

See discussions, stats, and author profiles for this publication at: <https://www.researchgate.net/publication/372788061>

Solving Some Integral and Fractional Differential Equations via Neutrosophic Pentagonal Metric Space

Article · August 2023

DOI: 10.3390/axioms12080758

CITATIONS

0

5 authors, including:



[Zoran Mitrovic](#)

University of Banja Luka

203 PUBLICATIONS 1,341 CITATIONS

[SEE PROFILE](#)

Some of the authors of this publication are also working on these related projects:



Fixed Point Results and Applications on Generalized Abstract Spaces [View project](#)



Metode funkcionalne analize u u varijacionim problemima kao i u analizi apstraktnih prostora [View project](#)

Article

Solving Some Integral and Fractional Differential Equations via Neutrosophic Pentagonal Metric Space

Gunaseelan Mani ¹ , Poornavel Subbarayan ¹, Zoran D. Mitrović ^{2,*} , Ahmad Aloqaily ^{3,4}  and Nabil Mlaiki ³ 

¹ Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai 602105, Tamil Nadu, India; mathsguna@yahoo.com (G.M.); poornavels.sse@saveetha.com (P.S.)

² Faculty of Electrical Engineering, University of Banja Luka, Patre 5, 78000 Banja Luka, Bosnia and Herzegovina

³ Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia; maloqaily@psu.edu.sa (A.A.)

⁴ School of Computer, Data and Mathematical Sciences, Western Sydney University, Sydney 2150, Australia

* Correspondence: zoran.mitrovic@etf.unibl.org

Abstract: In this paper, we first introduce the notion of neutrosophic pentagonal metric space. We prove several interesting results for some classes contraction mappings and prove some fixed point theorems in neutrosophic pentagonal metric space. Finally, we prove the uniqueness and existence of the integral equation and fractional differential equation to support our main result.

Keywords: neutrosophic metric space; neutrosophic pentagonal metric space; fixed point results; integral equation; fractional differential equation

MSC: 47H10; 54H25



Citation: Mani, G.; Subbarayan, P.; Mitrović, Z.D.; Aloqaily, A.; Mlaiki, N. Solving Some Integral and Fractional Differential Equations via Neutrosophic Pentagonal Metric Space. *Axioms* **2023**, *12*, 758. <https://doi.org/10.3390/axioms12080758>

Academic Editors: Vladimir Rakocevic and Hsien-Chung Wu

Received: 17 May 2023

Revised: 13 July 2023

Accepted: 27 July 2023

Published: 1 August 2023



Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

A fuzzy set is a category of items with a continuum of membership levels between zero and one. The concept of a fuzzy set was first introduced by Zadeh [1], who also provided a useful starting point for the development of a conceptual framework that, while similar to the framework used for sets in many ways, is more general and may have a wider range of applications, particularly in the fields of pattern classification and information processing. The study of statistical metric spaces and an examination of the continuous characteristics of the distance function were both continued by Schweizer and Sklar in [2]. Fuzzy metric spaces (shortly, Fuzzy MS) were proposed by Kramosil and Michálek in [3], who also expanded on the notion of convergence that is typically used to determine whether a generalization is appropriate. On fuzzy double-controlled MSs, Azmi [4] developed the novel idea of $(\alpha - \Pi)$ -fuzzy contractive mappings and illustrated various fixed-point results. In the framework of extended fuzzy b -MSs, various generalized fixed point findings of Banach and Ćirić type are established by Rome et al. [5]. A Hausdorff fuzzy b -MS is described by Batul et al. [6]. A few fixed point results for multivalued mappings in G -complete fuzzy b -MSs that satisfy an appropriate contractiveness criterion are established using the novel idea. Numerous fixed point theorems in fuzzy b -MSs make up Rakić et al. [7]. They provided a necessary condition for a sequence to be Cauchy in the fuzzy b -MS, which was a significant result. By using a control function $\alpha(x, y)$ of the right-hand side of the b -triangle inequality, Mlaiki [8] created a new extension of b -MSs known as controlled metric type spaces. Controlled fuzzy MS is a brand-new development of Sezen's [9] work on fuzzy metrics. Additionally, they demonstrated a new fixed point theorem and a Banach-type fixed point theorem for some fulfilling self-mappings. see [10–16]. Grabiec [17] extended two fixed point theorems of Banach and Edelstein to contractive mappings of complete and compact fuzzy MSs, respectively. In

order to established some fixed point theorems, Rehman et al. [18] defined α -admissible and α -II-fuzzy cone contraction in fuzzy cone MS. Finally, he used theoretical results to show that a nonlinear integral equation has a solution.

Park [19] created an intuitionistic fuzzy MS delta membership and nonmembership functions. The concept of an intuitionistic fuzzy b-MS was first proposed by Konwar [20], who also demonstrated a number of fixed point theorems. Neutrosophic MSs, which are initialized to handle membership, nonmembership, and naturalness, were introduced by Kirişci and Simsek in their [21] paper. Simsek and Kirişci [22] demonstrated various fixed point results in the context of neutrosophic MSs. Fixed point findings in neutrosophic MSs were demonstrated by Sowndrarajan et al. [23]. Itoh [24] showed a usage for random differential equations in Banach spaces.

In this article, we introduce the neutrosophic pentagonal MS (also known as NPMS) and demonstrate a few fixed point conclusions. The following are the primary goals of this work:

1. To introduce the notion of neutrosophic pentagonal MS;
2. To prove several fixed-point theorems for contraction mappings;
3. Show the existence of a unique solution of an integral equation;
4. Show the existence of a unique solution of a fractional differential equation.

2. Preliminaries

Here, we'll go over some fundamental terms that will be useful for the key results.

Definition 1 ([19]). Let $\star: [0, 1]^2 \rightarrow [0, 1]$ be a binary operation is said to be a continuous triangle norm if:

- (1) $\hat{\omega} \star \check{h} = \check{h} \star \hat{\omega}, \forall \hat{\omega}, \check{h} \in [0, 1];$
- (2) $\star$ is continuous;
- (3) $\hat{\omega} \star 1 = \hat{\omega}, \forall \hat{\omega} \in [0, 1];$
- (4) $(\hat{\omega} \star \check{h}) \star \tilde{\mu} = \hat{\omega} \star (\check{h} \star \tilde{\mu}), \forall \hat{\omega}, \check{h}, \tilde{\mu} \in [0, 1];$
- (5) If $\hat{\omega} \leq \tilde{\mu}$ and $\check{h} \leq \varsigma, \forall \hat{\omega}, \check{h}, \tilde{\mu}, \varsigma \in [0, 1]$, then $\hat{\omega} \star \check{h} \leq \tilde{\mu} \star \varsigma$.

Definition 2 ([19]). Let $\bullet: [0, 1]^2 \rightarrow [0, 1]$ be a binary operation is said to be a continuous triangle co-norm if:

- (1) $\hat{\omega} \bullet \check{h} = \check{h} \bullet \hat{\omega}, \forall \hat{\omega}, \check{h} \in [0, 1];$
- (2) $\bullet$ is continuous;
- (3) $\hat{\omega} \bullet 0 = 0, \forall \hat{\omega} \in [0, 1];$
- (4) $(\hat{\omega} \bullet \check{h}) \bullet \tilde{\mu} = \hat{\omega} \bullet (\check{h} \bullet \tilde{\mu}), \forall \hat{\omega}, \check{h}, \tilde{\mu} \in [0, 1];$
- (5) If $\hat{\omega} \leq \tilde{\mu}$ and $\check{h} \leq \varsigma$, with $\hat{\omega}, \check{h}, \tilde{\mu}, \varsigma \in [0, 1]$, then $\hat{\omega} \bullet \check{h} \leq \tilde{\mu} \bullet \varsigma$.

Definition 3 ([24]). Let $\chi, \mathbf{F}: K \times K \rightarrow [1, +\infty)$ be given two non-comparable functions, if $\partial: K \times K \rightarrow [0, +\infty)$ satisfies axioms:

- (a) $\partial(U, \mathcal{W}) = 0$ iff $U = \mathcal{W};$
- (b) $\partial(U, \mathcal{W}) = \partial(\mathcal{W}, U);$
- (c) $\partial(U, \mathcal{W}) \leq \chi(U, \mathcal{E})\partial(U, \mathcal{E}) + \mathbf{F}(\mathcal{E}, \mathcal{W})\partial(\mathcal{E}, \mathcal{W});$

$\forall U, \mathcal{W}, \mathcal{E} \in K$, then, (K, ∂) is known to be a double controlled MS (shortly, DCMS).

Definition 4 ([25]). Let $K \neq \emptyset$ and $\chi, \mathbf{F}: K \times K \rightarrow [1, +\infty)$ be known non-comparable functions. And $\star$ is a continuous t-norm also Λ be a fuzzy set on $K \times K \times (0, +\infty)$ is said to be fuzzy double controlled metric on $K, \forall U, \mathcal{W}, \mathcal{E} \in K$ if:

- (i) $\Lambda(U, \mathcal{W}, 0) = 0;$
- (ii) $\Lambda(U, \mathcal{W}, \Gamma) = 1 \forall \Gamma > 0, \Leftrightarrow U = \mathcal{W};$
- (iii) $\Lambda(U, \mathcal{W}, \Gamma) = \Lambda(\mathcal{W}, U, \Gamma);$

- (iv) $\Lambda(U, \mathcal{E}, \Gamma + \Gamma) \geq \Lambda\left(U, \mathcal{W}, \frac{\Gamma}{\chi(U, \mathcal{W})}\right) \star \Lambda\left(\mathcal{W}, \mathcal{E}, \frac{\Gamma}{\mathbb{F}(\mathcal{W}, \mathcal{E})}\right);$
(v) $\Lambda(U, \mathcal{W}, \cdot): (0, +\infty) \rightarrow [0, 1]$ is continuous on left.

Then, $(K, \Lambda, \mathcal{N}, \star)$ is known to be a fuzzy DCMS.

Definition 5 ([20]). Let $K \neq \emptyset$. Let $\star, \bullet$ are the continuous t -norm, continuous t -co-norm respectively, $\flat \geq 1$ and $\Lambda, \mathcal{N}$ be fuzzy sets on $K \times K \times (0, +\infty)$. If $(K, \Lambda, \mathcal{N}, \star, \bullet)$ fullfils all $U, \mathcal{W} \in K$ and $\check{\Gamma}, \Gamma > 0$:

- (I) $\Lambda(U, \mathcal{W}, \Gamma) + \Pi(U, \mathcal{W}, \Gamma) \leq 1;$
(II) $0 < \Lambda(U, \mathcal{W}, \Gamma);$
(III) $\Lambda(U, \mathcal{W}, \Gamma) = 1 \Leftrightarrow U = \mathcal{W};$
(IV) $\Lambda(U, \mathcal{W}, \Gamma) = \Lambda(\mathcal{W}, U, \Gamma);$
(V) $\Lambda(U, \mathcal{E}, \flat(\Gamma + \check{\Gamma})) \geq \Lambda(U, \mathcal{W}, \Gamma) \star \Lambda(\mathcal{W}, \mathcal{E}, \check{\Gamma});$
(VI) $\Lambda(U, \mathcal{W}, \cdot)$ is a non-decreasing function of $\mathbb{R}^+$ and $\lim_{\Gamma \rightarrow +\infty} \Lambda(U, \mathcal{W}, \Gamma) = 1;$
(VII) $0 < \Pi(U, \mathcal{W}, \Gamma);$
(VIII) $\Pi(U, \mathcal{W}, \Gamma) = 0 \Leftrightarrow U = \mathcal{W};$
(IX) $\Pi(U, \mathcal{W}, \Gamma) = \Pi(\mathcal{W}, U, \Gamma);$
(X) $\Pi(U, \mathcal{E}, \flat(\Gamma + \check{\Gamma})) \leq \Pi(U, \mathcal{W}, \Gamma) \bullet \Pi(\mathcal{W}, \mathcal{E}, \check{\Gamma});$
(XI) $\Pi(U, \mathcal{W}, \cdot)$ is a non-increasing function of $\mathbb{R}^+$ and $\lim_{\Gamma \rightarrow +\infty} \Pi(U, \mathcal{W}, \Gamma) = 0,$

Then, $(K, \Lambda, \Pi, \star, \bullet)$ is an intuitionistic fuzzy $\flat$ -MS.

Definition 6 ([21]). Let $K \neq \emptyset$, $\star, \bullet$ are the continuous t -norm, continuous t -co-norm respectively, and $\Lambda, \Pi, \mathcal{S}$ are neutrosophic sets (shortly, N -sets) on $K \times K \times (0, +\infty)$ is known to be a neutrosophic metric on K , if for all $U, \mathcal{W}, \mathcal{E} \in K$, the following axioms are fulfilled:

- (1) $\Lambda(U, \mathcal{W}, \Gamma) + \Pi(U, \mathcal{W}, \Gamma) + \mathcal{S}(U, \mathcal{W}, \Gamma) \leq 3;$
(2) $0 < \Lambda(U, \mathcal{W}, \Gamma);$
(3) $\Lambda(U, \mathcal{W}, \Gamma) = 1 \ \forall \ \Gamma > 0, \Leftrightarrow U = \mathcal{W};$
(4) $\Lambda(U, \mathcal{W}, \Gamma) = \Lambda(\mathcal{W}, U, \Gamma);$
(5) $\Lambda(U, \mathcal{E}, \Gamma + \check{\Gamma}) \geq \Lambda(U, \mathcal{W}, \Gamma) \star \Lambda(\mathcal{W}, \mathcal{E}, \check{\Gamma});$
(6) $\Lambda(U, \mathcal{W}, \cdot): (0, +\infty) \rightarrow [0, 1]$ is continuous and $\lim_{\Gamma \rightarrow +\infty} \Lambda(U, \mathcal{W}, \Gamma) = 1;$
(7) $1 < \Pi(U, \mathcal{W}, \Gamma);$
(8) $\Pi(U, \mathcal{W}, \Gamma) = 0 \ \forall \ \Gamma > 0, \Leftrightarrow U = \mathcal{W};$
(9) $\Pi(U, \mathcal{W}, \Gamma) = \Pi(\mathcal{W}, U, \Gamma);$
(10) $\Pi(U, \mathcal{E}, \Gamma + \check{\Gamma}) \leq \Pi(U, \mathcal{W}, \Gamma) \bullet \Pi(\mathcal{W}, \mathcal{E}, \check{\Gamma});$
(11) $\Pi(U, \mathcal{W}, \cdot): (0, +\infty) \rightarrow [0, 1]$ is continuous and $\lim_{\Gamma \rightarrow +\infty} \Pi(U, \mathcal{W}, \Gamma) = 0;$
(12) $1 < \mathcal{S}(U, \mathcal{W}, \Gamma);$
(13) $\mathcal{S}(U, \mathcal{W}, \Gamma) = 0 \ \forall \ \Gamma > 0, \Leftrightarrow U = \mathcal{W};$
(14) $\mathcal{S}(U, \mathcal{W}, \Gamma) = \mathcal{S}(\mathcal{W}, U, \Gamma);$
(15) $\mathcal{S}(U, \mathcal{E}, \Gamma + \check{\Gamma}) \leq \mathcal{S}(U, \mathcal{W}, \Gamma) \bullet \mathcal{S}(\mathcal{W}, \mathcal{E}, \check{\Gamma});$
(16) $\mathcal{S}(U, \mathcal{W}, \cdot): (0, +\infty) \rightarrow [0, 1]$ is continuous and $\lim_{\Gamma \rightarrow +\infty} \mathcal{S}(U, \mathcal{W}, \Gamma) = 0;$
(17) If $\Gamma \leq 0$, then $\Lambda(U, \mathcal{W}, \Gamma) = 0, \Pi(U, \mathcal{W}, \Gamma) = 0;$

Then, $(K, \Lambda, \Pi, \mathcal{S}, \star, \bullet)$ is known to be a neutrosophic MS.

In this article, we define NPMS and demonstrate fixed point theorems.

3. Main Results

This section presents NPMS and illustrates some fixed-point theorems.

