geq MVNULDNWBM(b_1, b_2, \cdots b_n).$$

(5) Boundedness. Let $a_i\,(i = 1, 2, \cdots, n)$ be a set of MVNULNs,.

Then

$$\min_i \{a_i\} \leq MVNULDNWBM(a_1, a_2, \cdots a_n) \leq \max_i \{a_i\}.$$

Proof: According to the property of idempotency, we can obtain

$$MVNULDNWBM(\min_i \{a_i\}, \min_i \{a_i\}, \cdots, \min_i \{a_i\}) = \min_i \{a_i\},$$
$$MVNULDNWBM(\max_i \{a_i\}, \max_i \{a_i\}, \cdots, \max_i \{a_i\}) = \max_i \{a_i\}.$$

Owing to $\min_i \{a_i\} \leq a_i \leq \max_i \{a_i\}\,(i = 1, 2, \cdots, n)$, then by the property of monotonicity, we can obtain

$$\min_i \{a_i\} \leq MVNULDNWBM(a_1, a_2, \cdots a_n) \leq \max_i \{a_i\}.$$

V. THE MADM APPROACH UTILIZING MVULDNWBM OPERATOR

In this section, an MADM method using the proposed aggregating operator is presented under multiple-valued neutrosophic uncertain linguistic environment.

Suppose $A = \{A_1, A_2, \cdots, A_m\}$ be a collection of m selected alternatives, and $C = \{C_1, C_2, \cdots, C_n\}$ be a collection of n attributes. Let $\omega = \{\omega_1, \omega_2, \cdots, \omega_n\}$ be corresponding weight vector for all attributes, where $\omega_j \geq 0(j = 1, 2, \cdots, n)$, and $\sum_{j=1}^{n} \omega_j = 1$. The evaluated value on alternative $A_i(i = 1, 2, \cdots, m)$ with respect to the criteria $C_j(j = 1, 2, \cdots, n)$ is given by expert, and each value is expressed in the form of MVNULN. Assume that $R = \left[a_{ij}\right]_{m\times n}$ is the multiple-valued neutrosophic uncertain linguistic matrix, $a_{ij} = \left\langle \left[s_{\theta(a_{ij})}, s_{\sigma(a_{ij})}\right], \left(\tilde{T}(a_{ij}), \tilde{I}(a_{ij}), \tilde{F}(a_{ij})\right)\right\rangle$ is the evaluation value which describes the assessment information of alternative $A_i(i = 1, 2, \cdots, m)$ on criteria $C_j(j = 1, 2, \cdots, n)$ belonging to the uncertain linguistic value $\left[s_{\theta(a_{ij})}, s_{\sigma(a_{ij})}\right]$, where $\tilde{T}(a_{ij})$ denotes the degree of satisfaction, $\tilde{I}(a_{ij})$ denotes the degree of uncertainty, and $\tilde{F}(a_{ij})$ denotes the degree of dissatisfaction.

Then, the typical approach for MADM problem is presented in the following.

Step 1 (The Decision Matrix Is Normalized): Generally, attribute in MADM problems consist of two kinds: cost attribute and benefit attribute, we need transform the cost attribute into the benefit attribute in order to diminish the influence of different attribute types. Assume $R = \left[a_{ij}\right]_{m\times n}$ be the original attribute matrix, which is transformed as below:

$$b_{ij} = \begin{cases} \left\langle \left[s_{\theta(a_{ij})}, s_{\sigma(a_{ij})}\right], \left(\tilde{T}(a_{ij}), \tilde{I}(a_{ij}), \tilde{F}(a_{ij})\right)\right\rangle, & \text{for benefit criteria} \\ \left\langle \left[s_{\iota-\sigma(a_{ij})}, s_{\iota-\theta(a_{ij})}\right], \left(\tilde{F}(a_{ij}), 1-\tilde{I}(a_{ij}), \tilde{T}(a_{ij})\right)\right\rangle, & \text{for cost criteria} \end{cases}$$

Then, the normalized matrix $B = \left[b_{ij}\right]_{m\times n}$ is obtained.

