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m-Polar Neutrosophic Topology with Applications to Multi-criteria Decision-Making in Medical Diagnosis and Clustering Analysis

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Abstract In this paper, we first introduce novel concepts of m-polar neutrosophic set (MPNS) and topological structure on m-polar neutrosophic set by combining the m-polar fuzzy set (MPFS) and neutrosophic set. Then, we investigate several characterizations of m-polar neutrosophic set and establish its various operations with the help of examples. We propose score functions for the comparison of m-polar neutrosophic numbers (MPNNs). We establish m-polar neutrosophic topology and define interior, closure, exterior, and frontier for m-polar neutrosophic sets (MPNSs) with illustrative examples. We discuss some results with counter examples, which hold for classical set theory, but do not hold for m-polar neutrosophic set theory. We introduce a cosine similarity measure and a set theoretic similarity measure for m-polar neutrosophic sets (MPNSs). Furthermore, we present three algorithms for multi-criteria decision-making (MCDM) in medical diagnosis and clustering analysis under uncertainty by using m-polar neutrosophic sets (MPNSs) and m-polar neutrosophic topology. Lastly, we present advantages, validity, flexibility, and comparison of our proposed algorithms with the existing techniques.

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1 Introduction and Background

Multi-criteria decision-making (MCDM) is a process that explicitly evaluates best alternative(s) among the feasible options. In archaic times, decisions were framed without handling the uncertainties in the data, which may lead to inadequate results to the real-life operational situations. If we amass the data and deduce the result without handling hesitations, then given results will be ambivalent, indefinite, or equivocal. MCDM is an integral part in modern management, business, medical diagnosis, and many other real-world problems. Essentially, rational or sound decision is necessary for a decision-maker. Every decision-maker takes hundreds of decisions subconsciously or consciously making it as the central part of his execution. Medical diagnosis with MCDM provides solutions for the doctors to determine symptoms of disease and kind of illness. MCDM is used in solving problems that contain complex and multiple criteria. In MCDM, we have to identify the problem by determining the possible alternatives, evaluate each alternative based upon the criteria given by the decision-maker or group of decision-makers and lastly select the best alternative. MCDM problems under fuzzy environment were first introduced by Bellman and Zadeh in (1970) [4]. A number of useful mathematical tools such as fuzzy sets, m-polar fuzzy sets, neutrosophic sets, and soft sets have been developed to deal with uncertainties and ambiguities for multi-criteria decision-making problems.

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Zadeh introduced fuzzy set [48] as a significant mathematical model to characterize and assembling of the objects whose boundary is ambiguous. A fuzzy set $\mathfrak{F}$ in the reference set $\mathcal{Q}$ is represented by a mapping $\sigma : \mathcal{Q} \to [0, 1]$. In real-life problems, we face various situations including uncertainties and ambiguities. For instance, if we speak about the "beautiful cities of a country" then the exact decision is ambiguous. Some cities are very beautiful, some of them are medium beautiful, and some are less beautiful. The criteria of being "beautiful" can be changed according to the decision-maker's choice. In these situations, the classical set theory fails and we use fuzzy set theory to treat these type of hesitations in the decision-making problems. We use linguistic terms to relate a real-world situation to the fuzzy numeric value and accumulate the input in the form of fuzzy numbers or fuzzy sets.

After Zadeh, many extensions of fuzzy sets have been presented and investigated such as, intuitionistic fuzzy sets (IFSs) [3], single valued neutrosophic sets (SVNSs) [28–30, 35], picture fuzzy sets [8], bipolar fuzzy sets (BPFSs) [50–52], m-polar fuzzy sets (MPFSs) [5], interval-valued fuzzy sets (IVFSs) [49], and Pythagorean fuzzy sets (PFSs) [42–44]. A neutrosophic set $\mathfrak{N}$ is defined by $\mathfrak{N} = \{\langle \varsigma, \mathfrak{A}(\varsigma), \mathfrak{S}(\varsigma), \mathfrak{Y}(\varsigma)\rangle, \varsigma \in \mathcal{Q}\}$, where $\mathfrak{A}, \mathfrak{S}, \mathfrak{Y} : \mathcal{Q} \to]^{-}0, 1^{+}[$ and $^{-}0 \leq \mathfrak{A}(\varsigma) + \mathfrak{S}(\varsigma) + \mathfrak{Y}(\varsigma) \leq 3^{+}$. The neutrosophic set yields the value from real standard or non-standard subsets of $]^{-}0, 1^{+}[$. It is difficult to utilize these values in daily life science and technology problems. Consequently, the neutrosophic set which takes the value from the subset of [0, 1] is to be regarded here. An abstraction of bipolar fuzzy set was inaugurated by Chen [5] named as MPFS. An MPFS $\mathfrak{C}$ in a non-empty universal set $\mathcal{Q}$ is a function $\mathfrak{C} : \mathcal{Q} \to [0, 1]^{m}$, symbolized by $\mathfrak{C} = \{\langle \varsigma, P_i o \Lambda(\varsigma)\rangle : \varsigma \in \mathcal{Q}; i = 1, 2, 3, \ldots, m\}$ where $P_i : [0, 1]^{m} \to [0, 1]$ is the ith projection mathematical function $(i \in m)$. $\mathfrak{C}_{\phi}(\varsigma) = (0, 0, \ldots, 0)$ is the smallest value in $[0, 1]^{m}$, and $\mathfrak{C}_{\widetilde{X}}(\varsigma) = (1, 1, \ldots, 1)$ is the greatest value in $[0, 1]^{m}$.

In the last few decades, many mathematicians worked on similarity measures, correlation coefficients, topological spaces, aggregation operators, and decision-making applications. These structures have different formulae according to the different sets and give better solutions to decision-making problems. It has numerous applications in the field of pattern recognition, medical diagnosis, artificial intelligence, social sciences, business, and multi-attribute decision-making problems.

Akram et al. [1] presented certain applications of m-polar fuzzy sets in the decision-making problems. Ali et al. [2] presented various properties of soft sets and rough sets with fuzzy soft sets. Garg [10] introduced new generalized Pythagorean fuzzy information aggregation using Einstein operations and established its application to decision-making problems. Garg [11] introduced generalized intuitionistic fuzzy interactive geometric interaction operators using Einstein t-norm and t-conorm and their application to decision-making. Karaaslan [15] introduced neutrosophic soft sets with its applications in decision-making. Xu et al. [41] established clustering algorithm for intuitionistic fuzzy sets and presented its applications for clustering. Jose and Kuriaskose [14] investigated aggregation operators with the corresponding score function for MCDM in the context of IFNs. Mahmood et al. [19] established generalized aggregation operators for cubic hesitant fuzzy numbers (CHFNs) and use it into MCDM problems. In 1968, Chang [7] introduced fuzzy topology on fuzzy sets. After fuzzy topology, many researchers have been introduced topologies and their properties on different hybrid structures of fuzzy sets. Pao-Ming and Ying-Ming [20, 21] introduced the structure of neighborhood of fuzzy-point. They provided the concept of fuzzy quasi-coincident and Q-neighborhood. They also discussed important properties of fuzzy topological space by using fuzzy Q-neighborhood. Shabir and Naz [31] established soft topological spaces. Deli et al. [9] introduced bipolar neutrosophic sets and their application based on multi-criteria decision-making problems. Riaz and Hashmi [23–25] developed fixed point theorems of fuzzy neutrosophic soft (FNS) mapping with its decision-making. They established multi-attribute group decision-making (MAGDM) for agribusiness by using various cubic m-polar fuzzy averaging aggregation operators. They introduced a novel structure of linear Diophantine fuzzy set as a generalization of intuitionistic fuzzy set, Pythagorean fuzzy set, and q-rung orthopair fuzzy set with its applications in multi-attribute decision-making problems. Riaz et al. [26, 27] introduced N-soft topology and its applications to multi-criteria group decision-making (MCGDM). They established cubic bipolar fuzzy ordered weighted geometric aggregation operators and presented their applications by using internal and external bipolar fuzzy information.