Definition 7. Let $K \neq \emptyset$ and function $\chi: K \times K \rightarrow [1, +\infty)$ be non-comparable, $\star, \bullet$ are the continuous t -norm, continuous t -co-norm respectively, and $\Lambda, \Pi, \mathcal{M}$ be N -sets on $K \times K \times (0, +\infty)$ is known to be a neutrosophic pentagonal metric on K , if for any $U, \mathcal{E} \in K$ and all distinct $\check{\mathcal{X}}, \mathcal{W}, \mathcal{E} \in K$, the following axioms are fulfilled:

- (i) $\Lambda(U, \mathcal{W}, \Gamma) + \Pi(U, \mathcal{W}, \Gamma) + \mathcal{M}(U, \mathcal{W}, \Gamma) \leq 3;$

- (ii) $0 < \Lambda(U, \mathcal{W}, \Gamma);$
- (iii) $\Lambda(U, \mathcal{W}, \Gamma) = 1 \ \forall \ \Gamma > 0, \Leftrightarrow U = \mathcal{W};$
- (iv) $\Lambda(U, \mathcal{W}, \Gamma) = \Lambda(\mathcal{W}, U, \Gamma);$
- (v) $\Lambda(U, \mathcal{L}, \Gamma + \check{g} + \check{k} + \check{e}) \geq \Lambda(U, \mathcal{W}, \Gamma) \star \Lambda(\mathcal{W}, \check{x}, \check{g}) \star \Lambda(\check{x}, \check{y}, \check{k})$
 $\star \Lambda(\check{y}, \mathcal{L}, \check{e});$
- (vi) $\Lambda(U, \mathcal{W}, \cdot): (0, +\infty) \rightarrow [0, 1]$ is continuous and $\lim_{\Gamma \rightarrow +\infty} \Lambda(U, \mathcal{W}, \Gamma) = 1;$
- (vii) $1 < \Pi(U, \mathcal{W}, \Gamma);$
- (viii) $\Pi(U, \mathcal{W}, \Gamma) = 0 \ \forall \ \Gamma > 0, \Leftrightarrow U = \mathcal{W};$
- (ix) $\Pi(U, \mathcal{W}, \Gamma) = \Pi(\mathcal{W}, U, \Gamma);$
- (x) $\Pi(U, \mathcal{L}, \Gamma + \check{g} + \check{k} + \check{e}) \leq \Pi(U, \mathcal{W}, \Gamma) \bullet \Pi(\mathcal{W}, \check{x}, \check{g}) \bullet \Pi(\check{x}, \check{y}, \check{k})$
 $\bullet \Pi(\check{y}, \mathcal{L}, \check{e});$
- (xi) $\Pi(U, \mathcal{W}, \cdot): (0, +\infty) \rightarrow [0, 1]$ is continuous and $\lim_{\Gamma \rightarrow +\infty} \Pi(U, \mathcal{W}, \Gamma) = 0;$
- (xii) $1 < \mathcal{M}(U, \mathcal{W}, \Gamma);$
- (xiii) $\mathcal{M}(U, \mathcal{W}, \Gamma) = 0 \ \forall \ \Gamma > 0, \Leftrightarrow U = \mathcal{W};$
- (xiv) $\mathcal{M}(U, \mathcal{W}, \Gamma) = \mathcal{M}(\mathcal{W}, U, \Gamma);$
- (xv) $\mathcal{M}(U, \mathcal{L}, \Gamma + \check{g} + \check{k} + \check{e}) \leq \mathcal{M}(U, \mathcal{W}, \Gamma) \bullet \mathcal{M}(\mathcal{W}, \check{x}, \check{g}) \bullet \mathcal{M}(\check{x}, \check{y}, \check{k})$
 $\bullet \mathcal{M}(\check{y}, \mathcal{L}, \check{e});$
- (xvi) $\mathcal{M}(U, \mathcal{W}, \cdot): (0, +\infty) \rightarrow [0, 1]$ is continuous and $\lim_{\Gamma \rightarrow +\infty} \mathcal{M}(U, \mathcal{W}, \Gamma) = 0;$
- (xvii) If $\Gamma \leq 0$, then $\Lambda(U, \mathcal{W}, \Gamma) = 0, \Pi(U, \mathcal{W}, \Gamma) = 1$ and $\mathcal{S}(U, \mathcal{W}, \Gamma) = 1.$

Then, $(K, \Lambda, \Pi, \mathcal{M}, \star, \bullet)$ is said to be a NPMS.

Example 1. Let $K = \{1, 2, 3, 4\}$. Define $\Lambda, \Pi, \mathcal{M}: K \times K \times (0, +\infty) \rightarrow [0, 1]$ as

$$\Lambda(U, \mathcal{W}, \Gamma) = \begin{cases} 1, & \text{if } U = \mathcal{W} \\ \frac{\Gamma}{\Gamma + \max\{U, \mathcal{W}\}}, & \text{otherwise,} \end{cases}$$

$$\Pi(U, \mathcal{W}, \Gamma) = \begin{cases} 0, & \text{if } U = \mathcal{W} \\ \frac{\max\{U, \mathcal{W}\}}{\Gamma + \max\{U, \mathcal{W}\}}, & \text{otherwise,} \end{cases}$$

and

$$\mathcal{M}(U, \mathcal{W}, \Gamma) = \begin{cases} 0, & \text{if } U = \mathcal{W} \\ \frac{\max\{U, \mathcal{W}\}}{\Gamma}, & \text{otherwise,} \end{cases}$$

Then, $(K, \Lambda, \Pi, \mathcal{M}, \star, \bullet)$ is a NPMS with continuous t -norm $\hat{\omega} \star \check{h} = \hat{\omega} \check{h}$ and continuous t -co-norm, $\hat{\omega} \bullet \check{f} = \max\{\hat{\omega}, \check{f}\}.$

Proof. Now, we prove the conditions (v), (x) and (xv) others are obvious.

Let $U = 1, \mathcal{W} = 2, \check{x} = 3, \mathcal{L} = 4$ and $\rho = 5$. Then

$$\Lambda(1, 5, \Gamma + \check{g} + \check{k} + \check{q}) = \frac{\Gamma + \check{g} + \check{k} + \check{q}}{\Gamma + \check{g} + \check{k} + \check{q} + \max\{1, 5\}} = \frac{\Gamma + \check{g} + \check{k} + \check{q}}{\Gamma + \check{g} + \check{k} + \check{q} + 5}.$$

On the other hand,

$$\Lambda(1, 2, \Gamma) = \frac{\Gamma}{\Gamma + \max\{1, 2\}} = \frac{\Gamma}{\Gamma + 2},$$

$$\Lambda(2, 3, \check{g}) = \frac{\check{g}}{\check{g} + \max\{2, 3\}} = \frac{\check{g}}{\check{g} + 3},$$

$$\Lambda(3, 4, \check{k}) = \frac{\check{k}}{\check{k} + \max\{3, 4\}} = \frac{\check{k}}{\check{k} + 4}$$

and

$$\Lambda(4, 5, \check{q}) = \frac{\check{q}}{\check{q} + \max\{4, 5\}} = \frac{\check{q}}{\check{q} + 5}.$$

i.e.,

$$\frac{\Gamma + \check{g} + \check{k} + \check{q}}{\Gamma + \check{g} + \check{k} + \check{q} + 5} \geq \frac{\Gamma}{\Gamma + 2} \frac{\check{g}}{\check{g} + 3} \frac{\check{k}}{\check{k} + 4} \frac{\check{q}}{\check{q} + 5}.$$

Then it satisfies all $\Gamma, \check{g}, \check{k}, \check{q} > 0$. Hence,

$$\Lambda(U, \rho, \Gamma + \check{g} + \check{k} + \check{q}) \geq \Lambda(U, \mathcal{W}, \Gamma) \star \Lambda(\mathcal{W}, \check{x}, \check{g}) \star \Lambda(\check{x}, \mathcal{L}, \check{k}) \star \Lambda(\mathcal{L}, \rho, \check{q}).$$

Now,

$$\Pi(1, 5, \Gamma + \check{g} + \check{k} + \check{q}) = \frac{\max\{1, 5\}}{\Gamma + \check{g} + \check{k} + \check{q} + \max\{1, 5\}} = \frac{5}{\Gamma + \check{g} + \check{k} + \check{q} + 5}.$$

On the other hand,

$$\Pi(1, 2, \Gamma) = \frac{\max\{1, 2\}}{\Gamma + \max\{1, 2\}} = \frac{2}{\Gamma + 2},$$

$$\Pi(2, 3, \check{g}) = \frac{\max\{2, 3\}}{\check{g} + \max\{2, 3\}} = \frac{3}{\check{g} + 3},$$

$$\Pi(3, 4, \check{k}) = \frac{\max\{3, 4\}}{\check{k} + \max\{3, 4\}} = \frac{4}{\check{k} + 4}$$

and

$$\Pi(4, 5, \check{q}) = \frac{\max\{4, 5\}}{\check{q} + \max\{4, 5\}} = \frac{5}{\check{q} + 5}$$

i.e.,

$$\frac{5}{\Gamma + \check{g} + \check{k} + \check{q} + 5} \leq \max \left\{ \frac{2}{\Gamma + 2}, \frac{3}{\check{g} + 3}, \frac{4}{\check{k} + 4}, \frac{5}{\check{q} + 5} \right\}.$$

Then it satisfies all $\Gamma, \check{g}, \check{k}, \check{q} > 0$. Hence,

$$\Pi(U, \rho, \Gamma + \check{g} + \check{k} + \check{q}) \leq \Pi(U, \mathcal{W}, \Gamma) \bullet \Pi(\check{x}, \mathcal{L}, \check{g}) \bullet \Pi(\check{x}, \mathcal{L}, \check{w}) \bullet \Pi(\mathcal{L}, \rho, \check{q}).$$

Now,

$$\mathcal{M}(1, 5, \Gamma + \check{g} + \check{k} + \check{q}) = \frac{\max\{1, 5\}}{\Gamma + \check{g} + \check{k} + \check{q}} = \frac{5}{\Gamma + \check{g} + \check{k} + \check{q}}.$$

On the other hand,

$$\mathcal{M}(1, 2, \Gamma) = \frac{\max\{1, 2\}}{\Gamma} = \frac{2}{\Gamma},$$

$$\mathcal{M}(2, 3, \check{g}) = \frac{\max\{2, 3\}}{\check{g}} = \frac{3}{\check{g}},$$

$$\mathcal{M}(3, 4, \check{k}) = \frac{\max\{3, 4\}}{\check{k}} = \frac{4}{\check{k}}$$

and

$$\mathcal{M}(4, 5, \check{q}) = \frac{\max\{4, 5\}}{\check{q}} = \frac{5}{\check{q}}$$

i.e.,

$$\frac{5}{\Gamma + \check{g} + \check{k} + \check{q}} \leq \max \left\{ \frac{2}{\Gamma}, \frac{3}{\check{g}}, \frac{4}{\check{k}}, \frac{5}{\check{q}} \right\}.$$

Then it satisfies all $\Gamma, \check{g}, \check{k}, \check{q} > 0$. Hence,

$$\mathcal{M}(U, \rho, \Gamma + \check{g} + \check{k} + \check{q}) \leq \mathcal{M}(U, \mathcal{W}, \Gamma) \bullet \mathcal{M}(\mathcal{W}, \check{x}, \check{g}) \bullet \mathcal{M}(\check{x}, \check{L}, \check{k}) \bullet \mathcal{M}(\check{L}, \rho, \check{q}).$$

Hence, $(K, \Lambda, \Pi, \mathcal{M}, \star, \bullet)$ is a NPMS. $\square$

Remark 1. The above example satisfies for continuous t -norm $\hat{\omega} \star \bar{f} = \min\{\hat{\omega}, \bar{f}\}$ and continuous t -co-norm $\hat{\omega} \bullet \bar{f} = \max\{\hat{\omega}, \bar{f}\}$.

Definition 8. Let $(K, \Lambda, \Pi, \mathcal{M}, \star, \bullet)$ is a NPMS, an open ball is then defined $\mathcal{M}(U, r, \Gamma)$ with center U , radius $r, 0 < r < 1$ and $\Gamma > 0$ as follows:

$$\mathcal{M}(U, r, \Gamma) = \{\mathcal{W} \in K: \Lambda(U, \mathcal{W}, \Gamma) > 1 - r, \Pi(U, \mathcal{W}, \Gamma) < r, \mathcal{M}(U, \mathcal{W}, \Gamma) < r\}.$$

Definition 9. Let $(K, \Lambda, \Pi, \mathcal{M}, \star, \bullet)$ is a NPMS and $\{U_{\beta_1}\}$ be a sequence in K . Then $\{U_{\beta_1}\}$ is called:

(a) a convergent if exists $U \in K$ such that

$$\lim_{\beta_1 \rightarrow +\infty} \Lambda(U_{\beta_1}, U, \Gamma) = 1, \lim_{\beta_1 \rightarrow +\infty} \Pi(U_{\beta_1}, U, \Gamma) = 0, \lim_{\beta_1 \rightarrow +\infty} \mathcal{M}(U_{\beta_1}, U, \Gamma) = 0, \forall \Gamma > 0,$$

(b) a Cauchy sequence, if for each $\bar{f} > 0, \Gamma > 0$, exists $\beta_{1_0} \in \mathbb{N}$ such that

$$\Lambda(U_{\beta_1}, U_{\beta_1+q}, \Gamma) \geq 1 - \bar{f}, \Pi(U_{\beta_1}, U_{\beta_1+q}, \Gamma) \leq \bar{f}, \Pi(U_{\beta_1}, U_{\beta_1+q}, \Gamma) \leq \bar{f}, \text{ for all } \beta_1, m \geq \beta_{1_0},$$

If every Cauchy sequence convergent in K , then $(K, \Lambda, \Pi, \mathcal{M}, \star, \bullet)$ is said to be complete NPMS.

Lemma 1. Let $\{U_{\beta_1}\}$ be a Cauchy sequence in NPMS $(K, \Lambda, \Pi, \mathcal{M}, \star, \bullet)$ such that $U_{\beta_1} \neq U_m$ whenever $m, \beta_1 \in \mathbb{N}$ with $\beta_1 \neq m$. Then the sequence $\{U_{\beta_1}\}$ can converge to, at most, one limit point.

Proof. Contrarily, suppose that $U_{\beta_1} \rightarrow U, U_{\beta_1} \rightarrow \mathcal{W}$, and $U \neq \mathcal{W}$.

Then, $\lim_{\beta_1 \rightarrow +\infty} \Lambda(U_{\beta_1}, U, \Gamma) = 1, \lim_{\beta_1 \rightarrow +\infty} \Pi(U_{\beta_1}, U, \Gamma) = 0,$

$\lim_{\beta_1 \rightarrow +\infty} \mathcal{M}(U_{\beta_1}, U, \Gamma) = 0$, and $\lim_{\beta_1 \rightarrow +\infty} \Lambda(U_{\beta_1}, \mathcal{W}, \Gamma) = 1,$

$\lim_{\beta_1 \rightarrow +\infty} \Pi(U_{\beta_1}, \mathcal{W}, \Gamma) = 0, \lim_{\beta_1 \rightarrow +\infty} \mathcal{M}(U_{\beta_1}, \mathcal{W}, \Gamma) = 0$, for all $\Gamma > 0$. Suppose

$$\begin{aligned}
\Lambda(U, \mathcal{W}, \Gamma) &\geq \Lambda\left(U, U_{\beta_1}, \frac{\Gamma}{4}\right) \star \Lambda\left(U_{\beta_1}, U_{\beta_1+1}, \frac{\Gamma}{4}\right) \star \Lambda\left(U_{\beta_1+1}, U_{\beta_1+2}, \frac{\Gamma}{4}\right) \star \Lambda\left(U_{\beta_1+2}, \mathcal{W}, \frac{\Gamma}{4}\right) \\
&\rightarrow 1 \star 1 \star 1 \star 1, \quad \text{as } \beta_1 \rightarrow +\infty, \\
\Pi(U, \mathcal{W}, \Gamma) &\leq \Pi\left(U, U_{\beta_1}, \frac{\Gamma}{4}\right) \bullet \Pi\left(U_{\beta_1}, U_{\beta_1+1}, \frac{\Gamma}{4}\right) \bullet \Pi\left(U_{\beta_1+1}, U_{\beta_1+2}, \frac{\Gamma}{4}\right) \bullet \Pi\left(U_{\beta_1+2}, \mathcal{W}, \frac{\Gamma}{4}\right) \\
&\rightarrow 0 \bullet 0 \bullet 0 \bullet 0, \quad \text{as } \beta_1 \rightarrow +\infty, \\
\mathcal{M}(U, \mathcal{W}, \Gamma) &\leq \mathcal{M}\left(U, U_{\beta_1}, \frac{\Gamma}{4}\right) \bullet \mathcal{M}\left(U_{\beta_1}, U_{\beta_1+1}, \frac{\Gamma}{4}\right) \bullet \mathcal{M}\left(U_{\beta_1+1}, U_{\beta_1+2}, \frac{\Gamma}{4}\right) \\
&\bullet \mathcal{M}\left(U_{\beta_1+2}, \mathcal{W}, \frac{\Gamma}{4}\right) \\
&\rightarrow 0 \bullet 0 \bullet 0 \bullet 0, \quad \text{as } \beta_1 \rightarrow +\infty.
\end{aligned}$$

That is $\Lambda(U, \mathcal{W}, \Gamma) \geq 1 \star 1 \star 1 \star 1 = 1$, $\Pi(U, \mathcal{W}, \Gamma) \leq 0 \bullet 0 \bullet 0 \bullet 0 = 0$ and $\mathcal{M}(U, \mathcal{W}, \Gamma) \leq 0 \bullet 0 \bullet 0 \bullet 0 = 0$. Hence $U = \mathcal{W}$, i.e., the sequence converges to at most one limit point. $\square$

Lemma 2. Let $(K, \Lambda, \Pi, \mathcal{M}, \star, \bullet)$ is a NPMS. If for some $0 < \natural < 1$ and for any $U, \mathcal{W} \in K$, $\Gamma > 0$,

$$\Lambda(U, \mathcal{W}, \Gamma) \geq \Lambda\left(U, \mathcal{W}, \frac{\Gamma}{\natural}\right), \Pi(U, \mathcal{W}, \Gamma) \leq \Pi\left(U, \mathcal{W}, \frac{\Gamma}{\natural}\right), \mathcal{M}(U, \mathcal{W}, \Gamma) \leq \mathcal{M}\left(U, \mathcal{W}, \frac{\Gamma}{\natural}\right) \quad (1)$$

then $U = \mathcal{W}$.