Step 2 (The Comprehensive Value of Each Alternative Is Calculated): The comprehensive value $b_i(i = 1, 2, \cdots, m)$ concerning alternative is calculated employing the MVNULDNWBM operator according to Definition 10, which can fuse multiple evaluation value about each alternative with concern to all criteria.

$$b_i = MVNULDNWBM(b_{i1}, b_{i2}, \cdots, b_{in})$$

Step 3 (The Value of Three Comparative Functions Is Calculated): According to the formula depicted in Definition 8, the score function E(b_i), the accuracy function H(b_i), and the certainty function C(b_i) are obtained based on the comprehensive value $b_i(i = 1, 2, \cdots, m)$ of each alternative $A_i(i = 1, 2, \cdots, m)$.

Step 4 (The Best Alternative Is Selected): According to Definition 9, all alternatives $A_i(i = 1, 2, \cdots, m)$ can be ranked in view of the value of E(b_i), H(b_i), and C(b_i), then the best alternative(s) can be obtained.

VI. AN ILLUSTRATIVE EXAMPLE

In this section, we will use an instance on investment alternative to demonstrate the effectiveness and flexibility of the proposed MADM approach under MVNUL environment.

An company wants to expand its business, and there are four investment alternatives can be chosen, A_1 is an auto corporation, A_2 is a food corporation, A_3 is a computer corporation, A_4 is an arm corporation. Each alternative is evaluated on the basis of all attributes, C_1 denotes the risk, C_2 denotes the growth, C_3 denotes the influence of environment, where C_3 is a cost criterion. The weighted vector of three criteria is $\omega = \{0.35, 0.25, 0.4\}$. Due to the complexity of practical problems, experts might be hesitant and provide multiple selecting values on the satisfaction, uncertainty and dissatisfaction concerning each alternative A_i regarding attribute C_j with the uncertain linguistic S. Thus, the evaluation value is described in the form of MVNULN, and the linguistic term S is shown as below:

$$S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\} = \{\text{extremely poor, very poor, poor, medium, good, very good, extremely good}\}.$$

Then, the corresponding multiple-valued neutrosophic uncertain linguistic attribute matrix is gained as below, $R = \left[a_{ij}\right]_{4\times 3}$, as shown at the bottom of the next page.

A. THE PROCESS UTILIZING THE NOVEL AGGREGATING OPERATOR

Step 1 (The Decision Matrix Need Be Converted): Because C_3 is a cost criterion, which needs to be normalized, then the normalized multiple-valued neutrosophic uncertain linguistic matrix $B = \left[b_{ij}\right]_{m\times n}$ can be obtained according to related transformation rules, $B = \left[b_{ij}\right]_{4\times 3}$, as shown at the bottom of this page.

Step 2 (The Comprehensive Value of Each Alternative Is Calculated): The comprehensive value $b_i (i = 1, 2, \cdots, m)$ of each alternative $A_i (i = 1, 2, \cdots, m)$ is computed by utilizing the MVNULNWBM operator in Definition 10. Assume $p = q = 1$, $\rho = 2$. The MVNULNWBM operator is presented as follows:

$$MVNULDNWBM(b_1, b_2, \cdots, b_n)$$

$$= \left\langle \left[s_{\left(\sum_{i,j=1, i\neq j}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\theta^p(b_i)\cdot\theta^q(b_j)\right)\right)^{\frac{1}{p+q}}}, s_{\left(\sum_{i,j=1, i\neq j}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\sigma^p(b_i)\cdot\sigma^q(b_j)\right)^{\frac{1}{p+q}}\right)} \right], \right.$$

$$\left(\bigcup_{\gamma_i\in\tilde{T}(b_i),\gamma_j\in\tilde{T}(b_j)} \left\{ \frac{\left((p+q)\cdot\sum_{i,j=1, i\neq j}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{1-\gamma_i}{\gamma_i}\right)^{\rho}+q\left(\frac{1-\gamma_j}{\gamma_j}\right)^{\rho}}\right)\right)^{\frac{1}{\rho}}}{1+\left((p+q)\cdot\sum_{i,j=1, i\neq j}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{1-\gamma_i}{\gamma_i}\right)^{\rho}+q\left(\frac{1-\gamma_j}{\gamma_j}\right)^{\rho}}\right)\right)^{\frac{1}{\rho}}} \right\}, \right.$$