Feng et al. [12, 13] introduced properties of soft sets combined with fuzzy soft sets and multi-attribute decision-making (MADM) models in the environment of generalized intuitionistic fuzzy soft sets and fuzzy soft sets. Liu et al. [16] established hesitant intuitionistic fuzzy linguistic operators and presented its MAGDM problems. Wei et al. [36] invented hesitant triangular fuzzy operators in MADM problems. Wei et al. [37, 38] worked on similarity measures on picture fuzzy sets and correlation coefficient to the interval-valued intuitionistic fuzzy sets with application in decision-making problems. Ye [45–47] introduced prioritized aggregation operators in the context of interval-valued hesitant fuzzy numbers (IVHFNs) and established it on MAGDM algorithms. He also established MCDM methods for interval neutrosophic sets and correlation coefficient under single-value neutrosophic environment. He established cosine similarity measures for intuitionistic fuzzy sets with application in decision-making problems. Zhang et al. [53] introduced aggregation operators with MCDM by using interval-valued fuzzy neutrosophic sets (IVFNSs). An extended TOPSIS method for decision-making was developed by Chi and Lui [6] on IVFNSs. Zhao et al. [55] introduced generalized aggregation operators in the context of intuitionistic fuzzy sets. Zhang et al. [54] established various results on clustering approach to intuitionistic fuzzy sets. Peng et al. [22] introduced Pythagorean fuzzy information measures and established interesting results on Pythagorean fuzzy sets. They introduced clustering algorithm for Pythagorean fuzzy sets and presented numerous applications on Pythagorean fuzzy input data. Li and Cheng [17] established new similarity measures of IFSs and its applications to pattern recognition. Lin et al. [18] studied hesitant fuzzy linguistic information and presented its application to models of selecting an ERP system. Salton and McGill [32] introduced modern information retrieval. Singh [33] established correlation coefficients of picture fuzzy sets. Son [34] inaugurated a novel distributed picture fuzzy clustering method on picture fuzzy sets. Xu and Chen [39, 40] established correlation, distance, and similarity measures on intuitionistic fuzzy sets.

In this era, experts think that the universe is moving towards multi-polarity. Therefore, it comes as no surprise that multi-polarity in data and information plays a vital role in various fields of science and technology. In neurobiology, multi-polar neurons in brain gather a great deal of information from other neurons. In information technology, multi-polar technology can be exploited to operate large-scale systems. In some real-life situations, we have to deal with the dissatisfaction and indeterminacy grades for the alternatives of the reference set. For instance, in the operation of throwing up a ballot, there exist some people who vote in favor, some of them vote against, and some abstain. In the area of electrical engineering, we deal with the conductors and non-conductors, but there also exist some substances which are insulators. These types of situations can easily handled by using neutrosophic set theory. In some real-life applications, we have to deal with multi-polarity, truth values, indeterminacy, and falsity grades of alternatives. To deal with these type of hesitations and uncertainties, we establish the idea of m-polar neutrosophic set (MPNS).

The motivation and objectives of this extended and hybrid work are given step by step in the whole manuscript. We establish that other hybrid structures of fuzzy sets become special cases of MPNS under some suitable conditions. We discuss about the robustness, flexibility, simplicity, and superiority of our suggested model and algorithms. This model is most generalized form and use to collect data at a large scale and applicable in medical, engineering, artificial intelligence, agriculture, and other daily life problems. In future, this work can be gone easily for other approaches and different types of hybrid structures.

The scheme of this manuscript is organized as follows. Section 2, implies a novel idea of m-polar neutrosophic set (MPNS). We establish some of its operations, score function, and improved score function. In Sect. 3, we use MPNS to establish m-polar neutrosophic topological space (MPNTS). We define various topological structures such as interior, closure, exterior, and frontier for MPNSs with the help of illustrations. We establish various results with their counter examples, which holds for classical set theory, but do not hold for m-polar neutrosophic set theory. We introduce cosine similarity measure and set theoretic similarity measure for MPNSs. In Sect. 4, we establish some methods for the solution of MCDM problems based on medical diagnosis and clustering analysis using MPNTS and MPNSs. We propose three algorithms with linguistic information based on m-polar neutrosophic data using MPNTS, similarity measures, and clustering analysis. It is interesting to note that first two algorithms for medical diagnosis yield the same result. Furthermore, we present advantages, simplicity, flexibility, and validity of the proposed algorithms. We give a brief discussion and comparative analysis of our proposed approach with some existing methodologies. In the end, the conclusion of this work is summarized in Sect. 5.

2 *m*-Polar Neutrosophic Set (MPNS)

Chen et al. [5] have proposed the concept of m-polar fuzzy set (MPFS) in 2014, which have the capability to deal with the data having vagueness and uncertainty under multi-criteria, multi-source, multi-sensor, and multi-polar information. Smarandache [30] extended the neutrosophic set, respectively, to neutrosophic overset (when some neutrosophic component is >1), neutrosophic underset (when some neutrosophic component is <0), and to neutrosophic offset (when some neutrosophic components are off the interval [0, 1], i.e., some neutrosophic component >1 and other neutrosophic component <0). In 2016, Smarandache introduced the neutrosophic tripolar set and neutrosophic multi-polar set, also the neutrosophic tripolar graph and neutrosophic multi-polar graph [30].

The membership grades of m-polar fuzzy sets range over the interval $[0,1]^m$, which represent m criteria of the object, but it cannot deal with the falsity and indeterminacy part of the object.

Neutrosophic set (NS) deals with truth, falsity, and indeterminacy for one criteria of the alternative, but cannot deal with the multi-criteria, multi-source, multi-polar information fusion of the alternatives. To overcome this problem, we introduce a new model of *m*-polar neutrosophic set (MPNS) by combining the concepts of *m*-polar fuzzy set (MPFS) and neutrosophic set (NS). MPNS has the ability to deal with the *m* criteria and to deal with the truth, falsity, and indeterminacy grades for each alternative. In fact, *m*-polar neutrosophic set is an extension of bipolar neutrosophic set introduced by Deli et al. [9]. We establish various properties and operations on *m*-polar neutrosophic sets. We propose score functions for the comparison of *m*-polar neutrosophic numbers (MPNNs). In the whole manuscript, we use $\mathcal{Q}$ as a fixed sample space and Δ as an indexing set. We use $\mathfrak{A}, \mathfrak{S}$ and $\mathfrak{Y}$ as membership, indeterminacy, and non-membership grades, respectively.

Definition 2.1 An object $\mathcal{M}_{\mathfrak{N}}$ in the reference set $\mathcal{Q}$ is called *m*-polar neutrosophic set (MPNS), if it can be expressed as

$$\mathcal{M}_{\mathfrak{N}} = \{(\varsigma, \langle \mathfrak{A}_\alpha(\varsigma), \mathfrak{S}_\alpha(\varsigma), \mathfrak{Y}_\alpha(\varsigma)\rangle) : \varsigma \in \mathcal{Q}, \quad \alpha = 1,2,3,\ldots,m\}$$

where $\mathfrak{A}_\alpha, \mathfrak{S}_\alpha, \mathfrak{Y}_\alpha : \mathcal{Q} \to [0,1]$ and $0 \le \mathfrak{A}_\alpha(\varsigma) + \mathfrak{S}_\alpha(\varsigma) + \mathfrak{Y}_\alpha(\varsigma) \le 3; \alpha = 1,2,3,\ldots,m$. This condition shows that all the three grades $\mathfrak{A}_\alpha, \mathfrak{S}_\alpha$ and $\mathfrak{Y}_\alpha$; $(\alpha = 1,2,3,\ldots,m)$ are independent and represents the truth, indeterminacy, and falsity of the considered object or alternative for multiple criteria, respectively. Simply an *m*-polar neutrosophic number (MPNN) can be represented as $\mathfrak{I} = (\langle \mathfrak{A}_\alpha, \mathfrak{S}_\alpha, \mathfrak{Y}_\alpha\rangle)$, where $0 \le \mathfrak{A}_\alpha + \mathfrak{S}_\alpha + \mathfrak{Y}_\alpha \le 3; \alpha = 1,2,3,\ldots,m$. In tabular form, the MPNS can be represented as Table 1.