Proof. By (1) follows that

$$\begin{aligned}
\Lambda(U, \mathcal{W}, \Gamma) &\geq \Lambda\left(U, \mathcal{W}, \frac{\Gamma}{\natural^{\beta_1}}\right), \Pi(U, \mathcal{W}, \Gamma) \leq \Pi\left(U, \mathcal{W}, \frac{\Gamma}{\natural^{\beta_1}}\right), \\
\mathcal{M}(U, \mathcal{W}, \Gamma) &\leq \mathcal{M}\left(U, \mathcal{W}, \frac{\Gamma}{\natural^{\beta_1}}\right), \beta_1 \in \mathbb{N}, \Gamma > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\Lambda(U, \mathcal{W}, \Gamma) &\geq \lim_{\beta_1 \rightarrow +\infty} \Lambda\left(U, \mathcal{W}, \frac{\Gamma}{\natural^{\beta_1}}\right) = 1, \\
\Pi(U, \mathcal{W}, \Gamma) &\leq \lim_{\beta_1 \rightarrow +\infty} \Pi\left(U, \mathcal{W}, \frac{\Gamma}{\natural^{\beta_1}}\right) = 0, \\
\mathcal{M}(U, \mathcal{W}, \Gamma) &\leq \lim_{\beta_1 \rightarrow +\infty} \mathcal{M}\left(U, \mathcal{W}, \frac{\Gamma}{\natural^{\beta_1}}\right) = 0, \Gamma > 0.
\end{aligned}$$

Also, by definition of (iii), (viii), (xiii), i.e., $U = \mathcal{W}$. $\square$

Theorem 1. Suppose $(K, \Lambda, \Pi, \mathcal{M}, \star, \bullet)$ is a complete NPMS, $\natural \in (0, 1)$ and assume that

$$\lim_{\Gamma \rightarrow +\infty} \Lambda(U, \mathcal{W}, \Gamma) = 1, \lim_{\Gamma \rightarrow +\infty} \Pi(U, \mathcal{W}, \Gamma) = 0 \quad \text{and} \quad \lim_{\Gamma \rightarrow +\infty} \mathcal{M}(U, \mathcal{W}, \Gamma) = 0, \quad (2)$$

for all $U, \mathcal{W} \in K$ and $\Gamma > 0$. Let $\wp: K \rightarrow K$ be a mapping satisfying

$$\begin{aligned}
\Lambda(\wp U, \wp \mathcal{W}, \natural \Gamma) &\geq \Lambda(U, \mathcal{W}, \Gamma), \\
\Pi(\wp U, \wp \mathcal{W}, \natural \Gamma) &\leq \Pi(U, \mathcal{W}, \Gamma) \quad \text{and} \quad \mathcal{M}(\wp U, \wp \mathcal{W}, \natural \Gamma) \leq \mathcal{M}(U, \mathcal{W}, \Gamma), \quad (3)
\end{aligned}$$

for all $U, \mathcal{W} \in K$ and $\Gamma > 0$. Then $\wp$ has a unique fixed point (shortly, ufp).

Proof. Consider a point U_0 of K and define a sequence U_{β_1} by $U_{\beta_1} = \wp^{\beta_1} U_0 = \wp U_{\beta_1-1}$, $\beta_1 \in \mathbb{N}$.

By utilising (3) for all $\Gamma > 0$, we obtain

$$\begin{aligned}\Lambda(U_{\beta_1}, U_{\beta_1+1}, \natural\Gamma) &= \Lambda(\wp U_{\beta_1-1}, \wp U_{\beta_1}, \natural\Gamma) \geq \Lambda(U_{\beta_1-1}, U_{\beta_1}, \Gamma) \geq \Lambda\left(U_{\beta_1-2}, U_{\beta_1-1}, \frac{\Gamma}{\natural}\right) \\ &\geq \Lambda\left(U_{\beta_1-3}, U_{\beta_1-2}, \frac{\Gamma}{\natural^2}\right) \geq \cdots \geq \Lambda\left(U_0, U_1, \frac{\Gamma}{\natural^{\beta_1-1}}\right), \\ \Pi(U_{\beta_1}, U_{\beta_1+1}, \natural\Gamma) &= \Pi(\wp U_{\beta_1-1}, \wp U_{\beta_1}, \natural\Gamma) \leq \Pi(U_{\beta_1-1}, U_{\beta_1}, \Gamma) \leq \Pi\left(U_{\beta_1-2}, U_{\beta_1-1}, \frac{\Gamma}{\natural}\right) \\ &\leq \Pi\left(U_{\beta_1-3}, U_{\beta_1-2}, \frac{\Gamma}{\natural^2}\right) \leq \cdots \leq \Pi\left(U_0, U_1, \frac{\Gamma}{\natural^{\beta_1-1}}\right).\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}(U_{\beta_1}, U_{\beta_1+1}, \natural\Gamma) &= \mathcal{M}(\wp U_{\beta_1-1}, \wp U_{\beta_1}, \Gamma) \leq \mathcal{M}(U_{\beta_1-1}, U_{\beta_1}, \Gamma) \leq \mathcal{M}\left(U_{\beta_1-2}, U_{\beta_1-1}, \frac{\Gamma}{\natural}\right) \\ &\leq \mathcal{M}\left(U_{\beta_1-3}, U_{\beta_1-2}, \frac{\Gamma}{\natural^2}\right) \leq \cdots \leq \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{\natural^{\beta_1-1}}\right).\end{aligned}$$

We obtain

$$\begin{aligned}\Lambda(U_{\beta_1}, U_{\beta_1+1}, \natural\Gamma) &\geq \Lambda\left(U_0, U_1, \frac{\Gamma}{\natural^{\beta_1-1}}\right), \\ \Pi(U_{\beta_1}, U_{\beta_1+1}, \natural\Gamma) &\leq \Pi\left(U_0, U_1, \frac{\Gamma}{\natural^{\beta_1-1}}\right) \quad \text{and} \quad \mathcal{M}(U_{\beta_1}, U_{\beta_1+1}, \natural\Gamma) \leq \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{\natural^{\beta_1-1}}\right).\end{aligned}\tag{4}$$

Consequently,

$$\begin{aligned}\Lambda(U_{\beta_1}, U_{\beta_1+2}, \natural\Gamma) &= \Lambda(\wp U_{\beta_1-1}, \wp U_{\beta_1+1}, \natural\Gamma) \geq \Lambda(U_{\beta_1-1}, U_{\beta_1+1}, \Gamma) \geq \Lambda\left(U_{\beta_1-2}, U_{\beta_1}, \frac{\Gamma}{\natural}\right) \\ &\geq \Lambda\left(U_{\beta_1-3}, U_{\beta_1-1}, \frac{\Gamma}{\natural^2}\right) \geq \cdots \geq \Lambda\left(U_0, U_2, \frac{\Gamma}{\natural^{\beta_1-1}}\right), \\ \Pi(U_{\beta_1}, U_{\beta_1+2}, \natural\Gamma) &= \Pi(\wp U_{\beta_1-1}, \wp U_{\beta_1+1}, \natural\Gamma) \leq \Pi(U_{\beta_1-1}, U_{\beta_1+1}, \Gamma) \leq \Pi\left(U_{\beta_1-2}, U_{\beta_1}, \frac{\Gamma}{\natural}\right) \\ &\leq \Pi\left(U_{\beta_1-3}, U_{\beta_1-1}, \frac{\Gamma}{\natural^2}\right) \leq \cdots \leq \Pi\left(U_0, U_2, \frac{\Gamma}{\natural^{\beta_1-1}}\right).\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}(U_{\beta_1}, U_{\beta_1+2}, \natural\Gamma) &= \mathcal{M}(\wp U_{\beta_1-1}, \wp U_{\beta_1+1}, \Gamma) \leq \mathcal{M}(U_{\beta_1-1}, U_{\beta_1+1}, \Gamma) \leq \mathcal{M}\left(U_{\beta_1-2}, U_{\beta_1}, \frac{\Gamma}{\natural}\right) \\ &\leq \mathcal{M}\left(U_{\beta_1-3}, U_{\beta_1-1}, \frac{\Gamma}{\natural^2}\right) \leq \cdots \leq \mathcal{M}\left(U_0, U_2, \frac{\Gamma}{\natural^{\beta_1-1}}\right).\end{aligned}$$

We obtain

$$\begin{aligned}\Lambda(U_{\beta_1}, U_{\beta_1+2}, \natural\Gamma) &\geq \Lambda\left(U_0, U_2, \frac{\Gamma}{\natural^{\beta_1-1}}\right), \\ \Pi(U_{\beta_1}, U_{\beta_1+2}, \natural\Gamma) &\leq \Pi\left(U_0, U_2, \frac{\Gamma}{\natural^{\beta_1-1}}\right) \quad \text{and} \quad \mathcal{M}(U_{\beta_1}, U_{\beta_1+2}, \natural\Gamma) \leq \mathcal{M}\left(U_0, U_2, \frac{\Gamma}{\natural^{\beta_1-1}}\right).\end{aligned}\tag{5}$$

It follows that

$$\begin{aligned}
\Lambda(U_{\beta_1}, U_{\beta_1+3}, \natural\Gamma) &= \Lambda(\wp U_{\beta_1-1}, \wp U_{\beta_1+2}, \natural\Gamma) \geq \Lambda(U_{\beta_1-1}, U_{\beta_1+2}, \Gamma) \geq \Lambda\left(U_{\beta_1-2}, U_{\beta_1+1}, \frac{\Gamma}{\natural}\right) \\
&\geq \Lambda\left(U_{\beta_1-3}, U_{\beta_1}, \frac{\Gamma}{\natural^2}\right) \geq \cdots \geq \Lambda\left(U_0, U_3, \frac{\Gamma}{\natural^{\beta_1-1}}\right), \\
\Pi(U_{\beta_1}, U_{\beta_1+3}, \natural\Gamma) &= \Pi(\wp U_{\beta_1-1}, \wp U_{\beta_1+2}, \natural\Gamma) \leq \Pi(U_{\beta_1-1}, U_{\beta_1+2}, \Gamma) \leq \Pi\left(U_{\beta_1-2}, U_{\beta_1+1}, \frac{\Gamma}{\natural}\right) \\
&\leq \Pi\left(U_{\beta_1-3}, U_{\beta_1}, \frac{\Gamma}{\natural^2}\right) \leq \cdots \leq \Pi\left(U_0, U_3, \frac{\Gamma}{\natural^{\beta_1-1}}\right).
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{M}(U_{\beta_1}, U_{\beta_1+3}, \natural\Gamma) &= \mathcal{M}(\wp U_{\beta_1-1}, \wp U_{\beta_1+2}, \Gamma) \leq \mathcal{M}(U_{\beta_1-1}, U_{\beta_1+2}, \Gamma) \leq \mathcal{M}\left(U_{\beta_1-2}, U_{\beta_1+1}, \frac{\Gamma}{\natural}\right) \\
&\leq \mathcal{M}\left(U_{\beta_1-3}, U_{\beta_1}, \frac{\Gamma}{\natural^2}\right) \leq \cdots \leq \mathcal{M}\left(U_0, U_3, \frac{\Gamma}{\natural^{\beta_1-1}}\right).
\end{aligned}$$

We obtain

$$\begin{aligned}
\Lambda(U_{\beta_1}, U_{\beta_1+3}, \natural\Gamma) &\geq \Lambda\left(U_0, U_3, \frac{\Gamma}{\natural^{\beta_1-1}}\right), \\
\Pi(U_{\beta_1}, U_{\beta_1+3}, \natural\Gamma) &\leq \Pi\left(U_0, U_3, \frac{\Gamma}{\natural^{\beta_1-1}}\right) \quad \text{and} \quad \mathcal{M}(U_{\beta_1}, U_{\beta_1+3}, \natural\Gamma) \leq \mathcal{M}\left(U_0, U_3, \frac{\Gamma}{\natural^{\beta_1-1}}\right).
\end{aligned}$$

Similarly, for $j = 1, 2, 3, \dots$, we obtain

$$\begin{aligned}
\Lambda(U_{\beta_1}, U_{\beta_1+3j+1}, \natural\Gamma) &\geq \Lambda\left(U_0, U_{3j+1}, \frac{\Gamma}{\natural^{\beta_1-1}}\right), \Pi(U_{\beta_1}, U_{\beta_1+3j+1}, \natural\Gamma) \leq \Pi\left(U_0, U_{3j+1}, \frac{\Gamma}{\natural^{\beta_1-1}}\right) \\
\text{and} \quad \mathcal{M}(U_{\beta_1}, U_{\beta_1+3j+1}, \natural\Gamma) &\leq \mathcal{M}\left(U_0, U_{3j+1}, \frac{\Gamma}{\natural^{\beta_1-1}}\right),
\end{aligned} \tag{6}$$

$$\begin{aligned}
\Lambda(U_{\beta_1}, U_{\beta_1+3j+2}, \natural\Gamma) &\geq \Lambda\left(U_0, U_{3j+2}, \frac{\Gamma}{\natural^{\beta_1-1}}\right), \Pi(U_{\beta_1}, U_{\beta_1+3j+2}, \natural\Gamma) \leq \Pi\left(U_0, U_{3j+2}, \frac{\Gamma}{\natural^{\beta_1-1}}\right) \\
\text{and} \quad \mathcal{M}(U_{\beta_1}, U_{\beta_1+3j+2}, \natural\Gamma) &\leq \mathcal{M}\left(U_0, U_{3j+2}, \frac{\Gamma}{\natural^{\beta_1-1}}\right),
\end{aligned}$$

$$\begin{aligned}
\Lambda(U_{\beta_1}, U_{\beta_1+3j+3}, \natural\Gamma) &\geq \Lambda\left(U_0, U_{3j+3}, \frac{\Gamma}{\natural^{\beta_1-1}}\right), \Pi(U_{\beta_1}, U_{\beta_1+3j+3}, \natural\Gamma) \leq \Pi\left(U_0, U_{3j+3}, \frac{\Gamma}{\natural^{\beta_1-1}}\right) \\
\text{and} \quad \mathcal{M}(U_{\beta_1}, U_{\beta_1+3j+3}, \natural\Gamma) &\leq \mathcal{M}\left(U_0, U_{3j+3}, \frac{\Gamma}{\natural^{\beta_1-1}}\right).
\end{aligned} \tag{7}$$

By using (4), we obtain for each $j = 1, 2, 3, \dots$,

$$\begin{aligned}
\Lambda(U_0, U_{3j+1}, \Gamma) &\geq \Lambda\left(U_0, U_1, \frac{\Gamma}{4}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4}\right) \\
&\star \Lambda\left(U_0, U_1, \frac{\Gamma}{4}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4}\right) \\
&\star \cdots \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4}\right),
\end{aligned}$$

$$\begin{aligned}\Pi(U_0, U_{3j+1}, \Gamma) &\leq \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^2}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^2}\right) \\ &\quad \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^3}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^4}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^5}\right) \\ &\quad \bullet \dots \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^{3j-1}}\right)\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}(U_0, U_{3j+1}, \Gamma) &\leq \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^2}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^2}\right) \\ &\quad \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^3}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^4}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^5}\right) \\ &\quad \bullet \dots \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^{3j-1}}\right).\end{aligned}$$

Now, from (6), we get

$$\begin{aligned}\Lambda(U_{\beta_1}, U_{\beta_1+3j+1}, \frac{\Gamma}{4^{\beta_1-1}}) &\geq \Lambda\left(U_0, U_{3j+1}, \frac{\Gamma}{4^{\beta_1-1}}\right) \\ &\geq \Lambda\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1-1}}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1-1}}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1}}\right) \\ &\quad \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+1}}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+2}}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+3}}\right) \\ &\quad \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+4}}\right) \star \dots \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4^{3j+\beta_1-2}}\right),\end{aligned}\tag{8}$$

$$\begin{aligned}\Pi(U_{\beta_1}, U_{\beta_1+3j+1}, \frac{\Gamma}{4^{\beta_1-1}}) &\leq \Pi\left(U_0, U_{3j+1}, \frac{\Gamma}{4^{\beta_1-1}}\right) \\ &\leq \Pi\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1-1}}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1-1}}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1}}\right) \\ &\quad \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+1}}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+2}}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+3}}\right) \\ &\quad \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+4}}\right) \bullet \dots \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^{3j+\beta_1-2}}\right)\end{aligned}\tag{9}$$

and

$$\begin{aligned}\mathcal{M}(U_{\beta_1}, U_{\beta_1+3j+1}, \frac{\Gamma}{4^{\beta_1-1}}) &\leq \mathcal{M}\left(U_0, U_{3j+1}, \frac{\Gamma}{4^{\beta_1-1}}\right) \\ &\leq \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1-1}}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1-1}}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1}}\right) \\ &\quad \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+1}}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+2}}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+3}}\right) \\ &\quad \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+4}}\right) \bullet \dots \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^{3j+\beta_1-2}}\right).\end{aligned}\tag{10}$$