$$\bigcup_{\delta_i\in\tilde{I}(b_i),\delta_j\in\tilde{I}(b_j)} \left\{ \frac{1}{1+\left((p+q)\cdot\sum_{i,j=1, i\neq j}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{\delta_i}{1-\delta_i}\right)^{\rho}+q\left(\frac{\delta_j}{1-\delta_j}\right)^{\rho}}\right)\right)^{\frac{1}{\rho}}} \right\},$$

$$\left.\left. \bigcup_{\eta_i\in\tilde{F}(b_i),\eta_j\in\tilde{F}(b_j)} \left\{ \frac{1}{1+\left((p+q)\cdot\sum_{i,j=1, i\neq j}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\frac{1}{p\left(\frac{\eta_i}{1-\eta_i}\right)^{\rho}+q\left(\frac{\eta_j}{1-\eta_j}\right)^{\rho}}\right)\right)^{\frac{1}{\rho}}} \right\} \right) \right\rangle$$

Then, we can get

$$b_1 = \langle [s_{3.4081}, s_{4.4350}], \left(\left\{ \begin{array}{l} 0.4183, 0.4456, 0.4444, 0.4886, 0.4609, 0.4873, \\ 0.4837, 0.5222, 0.5000, 0.5316, 0.5240, 0.5618 \end{array} \right\}, \{0.2771, 0.2782, 0.3802, 0.3836\}, \left\{ \begin{array}{l} 0.2681, 0.3000, 0.2965, 0.3280, 0.3089, 0.3372, \\ 0.3477, 0.3680 \end{array} \right\} \right) \rangle;$$

$$b_2 = \langle [s_{3.4515}, s_{4.4850}], \left(\left\{ \begin{array}{l} 0.4331, 0.4476, 0.4750, 0.4857, 0.4750, 0.4857, \\ 0.5408, 0.5472 \end{array} \right\}, \{0.1811, 0.1814, 0.2771, 0.2782\}, \{0.3794, 0.3968, 0.4152, 0.4389\} \right) \rangle;$$

$$b_3 = \langle [s_{3.9187}, s_{4.9303}], (\{0.4342, 0.4752\}, \{0.5250\}, \{0.3999, 0.4414, 0.4158, 0.4588\}) \rangle;$$

$$b_4 = \langle [s_{2.1231}, s_{3.1639}], \left(\left\{ \begin{array}{l} 0.4750, 0.4857, 0.5075, 0.5408, 0.5472, 0.5611, \\ 0.5012, 0.5100, 0.5285, 0.5894, 0.5937, 0.6032 \end{array} \right\}, \{0.1811\}, \{0.2125, 0.2592, 0.2393, 0.3003\} \right) \rangle.$$

Step 3 (The Value of Comparative Function Is Computed): By employing Definition 8, the score of $\mathrm{E(b}_i)$ $(i = 1, 2, 3, 4)$ are obtained as follows:

$$\mathrm{E(b_1)} = s_{2.4052};\quad \mathrm{E(b_2)} = s_{2.4461};$$
$$\mathrm{E(b_3)} = s_{2.2133};\quad \mathrm{E(b_4)} = s_{1.8531};$$

Step 4 (The Best Alternative Is Selected): By utilizing the comparative method in Definition 9, $\mathrm{E}(b_2) \succ \mathrm{E(b_1)} \succ \mathrm{E(b_3)} \succ \mathrm{E(b_4)}$ can be derived, then the eventual ranking order is $A_2 \succ A_1 \succ A_3 \succ A_4$. Consequently, it is obvious that the best investment alternative is A_2 and the worst investment alternative is A_4.