Example 2.2 Let $\mathcal{Q} = \{\varsigma_1, \varsigma_2, \varsigma_3\}$ be the collection of some well-known smart phones. Then 4-polar neutrosophic set in $\mathcal{Q}$ can be written as

$$\begin{aligned}\mathcal{M}_{\mathfrak{N}} = \Big\{&(\varsigma_1, \langle 0.512, 0.231, 0.321\rangle, \langle 0.653, 0.223, 0.116\rangle,\\ &\langle 0.875, 0.114, 0.243\rangle, \langle 0.961, 0.115, 0.431\rangle),\\ &(\varsigma_2, \langle 0.657, 0.114, 0.226\rangle, \langle 0.765, 0.224, 0.245\rangle,\\ &\langle 0.875, 0.465, 0.213\rangle, \langle 0.961, 0.141, 0.212\rangle),\\ &(\varsigma_3, \langle 0.876, 0.221, 0.321\rangle, \langle 0.657, 0.115, 0.116\rangle,\\ &\langle 0.987, 0.114, 0.322\rangle, \langle 0.675, 0.221, 0.423\rangle)\Big\}.\end{aligned}$$

In this set, multi-polarity (m = 1,2,3,4) of each altternative ς shows its characteristic or qualities according to the considered criteria such as

$$\alpha_1 = \text{affordable},\ \alpha_2 = \text{longlastingbattery},$$
$$\alpha_3 = \text{extrastorage},\ \alpha_4 = \text{goodcameraquality}.$$

For each ς and each of its criteria, we have neutrosophic values to represent the truth, indeterminacy, and falsity of corresponding alternative according to the considered criteria under the influence of expert's opinion. In the set $\mathcal{M}_{\mathfrak{N}}$ for ς_1 the first triplet $\langle 0.512, 0.231, 0.321\rangle$ shows that the smart phone ς_1 has 51.2% truth value, 23.1% indeterminacy, and 32.1% falsity value for the criteria "affordable." Similarly, we can see the values for all alternatives corresponding to the other criteria.

There is a relationship between MPNS and other hybrid structures of fuzzy set. This relationship can be elaborated in the given flow chart diagram of Fig. 1, where $\alpha = 1,2,3,\ldots,m$.

Definition 2.3 An MPNS $\mathcal{M}_{\mathfrak{N}}$ is said to be an empty MPNS, if $\mathfrak{A}_\alpha(\varsigma) = 0, \mathfrak{S}_\alpha(\varsigma) = 1$ and $\mathfrak{Y}_\alpha(\varsigma) = 1, \forall \alpha = 1,2,3,\ldots,m$ and it can be written as

$${}^{0}\mathcal{M}_{\mathfrak{N}} = \{\varsigma, (\langle 0,1,1\rangle, \langle 0,1,1\rangle, \cdots, \langle 0,1,1\rangle) : \varsigma \in \mathcal{Q}\}$$

and for absolute MPNS we have $\mathfrak{A}_\alpha(\varsigma) = 1, \mathfrak{S}_\alpha(\varsigma) = 0$ and $\mathfrak{Y}_\alpha(\varsigma) = 0, \forall \alpha = 1,2,3,\ldots,m$ and it can be written as

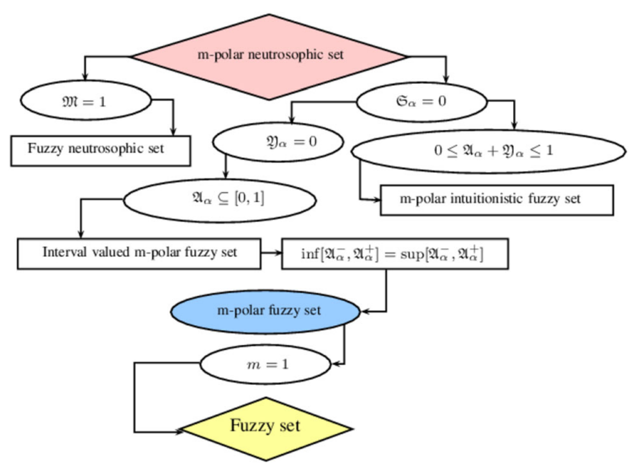

Fig. 1 Relationship between MPNS and other hybrid fuzzy sets

Table 1 Tabular representation of *m*-polar neutrosophic set

$\mathcal{M}_{\mathfrak{N}}$	*MPNS*
ς_1	$(\langle \mathfrak{A}_1(\varsigma_1), \mathfrak{S}_1(\varsigma_1), \mathfrak{Y}_1(\varsigma_1)\rangle, \langle \mathfrak{A}_2(\varsigma_1), \mathfrak{S}_2(\varsigma_1), \mathfrak{Y}_2(\varsigma_1)\rangle, \cdots, \langle \mathfrak{A}_m(\varsigma_1), \mathfrak{S}_m(\varsigma_1), \mathfrak{Y}_m(\varsigma_1)\rangle)$
ς_2	$(\langle \mathfrak{A}_1(\varsigma_2), \mathfrak{S}_1(\varsigma_2), \mathfrak{Y}_1(\varsigma_2)\rangle, \langle \mathfrak{A}_2(\varsigma_2), \mathfrak{S}_2(\varsigma_2), \mathfrak{Y}_2(\varsigma_2)\rangle, \cdots, \langle \mathfrak{A}_m(\varsigma_2), \mathfrak{S}_m(\varsigma_2), \mathfrak{Y}_m(\varsigma_2)\rangle)$
...	....
$\varsigma_{\mathfrak{N}}$	$(\langle \mathfrak{A}_1(\varsigma_{\mathfrak{N}}), \mathfrak{S}_1(\varsigma_{\mathfrak{N}}), \mathfrak{Y}_1(\varsigma_{\mathfrak{N}})\rangle, \langle \mathfrak{A}_2(\varsigma_{\mathfrak{N}}), \mathfrak{S}_2(\varsigma_{\mathfrak{N}}), \mathfrak{Y}_2(\varsigma_{\mathfrak{N}})\rangle, \cdots, \langle \mathfrak{A}_m(\varsigma_{\mathfrak{N}}), \mathfrak{S}_m(\varsigma_{\mathfrak{N}}), \mathfrak{Y}_m(\varsigma_{\mathfrak{N}})\rangle)$

$^1\mathcal{M}_{\mathfrak{R}} = \{\varsigma, (\langle 1,0,0\rangle, \langle 1,0,0\rangle, \cdots, \langle 1,0,0\rangle) : \varsigma \in \mathcal{Q}\}$

The assembling of all MPNSs in $\mathcal{Q}$ is represented as $\mathfrak{mpn}(\mathcal{Q})$.

Now we define some operations for MPNSs.

Definition 2.4 Let $\mathcal{M}_{\mathfrak{R}}, \mathcal{M}_{\mathfrak{R}_\wp} \in \mathfrak{mpn}(\mathcal{Q})$, where

$$\mathcal{M}_{\mathfrak{R}} = \{(\varsigma, \langle \mathfrak{A}_\alpha(\varsigma), \mathfrak{S}_\alpha(\varsigma), \mathfrak{Y}_\alpha(\varsigma)\rangle) : \varsigma \in \mathcal{Q}, \quad \alpha = 1,2,3,\ldots,m\}$$
$$\mathcal{M}_{\mathfrak{R}_\wp} = \{(\varsigma, \langle {}^\wp\mathfrak{A}_\alpha(\varsigma), {}^\wp\mathfrak{S}_\alpha(\varsigma), {}^\wp\mathfrak{Y}_\alpha(\varsigma)\rangle) : \varsigma \in \mathcal{Q}, \quad \alpha = 1,2,3,\ldots,m\}, \wp \in \Delta$$

then:

(i) $\mathcal{M}^c_{\mathfrak{R}} = \{(\varsigma, \langle \mathfrak{Y}_\alpha(\varsigma), 1-\mathfrak{S}_\alpha(\varsigma), \mathfrak{A}_\alpha(\varsigma)\rangle) : \varsigma \in \mathcal{Q}, \quad \alpha = 1,2,3,\ldots,m\}$