By using (4) and (5), we obtain for each $j = 1, 2, 3, \dots$,

$$\begin{aligned}\Lambda(U_0, U_{3j+2}, \Gamma) &\geq \Lambda\left(U_0, U_1, \frac{\Gamma}{4}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4^2}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4^2}\right) \\ &\quad \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4^3}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4^4}\right) \star \Lambda\left(U_0, U_2, \frac{\Gamma}{4^5}\right) \\ &\quad \star \dots \star \Lambda\left(U_0, U_2, \frac{\Gamma}{4^{3j-1}}\right),\end{aligned}$$

$$\begin{aligned}\Pi(U_0, U_{3j+2}, \Gamma) &\leq \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^2}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^2}\right) \\ &\quad \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^3}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^4}\right) \bullet \Pi\left(U_0, U_2, \frac{\Gamma}{4^5}\right) \\ &\quad \bullet \dots \bullet \Pi\left(U_0, U_2, \frac{\Gamma}{4^{3j-1}}\right)\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}(U_0, U_{3j+2}, \Gamma) &\leq \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^2}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^2}\right) \\ &\quad \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^3}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^4}\right) \bullet \mathcal{M}\left(U_0, U_2, \frac{\Gamma}{4^5}\right) \\ &\quad \bullet \dots \bullet \mathcal{M}\left(U_0, U_2, \frac{\Gamma}{4^{3j-1}}\right).\end{aligned}$$

Now, from (6), we get

$$\begin{aligned}\Lambda(U_{\beta_1}, U_{\beta_1+3j+2}, \frac{\Gamma}{4^{\beta_1-1}}) &\geq \Lambda\left(U_0, U_{3j+2}, \frac{\Gamma}{4^{\beta_1-1}}\right) \\ &\geq \Lambda\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1-1}}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1-1}}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1}}\right) \\ &\quad \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+1}}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+2}}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+3}}\right) \\ &\quad \star \Lambda\left(U_0, U_2, \frac{\Gamma}{4^{\beta_1+4}}\right) \star \dots \star \Lambda\left(U_0, U_2, \frac{\Gamma}{4^{3j+\beta_1-2}}\right),\end{aligned}\tag{11}$$

$$\begin{aligned}\Pi(U_{\beta_1}, U_{\beta_1+3j+2}, \frac{\Gamma}{4^{\beta_1-1}}) &\leq \Pi\left(U_0, U_{3j+2}, \frac{\Gamma}{4^{\beta_1-1}}\right) \\ &\leq \Pi\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1-1}}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1-1}}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1}}\right) \\ &\quad \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+1}}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+2}}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+3}}\right) \\ &\quad \bullet \Pi\left(U_0, U_2, \frac{\Gamma}{4^{\beta_1+4}}\right) \bullet \dots \bullet \Pi\left(U_0, U_2, \frac{\Gamma}{4^{3j+\beta_1-2}}\right)\end{aligned}\tag{12}$$

and

$$\begin{aligned}
\mathcal{M}(U_{\beta_1}, U_{\beta_1+3j+2}, \frac{\Gamma}{4^{\beta_1-1}}) &\leq \mathcal{M}\left(U_0, U_{3j+2}, \frac{\Gamma}{4^{\beta_1-1}}\right) \\
&\leq \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1-1}}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1-1}}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1}}\right) \\
&\bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+1}}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+2}}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+3}}\right) \\
&\bullet \mathcal{M}\left(U_0, U_2, \frac{\Gamma}{4^{\beta_1+4}}\right) \bullet \dots \bullet \mathcal{M}\left(U_0, U_2, \frac{\Gamma}{4^{\beta_1+3j-2}}\right).
\end{aligned} \tag{13}$$

By using (4) and (5), we obtain for each $j = 1, 2, 3, \dots$,

$$\begin{aligned}
\Lambda(U_0, U_{3j+3}, \Gamma) &\geq \Lambda\left(U_0, U_1, \frac{\Gamma}{4}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4^2}\right) \\
&\star \Lambda\left(U_0, U_1, \frac{\Gamma}{4^3}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4^4}\right) \star \Lambda\left(U_0, U_3, \frac{\Gamma}{4^5}\right) \\
&\star \dots \star \Lambda\left(U_0, U_3, \frac{\Gamma}{4^{3j-1}}\right),
\end{aligned}$$

$$\begin{aligned}
\Pi(U_0, U_{3j+3}, \Gamma) &\leq \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^2}\right) \\
&\bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^3}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^4}\right) \bullet \Pi\left(U_0, U_3, \frac{\Gamma}{4^5}\right) \\
&\bullet \dots \bullet \Pi\left(U_0, U_3, \frac{\Gamma}{4^{3j-1}}\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{M}(U_0, U_{3j+3}, \Gamma) &\leq \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^2}\right) \\
&\bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^3}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^4}\right) \bullet \mathcal{M}\left(U_0, U_3, \frac{\Gamma}{4^5}\right) \\
&\bullet \dots \bullet \mathcal{M}\left(U_0, U_3, \frac{\Gamma}{4^{3j-1}}\right).
\end{aligned}$$

Now, from (6), we get

$$\begin{aligned}
\Lambda(U_{\beta_1}, U_{\beta_1+3j+3}, \frac{\Gamma}{4^{\beta_1-1}}) &\geq \Lambda\left(U_0, U_{3j+3}, \frac{\Gamma}{4^{\beta_1-1}}\right) \\
&\geq \Lambda\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1-1}}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1-1}}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1}}\right) \\
&\star \Lambda\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+1}}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+2}}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+3}}\right) \\
&\star \Lambda\left(U_0, U_3, \frac{\Gamma}{4^{\beta_1+4}}\right) \star \dots \star \Lambda\left(U_0, U_3, \frac{\Gamma}{4^{\beta_1+3j-2}}\right),
\end{aligned} \tag{14}$$

$$\begin{aligned}
\Pi(U_{\beta_1}, U_{\beta_1+3j+3}, \frac{\Gamma}{4^{\beta_1-1}}) &\leq \Pi\left(U_0, U_{3j+3}, \frac{\Gamma}{4^{\beta_1-1}}\right) \\
&\leq \Pi\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1-1}}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1-1}}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1}}\right) \\
&\bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+1}}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+1}}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+3}}\right) \\
&\bullet \Pi\left(U_0, U_3, \frac{\Gamma}{4^{\beta_1+4}}\right) \bullet \dots \bullet \Pi\left(U_0, U_3, \frac{\Gamma}{4^{\beta_1+3j+2}}\right)
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
\mathcal{M}(U_{\beta_1}, U_{\beta_1+3j+3}, \frac{\Gamma}{4^{\beta_1-1}}) &\leq \mathcal{M}\left(U_0, U_{3j+3}, \frac{\Gamma}{4^{\beta_1-1}}\right) \\
&\leq \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1-1}}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1-1}}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1}}\right) \\
&\bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+1}}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+1}}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4^{\beta_1+3}}\right) \\
&\bullet \mathcal{M}\left(U_0, U_2, \frac{\Gamma}{4^{\beta_1+4}}\right) \bullet \dots \bullet \mathcal{M}\left(U_0, U_2, \frac{\Gamma}{4^{\beta_1+3j+2}}\right).
\end{aligned} \tag{16}$$

Using (8)–(16), for each case $\beta_1 \rightarrow +\infty$, we deduce that

$$\begin{aligned}
\lim_{\beta_1 \rightarrow +\infty} \Lambda(U_{\beta_1}, U_{\beta_1+i}, \Gamma) &= 1 \star 1 \star \dots \star 1 = 1, \\
\lim_{\beta_1 \rightarrow +\infty} \Pi(U_{\beta_1}, U_{\beta_1+i}, \Gamma) &= 0 \bullet 0 \bullet \dots \bullet 0 = 0
\end{aligned}$$

and

$$\lim_{\beta_1 \rightarrow +\infty} \mathcal{M}(U_{\beta_1}, U_{\beta_1+i}, \Gamma) = 0 \bullet 0 \bullet \dots \bullet 0 = 0.$$

Which implies that $\{U_{\beta_1}\}$ is a Cauchy sequence. Since $(K, \Lambda, \Pi, \mathcal{M}, \star, \bullet)$ is a complete NPMS, we have

$$\lim_{\beta_1 \rightarrow +\infty} U_{\beta_1} = U.$$

Now, U is a fixed point of $\wp$, using conditions (v), (x), (xv) and Equation (2), we obtained

$$\begin{aligned}
\Lambda(U, \wp U, \Gamma) &\geq \Lambda\left(U, U_{\beta_1}, \frac{\Gamma}{4}\right) \star \Lambda\left(U_{\beta_1}, U_{\beta_1+1}, \frac{\Gamma}{4}\right) \star \Lambda\left(U_{\beta_1+1}, U_{\beta_1+2}, \frac{\Gamma}{4}\right) \star \Lambda\left(U_{\beta_1+2}, \wp U, \frac{\Gamma}{4}\right) \\
&= \Lambda\left(U, U_{\beta_1}, \frac{\Gamma}{4}\right) \star \Lambda\left(\wp U_{\beta_1-1}, \wp U_{\beta_1}, \frac{\Gamma}{4}\right) \star \Lambda\left(\wp U_{\beta_1}, \wp U_{\beta_1+1}, \frac{\Gamma}{4}\right) \\
&\star \Lambda\left(\wp U_{\beta_1+1}, \wp U, \frac{\Gamma}{4}\right) \\
&\geq \Lambda\left(U, U_{\beta_1}, \frac{\Gamma}{4}\right) \star \Lambda\left(U_{\beta_1-1}, U_{\beta_1}, \frac{\Gamma}{4^{\beta_1-2}}\right) \star \Lambda\left(U_{\beta_1}, U_{\beta_1+1}, \frac{\Gamma}{4^{\beta_1-1}}\right) \\
&\star \Lambda\left(U_{\beta_1+1}, U, \frac{\Gamma}{4}\right) \\
&\rightarrow 1 \star 1 \star 1 \star 1 = 1 \quad \text{as } \beta_1 \rightarrow +\infty,
\end{aligned}$$

$$\begin{aligned}
\Pi(U, \wp U, \Gamma) &\leq \Pi\left(U, U_{\beta_1}, \frac{\Gamma}{4}\right) \bullet \Pi\left(U_{\beta_1}, U_{\beta_1+1}, \frac{\Gamma}{4}\right) \bullet \Pi\left(U_{\beta_1+1}, U_{\beta_1+2}, \frac{\Gamma}{4}\right) \bullet \Pi\left(U_{\beta_1+2}, \wp U, \frac{\Gamma}{4}\right) \\
&= \Pi\left(U, U_{\beta_1}, \frac{\Gamma}{4}\right) \bullet \Pi\left(\wp U_{\beta_1-1}, \wp U_{\beta_1}, \frac{\Gamma}{4}\right) \bullet \Pi\left(\wp U_{\beta_1}, \wp U_{\beta_1+1}, \frac{\Gamma}{4}\right) \\
&\bullet \Pi\left(\wp U_{\beta_1+1}, \wp U, \frac{\Gamma}{4}\right) \\
&\leq \Pi\left(U, U_{\beta_1}, \frac{\Gamma}{4}\right) \bullet \Pi\left(U_{\beta_1-1}, U_{\beta_1}, \frac{\Gamma}{4\mathfrak{q}^{\beta_1-2}}\right) \bullet \Pi\left(U_{\beta_1}, U_{\beta_1+1}, \frac{\Gamma}{4\mathfrak{q}^{\beta_1-1}}\right) \\
&\bullet \Pi\left(U_{\beta_1+1}, U, \frac{\Gamma}{4\mathfrak{q}}\right) \\
&\rightarrow 0 \bullet 0 \bullet 0 \bullet 0 = 0 \quad \text{as } \beta_1 \rightarrow +\infty,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{M}(U, \wp U, \Gamma) &\leq \mathcal{M}\left(U, U_{\beta_1}, \frac{\Gamma}{4}\right) \bullet \mathcal{M}\left(U_{\beta_1}, U_{\beta_1+1}, \frac{\Gamma}{4}\right) \\
&\bullet \mathcal{M}\left(U_{\beta_1+1}, U_{\beta_1+2}, \frac{\Gamma}{4}\right) \bullet \mathcal{M}\left(U_{\beta_1+2}, \wp U, \frac{\Gamma}{4}\right) \\
&= \mathcal{M}\left(U, U_{\beta_1}, \frac{\Gamma}{4}\right) \bullet \mathcal{M}\left(\wp U_{\beta_1-1}, \wp U_{\beta_1}, \frac{\Gamma}{4}\right) \bullet \mathcal{M}\left(\wp U_{\beta_1}, \wp U_{\beta_1+1}, \frac{\Gamma}{4}\right) \\
&\bullet \mathcal{M}\left(\wp U_{\beta_1+1}, \wp U, \frac{\Gamma}{4}\right) \\
&\leq \mathcal{M}\left(U, U_{\beta_1}, \frac{\Gamma}{4}\right) \bullet \mathcal{M}\left(U_{\beta_1-1}, U_{\beta_1}, \frac{\Gamma}{4\mathfrak{q}^{\beta_1-2}}\right) \bullet \mathcal{M}\left(U_{\beta_1}, U_{\beta_1+1}, \frac{\Gamma}{4\mathfrak{q}^{\beta_1-1}}\right) \\
&\bullet \mathcal{M}\left(U_{\beta_1+1}, U, \frac{\Gamma}{4\mathfrak{q}}\right) \\
&\rightarrow 0 \bullet 0 \bullet 0 \bullet 0 = 0 \quad \text{as } \beta_1 \rightarrow +\infty,
\end{aligned}$$

Hence, $\wp U = U$. Let $\wp \tilde{\mu} = \tilde{\mu}$ for some $\tilde{\mu} \in K$, then

$$\begin{aligned}
1 &\geq \Lambda(\tilde{\mu}, U, \Gamma) = \Lambda(\wp \tilde{\mu}, \wp U, \Gamma) \geq \Lambda\left(\tilde{\mu}, U, \frac{\Gamma}{\mathfrak{q}}\right) = \Lambda\left(\wp \tilde{\mu}, \wp U, \frac{\Gamma}{\mathfrak{q}}\right) \\
&\geq \Lambda\left(\tilde{\mu}, U, \frac{\Gamma}{\mathfrak{q}^2}\right) \geq \cdots \geq \Lambda\left(\tilde{\mu}, U, \frac{\Gamma}{\mathfrak{q}^{\beta_1}}\right) \rightarrow 1 \quad \text{as } \beta_1 \rightarrow +\infty, \\
0 &\leq \Pi(\tilde{\mu}, U, \Gamma) = \Pi(\wp \tilde{\mu}, \wp U, \Gamma) \leq \Pi\left(\tilde{\mu}, U, \frac{\Gamma}{\mathfrak{q}}\right) = \Pi\left(\wp \tilde{\mu}, \wp U, \frac{\Gamma}{\mathfrak{q}}\right) \\
&\leq \Pi\left(\tilde{\mu}, U, \frac{\Gamma}{\mathfrak{q}^2}\right) \leq \cdots \leq \Pi\left(\tilde{\mu}, U, \frac{\Gamma}{\mathfrak{q}^{\beta_1}}\right) \rightarrow 0 \quad \text{as } \beta_1 \rightarrow +\infty,
\end{aligned}$$

and

$$\begin{aligned}
0 &\leq \mathcal{M}(\tilde{\mu}, U, \Gamma) = \mathcal{M}(\wp \tilde{\mu}, \wp U, \Gamma) \leq \mathcal{M}\left(\tilde{\mu}, U, \frac{\Gamma}{\mathfrak{q}}\right) = \mathcal{M}\left(\wp \tilde{\mu}, \wp U, \frac{\Gamma}{\mathfrak{q}}\right) \\
&\leq \mathcal{M}\left(\tilde{\mu}, U, \frac{\Gamma}{\mathfrak{q}^2}\right) \leq \cdots \leq \mathcal{M}\left(\tilde{\mu}, U, \frac{\Gamma}{\mathfrak{q}^{\beta_1}}\right) \rightarrow 0 \quad \text{as } \beta_1 \rightarrow +\infty,
\end{aligned}$$

using by (iii), (viii) and (xiii), $U = \tilde{\mu}$. $\square$

Definition 10. Let $(K, \Lambda, \Pi, \mathcal{M}, \star, \bullet)$ be a NPMS. A function $\wp: K \rightarrow K$ is an NPC (neutrosophic pentagonal contraction) if $\exists 0 < \natural < 1$, such that

$$\frac{1}{\Lambda(\mathcal{P}U, \mathcal{P}W, \Gamma)} - 1 \leq \natural \left[\frac{1}{\Lambda(U, W, \Gamma)} - 1 \right] \quad (17)$$

$$\Pi(\mathcal{P}U, \mathcal{P}W, \Gamma) \leq \natural \Pi(U, W, \Gamma), \quad (18)$$

and

$$\mathcal{M}(\mathcal{P}U, \mathcal{P}W, \Gamma) \leq \natural \mathcal{M}(U, W, \Gamma), \quad (19)$$

for all $U, W \in K$ and $\Gamma > 0$.