B. SENSITIVITY ANALYSIS

In this section, we shall discuss the impact of parameters p, q, and ρ with different values regarding the score value

$$R = \left[a_{ij}\right]_{4\times 3} = \begin{bmatrix} \langle [s_4, s_5], (\{0.3, 0.4, 0.5\}, \{0.1\}, \{0.3, 0.4\})\rangle & \langle [s_5, s_6], (\{0.5, 0.6\}, \{0.2, 0.3\}, \{0.3, 0.4\})\rangle & \langle [s_4, s_5], (\{0.2, 0.3\}, \{0.1, 0.2\}, \{0.5, 0.6\})\rangle \\ \langle [s_5, s_6], (\{0.6, 0.7\}, \{0.1, 0.2\}, \{0.2, 0.3\})\rangle & \langle [s_4, s_5], (\{0.6, 0.7\}, \{0.1\}, \{0.3\})\rangle & \langle [s_4, s_5], (\{0.6, 0.7\}, \{0.1, 0.2\}, \{0.1, 0.2\})\rangle \\ \langle [s_5, s_6], (\{0.5, 0.6\}, \{0.4\}, \{0.2, 0.3\})\rangle & \langle [s_4, s_5], (\{0.6\}, \{0.3\}, \{0.4\})\rangle & \langle [s_3, s_4], (\{0.5, 0.6\}, \{0.1\}, \{0.3\})\rangle \\ \langle [s_3, s_4], (\{0.7, 0.8\}, \{0.1\}, \{0.1, 0.2\})\rangle & \langle [s_3, s_4], (\{0.6, 0.7\}, \{0.1\}, \{0.2\})\rangle & \langle [s_5, s_6], (\{0.3, 0.5\}, \{0.2\}, \{0.1, 0.2, 0.3\})\rangle \end{bmatrix}$$

$$B = \left[b_{ij}\right]_{4\times 3} = \begin{bmatrix} \langle [s_4, s_5], (\{0.3, 0.4, 0.5\}, \{0.1\}, \{0.3, 0.4\})\rangle & \langle [s_5, s_6], (\{0.5, 0.6\}, \{0.2, 0.3\}, \{0.3, 0.4\})\rangle & \langle [s_2, s_3], (\{0.5, 0.6\}, \{0.8, 0.9\}, \{0.2, 0.3\})\rangle \\ \langle [s_5, s_6], (\{0.6, 0.7\}, \{0.1, 0.2\}, \{0.2, 0.3\})\rangle & \langle [s_4, s_5], (\{0.6, 0.7\}, \{0.1\}, \{0.3\})\rangle & \langle [s_2, s_3], (\{0.1, 0.2\}, \{0.8, 0.9\}, \{0.6, 0.7\})\rangle \\ \langle [s_5, s_6], (\{0.5, 0.6\}, \{0.4\}, \{0.2, 0.3\})\rangle & \langle [s_4, s_5], (\{0.6\}, \{0.3\}, \{0.4\})\rangle & \langle [s_3, s_4], (\{0.3\}, \{0.9\}, \{0.5, 0.6\})\rangle \\ \langle [s_3, s_4], (\{0.7, 0.8\}, \{0.1\}, \{0.1, 0.2\})\rangle & \langle [s_3, s_4], (\{0.6, 0.7\}, \{0.1\}, \{0.2\})\rangle & \langle [s_1, s_2], (\{0.1, 0.2, 0.3\}, \{0.8\}, \{0.3, 0.5\})\rangle \end{bmatrix}$$

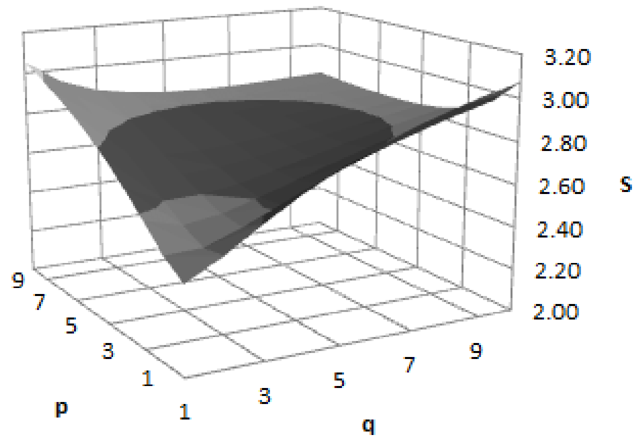

FIGURE 1. Score of alternative A_1 when $p, q \in [1, 10]$, $\rho = 2$ based on MVNULNWBM operator.