(ii) $\mathcal{M}_{\mathfrak{R}_1} = \mathcal{M}_{\mathfrak{R}_2} \Leftrightarrow \langle {}^1\mathfrak{A}_\alpha(\varsigma), {}^1\mathfrak{S}_\alpha(\varsigma), {}^1\mathfrak{Y}_\alpha(\varsigma)\rangle = \langle {}^2\mathfrak{A}_\alpha(\varsigma), {}^2\mathfrak{S}_\alpha(\varsigma), {}^2\mathfrak{Y}_\alpha(\varsigma)\rangle; \varsigma \in \mathcal{Q}, \alpha = 1,2,3,\ldots,m$

(iii) $\mathcal{M}_{\mathfrak{R}_1} \subseteq \mathcal{M}_{\mathfrak{R}_2} \Leftrightarrow {}^1\mathfrak{A}_\alpha(\varsigma) \le {}^2\mathfrak{A}_\alpha(\varsigma), {}^1\mathfrak{S}_\alpha(\varsigma) \ge {}^2\mathfrak{S}_\alpha(\varsigma), {}^1\mathfrak{Y}_\alpha(\varsigma) \ge {}^2\mathfrak{Y}_\alpha(\varsigma); \varsigma \in \mathcal{Q}, \alpha = 1,2,3,\ldots,m$

(iv) $\bigcup_\wp \mathcal{M}_{\mathfrak{R}_\wp} = \{(\varsigma, \langle \sup_\wp {}^\wp\mathfrak{A}_\alpha(\varsigma), \inf_\wp {}^\wp\mathfrak{S}_\alpha(\varsigma), \inf_\wp {}^\wp\mathfrak{Y}_\alpha(\varsigma)\rangle); \varsigma \in \mathcal{Q}, \wp \in \Delta, \alpha = 1,2,3,\ldots,m\}$

(v) $\bigcap_\wp \mathcal{M}_{\mathfrak{R}_\wp} = \{(\varsigma, \langle \inf_\wp {}^\wp\mathfrak{A}_\alpha(\varsigma), \sup_\wp {}^\wp\mathfrak{S}_\alpha(\varsigma), \sup_\wp {}^\wp\mathfrak{Y}_\alpha(\varsigma)\rangle); \varsigma \in \mathcal{Q}, \wp \in \Delta, \quad \alpha = 1,2,3,\ldots,m\}$

Example 2.5 Consider two 4-polar neutrosophic sets $\mathcal{M}_{\mathfrak{R}_1}$ and $\mathcal{M}_{\mathfrak{R}_2}$ given in tabular form as Table 2.

Now we calculate complement, union, and intersection by using Definition 2.4 and results can be seen in tabular form as Table 3.

In order to deal with multi-criteria decision-making problems with m-polar neutrosophic numbers (MPNNs), we define some score functions for the ranking of MPNNs.

Definition 2.6 Let $\mathfrak{I} = (\langle \mathfrak{A}_\alpha, \mathfrak{S}_\alpha, \mathfrak{Y}_\alpha\rangle; \alpha = 1,2,3,\ldots,m)$ be an MPNN, then its score functions are given as:

$$\pounds_1(\mathfrak{I}) = \frac{1}{2m}\left(m + \sum_{\alpha=1}^{m}(\mathfrak{A}_\alpha - 2\mathfrak{S}_\alpha - \mathfrak{Y}_\alpha)\right); \pounds_1(\mathfrak{I}) \in [0,1]$$

$$\pounds_2(\mathfrak{I}) = \frac{1}{m}\sum_{\alpha=1}^{m}(\mathfrak{A}_\alpha - 2\mathfrak{S}_\alpha - \mathfrak{Y}_\alpha); \pounds_2(\mathfrak{I}) \in [-1,1]$$

In the case, when score value of two MPNNs is same, we define an improved score function for the ranking of MPNNs given as

$$\pounds_3(\mathfrak{I}) = \frac{1}{2m}\left(m + \sum_{\alpha=1}^{m}((\mathfrak{A}_\alpha - 2\mathfrak{S}_\alpha - \mathfrak{Y}_\alpha)(2 - \mathfrak{A}_\alpha - \mathfrak{Y}_\alpha))\right);$$
$$\pounds_3(\mathfrak{I}) \in [-1,1].$$

In the case, when $\mathfrak{A}_\alpha + \mathfrak{Y}_\alpha = 1; \forall\, \alpha = 1,2,\ldots,m$, then $\pounds_3(\mathfrak{I})$ reduces to $\pounds_1(\mathfrak{I})$.

Definition 2.7 Let $\mathfrak{I}_1$ and $\mathfrak{I}_2$ be two MPNNs, then the following order relation between the score values of MPNNs hold:

(a) If $\pounds_1(\mathfrak{I}_1) \succ \pounds_1(\mathfrak{I}_2)$ then $\mathfrak{I}_1 \succ \mathfrak{I}_2$.
(b) If $\pounds_1(\mathfrak{I}_1) = \pounds_1(\mathfrak{I}_2)$ then
(1) If $\pounds_2(\mathfrak{I}_1) \succ \pounds_2(\mathfrak{I}_2)$ then $\mathfrak{I}_1 \succ \mathfrak{I}_2$.
(2) If $\pounds_2(\mathfrak{I}_1) = \pounds_2(\mathfrak{I}_2)$ then
(i) If $\pounds_3(\mathfrak{I}_1) \succ \pounds_3(\mathfrak{I}_2)$ then $\mathfrak{I}_1 \succ \mathfrak{I}_2$.
(ii) If $\pounds_3(\mathfrak{I}_1) \prec \pounds_3(\mathfrak{I}_2)$ then $\mathfrak{I}_1 \prec \mathfrak{I}_2$.
(iii) If $\pounds_3(\mathfrak{I}_1) = \pounds_3(\mathfrak{I}_2)$ then $\mathfrak{I}_1 \sim \mathfrak{I}_2$.

Example 2.8 Consider two 2-polar neutrosophic numbers $\mathfrak{I}_1$ and $\mathfrak{I}_2$ given in tabular form as Table 4.

Then by using Definition 2.6 $\pounds_1(\mathfrak{I}_1) = \frac{1}{2(2)}[2 + 0.5 - 2(0.3) - 0.4 + 0.5 - 2(0.1) - 0.8] = 0.25.$ Similarly, $\pounds_1(\mathfrak{I}_2) = 0.25$. This shows that $\pounds_1$ fails to give the ranking between both 2PNNs. Now we will use second score function $\pounds_2$. By using Definition 2.6, we obtain the score values $\pounds_2(\mathfrak{I}_1) = -0.5 = \pounds_2(\mathfrak{I}_2)$. This shows that $\pounds_2$ also fails to evaluate the ranking. Now we will use improved score function for the ranking of 2PNNs. After calcula-

Table 4 2-polar neutrosophic numbers $\mathfrak{I}_1$ and $\mathfrak{I}_2$

$\mathcal{Q}$	*2PNNs*
$\mathfrak{I}_1$	$(\langle 0.5, 0.3, 0.4\rangle, \langle 0.5, 0.1, 0.8\rangle)$
$\mathfrak{I}_2$	$(\langle 0.2, 0.3, 0.1\rangle, \langle 0.2, 0.1, 0.5\rangle)$

Table 2 4-polar neutrosophic sets $\mathcal{M}_{\mathfrak{R}_1}$ and $\mathcal{M}_{\mathfrak{R}_2}$

$\mathcal{Q}$	*4PNSs*
$\mathcal{M}_{\mathfrak{R}_1}$	$(\langle 0.611, 0.111, 0.251\rangle, \langle 0.821, 0.631, 0.111\rangle, \langle 0.721, 0.381, 0.591\rangle, \langle 0.211, 0.321, 0.411\rangle)$
$\mathcal{M}_{\mathfrak{R}_2}$	$(\langle 0.321, 0.621, 0.511\rangle, \langle 0.831, 0.111, 0.921\rangle, \langle 0.521, 0.431, 0.391\rangle, \langle 0.181, 0.931, 0.821\rangle)$