Theorem 2. Let $(K, \Lambda, \Pi, \mathcal{M}, \star, \bullet)$ be a complete NPMS with $\chi: K \times K \rightarrow [1, +\infty)$ and assume that

$$\lim_{\Gamma \rightarrow +\infty} \mathcal{M}(U, W, \Gamma) = 0, \quad \lim_{\Gamma \rightarrow +\infty} \Pi(U, W, \Gamma) = 0 \quad \text{and} \quad \lim_{\Gamma \rightarrow +\infty} \Lambda(U, W, \Gamma) = 1, \quad (20)$$

for all $U, W \in K$ and $\Gamma > 0$. Let $\wp: K \rightarrow K$ be a ND-controlled contraction. Furthermore, assume that for an arbitrary $U_0 \in K$, and $\beta_1, q \in \mathbb{N}$, where $U_{\beta_1} = \wp^{\beta_1} U_0 = \wp U_{\beta_1-1}$. Then, $\wp$ has a ufp.

Proof. Suppose U_0 be a point of K and define a sequence $\{U_{\beta_1}\}$ by $U_{\beta_1} = \wp^{\beta_1} U_0 = \wp U_{\beta_1-1}$, $\beta_1 \in \mathbb{N}$. Using by (17)–(19) for all $\Gamma > 0$, $\beta_1 > q$, we deduce

$$\begin{aligned} \frac{1}{\Lambda(U_{\beta_1}, U_{\beta_1+1}, \Gamma)} - 1 &= \frac{1}{\Lambda(\wp U_{\beta_1-1}, \wp U_{\beta_1}, \Gamma)} - 1 \\ &\leq \natural \left[\frac{1}{\Lambda(U_{\beta_1-1}, U_{\beta_1}, \Gamma)} - 1 \right] = \frac{\natural}{\Lambda(U_{\beta_1-1}, U_{\beta_1}, \Gamma)} - \natural. \end{aligned}$$

This implies

$$\begin{aligned} \frac{1}{\Lambda(U_{\beta_1}, U_{\beta_1+1}, \Gamma)} &\leq \frac{\natural}{\Lambda(U_{\beta_1-1}, U_{\beta_1}, \Gamma)} + (1 - \natural) \\ &\leq \frac{\natural^2}{\Lambda(U_{\beta_1-2}, U_{\beta_1-1}, \Gamma)} + \natural(1 - \natural) + (1 - \natural). \end{aligned}$$

In this manner, we conclude that

$$\begin{aligned} \frac{1}{\Lambda(U_{\beta_1}, U_{\beta_1+1}, \Gamma)} &\leq \frac{\natural^{\beta_1}}{\Lambda(U_0, U_1, \Gamma)} + \natural^{\beta_1-1}(1 - \natural) + \natural^{\beta_1-2}(1 - \natural) \\ &\quad + \dots + \natural(1 - \natural) + (1 - \natural) \\ &\leq \frac{\natural^{\beta_1}}{\Lambda(U_0, U_1, \Gamma)} + (\natural^{\beta_1-1} + \natural^{\beta_1-2} + \dots + 1)(1 - \natural) \\ &\leq \frac{\natural^{\beta_1}}{\Lambda(U_0, U_1, \Gamma)} + (1 - \natural^{\beta_1}). \end{aligned}$$

We obtain

$$\frac{1}{\frac{\natural^{\beta_1}}{\Lambda(U_0, U_1, \Gamma)} + (1 - \natural^{\beta_1})} \leq \Lambda(U_{\beta_1}, U_{\beta_1+1}, \Gamma)$$

$$\begin{aligned}\Pi(U_{\beta_1}, U_{\beta_1+1}, \Gamma) &= \Pi(\wp U_{\beta_1-1}, \wp U_{\beta_1}, \Gamma) \leq \natural \Pi(U_{\beta_1-1}, U_{\beta_1}, \Gamma) = \Pi(\wp U_{\beta_1-2}, \wp U_{\beta_1-1}, \Gamma) \\ &\leq \natural^2 \Pi(U_{\beta_1-2}, U_{\beta_1-1}, \Gamma) \leq \dots \leq \natural^{\beta_1} \Pi(U_0, U_1, \Gamma)\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}(U_{\beta_1}, U_{\beta_1+1}, \Gamma) &= \mathcal{M}(\wp U_{\beta_1-1}, \wp U_{\beta_1}, \Gamma) \leq \natural \mathcal{M}(U_{\beta_1-1}, U_{\beta_1}, \Gamma) = \mathcal{M}(\wp U_{\beta_1-2}, \wp U_{\beta_1-1}, \Gamma) \\ &\leq \natural^2 \mathcal{M}(U_{\beta_1-2}, U_{\beta_1-1}, \Gamma) \leq \dots \leq \natural^{\beta_1} \mathcal{M}(U_0, U_1, \Gamma).\end{aligned}\quad (21)$$

It again follows that

$$\begin{aligned}\frac{1}{\Lambda(U_{\beta_1}, U_{\beta_1+2}, \Gamma)} - 1 &= \frac{1}{\Lambda(\wp U_{\beta_1-1}, \wp U_{\beta_1+1}, \Gamma)} - 1 \\ &\leq \natural \left[\frac{1}{\Lambda(U_{\beta_1-1}, U_{\beta_1+1}, \Gamma)} \right] \\ &= \frac{\natural}{\Lambda(U_{\beta_1-1}, U_{\beta_1+1}, \Gamma)} - \natural \\ \Rightarrow \frac{1}{\Lambda(U_{\beta_1}, U_{\beta_1+2}, \Gamma)} &\leq \frac{\natural}{\Lambda(U_{\beta_1-1}, U_{\beta_1+1}, \Gamma)} + (1 - \natural) \\ &\leq \frac{\natural^2}{\Lambda(U_{\beta_1-2}, U_{\beta_1}, \Gamma)} + \natural(1 - \natural) + (1 - \natural).\end{aligned}$$

In this manner, we conclude that

$$\begin{aligned}\frac{1}{\Lambda(U_{\beta_1}, U_{\beta_1+2}, \Gamma)} &\leq \frac{\natural^{\beta_1}}{\Lambda(U_0, U_2, \Gamma)} + \natural^{\beta_1-1}(1 - \natural) + \natural^{\beta_1-2}(1 - \natural) \\ &\quad + \dots + \natural(1 - \natural) + (1 - \natural) \\ &\leq \frac{\natural^{\beta_1}}{\Lambda(U_0, U_2, \Gamma)} + (\natural^{\beta_1-1} + \natural^{\beta_1-2} + \dots + 1)(1 - \natural) \\ &\leq \frac{\natural^{\beta_1}}{\Lambda(U_0, U_2, \Gamma)} + (1 - \natural^{\beta_1}).\end{aligned}$$

We obtain

$$\frac{1}{\frac{\natural^{\beta_1}}{\Lambda(U_0, U_2, \Gamma)} + (1 - \natural^{\beta_1})} \leq \Lambda(U_{\beta_1}, U_{\beta_1+2}, \Gamma)$$

$$\begin{aligned}\Pi(U_{\beta_1}, U_{\beta_1+2}, \Gamma) &= \Pi(\wp U_{\beta_1-1}, \wp U_{\beta_1+1}, \Gamma) \leq \natural \Pi(U_{\beta_1-1}, U_{\beta_1+1}, \Gamma) = \Pi(\wp U_{\beta_1-2}, \wp U_{\beta_1}, \Gamma) \\ &\leq \natural^2 \Pi(U_{\beta_1-2}, U_{\beta_1}, \Gamma) \leq \dots \leq \natural^{\beta_1} \Pi(U_0, U_2, \Gamma)\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}(U_{\beta_1}, U_{\beta_1+2}, \Gamma) &= \mathcal{M}(\wp U_{\beta_1-1}, \wp U_{\beta_1+1}, \Gamma) \leq \natural \mathcal{M}(U_{\beta_1-1}, U_{\beta_1+1}, \Gamma) = \mathcal{M}(\wp U_{\beta_1-2}, \wp U_{\beta_1}, \Gamma) \\ &\leq \natural^2 \mathcal{M}(U_{\beta_1-2}, U_{\beta_1}, \Gamma) \leq \dots \leq \natural^{\beta_1} \mathcal{M}(U_0, U_2, \Gamma).\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{1}{\Lambda(U_{\beta_1}, U_{\beta_1+3}, \Gamma)} - 1 &= \frac{1}{\Lambda(\wp U_{\beta_1-1}, \wp U_{\beta_1+2}, \Gamma)} - 1 \\ &\leq \natural \left[\frac{1}{\Lambda(U_{\beta_1-1}, U_{\beta_1+2}, \Gamma)} \right] \\ &= \frac{\natural}{\Lambda(U_{\beta_1-1}, U_{\beta_1+2}, \Gamma)} - \natural.\end{aligned}$$

This implies

$$\begin{aligned}\frac{1}{\Lambda(U_{\beta_1}, U_{\beta_1+3}, \Gamma)} &\leq \frac{\natural}{\Lambda(U_{\beta_1-1}, U_{\beta_1+2}, \Gamma)} + (1 - \natural) \\ &\leq \frac{\natural^2}{\Lambda(U_{\beta_1-2}, U_{\beta_1+1}, \Gamma)} + \natural(1 - \natural) + (1 - \natural).\end{aligned}$$

In this manner, we conclude that

$$\begin{aligned}\frac{1}{\Lambda(U_{\beta_1}, U_{\beta_1+3}, \Gamma)} &\leq \frac{\natural^{\beta_1}}{\Lambda(U_0, U_3, \Gamma)} + \natural^{\beta_1-1}(1 - \natural) + \natural^{\beta_1-2}(1 - \natural) \\ &\quad + \cdots + \natural(1 - \natural) + (1 - \natural) \\ &\leq \frac{\natural^{\beta_1}}{\Lambda(U_0, U_3, \Gamma)} + (\natural^{\beta_1-1} + \natural^{\beta_1-2} + \cdots + 1)(1 - \natural) \\ &\leq \frac{\natural^{\beta_1}}{\Lambda(U_0, U_3, \Gamma)} + (1 - \natural^{\beta_1}).\end{aligned}$$

We obtain

$$\frac{1}{\frac{\natural^{\beta_1}}{\Lambda(U_0, U_3, \Gamma)} + (1 - \natural^{\beta_1})} \leq \Lambda(U_{\beta_1}, U_{\beta_1+3}, \Gamma),$$

$$\begin{aligned}\Pi(U_{\beta_1}, U_{\beta_1+3}, \Gamma) &= \Pi(\wp U_{\beta_1-1}, \wp U_{\beta_1+2}, \Gamma) \leq \natural \Pi(U_{\beta_1-1}, U_{\beta_1+2}, \Gamma) = \Pi(\wp U_{\beta_1-2}, \wp U_{\beta_1+1}, \Gamma) \\ &\leq \natural^2 \Pi(U_{\beta_1-2}, U_{\beta_1+1}, \Gamma) \leq \cdots \leq \natural^{\beta_1} \Pi(U_0, U_3, \Gamma)\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}(U_{\beta_1}, U_{\beta_1+3}, \Gamma) &= \mathcal{M}(\wp U_{\beta_1-1}, \wp U_{\beta_1+2}, \Gamma) \leq \natural \mathcal{M}(U_{\beta_1-1}, U_{\beta_1+2}, \Gamma) = \mathcal{M}(\wp U_{\beta_1-2}, \wp U_{\beta_1+1}, \Gamma) \\ &\leq \natural^2 \mathcal{M}(U_{\beta_1-2}, U_{\beta_1+1}, \Gamma) \leq \cdots \leq \natural^{\beta_1} \mathcal{M}(U_0, U_3, \Gamma).\end{aligned}$$

Similarly, for $j = 1, 2, 3, \dots$, we obtain

$$\begin{aligned}\frac{1}{\frac{\natural^{\beta_1}}{\Lambda(U_0, U_{3j+1}, \Gamma)} + (1 - \natural^{\beta_1})} &\leq \Lambda(U_{\beta_1}, U_{\beta_1+3j+1}, \Gamma) \\ \Pi(U_{\beta_1}, U_{\beta_1+3j+1}, \natural \Gamma) &\leq \natural^{\beta_1} \Pi(U_0, U_{3j+1}, \Gamma) \quad \text{and} \quad \mathcal{M}(U_{\beta_1}, U_{\beta_1+3j+1}, \natural \Gamma) \leq \natural^{\beta_1} \mathcal{M}(U_0, U_{3j+1}, \Gamma), \\ \frac{1}{\frac{\natural^{\beta_1}}{\Lambda(U_0, U_{3j+2}, \Gamma)} + (1 - \natural^{\beta_1})} &\leq \Lambda(U_{\beta_1}, U_{\beta_1+3j+2}, \Gamma) \\ \Pi(U_{\beta_1}, U_{\beta_1+3j+2}, \natural \Gamma) &\leq \natural^{\beta_1} \Pi(U_0, U_{3j+2}, \Gamma) \quad \text{and} \quad \mathcal{M}(U_{\beta_1}, U_{\beta_1+3j+2}, \natural \Gamma) \leq \natural^{\beta_1} \mathcal{M}(U_0, U_{3j+2}, \Gamma),\end{aligned}$$

$$\frac{1}{\frac{\mathfrak{h}^{\beta_1}}{\Lambda(U_0, U_{3j+3}, \Gamma)} + (1 - \mathfrak{h}^{\beta_1})} \leq \Lambda(U_{\beta_1}, U_{\beta_1+3j+3}, \Gamma)$$

$$\Pi(U_{\beta_1}, U_{\beta_1+3j+3}, \mathfrak{h}\Gamma) \leq \mathfrak{h}^{\beta_1} \Pi(U_0, U_{3j+3}, \Gamma) \quad \text{and} \quad \mathcal{M}(U_{\beta_1}, U_{\beta_1+3j+3}, \mathfrak{h}\Gamma) \leq \mathfrak{h}^{\beta_1} \mathcal{M}(U_0, U_{3j+3}, \Gamma).$$

By using (21), we obtain for each $j = 1, 2, 3, \dots$,

$$\begin{aligned} \Lambda(U_0, U_{3j+1}, \Gamma) &\geq \frac{1}{\frac{1}{\Lambda(U_0, U_1, \frac{\Gamma}{4})} + (1 - \mathfrak{h})} \star \frac{1}{\frac{\mathfrak{h}}{\Lambda(U_0, U_1, \frac{\Gamma}{4})} + (1 - \mathfrak{h})} \star \frac{1}{\frac{\mathfrak{h}^2}{\Lambda(U_0, U_1, \frac{\Gamma}{4})} + (1 - \mathfrak{h}^2)} \\ &\star \frac{1}{\frac{\mathfrak{h}^3}{\Lambda(U_0, U_1, \frac{\Gamma}{4})} + (1 - \mathfrak{h}^3)} \star \frac{1}{\frac{\mathfrak{h}^4}{\Lambda(U_0, U_1, \frac{\Gamma}{4})} + (1 - \mathfrak{h}^4)} \\ &\star \frac{1}{\frac{\mathfrak{h}^5}{\Lambda(U_0, U_1, \frac{\Gamma}{4})} + (1 - \mathfrak{h}^5)} \star \frac{1}{\frac{\mathfrak{h}^6}{\Lambda(U_0, U_1, \frac{\Gamma}{4})} + (1 - \mathfrak{h}^6)} \\ &\star \dots \star \frac{1}{\frac{\mathfrak{h}^{3j}}{\Lambda(U_0, U_1, \frac{\Gamma}{4})} + (1 - \mathfrak{h}^{3j})} \\ \Pi(U_0, U_{3j+1}, \Gamma) &\leq \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \mathfrak{h} \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \mathfrak{h}^2 \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \\ &\bullet \mathfrak{h}^3 \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \mathfrak{h}^4 \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \mathfrak{h}^5 \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \\ &\bullet \mathfrak{h}^6 \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \dots \bullet \mathfrak{h}^{3j} \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}(U_0, U_{3j+1}, \Gamma) &\leq \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \mathfrak{h} \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \mathfrak{h}^2 \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \\ &\bullet \mathfrak{h}^3 \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \mathfrak{h}^4 \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \mathfrak{h}^5 \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \\ &\bullet \mathfrak{h}^6 \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \dots \bullet \mathfrak{h}^{3j} \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right). \end{aligned}$$