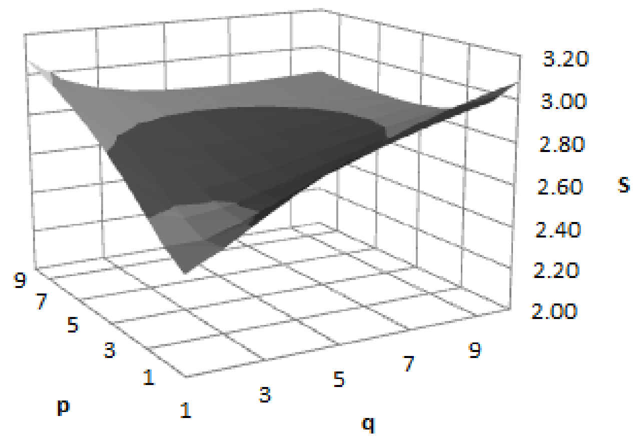

FIGURE 2. Score of alternative A_2 when $p, q \in [1, 10]$, $\rho = 2$ based on MVNULNWBM operator.

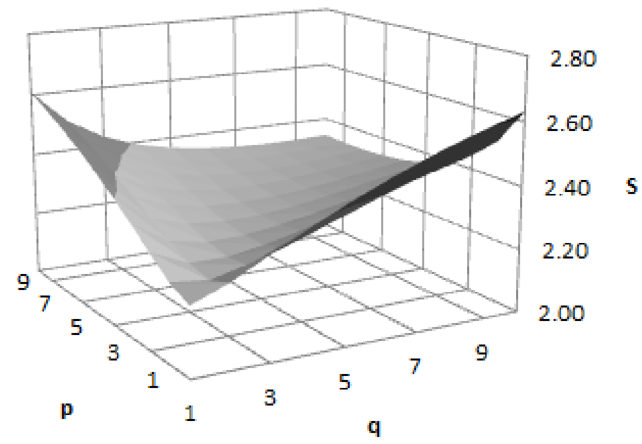

FIGURE 3. Score of alternative A_3 when $p, q \in [1, 10]$, $\rho = 2$ based on MVNULNWBM operator.

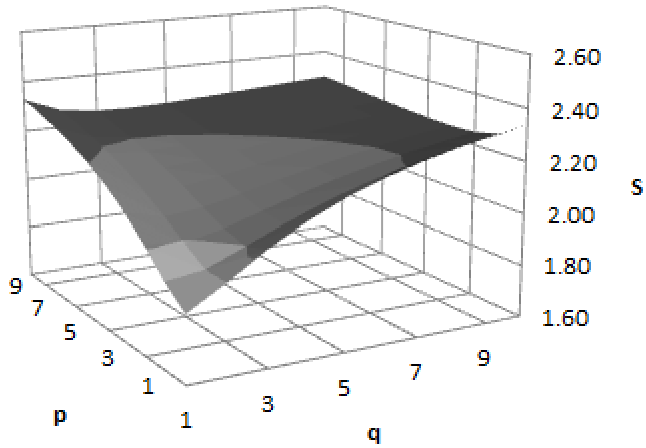

FIGURE 4. Score of alternative A_4 when $p, q \in [1, 10]$, $\rho = 2$ based on MVNULNWBM operator.

and ranking orders utilizing the proposed MVNULNWBM operator.

The scores of alternative $A_i (i = 1, 2, 3, 4)$ are shown in Figures 1-4, where we set ρ be a fixed value, that is $\rho = 2$, and p, q belonging to [1,10] respectively. From Figures 1-4, we can see that the values of score function of all alternatives may be different with different parameters p and q, the appropriate values for parameters p and q can be given by decision makers according to actual needs, which reflects the flexibility of the proposed operator and the MADM method.

The scores and ranking orders of all alternative are presented in Figures 5-6. In Figure 5, we set p and ρ be a fixed value respectively, that is $p = 1, \rho = 2$, and q belonging to [1, 10]. In Figure 6, we set q and ρ be a fixed value respectively, that is $q = 1, \rho = 2$, and p belonging to [1, 10]. From Figures 5-6, we can draw the conclusions that the value of score function on alternative $A_i (i = 1, 2, 3, 4)$ becomes greater and greater with the increasing of parameter p or q, and the score values of different alternative might be different with the same parameters p, q, and ρ, but the ranking orders of all alternatives are same, which is always $A_2 \succ A_1 \succ A_3 \succ A_4$, which reflects the robustness of the proposed operator and the MADM method.