Table 3 Complement, union, and intersection of 4-polar neutrosophic sets

$\mathcal{Q}$	*4PNSs*
$\mathcal{M}^c_{\mathfrak{R}}$	$(\langle 0.251, 0.889, 0.611\rangle, \langle 0.111, 0.369, 0.821\rangle, \langle 0.591, 0.619, 0.721\rangle, \langle 0.411, 0.679, 0.211\rangle)$
$\mathcal{M}_{\mathfrak{R}_1} \cup \mathcal{M}_{\mathfrak{R}_2}$	$(\langle 0.611, 0.111, 0.251\rangle, \langle 0.831, 0.111, 0.111\rangle, \langle 0.721, 0.381, 0.391\rangle, \langle 0.211, 0.321, 0.411\rangle)$
$\mathcal{M}_{\mathfrak{R}_1} \cap \mathcal{M}_{\mathfrak{R}_2}$	$(\langle 0.321, 0.621, 0.511\rangle, \langle 0.821, 0.631, 0.921\rangle, \langle 0.521, 0.431, 0.591\rangle, \langle 0.181, 0.931, 0.821\rangle)$

tions, we get $\pounds_3(\mathfrak{I}_1) = 0.275$ and $\pounds_3(\mathfrak{I}_2) = 0.125$. Hence $\pounds_3(\mathfrak{I}_1) \succ \pounds_3(\mathfrak{I}_2)$, so $\mathfrak{I}_1 \succ \mathfrak{I}_2$.

Remark

- For null MPNN ${}^0\mathfrak{I}$ we have $\pounds_3({}^0\mathfrak{I}) = -1$.
- For absolute MPNN ${}^1\mathfrak{I}$ we have $\pounds_3({}^1\mathfrak{I}) = 1$.

Proposition 2.9 *Let $\mathcal{M}_{\mathfrak{N}} \in \mathfrak{mpn}(\mathcal{Q})$, and ${}^0\mathcal{M}_{\mathfrak{N}}$ and ${}^1\mathcal{M}_{\mathfrak{N}}$ be null and absolute MPNSs. Then the following axioms hold:*

(i) $\mathcal{M}_{\mathfrak{N}} \subseteq \mathcal{M}_{\mathfrak{N}} \cup \mathcal{M}_{\mathfrak{N}}$,
(ii) $\mathcal{M}_{\mathfrak{N}} \cap \mathcal{M}_{\mathfrak{N}} \subseteq \mathcal{M}_{\mathfrak{N}}$,
(iii) $\mathcal{M}_{\mathfrak{N}} \cup {}^0\mathcal{M}_{\mathfrak{N}} = \mathcal{M}_{\mathfrak{N}}$,
(iv) $\mathcal{M}_{\mathfrak{N}} \cap {}^0\mathcal{M}_{\mathfrak{N}} = {}^0\mathcal{M}_{\mathfrak{N}}$,
(v) $\mathcal{M}_{\mathfrak{N}} \cup {}^1\mathcal{M}_{\mathfrak{N}} = {}^1\mathcal{M}_{\mathfrak{N}}$,
(vi) $\mathcal{M}_{\mathfrak{N}} \cap {}^1\mathcal{M}_{\mathfrak{N}} = \mathcal{M}_{\mathfrak{N}}$

Proof The proof is obvious and can be proved by Definition 2.4. □

Proposition 2.10 *Let $\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_3} \in \mathfrak{mpn}(\mathcal{Q})$, then the following results hold:*

(i) $\mathcal{M}_{\mathfrak{N}_1} \cup \mathcal{M}_{\mathfrak{N}_2} = \mathcal{M}_{\mathfrak{N}_2} \cup \mathcal{M}_{\mathfrak{N}_1}$,
(ii) $\mathcal{M}_{\mathfrak{N}_1} \cap \mathcal{M}_{\mathfrak{N}_2} = \mathcal{M}_{\mathfrak{N}_2} \cap \mathcal{M}_{\mathfrak{N}_1}$,
(iii) $\mathcal{M}_{\mathfrak{N}_1} \cup (\mathcal{M}_{\mathfrak{N}_2} \cup \mathcal{M}_{\mathfrak{N}_3}) = (\mathcal{M}_{\mathfrak{N}_1} \cup \mathcal{M}_{\mathfrak{N}_2}) \cup \mathcal{M}_{\mathfrak{N}_3}$,
(iv) $\mathcal{M}_{\mathfrak{N}_1} \cap (\mathcal{M}_{\mathfrak{N}_2} \cap \mathcal{M}_{\mathfrak{N}_3}) = (\mathcal{M}_{\mathfrak{N}_1} \cap \mathcal{M}_{\mathfrak{N}_2}) \cap \mathcal{M}_{\mathfrak{N}_3}$,
(v) $(\mathcal{M}_{\mathfrak{N}_1} \cup \mathcal{M}_{\mathfrak{N}_2})^c = \mathcal{M}^c_{\mathfrak{N}_1} \cap \mathcal{M}^c_{\mathfrak{N}_2}$,
(vi) $(\mathcal{M}_{\mathfrak{N}_1} \cap \mathcal{M}_{\mathfrak{N}_2})^c = \mathcal{M}^c_{\mathfrak{N}_1} \cup \mathcal{M}^c_{\mathfrak{N}_2}$

Proof The proof is obvious and can be proved by Definition 2.4. □

3 *m*-Polar Neutrosophic Topology

In this section, we introduce the *m*-polar neutrosophic topology on *m*-polar neutrosophic set and discuss interior, closure, exterior, and frontier of MPNSs with the help of illustrations. We introduce various results which hold for classical set theory, but do not hold for MPN data. We present a cosine similarity measure and set theoretic similarity measure to find the similarity between MPNSs.

3.1 *m*-Polar Neutrosophic Topological Space

In mathematics, topology is concerned with the alternatives of a geometric object that are kept under continuous deformations, such as stretching, twisting, crumpling, and bending, but not tearing or gluing. "A topological space is a set endowed with a structure, called a topology, which allows defining continuous deformation of subspaces and more broadly, all kinds of continuity." The concept of topology can be defined by using sets, continuous functions, manifolds, algebra, differentiable functions, differential geometry, etc. It has numerous applications in biology, medical diagnosis, physics, computer science, robotics, game theory, and fiber art.

The question arises here that why we use *m*-polar neutrosophic topological space? Crisp topological space cannot deal with the uncertainties and imprecision in the decision-making problems. To handle these ambiguities, Chang [7] introduced fuzzy topological spaces in 1968. After that, many mathematicians established topological spaces on other hybrid structures of fuzzy sets. Every topological space has its own boundaries, e.g., neutrosophic topological space cannot deals with the multiple criteria or multi-polarity of alternatives. *m*-polar topological space cannot deal with the indeterminacy part and dissatisfaction part of alternatives in decision-making problems. To remove these restrictions, we introduce *m*-polar neutrosophic topological space (MPNTS) by combining the *m*-polar fuzzy sets and neutrosophic sets. MPNTS handle these hesitations in the input data by treating with the multi-polarity, membership, non-membership, and indeterminacy grades for the decision-making problems. The motivation of our projected model is given step by step in the whole manuscript, especially in Sect. 4.

Definition 3.1 Let $\mathcal{Q}$ be the non-empty reference set and $\mathfrak{mpn}(\mathcal{Q})$ be the collection of all MPNSs in $\mathcal{Q}$. Then the collection $\mathcal{T}_{\mathcal{M}_{\mathfrak{N}}}$ containing MPNSs is called *m*-polar neutrosophic topology (MPNT) if it satisfies the following properties:

(i) ${}^0\mathcal{M}_{\mathfrak{N}}, {}^1\mathcal{M}_{\mathfrak{N}} \in \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}}$.
(ii) If $(\mathcal{M}_{\mathfrak{N}})_{\wp} \in \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}}, \forall \wp \in \Delta$, then $\bigcup_{\wp \in \Delta} (\mathcal{M}_{\mathfrak{N}})_{\wp} \in \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}}$.
(iii) If $\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2} \in \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}}$, then $\mathcal{M}_{\mathfrak{N}_1} \cap \mathcal{M}_{\mathfrak{N}_2} \in \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}}$.