Now, from (21), we get

$$\begin{aligned} \Lambda(U_{\beta_1}, U_{\beta_1+3j+1}, \mathfrak{h}\Gamma) &\geq \Lambda\left(U_0, U_{3j+1}, \frac{\Gamma}{\mathfrak{h}^{\beta_1-1}}\right) \\ &\geq \frac{1}{\frac{1}{\Lambda(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}^{\beta_1-1}})} + (1 - \mathfrak{h})} \star \frac{1}{\frac{\mathfrak{h}}{\Lambda(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}^{\beta_1-1}})} + (1 - \mathfrak{h})} \\ &\star \frac{1}{\frac{\mathfrak{h}^2}{\Lambda(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}^{\beta_1-1}})} + (1 - \mathfrak{h}^2)} \\ &\star \frac{1}{\frac{\mathfrak{h}^3}{\Lambda(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}^{\beta_1-1}})} + (1 - \mathfrak{h}^3)} \star \frac{1}{\frac{\mathfrak{h}^4}{\Lambda(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}^{\beta_1-1}})} + (1 - \mathfrak{h}^4)} \\ &\star \frac{1}{\frac{\mathfrak{h}^5}{\Lambda(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}^{\beta_1-1}})} + (1 - \mathfrak{h}^5)} \star \frac{1}{\frac{\mathfrak{h}^6}{\Lambda(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}^{\beta_1-1}})} + (1 - \mathfrak{h}^6)} \\ &\star \dots \star \frac{1}{\frac{\mathfrak{h}^{3j}}{\Lambda(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}^{\beta_1-1}})} + (1 - \mathfrak{h}^{3j})}, \end{aligned} \tag{22}$$

$$\begin{aligned}
\Pi(U_{\beta_1}, U_{\beta_1+3j+1}, \natural\Gamma) &\leq \Pi\left(U_0, U_{3j+1}, \frac{\Gamma}{\natural^{\beta_1-1}}\right) \\
&\leq \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \natural\Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \natural^2\Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \\
&\bullet \natural^3\Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \natural^4\Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \\
&\bullet \natural^5\Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \natural^6\Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \\
&\bullet \dots \bullet \natural^{3j}\Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right)
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
\mathcal{M}(U_{\beta_1}, U_{\beta_1+3j+1}, \natural\Gamma) &\leq \mathcal{M}\left(U_0, U_{3j+1}, \frac{\Gamma}{\natural^{\beta_1-1}}\right) \\
&\leq \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \natural\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \\
&\bullet \natural^2\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \natural^3\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \\
&\bullet \natural^4\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \natural^5\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \\
&\bullet \natural^6\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \dots \bullet \natural^{3j}\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right).
\end{aligned}$$

By using (21), we obtain for each $j = 1, 2, 3, \dots$,

$$\begin{aligned}
\Lambda(U_0, U_{3j+2}, \Gamma) &\geq \frac{1}{\frac{1}{\Lambda(U_0, U_1, \frac{\Gamma}{4})} + (1 - \natural)} \star \frac{1}{\frac{\natural}{\Lambda(U_0, U_1, \frac{\Gamma}{4})} + (1 - \natural)} \star \frac{1}{\frac{\natural^2}{\Lambda(U_0, U_1, \frac{\Gamma}{4})} + (1 - \natural^2)} \\
&\star \frac{1}{\frac{\natural^3}{\Lambda(U_0, U_1, \frac{\Gamma}{4})} + (1 - \natural^3)} \star \frac{1}{\frac{\natural^4}{\Lambda(U_0, U_1, \frac{\Gamma}{4})} + (1 - \natural^4)} \\
&\star \frac{1}{\frac{\natural^5}{\Lambda(U_0, U_1, \frac{\Gamma}{4})} + (1 - \natural^5)} \star \frac{1}{\frac{\natural^6}{\Lambda(U_0, U_2, \frac{\Gamma}{4})} + (1 - \natural^6)} \\
&\star \dots \star \frac{1}{\frac{\natural^{3j}}{\Lambda(U_0, U_1, \frac{\Gamma}{4})} + (1 - \natural^{3j})} \\
\Pi(U_0, U_{3j+2}, \Gamma) &\leq \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural\Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural^2\Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \\
&\bullet \natural^3\Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural^4\Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural^5\Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \\
&\bullet \natural^6\Pi\left(U_0, U_2, \frac{\Gamma}{4}\right) \bullet \dots \bullet \natural^{3j}\Pi\left(U_0, U_2, \frac{\Gamma}{4}\right)
\end{aligned}$$

and

$$\begin{aligned} \mathcal{M}(U_0, U_{3j+2}, \Gamma) &\leq \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \mathfrak{h} \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \mathfrak{h}^2 \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \\ &\bullet \mathfrak{h}^3 \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \mathfrak{h}^4 \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \mathfrak{h}^5 \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \\ &\bullet \mathfrak{h}^6 \mathcal{M}\left(U_0, U_2, \frac{\Gamma}{4}\right) \bullet \dots \bullet \mathfrak{h}^{3j} \mathcal{M}\left(U_0, U_2, \frac{\Gamma}{4}\right). \end{aligned}$$

Now, from (21), we get

$$\begin{aligned} \Lambda(U_{\beta_1}, U_{\beta_1+3j+2}, \mathfrak{h}\Gamma) &\geq \Lambda\left(U_0, U_{3j+2}, \frac{\Gamma}{\mathfrak{h}\beta_1-1}\right) \\ &\geq \frac{1}{\frac{1}{\Lambda(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}\beta_1-1})}} \star \frac{1}{\frac{\mathfrak{h}}{\Lambda(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}\beta_1-1})} + (1-\mathfrak{h})} \\ &\star \frac{1}{\frac{\mathfrak{h}^2}{\Lambda(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}\beta_1-1})} + (1-\mathfrak{h}^2)} \star \frac{1}{\frac{\mathfrak{h}^3}{\Lambda(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}\beta_1-1})} + (1-\mathfrak{h}^3)} \\ &\star \frac{1}{\frac{\mathfrak{h}^4}{\Lambda(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}\beta_1-1})} + (1-\mathfrak{h}^4)} \star \frac{1}{\frac{\mathfrak{h}^5}{\Lambda(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}\beta_1-1})} + (1-\mathfrak{h}^5)} \\ &\star \frac{1}{\frac{\mathfrak{h}^6}{\Lambda(U_0, U_2, \frac{\Gamma}{4\mathfrak{h}\beta_1-1})} + (1-\mathfrak{h}^6)} \star \dots \star \frac{1}{\frac{\mathfrak{h}^{3j}}{\Lambda(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}\beta_1-1})} + (1-\mathfrak{h}^{3j})} \end{aligned} \quad (24)$$

$$\begin{aligned} \Pi(U_{\beta_1}, U_{\beta_1+3j+2}, \mathfrak{h}\Gamma) &\leq \Pi\left(U_0, U_{3j+2}, \frac{\Gamma}{\mathfrak{h}\beta_1-1}\right) \\ &\leq \Pi\left(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}\beta_1-1}\right) \bullet \mathfrak{h} \Pi\left(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}\beta_1-1}\right) \\ &\bullet \mathfrak{h}^2 \Pi\left(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}\beta_1-1}\right) \bullet \mathfrak{h}^3 \Pi\left(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}\beta_1-1}\right) \\ &\bullet \mathfrak{h}^4 \Pi\left(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}\beta_1-1}\right) \bullet \mathfrak{h}^5 \Pi\left(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}\beta_1-1}\right) \\ &\bullet \mathfrak{h}^6 \Pi\left(U_0, U_2, \frac{\Gamma}{4\mathfrak{h}\beta_1-1}\right) \bullet \dots \bullet \mathfrak{h}^{3j} \Pi\left(U_0, U_2, \frac{\Gamma}{4\mathfrak{h}\beta_1-1}\right) \end{aligned} \quad (25)$$

and

$$\begin{aligned} \mathcal{M}(U_{\beta_1}, U_{\beta_1+3j+2}, \mathfrak{h}\Gamma) &\leq \mathcal{M}\left(U_0, U_{3j+2}, \frac{\Gamma}{\mathfrak{h}\beta_1-1}\right) \\ &\leq \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}\beta_1-1}\right) \bullet \mathfrak{h} \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}\beta_1-1}\right) \\ &\bullet \mathfrak{h}^2 \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}\beta_1-1}\right) \bullet \mathfrak{h}^3 \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}\beta_1-1}\right) \\ &\bullet \mathfrak{h}^4 \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}\beta_1-1}\right) \bullet \mathfrak{h}^5 \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}\beta_1-1}\right) \\ &\bullet \mathfrak{h}^6 \mathcal{M}\left(U_0, U_2, \frac{\Gamma}{4\mathfrak{h}\beta_1-1}\right) \bullet \dots \bullet \mathfrak{h}^{3j} \mathcal{M}\left(U_0, U_2, \frac{\Gamma}{4\mathfrak{h}\beta_1-1}\right). \end{aligned} \quad (26)$$

By using (21), we obtain for each $j = 1, 2, 3, \dots$,

$$\begin{aligned} \Lambda(U_0, U_{3j+3}, \Gamma) &\geq \frac{1}{\frac{1}{\Lambda(U_0, U_1, \frac{\Gamma}{4})}} \star \frac{1}{\frac{\natural}{\Lambda(U_0, U_1, \frac{\Gamma}{4})} + (1 - \natural)} \star \frac{1}{\frac{\natural^2}{\Lambda(U_0, U_1, \frac{\Gamma}{4})} + (1 - \natural^2)} \\ &\star \frac{1}{\frac{\natural^3}{\Lambda(U_0, U_1, \frac{\Gamma}{4})} + (1 - \natural^3)} \star \frac{1}{\frac{\natural^4}{\Lambda(U_0, U_1, \frac{\Gamma}{4})} + (1 - \natural^4)} \\ &\star \frac{1}{\frac{\natural^5}{\Lambda(U_0, U_1, \frac{\Gamma}{4})} + (1 - \natural^5)} \star \frac{1}{\frac{\natural^6}{\Lambda(U_0, U_3, \frac{\Gamma}{4})} + (1 - \natural^6)} \\ &\star \dots \star \frac{1}{\frac{\natural^{3j}}{\Lambda(U_0, U_3, \frac{\Gamma}{4})} + (1 - \natural^{3j})} \end{aligned}$$

$$\begin{aligned} \Pi(U_0, U_{3j+3}, \Gamma) &\leq \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural^2 \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \\ &\bullet \natural^3 \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural^4 \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural^5 \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \\ &\bullet \natural^6 \Pi\left(U_0, U_3, \frac{\Gamma}{4}\right) \bullet \dots \bullet \natural^{3j} \Pi\left(U_0, U_3, \frac{\Gamma}{4}\right) \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}(U_0, U_{3j+3}, \Gamma) &\leq \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural^2 \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \\ &\bullet \natural^3 \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural^4 \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural^5 \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \\ &\bullet \natural^6 \mathcal{M}\left(U_0, U_3, \frac{\Gamma}{4}\right) \bullet \dots \bullet \natural^{3j} \mathcal{M}\left(U_0, U_3, \frac{\Gamma}{4}\right). \end{aligned}$$

Now, from (21), we get

$$\begin{aligned} \Lambda(U_{\beta_1}, U_{\beta_1+3j+3}, \natural\Gamma) &\geq \Lambda\left(U_0, U_{3j+3}, \frac{\Gamma}{\natural^{\beta_1-1}}\right) \\ &\geq \frac{1}{\frac{1}{\Lambda(U_0, U_1, \frac{\Gamma}{4^{\beta_1-1}})}} \star \frac{1}{\frac{\natural}{\Lambda(U_0, U_1, \frac{\Gamma}{4^{\beta_1-1}})} + (1 - \natural)} \\ &\star \frac{1}{\frac{\natural^2}{\Lambda(U_0, U_1, \frac{\Gamma}{4^{\beta_1-1}})} + (1 - \natural^2)} \star \frac{1}{\frac{\natural^3}{\Lambda(U_0, U_1, \frac{\Gamma}{4^{\beta_1-1}})} + (1 - \natural^3)} \\ &\star \frac{1}{\frac{\natural^4}{\Lambda(U_0, U_1, \frac{\Gamma}{4^{\beta_1-1}})} + (1 - \natural^4)} \star \frac{1}{\frac{\natural^5}{\Lambda(U_0, U_1, \frac{\Gamma}{4^{\beta_1-1}})} + (1 - \natural^5)} \\ &\star \frac{1}{\frac{\natural^6}{\Lambda(U_0, U_3, \frac{\Gamma}{4^{\beta_1-1}})} + (1 - \natural^6)} \\ &\star \dots \star \frac{1}{\frac{\natural^{3j}}{\Lambda(U_0, U_3, \frac{\Gamma}{4^{\beta_1-1}})} + (1 - \natural^{3j})}, \end{aligned} \tag{27}$$

$$\begin{aligned}
 \Pi(U_{\beta_1}, U_{\beta_1+3j+3}, \mathfrak{h}\Gamma) &\leq \Pi\left(U_0, U_{3j+3}, \frac{\Gamma}{\mathfrak{h}^{\beta_1-1}}\right) \\
 &\leq \Pi\left(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}^{\beta_1-1}}\right) \bullet \mathfrak{h}\Pi\left(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}^{\beta_1-1}}\right) \\
 &\quad \bullet \mathfrak{h}^2\Pi\left(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}^{\beta_1-1}}\right) \bullet \mathfrak{h}^3\Pi\left(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}^{\beta_1-1}}\right) \\
 &\quad \bullet \mathfrak{h}^4\Pi\left(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}^{\beta_1-1}}\right) \bullet \mathfrak{h}^5\Pi\left(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}^{\beta_1-1}}\right) \\
 &\quad \bullet \mathfrak{h}^6\Pi\left(U_0, U_3, \frac{\Gamma}{4\mathfrak{h}^{\beta_1-1}}\right) \bullet \dots \bullet \mathfrak{h}^{3j}\Pi\left(U_0, U_3, \frac{\Gamma}{4\mathfrak{h}^{\beta_1-1}}\right) \quad (28)
 \end{aligned}$$

and

$$\begin{aligned}
 \mathcal{M}(U_{\beta_1}, U_{\beta_1+3j+3}, \mathfrak{h}\Gamma) &\leq \mathcal{M}\left(U_0, U_{3j+3}, \frac{\Gamma}{\mathfrak{h}^{\beta_1-1}}\right) \\
 &\leq \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}^{\beta_1-1}}\right) \bullet \mathfrak{h}\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}^{\beta_1-1}}\right) \\
 &\quad \bullet \mathfrak{h}^2\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}^{\beta_1-1}}\right) \bullet \mathfrak{h}^3\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}^{\beta_1-1}}\right) \\
 &\quad \bullet \mathfrak{h}^4\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}^{\beta_1-1}}\right) \bullet \mathfrak{h}^5\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\mathfrak{h}^{\beta_1-1}}\right) \\
 &\quad \bullet \mathfrak{h}^6\mathcal{M}\left(U_0, U_3, \frac{\Gamma}{4\mathfrak{h}^{\beta_1-1}}\right) \bullet \dots \bullet \mathfrak{h}^{3j}\mathcal{M}\left(U_0, U_3, \frac{\Gamma}{4\mathfrak{h}^{\beta_1-1}}\right). \quad (29)
 \end{aligned}$$

Using (22)–(29), for each case $\beta_1 \rightarrow +\infty$, we deduce

$$\begin{aligned}
 \lim_{\beta_1 \rightarrow +\infty} \Lambda(U_{\beta_1}, U_{\beta_1+q}, \Gamma) &= 1 \star 1 \star \dots \star 1 = 1, \\
 \lim_{\beta_1 \rightarrow +\infty} \Pi(U_{\beta_1}, U_{\beta_1+q}, \Gamma) &= 0 \bullet 0 \bullet \dots \bullet 0 = 0,
 \end{aligned}$$

and

$$\lim_{\beta_1 \rightarrow +\infty} \mathcal{M}(U_{\beta_1}, U_{\beta_1+q}, \Gamma) = 0 \bullet 0 \bullet \dots \bullet 0 = 0.$$

It follows that $\{U_{\beta_1}\}$ is a Cauchy sequence. Since $(K, \Lambda, \Pi, \mathcal{M}, \star, \bullet)$ be a complete NPMS, there exists

$$\lim_{\beta_1 \rightarrow +\infty} U_{\beta_1} = U.$$

Since (v), (x) and (xv), we get

$$\begin{aligned}
 \frac{1}{\Lambda(\wp U_{\beta_1}, \wp U, \Gamma)} - 1 &\leq \mathfrak{h}\left[\frac{1}{\Lambda(U_{\beta_1}, U, \Gamma)} - 1\right] = \frac{\mathfrak{h}}{\Lambda(U_{\beta_1}, U, \Gamma)} - \mathfrak{h} \\
 &\Rightarrow \frac{1}{\frac{\mathfrak{h}}{\Lambda(U_{\beta_1}, U, \Gamma)} + (1 - \mathfrak{h})} \leq \Lambda(\wp U_{\beta_1}, \wp U, \Gamma).
 \end{aligned}$$