In this subsection, we shall investigate the impact of different parameter ρ regarding the score value and ranking orders when p and q be assigned a fixed value. The results are shown in Figure 7, where we set $p = q = 1$, and ρ is varying from 0.5 to 15.

From Figure 7, we can derived that the value of score function for alternative $A_i (i = 1, 2, 3, 4)$ becomes greater and greater with the increasing of parameter ρ, and the score values of different alternative may differ along with same parameters p and q, meanwhile, the ranking orders of all alternatives may be vary with different parameter ρ. As we can see from Figure 7, the ranking orders of all alternatives is $A_1 \succ A_2 \succ A_3 \succ A_4$ when parameter $\rho \leq 1.7$, the ranking orders of all alternatives is $A_2 \succ A_1 \succ A_3 \succ A_4$ when parameter $\rho \in (1.7, 15)$, and the ranking orders of all alternatives is $A_2 \succ A_3 \succ A_1 \succ A_4$ when parameter $\rho \geq 15$,which reflects the flexibility of the proposed operator based on Dombi operations with a general parameter ρ.

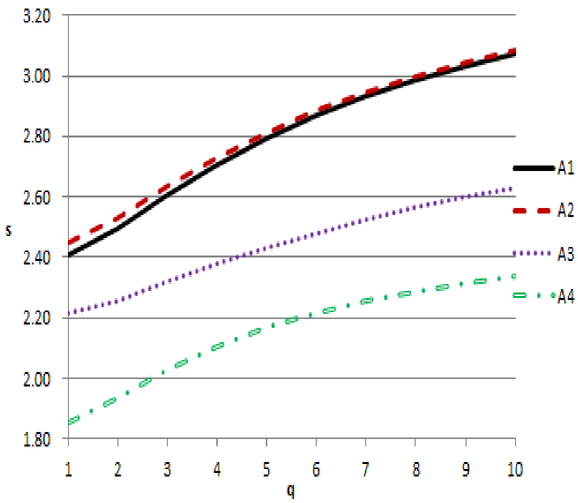

FIGURE 5. Score of all alternatives when $p = 1, q \in [1, 10], \rho = 2$ based on MVNULNWBM operator.

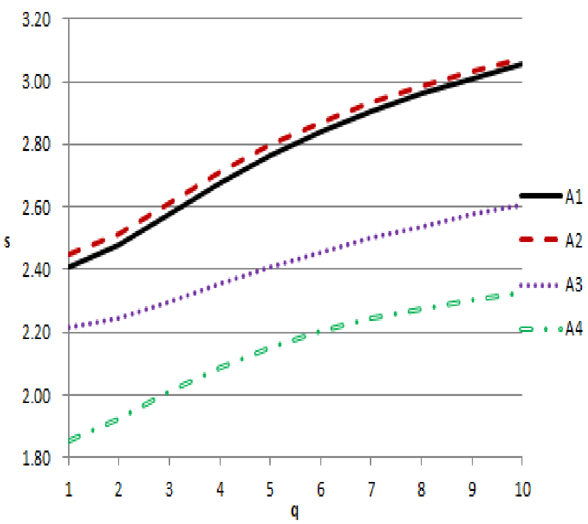

FIGURE 6. Score of all alternatives when $q = 1, p \in [1, 10], \rho = 2$ based on MVNULNWBM operator.

C. COMPARATIVE ANALYSIS

To further demonstrate the effectiveness and flexibility of the proposed aggregation operator for solving MADM problem, a comparative analysis between the proposed method and the other methods [53, 19, 16] is conducted.

The MADGM method proposed by Liu [52] adopted NULNIGWHM and NULNIGGWHM operators, which are on the basis of algebraic operations, and the method is applied under the single valued neutrosophic uncertain linguistic environment. The developed Heronian mean operator by Liu can catch the interrelationships among attributes. However, the algebraic operations are a special case of Dombi operations utilized in our research, and the single valued neutrosophic uncertain linguistic set is also a special case of multiple-valued neutrosophic uncertain linguistic set defined in our paper. Thus, our approach proposed is more generalization than that proposed by Liu.