Then the pair $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}})$ is called MPNTS. The members of $\mathcal{T}_{\mathcal{M}_{\mathfrak{N}}}$ are called open MPNSs and their complements are called closed MPNSs.

Theorem 3.2 *Let $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}})$ be an MPNTS. Then the following conditions are satisfied:*

(i) *${}^0\mathcal{M}_{\mathfrak{N}}$ and ${}^1\mathcal{M}_{\mathfrak{N}}$ are open MPNSs.*
(ii) Union of any number of open MPNSs is open.

(iii) Intersection of finite number of closed MPNSs is closed.

Proof The proof is obvious. □

Example 3.3 Let $\mathcal{Q} = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$ be an assembling of books. Then $\mathfrak{mpn}(\mathcal{Q})$ be the collection of all MPNSs in $\mathcal{Q}$. We consider two 3-polar neutrosophic subsets of $\mathfrak{mpn}(\mathcal{Q})$ given as

$$\mathcal{M}_{\mathfrak{R}_1} = \Big\{(\varsigma_1, \langle 0.871, 0.451, 0.412\rangle, \langle 0.317, 0.412, 0.321\rangle, \langle 0.187, 0.213, 0.118\rangle), (\varsigma_2, \langle 0.547, 0.158, 0.413\rangle, \langle 0.518, 0.152, 0.118\rangle, \langle 0.618, 0.418, 0.321\rangle), (\varsigma_3, \langle 0.618, 0.341, 0.231\rangle, \langle 0.815, 0.118, 0.527\rangle, \langle 0.511, 0.431, 0.215\rangle), (\varsigma_4, \langle 0.518, 0.391, 0.812\rangle, \langle 0.815, 0.321, 0.415\rangle, \langle 0.911, 0.321, 0.512\rangle)\Big\}$$

$$\mathcal{M}_{\mathfrak{R}_2} = \Big\{(\varsigma_1, \langle 0.611, 0.512, 0.611\rangle, \langle 0.218, 0.531, 0.415\rangle, \langle 0.035, 0.311, 0.211\rangle), (\varsigma_2, \langle 0.212, 0.218, 0.513\rangle, \langle 0.435, 0.218, 0.315\rangle, \langle 0.519, 0.511, 0.438\rangle), (\varsigma_3, \langle 0.418, 0.432, 0.321\rangle, \langle 0.639, 0.218, 0.357\rangle, \langle 0.211, 0.531, 0.316\rangle), (\varsigma_4, \langle 0.219, 0.491, 0.815\rangle, \langle 0.716, 0.421, 0.518\rangle, \langle 0.712, 0.421, 0.618\rangle)\Big\}$$

Then clearly the collection $\mathcal{T}_{\mathcal{M}_{\mathfrak{R}}} = \{{}^0\mathcal{M}_{\mathfrak{R}}, {}^1\mathcal{M}_{\mathfrak{R}}, \mathcal{M}_{\mathfrak{R}_1}, \mathcal{M}_{\mathfrak{R}_2}\}$ is 3-polar neutrosophic topological space.

Definition 3.4 Let $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{R}}})$ and $(\mathcal{Q}, \mathcal{T}'_{\mathcal{M}_{\mathfrak{R}}})$ be two MPNTSs in $\mathcal{Q}$. Two MPNTSs are said to be comparable if $\mathcal{T}_{\mathcal{M}_{\mathfrak{R}}} \subseteq \mathcal{T}'_{\mathcal{M}_{\mathfrak{R}}}$ or $\mathcal{T}'_{\mathcal{M}_{\mathfrak{R}}} \subseteq \mathcal{T}_{\mathcal{M}_{\mathfrak{R}}}$.

If $\mathcal{T}_{\mathcal{M}_{\mathfrak{R}}} \subseteq \mathcal{T}'_{\mathcal{M}_{\mathfrak{R}}}$, then $\mathcal{T}_{\mathcal{M}_{\mathfrak{R}}}$ is courser or weaker than $\mathcal{T}'_{\mathcal{M}_{\mathfrak{R}}}$ and $\mathcal{T}'_{\mathcal{M}_{\mathfrak{R}}}$ is stronger and finer than $\mathcal{T}_{\mathcal{M}_{\mathfrak{R}}}$.

Theorem 3.5 *Let $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{R}}})$ be an MPNTS. Then the following conditions are satisfied:*

(i) *${}^0\mathcal{M}_{\mathfrak{R}}$ and ${}^1\mathcal{M}_{\mathfrak{R}}$ are closed MPNSs.*
(ii) *Intersection of any number of closed MPNSs is closed.*
(iii) *Union of finite number of closed MPNSs is closed.*

Proof

(i) $({}^1\mathcal{M}_{\mathfrak{R}})^c = {}^0\mathcal{M}_{\mathfrak{R}}$ and $({}^0\mathcal{M}_{\mathfrak{R}})^c = {}^1\mathcal{M}_{\mathfrak{R}}$ are both open and closed MPNSs.
(ii) If $\{\mathcal{M}_{\mathfrak{R}_\alpha} : \mathcal{M}^c_{\mathfrak{R}_\alpha} \in \mathcal{T}_{\mathcal{M}_{\mathfrak{R}}}, \alpha \in \Delta\}$ is an assembling of closed MPNSs then $(\bigcap_{\alpha\in\Delta} \mathcal{M}_{\mathfrak{R}_\alpha})^c = \bigcup_{\alpha\in\Delta} \mathcal{M}^c_{\mathfrak{R}_\alpha}$ is open. This shows that $\bigcap_{\alpha\in\Delta} \mathcal{M}_{\mathfrak{R}_\alpha}$ is closed MPNS.
(iii) Since $\mathcal{M}_{\mathfrak{R}_\beta}$ is closed for $\beta = 1, 2, \ldots, z$, then $(\bigcup_{\beta=1}^{z} \mathcal{M}_{\mathfrak{R}_\beta})^c = \bigcap_{\beta=1}^{z} \mathcal{M}^c_{\mathfrak{R}_\beta}$ is open MPNS. Thus $\bigcup_{\beta=1}^{z} \mathcal{M}_{\mathfrak{R}_\beta}$ is closed MPNS.

□

Definition 3.6 Let $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{R}}})$ be MPNTS and $\mathcal{M}_{\mathfrak{R}} \in \mathfrak{mpn}({}^1\mathcal{M}_{\mathfrak{R}})$, then interior of $\mathcal{M}_{\mathfrak{R}}$ is denoted as $\mathcal{M}^o_{\mathfrak{R}}$ and defined as the union of all open MPN subsets contained in $\mathcal{M}_{\mathfrak{R}}$. It is the greatest open MPNS contained in $\mathcal{M}_{\mathfrak{R}}$.

Example 3.7 We consider the 3-polar neutrosophic topological space constructed in Example 3.3 and let $\mathcal{M}_{\mathfrak{R}_3} \in \mathfrak{mpn}(\mathcal{Q})$ given as

$$\mathcal{M}_{\mathfrak{R}_3} = \{(\varsigma_1, \langle 0.713, 0.412, 0.311\rangle, \langle 0.318, 0.418, 0.311\rangle, \langle 0.451, 0.211, 0.218\rangle), (\varsigma_2, \langle 0.312, 0.117, 0.418\rangle, \langle 0.513, 0.212, 0.218\rangle, \langle 0.613, 0.411, 0.438\rangle), (\varsigma_3, \langle 0.518, 0.321, 0.311\rangle, \langle 0.718, 0.118, 0.257\rangle, \langle 0.317, 0.461, 0.217\rangle), (\varsigma_1, \langle 0.319, 0.219, 0.615\rangle, \langle 0.719, 0.321, 0.418\rangle, \langle 0.811, 0.321, 0.417\rangle)\}$$

Then $\mathcal{M}^o_{\mathfrak{R}_3} = {}^o\mathcal{M}_{\mathfrak{R}} \cup \mathcal{M}_{\mathfrak{R}_2} = \mathcal{M}_{\mathfrak{R}_2}$ is open MPNS.