By the above inequality, we have

$$\begin{aligned}
\Lambda(U, \wp U, \Gamma) &\geq \Lambda\left(U, U_{\beta_1}, \frac{\Gamma}{4}\right) \star \Lambda\left(U_{\beta_1}, U_{\beta_1+1}, \frac{\Gamma}{4}\right) \star \Lambda\left(U_{\beta_1+1}, \wp U_{\beta_1+2}, \frac{\Gamma}{4}\right) \\
&\quad \star \Lambda\left(U_{\beta_1+2}, \wp U, \frac{\Gamma}{4}\right) \\
&\geq \Lambda\left(U, U_{\beta_1}, \frac{\Gamma}{4}\right) \star \Lambda\left(\wp U_{\beta_1-1}, \wp U_{\beta_1}, \frac{\Gamma}{4}\right) \star \Lambda\left(\wp U_{\beta_1}, \wp U_{\beta_1+1}, \frac{\Gamma}{4}\right) \\
&\quad \star \Lambda\left(\wp U_{\beta_1+1}, \wp U, \frac{\Gamma}{4}\right) \\
&\geq \Lambda\left(U, U_{\beta_1}, \frac{\Gamma}{4}\right) \star \frac{1}{\frac{\mathfrak{z}^{\beta_1-1}}{\Lambda(U_{\beta_1-1}, U_{\beta_1}, \frac{\Gamma}{4})} + (1 - \mathfrak{z}^{\beta_1-1})} \star \frac{1}{\frac{\mathfrak{z}^{\beta_1}}{\Lambda(U_{\beta_1}, U_{\beta_1+1}, \frac{\Gamma}{4})} + (1 - \mathfrak{z}^{\beta_1})} \\
&\quad \star \frac{1}{\frac{\mathfrak{z}}{\Lambda(U_{\beta_1+1}, U, \frac{\Gamma}{4})} + (1 - \mathfrak{z})} \\
&\rightarrow 1 \star 1 \star 1 \star 1 = 1 \text{ as } \beta_1 \rightarrow +\infty,
\end{aligned}$$

$$\begin{aligned}
\Pi(U, \wp U, \Gamma) &\leq \Pi\left(U, U_{\beta_1}, \frac{\Gamma}{4}\right) \bullet \Pi\left(U_{\beta_1}, U_{\beta_1+1}, \frac{\Gamma}{4}\right) \bullet \Pi\left(U_{\beta_1+1}, \wp U_{\beta_1+2}, \frac{\Gamma}{4}\right) \\
&\quad \bullet \Pi\left(U_{\beta_1+2}, \wp U, \frac{\Gamma}{4}\right) \\
&\leq \Pi\left(U, U_{\beta_1}, \frac{\Gamma}{4}\right) \bullet \Pi\left(\wp U_{\beta_1-1}, \wp U_{\beta_1}, \frac{\Gamma}{4}\right) \bullet \Pi\left(\wp U_{\beta_1}, \wp U_{\beta_1+1}, \frac{\Gamma}{4}\right) \\
&\quad \bullet \Pi\left(\wp U_{\beta_1+1}, \wp U, \frac{\Gamma}{4}\right) \\
&\leq \Pi\left(U, U_{\beta_1}, \frac{\Gamma}{4}\right) \bullet \mathfrak{z}^{\beta_1-1} \Pi\left(U_{\beta_1-1}, U_{\beta_1}, \frac{\Gamma}{4}\right) \bullet \mathfrak{z}^{\beta_1} \Pi\left(U_{\beta_1}, U_{\beta_1+1}, \frac{\Gamma}{4}\right) \\
&\quad \bullet \mathfrak{z} \Pi\left(U_{\beta_1+1}, U, \frac{\Gamma}{4}\right) \\
&\rightarrow 0 \bullet 0 \bullet 0 \bullet 0 = 0 \text{ as } \beta_1 \rightarrow +\infty
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{M}(U, \wp U, \Gamma) &\leq \mathcal{M}\left(U, U_{\beta_1}, \frac{\Gamma}{4}\right) \bullet \mathcal{M}\left(U_{\beta_1}, U_{\beta_1+1}, \frac{\Gamma}{4}\right) \bullet \mathcal{M}\left(U_{\beta_1+1}, U_{\beta_1+2}, \frac{\Gamma}{4}\right) \\
&\quad \bullet \mathcal{M}\left(U_{\beta_1+2}, \wp U, \frac{\Gamma}{4}\right) \\
&\leq \mathcal{M}\left(U, U_{\beta_1}, \frac{\Gamma}{4}\right) \bullet \mathcal{M}\left(\wp U_{\beta_1-1}, \wp U_{\beta_1}, \frac{\Gamma}{4}\right) \bullet \mathcal{M}\left(\wp U_{\beta_1}, \wp U_{\beta_1+1}, \frac{\Gamma}{4}\right) \\
&\quad \bullet \mathcal{M}\left(\wp U_{\beta_1+1}, \wp U, \frac{\Gamma}{4}\right) \\
&\leq \mathcal{M}\left(U, U_{\beta_1}, \frac{\Gamma}{4}\right) \bullet \mathfrak{z}^{\beta_1-1} \mathcal{M}\left(U_{\beta_1-1}, U_{\beta_1}, \frac{\Gamma}{4}\right) \bullet \mathfrak{z}^{\beta_1} \mathcal{M}\left(U_{\beta_1}, U_{\beta_1+1}, \frac{\Gamma}{4}\right) \\
&\quad \bullet \mathfrak{z} \mathcal{M}\left(U_{\beta_1+1}, U, \frac{\Gamma}{4}\right) \\
&\rightarrow 0 \bullet 0 \bullet 0 \bullet 0 = 0 \text{ as } \beta_1 \rightarrow +\infty.
\end{aligned}$$

Hence, $\wp U = U$. Let $\wp \tilde{\mu} = \tilde{\mu}$ for some $\tilde{\mu} \in K$, then

$$\begin{aligned} \frac{1}{\Lambda(U, \tilde{\mu}, \Gamma)} - 1 &= \frac{1}{\Lambda(\wp U, \wp \tilde{\mu}, \Gamma)} - 1 \\ &\leq \natural \left[\frac{1}{\Lambda(U, \tilde{\mu}, \Gamma)} - 1 \right] < \frac{1}{\Lambda(U, \tilde{\mu}, \Gamma)} - 1, \end{aligned}$$

which is a contradiction.

$$\Pi(U, \tilde{\mu}, \Gamma) = \Pi(\wp U, \wp \tilde{\mu}, \Gamma) \leq \natural \Pi(U, \tilde{\mu}, \Gamma) < \Pi(U, \tilde{\mu}, \Gamma),$$

which is a contradiction and

$$\mathcal{M}(U, \tilde{\mu}, \Gamma) = \mathcal{M}(\wp U, \wp \tilde{\mu}, \Gamma) \leq \natural \mathcal{M}(U, \tilde{\mu}, \Gamma) < \mathcal{M}(U, \tilde{\mu}, \Gamma),$$

which is a contradiction. Therefore, we obtain $\Lambda(U, \tilde{\mu}, \Gamma) = 1, \Pi(U, \tilde{\mu}, \Gamma) = 0$ and $\mathcal{M}(U, \tilde{\mu}, \Gamma) = 0$, hence, $U = \tilde{\mu}$. $\square$

Example 2. Let $K = [0, 1]$. Define $\Lambda, \Pi, \mathcal{M}: K \times K \times (0, +\infty) \rightarrow [0, 1]$ as

$$\begin{aligned} \Lambda(U, \mathcal{W}, \Gamma) &= \frac{\Gamma}{\Gamma + |U - \mathcal{W}|}, \\ \Pi(U, \mathcal{W}, \Gamma) &= \frac{|U - \mathcal{W}|}{\Gamma + |U - \mathcal{W}|}, \\ \mathcal{M}(U, \mathcal{W}, \Gamma) &= \frac{|U - \mathcal{W}|}{\Gamma}. \end{aligned}$$

Then, $(K, \Lambda, \Pi, \mathcal{M}, \star, \bullet)$ is a complete NPMS with continuous t -norm and t -co-norm, i.e., $\hat{\omega} \star \check{h} = \hat{\omega} \check{h}$ and $\hat{\omega} \bullet \check{h} = \max\{\hat{\omega}, \check{h}\}$.

Define $\wp: K \rightarrow K$ by $\wp(U) = \frac{1-3^{-U}}{11}$ and set $\natural \in [\frac{1}{2}, 1)$, then

$$\begin{aligned} \Lambda(\wp U, \wp \mathcal{W}, \natural \Gamma) &= \Lambda\left(\frac{1-3^{-U}}{11}, \frac{1-3^{-\mathcal{W}}}{11}, \natural \Gamma\right) \\ &= \frac{\natural \Gamma}{\natural \Gamma + \left| \frac{1-3^{-U}}{11} - \frac{1-3^{-\mathcal{W}}}{11} \right|} = \frac{\natural \Gamma}{\natural \Gamma + \frac{|3^{-U} - 3^{-\mathcal{W}}|}{11}} \\ &\geq \frac{\natural \Gamma}{\natural \Gamma + \frac{|U - \mathcal{W}|}{11}} = \frac{11 \natural \Gamma}{11 \natural \Gamma + |U - \mathcal{W}|} \geq \frac{\Gamma}{\Gamma + |U - \mathcal{W}|} = \Lambda(U, \mathcal{W}, \Gamma), \end{aligned}$$

$$\begin{aligned} \Pi(\wp U, \wp \mathcal{W}, \natural \Gamma) &= \Pi\left(\frac{1-3^{-U}}{11}, \frac{1-3^{-\mathcal{W}}}{11}, \natural \Gamma\right) \\ &= \frac{\left| \frac{1-3^{-U}}{11} - \frac{1-3^{-\mathcal{W}}}{11} \right|}{\natural \Gamma + \left| \frac{1-3^{-U}}{11} - \frac{1-3^{-\mathcal{W}}}{11} \right|} = \frac{\frac{|3^{-U} - 3^{-\mathcal{W}}|}{11}}{\natural \Gamma + \frac{|3^{-U} - 3^{-\mathcal{W}}|}{11}} \\ &= \frac{|3^{-U} - 3^{-\mathcal{W}}|}{11 \natural \Gamma + |3^{-U} - 3^{-\mathcal{W}}|} \leq \frac{|U - \mathcal{W}|}{11 \natural \Gamma + |U - \mathcal{W}|} \leq \frac{|U - \mathcal{W}|}{\Gamma + |U - \mathcal{W}|} = \Pi(U, \mathcal{W}, \Gamma) \end{aligned}$$

and

$$\begin{aligned}\mathcal{M}(\wp U, \wp W, \natural \Gamma) &= \mathcal{M}\left(\frac{1-3^{-U}}{11}, \frac{1-3^{-W}}{11}, \natural \Gamma\right) \\ &= \frac{\left|\frac{1-3^{-U}}{11} - \frac{1-3^{-W}}{11}\right|}{\natural \Gamma} = \frac{|3^{-U}-3^{-W}|}{\natural \Gamma} \\ &= \frac{|3^{-U}-3^{-W}|}{11 \natural \Gamma} \leq \frac{|U-W|}{11 \natural \Gamma} \leq \frac{|U-W|}{\Gamma} = \mathcal{M}(U, W, \Gamma).\end{aligned}$$

As a result, all of Theorem 1 criteria are satisfied, and 0 is the only fixed point for $\wp$.

4. Applications

4.1. Application to Fredholm Integral Equation

Let $K = \mathcal{C}([j, \delta], \mathbb{R})$ be the set of real value continuous functions on $[j, \delta]$. Consider the integral equation:

$$U(\aleph) = \wedge(\aleph) + \ell \int_j^\delta \mathcal{U}(\aleph, \kappa) U(\aleph) \zeta \kappa \quad \text{for } \aleph, \kappa \in [j, \delta], \quad (30)$$

where $\wedge(\kappa)$ is a fuzzy function of $\kappa \in [j, \delta]$, $\ell > 0$ and $\mathcal{U}: \mathcal{C}([j, \delta] \times \mathbb{R}) \rightarrow \mathbb{R}^+$. Define Λ, Π and $\mathcal{M}$ by means of

$$\begin{aligned}\Lambda(U(\aleph), W(\aleph), \Gamma) &= \sup_{\aleph \in [j, \delta]} \frac{\Gamma}{\Gamma + |U(\aleph) - W(\aleph)|} \quad \forall \quad U, W \in K \text{ and } \Gamma > 0, \\ \Pi(U(\aleph), W(\aleph), \Gamma) &= 1 - \sup_{\aleph \in [j, \delta]} \frac{\Gamma}{\Gamma + |U(\aleph) - W(\aleph)|} \quad \forall \quad U, W \in K \text{ and } \Gamma > 0,\end{aligned}$$

and

$$\mathcal{M}(U(\aleph), W(\aleph), \Gamma) = \sup_{\aleph \in [j, \delta]} \frac{|U(\aleph) - W(\aleph)|}{\Gamma} \quad \forall \quad U, W \in K \text{ and } \Gamma > 0,$$

by continuous t-norm and continuous t-co-norm define by $\hat{\omega} \star \check{h} = \hat{\omega} \cdot \check{h}$ and $\hat{\omega} \bullet \check{h} = \max\{\hat{\omega}, \check{h}\}$. Then $(K, \Lambda, \Pi, \mathcal{M}, \star, \bullet)$ is a complete NPMS. Suppose that $|\mathcal{U}(\aleph, \kappa) U(\aleph) - \mathcal{U}(\aleph, \kappa) W(\aleph)| \leq |U(\aleph) - W(\aleph)|$ for $U, W \in K, 0 < \natural < 1$ and $\forall \aleph, \kappa \in [j, \delta]$. Let $\mathcal{U}(\aleph, \kappa)(\ell \int_j^\delta d\kappa) \leq \natural < 1$. Then Equation (30) has a unique solution.

Proof. Define $\wp: K \rightarrow K$ by

$$\wp U(\aleph) = \wedge(\aleph) + \ell \int_j^\delta \mathcal{U}(\aleph, \kappa) U(\aleph) \zeta \kappa, \text{ for all } \aleph, \kappa \in [j, \delta].$$

Now, for all $U, W \in K$, we deduce

$$\begin{aligned}\Lambda(\wp U(\aleph), \wp W(\aleph), \natural \Gamma) &= \sup_{\aleph \in [j, \delta]} \frac{\natural \Gamma}{\natural \Gamma + |\wp U(\aleph) - \wp W(\aleph)|} \\ &= \sup_{\aleph \in [j, \delta]} \frac{\natural \Gamma}{\natural \Gamma + |\wedge(\aleph) + \ell \int_j^\delta \mathcal{U}(\aleph, \kappa) U(\aleph) \zeta \kappa - \wedge(\aleph) - \ell \int_j^\delta \mathcal{U}(\aleph, \kappa) W(\aleph) \zeta \kappa|} \\ &= \sup_{\aleph \in [j, \delta]} \frac{\natural \Gamma}{\natural \Gamma + |\ell \int_j^\delta \mathcal{U}(\aleph, \kappa) U(\aleph) \zeta \kappa - \ell \int_j^\delta \mathcal{U}(\aleph, \kappa) W(\aleph) \zeta \kappa|}\end{aligned}$$

$$\begin{aligned}
&= \sup_{\aleph \in [j, \delta]} \frac{\aleph \Gamma}{\aleph \Gamma + |\mathcal{U}(\aleph, \kappa)U(\aleph) - \mathcal{U}(\aleph, \kappa)\mathcal{W}(\aleph)|(\ell \int_j^\delta \zeta \kappa)} \\
&\geq \sup_{\aleph \in [j, \delta]} \frac{\Gamma}{\Gamma + |U(\aleph) - \mathcal{W}(\aleph)|} \\
&\geq \Lambda(U(\aleph), \mathcal{W}(\aleph), \Gamma), \\
\Pi(\wp U(\aleph), \wp \mathcal{W}(\aleph), \aleph \Gamma) &= 1 - \sup_{\aleph \in [j, \delta]} \frac{\aleph \Gamma}{\aleph \Gamma + |\wp U(\aleph) - \wp \mathcal{W}(\aleph)|} \\
&= 1 - \sup_{\aleph \in [j, \delta]} \frac{\aleph \Gamma}{\aleph \Gamma + |\wedge(\aleph) + \ell \int_j^\delta \mathcal{U}(\aleph, \kappa)U(\aleph)\zeta \kappa - \wedge(\aleph) - \ell \int_j^\delta \mathcal{U}(\aleph, \kappa)U(\aleph)\zeta \kappa|} \\
&= 1 - \sup_{\aleph \in [j, \delta]} \frac{\aleph \Gamma}{\aleph \Gamma + |\ell \int_j^\delta \mathcal{U}(\aleph, \kappa)U(\aleph)\zeta \kappa - \ell \int_j^\delta \mathcal{U}(\aleph, \kappa)U(\aleph)\zeta \kappa|} \\
&= 1 - \sup_{\aleph \in [j, \delta]} \frac{\aleph \Gamma}{\aleph \Gamma + |\mathcal{U}(\aleph, \kappa)U(\aleph) - \mathcal{U}(\aleph, \kappa)\mathcal{W}(\aleph)|(\ell \int_j^\delta \zeta \kappa)} \\
&\leq 1 - \sup_{\aleph \in [j, \delta]} \frac{\Gamma}{\Gamma + |U(\aleph) - \mathcal{W}(\aleph)|} \\
&\leq \Pi(U(\aleph), \mathcal{W}(\aleph), \Gamma),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{M}(\wp U(\aleph), \wp \mathcal{W}(\aleph), \aleph \Gamma) &= \sup_{\aleph \in [j, \delta]} \frac{|\wp U(\aleph) - \wp \mathcal{W}(\aleph)|}{\aleph \Gamma} \\
&= \sup_{\aleph \in [j, \delta]} \frac{|\wedge(\aleph) + \ell \int_j^\delta \mathcal{U}(\aleph, \kappa)U(\aleph)\zeta \kappa - \wedge(\aleph) - \ell \int_j^\delta \mathcal{U}(\aleph, \kappa)U(\aleph)\zeta \kappa|}{\aleph \Gamma} \\
&= \sup_{\aleph \in [j, \delta]} \frac{|\ell \int_j^\delta \mathcal{U}(\aleph, \kappa)U(\aleph)\zeta \kappa - \ell \int_j^\delta \mathcal{U}(\aleph, \kappa)U(\aleph)\zeta \kappa|}{\aleph \Gamma} \\
&= \sup_{\aleph \in [j, \delta]} \frac{|\mathcal{U}(\aleph, \kappa)U(\aleph) - \mathcal{U}(\aleph, \kappa)\mathcal{W}(\aleph)|(\ell \int_j^\delta \zeta \kappa)}{\aleph \Gamma} \\
&\leq \sup_{\aleph \in [j, \delta]} \frac{|U(\aleph) - \mathcal{W}(\aleph)|}{\Gamma} \\
&\leq \mathcal{M}(U(\aleph), \mathcal{W}(\aleph), \Gamma),
\end{aligned}$$

As a result, $\wp$ has a ufp and all of the requirements of Theorem 1 are fulfilled. It is obvious that the Equation (30) has only one solution. $\square$

Example 3. Consider the non-linear integral equation.

$$U(\aleph) = |\cos \aleph| + \frac{1}{11} \int_0^1 \kappa U(\kappa) \zeta \kappa, \text{ for all } \kappa \in [0, 1].$$

Then it has a solution in K .