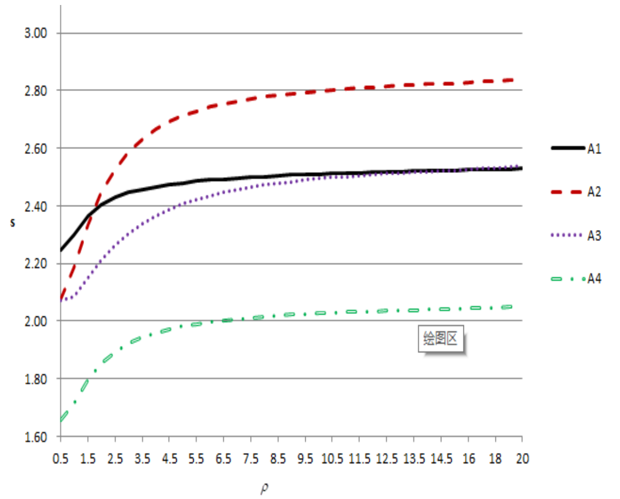

FIGURE 7. Score of all alternatives when $p = q = 1, \rho \in [0.5, 15]$ based on MVNULNWBM operator.

The MADGM method proposed by Ye [19] developed INULWAA and INULWGA operators, which are also on the basis of algebraic operations, and the method is utilized under the interval neutrosophic uncertain linguistic environment. The corresponding weighted arithmetic or geometric averaging operators proposed by Liu cannot capture the interrelationships among attributes. Similarly, the algebraic operational rules are a particular situation of Dombi operational rules utilized in our research. Thus, our method is more general and practical than that proposed by Ye.

The MCDM approach proposed by Li [16] defined MVNLNWBMH operator, which are on the basis of Hamacher operations, and the method is conducted with the multiple-valued neutrosophic linguistic information. The corresponding NWBM operator used by Li can also capture the interrelationships among attributes. The Hamacher operational rules are generalization of the Algebraic operations and Einstein operations, thus, the method proposed by Li is flexible, whereas Li's method can only handle multiple-valued neutrosophic linguistic elements, and our method proposed in this paper can handle more complex multiple-valued neutrosophic uncertain linguistic information. Therefore, our method conducted in this paper is more effective and powerful than that proposed by Li.

The key characteristics of the proposed approach are summarized as below: (1) an MVNULS is more suitable for expressing the indeterminate, inconsistent information in practical MADM or MAGDM problems, the reason is that it combined the multiple-valued neutrosophic set and uncertain linguistic variables takes into account both quantitative and qualitative information. (2) the proposed MVNULNWBM operator is more general and flexible in the aggregating process, the reason is that it is based on the operations of Dombi with a general parameter ρ. (3) the proposed approach is more effective and practical in solving MADM problem, the reason is that it utilizing NWBM operator can capture the interrelationships among attributes, and the parameters p

and q can be determined by decision makers according to the actual situation.

VII. CONCLUSION

Multiple-valued neutrosophic uncertain linguistic set (MVNULS) has virtues of both multiple-valued neutrosophic set (MVNS) and uncertain linguistic set (ULS), therefore, it is more practical and elastic in expressing the increasing complex decision making information in real life. In this paper, we firstly proposed some new notions, that are, multiple-valued neutrosophic uncertain linguistic set (MVNULS) and MVNULN. Then, we explored the operational rules of MVNULNs based on the operations of Dombi, and introduced the score, accuracy, and certainty comparative functions for ranking MVNULNs. Considering the advantages of NWBM operator, we extended it to multiple-valued neutrosophic uncertain linguistic environment and developed MVNULNWBM operator, and discussed the promising properties of the proposed operator. According to the aforementioned content, we investigated a new MADM method for managing multiple-valued neutrosophic uncertain linguistic information. Finally, an illustrative instance was applied to manage multiple-valued neutrosophic uncertain linguistic information and demonstrated the flexibility and effectiveness of the proposed approach, meanwhile, the sensitivity analysis of different parameters and comparative analysis of the proposed approach with the existing approaches were also conducted.

In future research, we shall further investigate the application of different aggregating operator for solving various decision-making problems, for instance pattern recognition, medical diagnosis,information fusion, and so on.

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