Theorem 3.8 *Let $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{R}}})$ be MPNTS and $\mathcal{M}_{\mathfrak{R}} \in \mathfrak{mpn}(\mathcal{Q})$. Then $\mathcal{M}_{\mathfrak{R}}$ is open MPNS $\Leftrightarrow \mathcal{M}^o_{\mathfrak{R}} = \mathcal{M}_{\mathfrak{R}}$.*

Proof If $\mathcal{M}_{\mathfrak{R}}$ is open MPNS then greatest open MPNS contained in $\mathcal{M}_{\mathfrak{R}}$ is itself $\mathcal{M}_{\mathfrak{R}}$. Thus $\mathcal{M}^o_{\mathfrak{R}} = \mathcal{M}_{\mathfrak{R}}$.

Conversely, if $\mathcal{M}^o_{\mathfrak{R}} = \mathcal{M}_{\mathfrak{R}}$ then $\mathcal{M}^o_{\mathfrak{R}}$ is open MPNS. This implies that $\mathcal{M}_{\mathfrak{R}}$ is open MPNS. □

Theorem 3.9 *Let $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{R}}})$ be MPNTS and $\mathcal{M}_{\mathfrak{R}_1}, \mathcal{M}_{\mathfrak{R}_2} \in \mathfrak{mpn}({}^1\mathcal{M}_{\mathfrak{R}})$, then*

(i) $(\mathcal{M}^o_{\mathfrak{R}_1})^o = \mathcal{M}^o_{\mathfrak{R}_1}$,
(ii) $\mathcal{M}_{\mathfrak{R}_1} \subseteq \mathcal{M}_{\mathfrak{R}_2} \Rightarrow \mathcal{M}^o_{\mathfrak{R}_1} \subseteq \mathcal{M}^o_{\mathfrak{R}_2}$,
(iii) $(\mathcal{M}_{\mathfrak{R}_1} \cap \mathcal{M}_{\mathfrak{R}_2})^o = \mathcal{M}^o_{\mathfrak{R}_1} \cap \mathcal{M}^o_{\mathfrak{R}_2}$,
(iv) $(\mathcal{M}_{\mathfrak{R}_1} \cup \mathcal{M}_{\mathfrak{R}_2})^o \supseteq \mathcal{M}^o_{\mathfrak{R}_1} \cup \mathcal{M}^o_{\mathfrak{R}_2}$.

Proof The proof is obvious. □

Definition 3.10 Let $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}})$ be MPNTS and $\mathcal{M}_{\mathfrak{N}} \in \mathfrak{mpn}(\mathcal{Q})$, then the closure of $\mathcal{M}_{\mathfrak{N}}$ is denoted by $\overline{\mathcal{M}_{\mathfrak{N}}}$ and defined by intersection of all closed-MPN supersets of $\mathcal{M}_{\mathfrak{N}}$. It is the smallest closed-MPN superset of $\mathcal{M}_{\mathfrak{N}}$.

Example 3.11 We consider the 3-polar neutrosophic topological space constructed in Example 3.3, then closed MPNSs are given as,

$$\begin{aligned}
{}^{o}\mathcal{M}^{c}_{\mathfrak{N}} &= {}^{1}\mathcal{M}_{\mathfrak{N}}, {}^{1}\mathcal{M}^{c}_{\mathfrak{N}} = {}^{o}\mathcal{M}_{\mathfrak{N}},\\
\mathcal{M}^{c}_{\mathfrak{N}_1} &= \{(\varsigma_1, \langle 0.412, 0.549, 0.871\rangle, \langle 0.321, 0.588, 0.317\rangle,\\
&\langle 0.118, 0.787, 0.187\rangle), (\varsigma_2, \langle 0.413, 0.842, 0.547\rangle,\\
&\langle 0.118, 0.848, 0.518\rangle, \langle 0.321, 0.582, 0.618\rangle),\\
&(\varsigma_3, \langle 0.231, 0.659, 0.618\rangle, \langle 0.257, 0.882, 0.815\rangle,\\
&\langle 0.215, 0.569, 0.511\rangle), (\varsigma_4, \langle 0.812, 0.609, 0.518\rangle,\\
&\langle 0.415, 0.679, 0.815\rangle, \langle 0.512, 0.679, 0.911\rangle)\}\\
\mathcal{M}^{c}_{\mathfrak{N}_2} &= \{(\varsigma_1, \langle 0.611, 0.488, 0.611\rangle, \langle 0.415, 0.487, 0.218\rangle,\\
&\langle 0.211, 0.689, 0.035\rangle), (\varsigma_2, \langle 0.513, 0.782, 0.212\rangle,\\
&\langle 0.315, 0.782, 0.435\rangle, \langle 0.438, 0.489, 0.519\rangle),\\
&(\varsigma_3, \langle 0.321, 0.568, 0.418\rangle, \langle 0.357, 0.782, 0.639\rangle,\\
&\langle 0.316, 0.469, 0.211\rangle), (\varsigma_4, \langle 0.815, 0.509, 0.219\rangle,\\
&\langle 0.518, 0.579, 0.716\rangle, \langle 0.618, 0.579, 0.712\rangle)\}
\end{aligned}$$

Let $\mathcal{M}_{\mathfrak{N}_4} \in \mathfrak{mpn}({}^{1}\mathcal{M}_{\mathfrak{N}})$ given as

$$\begin{aligned}
\mathcal{M}_{\mathfrak{N}_4} &= \{(\varsigma_1, \langle 0.319, 0.615, 0.888\rangle, \langle 0.217, 0.618, 0.411\rangle,\\
&\langle 0.115, 0.817, 0.345\rangle), (\varsigma_2, \langle 0.312, 0.888, 0.617\rangle,\\
&\langle 0.113, 0.878, 0.678\rangle, \langle 0.231, 0.598, 0.765\rangle),\\
&(\varsigma_3, \langle 0.112, 0.767, 0.887\rangle, \langle 0.213, 0.889, 0.889\rangle,\\
&\langle 0.114, 0.667, 0.665\rangle), (\varsigma_4, \langle 0.319, 0.768, 0.615\rangle,\\
&\langle 0.321, 0.778, 0.898\rangle, \langle 0.435, 0.767, 0.987\rangle)\}
\end{aligned}$$

Then $\overline{\mathcal{M}_{\mathfrak{N}_4}} = {}^{1}\mathcal{M}_{\mathfrak{N}} \cap \mathcal{M}^{c}_{\mathfrak{N}_1} \cap \mathcal{M}^{c}_{\mathfrak{N}_2} = \mathcal{M}^{c}_{\mathfrak{N}_1}$ is closed MPNS.

Theorem 3.12 *Let* $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}})$ *be MPNTS and* $\mathcal{M}_{\mathfrak{N}} \in \mathfrak{mpn}(\mathcal{Q})$. $\mathcal{M}_{\mathfrak{N}}$ *is closed MPNS* $\Leftrightarrow \overline{\mathcal{M}_{\mathfrak{N}}} = \mathcal{M}_{\mathfrak{N}}$.

Proof The proof is obvious. □

Definition 3.13 Let $\mathcal{M}_{\mathfrak{N}}$ be an MPN-subset of $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}})$, then its frontier or boundary is defined by $F_r(\mathcal{M}_{\mathfrak{N}}) = \overline{\mathcal{M}_{\mathfrak{N}}} \cap \overline{\mathcal{M}^{c}_{\mathfrak{N}}}$.

Definition 3.14 Let $\mathcal{M}_{\mathfrak{N}}$ be an MPN-subset of $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{N}}})$, then its exterior can be represented as $Ext(\mathcal{M}_{\mathfrak{N}})$ and defined as $Ext(\mathcal{M}_{\mathfrak{N}}) = (\overline{\mathcal{M}_{\mathfrak{N}}})^{c} = (\mathcal{M}^{c}_{\mathfrak{N}})^{o}$.