Proof. Let $\wp: K \rightarrow K$ be defined by

$$\wp U(\aleph) = |\cos \aleph| + \frac{1}{11} \int_0^1 \kappa U(\kappa) \zeta \kappa,$$

and set $\mathcal{U}(\aleph, \kappa)U(\aleph) = \frac{1}{\Gamma(\ell)}\kappa U(\kappa)$ and $\mathcal{U}(\aleph, \kappa)\mathcal{W}(\aleph) = \frac{1}{\Gamma(\ell)}\kappa\mathcal{W}(\kappa)$, where $U, \mathcal{W} \in K$, and $\forall \aleph, \kappa \in [0, 1]$. Then we obtain

$$\begin{aligned} |\mathcal{U}(\aleph, \kappa)U(\aleph) - \mathcal{U}(\aleph, \kappa)\mathcal{W}(\aleph)| &= \left| \frac{1}{\Gamma(\ell)}\kappa U(\kappa) - \frac{1}{\Gamma(\ell)}\kappa\mathcal{W}(\kappa) \right| \\ &= \frac{\kappa}{\Gamma(\ell)}|U(\kappa) - \mathcal{W}(\kappa)| \leq |U(\kappa) - \mathcal{W}(\kappa)|. \end{aligned}$$

Furthermore, see that $\frac{1}{\Gamma(\ell)} \int_0^1 \kappa \zeta \kappa = \frac{1}{\Gamma(\ell)} \left(\frac{(1)^2}{2} - \frac{(0)^2}{2} \right) = \frac{1}{\Gamma(\ell)} = \mathfrak{h} \leq 1$, where $\ell = \frac{1}{\Gamma(\ell)}$. Then, it follows that all criteria of the above application are easily verified and the above problem has a solution in K . $\square$

4.2. Application to Fractional Differential Equations

In order to start, we need to review some basic definitions from the theory of fractional calculus.

For a function $U \in \mathcal{C}[0, 1]$, the Reiman-Liouville fractional derivative of order $\ell > 0$ is given by

$$\frac{1}{\Gamma(\beta_1 - \ell)} \frac{d^{\beta_1}}{d\aleph^{\beta_1}} \int_0^{\aleph} \frac{U(j) dj}{(\aleph - j)^{\ell - \beta_1 + 1}} = \mathcal{D}^{\ell} U(\aleph),$$

For as long as the right hand side is pointwise defined on $[0, 1]$, $[\ell]$ is the integer portion of the number ℓ , Γ is the Euler gamma function.

Consider the following fractional differential equation

$$\begin{aligned} {}^J\mathcal{D}^{\chi} U(\aleph) + \mathfrak{f}(\aleph, U(\aleph)) &= 0, \quad 1 \leq \aleph \leq 0, \quad 2 \leq \chi > 1; \\ U(0) &= U(1) = 0, \end{aligned} \quad (31)$$

where $\mathfrak{f}$ is a continuous function from $[0, 1] \times \mathbb{R}$ to $\mathbb{R}$ and ${}^J\mathcal{D}^{\chi}$ represents the Caputo fractional derivative of order χ and it is defined by

$${}^J\mathcal{D}^{\chi} = \frac{1}{\Gamma(\beta_1 - \chi)} \int_0^{\aleph} \frac{U^{\beta_1}(j) dj}{(\aleph - j)^{\chi - \beta_1 + 1}}.$$

The given fractional differential Equation (31) is equivalent

$$U(\aleph) = \int_0^1 \Upsilon(\aleph, j) \mathfrak{f}(\aleph, U(j)) dj,$$

for all $U \in \mathcal{Y}$ and $\aleph \in [0, 1]$, where

$$\Upsilon(\aleph, j) = \begin{cases} \frac{[\aleph(1-j)]^{\chi-1} - (\aleph-j)^{\chi-1}}{\Gamma(\chi)}, & 0 \leq j \leq \aleph \leq 1, \\ \frac{[\aleph(1-j)]^{\chi-1}}{\Gamma(\chi)}, & 0 \leq \aleph \leq j \leq 1. \end{cases}$$

Let $\mathcal{C}([0, 1], \mathbb{R}) = K$ be the space of all continuous functions defined on $[0, 1]$. Define Λ , Π and $\mathcal{M}$ by means of

$$\begin{aligned} \Lambda(U(\aleph), \mathcal{W}(\aleph), \Gamma) &= \sup_{\aleph \in [0, 1]} \frac{\Gamma}{\Gamma + |U(\aleph) - \mathcal{W}(\aleph)|}, \quad \text{for all } U, \mathcal{W} \in K \text{ and } \Gamma > 0, \\ \Pi(U(\aleph), \mathcal{W}(\aleph), \Gamma) &= 1 - \sup_{\aleph \in [0, 1]} \frac{\Gamma}{\Gamma + |U(\aleph) - \mathcal{W}(\aleph)|}, \quad \text{for all } U, \mathcal{W} \in K \text{ and } \Gamma > 0 \end{aligned}$$

and

$$\mathcal{M}(U(\aleph), \mathcal{W}(\aleph), \Gamma) = \sup_{\aleph \in [0,1]} \frac{|U(\aleph) - \mathcal{W}(\aleph)|}{\Gamma}, \text{ for all } U, \mathcal{W} \in K \text{ and } \Gamma > 0,$$

with continuous t-norm and continuous t-co-norm define by $\hat{\omega} \star \check{h} = \hat{\omega} \cdot \check{h}$ and $\hat{\omega} \bullet \check{h} = \max\{\hat{\omega}, \check{h}\}$. Then $(K, \Lambda, \Pi, \mathcal{M}, \star, \bullet)$ is a complete NPMS.

Theorem 3. Consider the nonlinear fractional differential Equation (31). Let us assume that the following claims are true:

(i) there exist $\aleph \in [0, 1]$ and $U, \mathcal{W} \in \mathcal{C}([0, 1], \mathbb{R})$ such that

$$|\mathfrak{f}(\aleph, U) - \mathfrak{f}(\aleph, \mathcal{W})| \leq |U(\aleph) - \mathcal{W}(\aleph)|;$$

(ii) $\sup_{\aleph \in [0,1]} \int_0^1 \Upsilon(\aleph, j) \varsigma \aleph \leq \natural < 1$.

Then the fractional differential Equation (31) has a unique solution in K .

Proof. Let us consider the function $\wp: K \rightarrow K$ defined by

$$KU(\aleph) = \int_0^1 \Upsilon(\aleph, j) \mathfrak{f}(\aleph, U(j)) dj.$$

It is obvious that U^* is a solution to the problem (31) if U^* is a fixed point of $\wp$. Now, for all $U, \mathcal{W} \in K$, we deduce

$$\begin{aligned} \Lambda(\wp U(\aleph), \wp \mathcal{W}(\aleph), \natural \Gamma) &= \sup_{\aleph \in [0,1]} \frac{\natural \Gamma}{\natural \Gamma + |\wp U(\aleph) - \wp \mathcal{W}(\aleph)|} \\ &= \sup_{\aleph \in [0,1]} \frac{\natural \Gamma}{\natural \Gamma + |\int_0^1 \Upsilon(\aleph, j) \mathfrak{f}(\aleph, U(j)) dj - \int_0^1 \Upsilon(\aleph, j) \mathfrak{f}(\aleph, \mathcal{W}(j)) dj|} \\ &= \sup_{\aleph \in [0,1]} \frac{\natural \Gamma}{\natural \Gamma + \int_0^1 \Upsilon(\aleph, j) |\mathfrak{f}(\aleph, U(j)) - \mathfrak{f}(\aleph, \mathcal{W}(j))| dj} \\ &\geq \sup_{\aleph \in [0,1]} \frac{\Gamma}{\Gamma + |U(\aleph) - \mathcal{W}(\aleph)|} \\ &\geq \Lambda(U(\aleph), \mathcal{W}(\aleph), \Gamma), \end{aligned}$$

$$\begin{aligned} \Pi(\wp U(\aleph), \wp \mathcal{W}(\aleph), \natural \Gamma) &= 1 - \sup_{\aleph \in [0,1]} \frac{\natural \Gamma}{\natural \Gamma + |\wp U(\aleph) - \wp \mathcal{W}(\aleph)|} \\ &= 1 - \sup_{\aleph \in [0,1]} \frac{\natural \Gamma}{\natural \Gamma + |\int_0^1 \Upsilon(\aleph, j) \mathfrak{f}(\aleph, U(j)) dj - \int_0^1 \Upsilon(\aleph, j) \mathfrak{f}(\aleph, \mathcal{W}(j)) dj|} \\ &= 1 - \sup_{\aleph \in [0,1]} \frac{\natural \Gamma}{\natural \Gamma + \int_0^1 \Upsilon(\aleph, j) |\mathfrak{f}(\aleph, U(j)) - \mathfrak{f}(\aleph, \mathcal{W}(j))| dj} \\ &\leq 1 - \sup_{\aleph \in [0,1]} \frac{\Gamma}{\Gamma + |U(\aleph) - \mathcal{W}(\aleph)|} \\ &\leq \Pi(U(\aleph), \mathcal{W}(\aleph), \Gamma) \end{aligned}$$

and

$$\begin{aligned}\mathcal{M}(\wp U(\aleph), \wp W(\aleph), \natural\Gamma) &= \sup_{\aleph \in [0,1]} \frac{|\wp U(\aleph) - \wp W(\aleph)|}{\natural\Gamma} \\ &= \sup_{\aleph \in [0,1]} \frac{|\int_0^1 Y(\aleph, j) f(\aleph, U(j)) dj - \int_0^1 Y(\aleph, j) f(\aleph, W(j)) dj|}{\natural\Gamma} \\ &= \sup_{\aleph \in [0,1]} \frac{\int_0^1 Y(\aleph, j) |f(\aleph, U(j)) - f(\aleph, W(j))| dj}{\natural\Gamma} \\ &\leq \sup_{\aleph \in [0,1]} \frac{|U(\aleph) - W(\aleph)|}{\Gamma} \\ &\leq \mathcal{M}(U(\aleph), W(\aleph), \Gamma).\end{aligned}$$

As a result, $\wp$ has a ufp and all of Theorem 1 requirements are satisfied. It implies that there is only one solution to the Equation (30). $\square$

5. Conclusions

We proposed the idea of neutrosophic pentagonal MS in this study and proved new varieties of fixed-point theorems. By applying a new methodology to an application based on the literature, we have shown that it outperforms our results.

Author Contributions: Conceptualization, G.M., P.S., Z.D.M., A.A. and N.M.; formal analysis, Z.D.M., A.A. and N.M.; writing—original draft preparation, G.M., P.S., Z.D.M., A.A. and N.M.; writing—review and editing, G.M., P.S., Z.D.M., A.A. and N.M.; funding acquisition, A.A. and N.M. All authors have and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors A. Aloqaily and N. Mlaiki thank Prince Sultan University for paying the APC and for the support from the TAS research lab.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353. [\[CrossRef\]](#)
2. Schweizer, B.; Sklar, A. Statistical metric spaces. *Pac. J. Math.* **1960**, *10*, 314–334. [\[CrossRef\]](#)
3. Kramosil, I.; Michálek, J. Fuzzy metric and statistical metric spaces. *Kybernetika* **1975**, *11*, 336–344.
4. Azmi, F.M. Fixed-Point Results for α - ψ -fuzzy Contractive Mappings on Fuzzy Double-Controlled Metric Spaces. *Symmetry* **2023**, *15*, 716. [\[CrossRef\]](#)
5. Rome, B.-E.; Sarwar, M.; Shah, K.; Abdalla, B.; Abdeljawad, T. Some generalized fixed point results of Banach and Ćirić type in extended fuzzy b -metric spaces with applications. *AIMS Math.* **2022**, *7*, 14029–14050. [\[CrossRef\]](#)
6. Batul, S.; Mehmood, F.; Hussain, A.; Sagheer, D.-e-S.; Aydi, H.; Mukheimer, A. Multivalued contraction maps on fuzzy b -metric spaces and an application. *AIMS Math.* **2022**, *7*, 5925–5942. [\[CrossRef\]](#)
7. Rakić, D.; Mukheimer, A.; Došenović, T.; Mitrović, Z.D.; Radenović, S. On some new fixed point results in fuzzy b -metric spaces. *J. Inequal. Appl.* **2020**, *2020*, 14. [\[CrossRef\]](#)
8. Mlaiki, N. Controlled metric type spaces and the related contraction principle. *Mathematics* **2018**, *6*, 194. [\[CrossRef\]](#)
9. Sezen, M.S. Controlled fuzzy metric spaces and some related fixed point results. *Numer. Methods Partial Differ. Equ.* **2020**, *37*, 583–593. [\[CrossRef\]](#)
10. Rafi, M.; Noorani, M.S.M. Fixed theorems on intuitionistic fuzzy metric space. *Iran. J. Fuzzy Syst.* **2006**, *3*, 23–29.
11. Sintunavarat, W.; Kumam, P. Fixed Theorems for a Generalized Intuitionistic Fuzzy Contraction in Intuitionistic Fuzzy Metric Spaces. *Thai J. Math.* **2012**, *10*, 123–135.
12. Alaca, C.; Turkoglu, D.; Yildiz, C. Fixed points in intuitionistic fuzzy metric spaces. *Chaos Solitons Fractals* **2005**, *29*, 1073–1078. [\[CrossRef\]](#)

13. Javed, K.; Uddin, F.; Aydi, H.; Arshad, M.; Ishtiaq, U.; Alsamir, M. On Fuzzy b -Metric-Like Spaces. *J. Funct. Spaces* **2021**, *2021*, 6615976. [[CrossRef](#)]
14. Mohamad, A. Fixed-point theorems in intuitionistic fuzzy metric spaces. *Chaos Solitons Fractals* **2007**, *34*, 1689–1695. [[CrossRef](#)]
15. Dey, D.; Saha, M. An extension of Banach fixed point theorem in fuzzy metric space. *Bol. Soc. Parana. Mat.* **2014**, *32*, 299–304. [[CrossRef](#)]
16. Uddin, F.; Ishtiaq, U.; Hussain, A.; Javed, K.; Al Sulami, H.; Ahmed, K. Neutrosophic Double Controlled Metric Spaces and Related Results with Application. *Fractal Fract.* **2022**, *6*, 318. [[CrossRef](#)]
17. Grabiec, M. Fixed points in fuzzy metric spaces. *Fuzzy Sets Syst.* **1988**, *27*, 385–389. [[CrossRef](#)]
18. Rehman, S.U.; Jabeen, S.; Khan, S.U.; Jaradat, M.M.M. Some α - ϕ -Fuzzy Cone Contraction Results with Integral Type Application. *J. Math.* **2021**, *2021*, 1566348.
19. Park, J.H. Intuitionistic fuzzy metric spaces. *Chaos Solitons Fractals* **2004**, *22*, 1039–1046. [[CrossRef](#)]
20. Konwar, W. Extension of fixed results in intuitionistic fuzzy b -metric spaces. *J. Intell. Fuzzy Syst.* **2020**, *39*, 7831–7841. [[CrossRef](#)]
21. Kirişçi, M.; Simsek, N. Neutrosophic metric spaces. *Math. Sci.* **2020**, *14*, 241–248. [[CrossRef](#)]
22. Simsek, N.; Kirişçi, M. Fixed point theorems in Neutrosophic metric spaces. *Sigma J. Eng. Nat. Sci.* **2019**, *10*, 221–230.
23. Sowndrarajan, S.; Jeyarama, M.; Smarandache, F. Fixed point Results for Contraction Theorems in Neutrosophic Metric Spaces. *Neutrosophic Sets. Syst.* **2020**, *36*, 23.
24. Itoh, S. Random fixed point theorems with an application to random differential equations in Banach spaces. *J. Math. Anal. Appl.* **1979**, *67*, 261–273. [[CrossRef](#)]
25. Saleem, N.; Isik, H.; Furqan, S.; Park, C. Fuzzy double controlled metric spaces. *J. Intell. Fuzzy Syst.* **2021**, *40*, 9977–9985. [[CrossRef](#)]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.