Example 3.15 We consider the MPNTS constructed in Example 3.3 and consider the MPNSs $\mathcal{M}_{\mathfrak{N}_3}$ and $\mathcal{M}_{\mathfrak{N}_4}$ given in Examples 3.7 and 3.11. Then by using previous definitions we can write that

$$\begin{aligned}
\mathcal{M}^{o}_{\mathfrak{N}_3} &= \mathcal{M}_{\mathfrak{N}_2}, \overline{\mathcal{M}_{\mathfrak{N}_3}} = {}^{1}\mathcal{M}_{\mathfrak{N}},\\
F_r(\mathcal{M}_{\mathfrak{N}_3}) &= {}^{1}\mathcal{M}_{\mathfrak{N}}, Ext(\mathcal{M}_{\mathfrak{N}_3}) = {}^{0}\mathcal{M}_{\mathfrak{N}},\\
\mathcal{M}^{o}_{\mathfrak{N}_4} &= {}^{0}\mathcal{M}_{\mathfrak{N}}, \overline{\mathcal{M}_{\mathfrak{N}_4}} = \mathcal{M}^{c}_{\mathfrak{N}_1},\\
F_r(\mathcal{M}_{\mathfrak{N}_4}) &= \mathcal{M}^{c}_{\mathfrak{N}_1}, Ext(\mathcal{M}_{\mathfrak{N}_4}) = \mathcal{M}_{\mathfrak{N}_1}.
\end{aligned}$$

Now, we present some results which do not hold in MPNTS but hold in crisp set theory due to the law of contradiction and law of excluded middle.

Remark

(i) In MPNTS, the members of discrete topology are infinite due to the infinite subsets of an arbitrary MPNS.

(ii) In MPNTS law of contradiction $\mathcal{M}_{\mathfrak{N}} \cap \mathcal{M}^{c}_{\mathfrak{N}} = {}^{0}\mathcal{M}_{\mathfrak{N}}$ and law of excluded middle $\mathcal{M}_{\mathfrak{N}} \cup \mathcal{M}^{c}_{\mathfrak{N}} = {}^{1}\mathcal{M}_{\mathfrak{N}}$ do not hold in general. In Example 3.15, we can observe that $\mathcal{M}_{\mathfrak{N}_3} \cap \mathcal{M}^{c}_{\mathfrak{N}_3} \neq {}^{0}\mathcal{M}_{\mathfrak{N}}$ and $\mathcal{M}_{\mathfrak{N}_3} \cup \mathcal{M}^{c}_{\mathfrak{N}_3} \neq {}^{1}\mathcal{M}_{\mathfrak{N}}$.

(iii) In *m*-polar neutrosophic set theory, an assembling $\mathcal{T}_{\mathcal{M}_{\mathfrak{N}}} = \{{}^{0}\mathcal{M}_{\mathfrak{N}}, {}^{1}\mathcal{M}_{\mathfrak{N}}, \mathcal{M}_{\mathfrak{N}}, \mathcal{M}^{c}_{\mathfrak{N}}\}$ is not an MPNTS in general. But this result hold in classical set theory. This result can be easily seen by using Example 3.15.

Theorem 3.16 *Let* $\mathcal{M}_{\mathfrak{N}} \in \mathfrak{mpn}({}^{1}\mathcal{M}_{\mathfrak{N}})$, *then*

(1) $(\mathcal{M}^{o}_{\mathfrak{N}})^{c} = \overline{(\mathcal{M}^{c}_{\mathfrak{N}})}$,
(2) $(\overline{\mathcal{M}_{\mathfrak{N}}})^{c} = (\mathcal{M}^{c}_{\mathfrak{N}})^{o}$,
(3) $Ext(\mathcal{M}^{c}_{\mathfrak{N}}) = \mathcal{M}^{o}_{\mathfrak{N}}$,
(4) $Ext(\mathcal{M}_{\mathfrak{N}}) = (\mathcal{M}^{c}_{\mathfrak{N}})^{o}$,
(5) $Ext(\mathcal{M}_{\mathfrak{N}}) \cup F_r(\mathcal{M}_{\mathfrak{N}}) \cup \mathcal{M}^{o}_{\mathfrak{N}} \neq {}^{1}\mathcal{M}_{\mathfrak{N}}$,
(6) $F_r(\mathcal{M}_{\mathfrak{N}}) = F_r(\mathcal{M}^{c}_{\mathfrak{N}})$,
(7) $\mathcal{M}^{o}_{\mathfrak{N}} \cap F_r(\mathcal{M}_{\mathfrak{N}}) \neq {}^{0}\mathcal{M}_{\mathfrak{N}}$.

Proof

(1) and (2): are obvious.
(3) $Ext(\mathcal{M}^{c}_{\mathfrak{N}}) = (\overline{\mathcal{M}^{c}_{\mathfrak{N}}})^{c}$
$\Rightarrow Ext(\mathcal{M}^{c}_{\mathfrak{N}}) = [(\mathcal{M}^{c}_{\mathfrak{N}})^{c}]^{o}$
$\Rightarrow Ext(\mathcal{M}^{c}_{\mathfrak{N}}) = \mathcal{M}^{o}_{\mathfrak{N}}$.
(4) $Ext(\mathcal{M}_{\mathfrak{N}}) = (\overline{\mathcal{M}_{\mathfrak{N}}})^{c}$
$\Rightarrow Ext(\mathcal{M}_{\mathfrak{N}}) = (\mathcal{M}^{c}_{\mathfrak{N}})^{o}$.
(5) $Ext(\mathcal{M}_{\mathfrak{N}}) \cup F_r(\mathcal{M}_{\mathfrak{N}}) \cup \mathcal{M}^{o}_{\mathfrak{N}} \neq {}^{1}\mathcal{M}_{\mathfrak{N}}$. By Example 3.15, we can see that $\mathcal{M}_{\mathfrak{N}_1} \cup \mathcal{M}^{c}_{\mathfrak{N}_1} \cup {}^{0}\mathcal{M}_{\mathfrak{N}} \neq {}^{1}\mathcal{M}_{\mathfrak{N}}$.
(6) $F_r(\mathcal{M}^{c}_{\mathfrak{N}}) = \overline{(\mathcal{M}^{c}_{\mathfrak{N}})} \cap \overline{[(\mathcal{M}^{c}_{\mathfrak{N}})]^{c}}$

$\Rightarrow F_r(\mathcal{M}_{\mathfrak{N}}^c) = \overline{(\mathcal{M}_{\mathfrak{N}}^c)} \cap \overline{(\mathcal{M}_{\mathfrak{N}})} = F_r(\mathcal{M}_{\mathfrak{N}})$.
(7) $\mathcal{M}_{\mathfrak{N}}^o \cap F_r(\mathcal{M}_{\mathfrak{N}}) \neq {}^0\mathcal{M}_{\mathfrak{N}}$. Example 3.15 shows that $\mathcal{M}_{\mathfrak{N}_2} \cap {}^1\mathcal{M}_{\mathfrak{N}} \neq {}^0\mathcal{M}_{\mathfrak{N}}$.

□

3.2 Similarity Measures

In this part, we present two different formulae for similarity measures between MPNSs. This concept will help us in the forthcoming section of multi-criteria decision-making.

Definition 3.17 **(Cosine similarity measure for MPNSs)** We define the cosine similarity measure for *m*-polar neutrosophic sets based on Bhattacharyas distance [32, 47]. Suppose that $\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2} \in \mathfrak{mpn}(\mathcal{M}_{\mathfrak{N}})$, in $\mathcal{Q} = \{\varsigma_1, \varsigma_2, \ldots, \varsigma_l\}$. A cosine similarity measure between $\mathcal{M}_{\mathfrak{N}_1}$ $\mathcal{M}_{\mathfrak{N}_2}$ is given as

$$\mathfrak{C}^1_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2}) = \frac{1}{ml}\sum_{\eta=1}^{l}\sum_{\alpha=1}^{m} \frac{{}^1\mathfrak{A}_\alpha(\varsigma_\eta)^2\mathfrak{A}_\alpha(\varsigma_\eta) + {}^1\mathfrak{S}_\alpha(\varsigma_\eta)^2\mathfrak{S}_\alpha(\varsigma_\eta) + {}^1\mathfrak{Y}_\alpha(\varsigma_\eta)^2\mathfrak{Y}_\alpha(\varsigma_\eta)}{\sqrt{({}^1\mathfrak{A}_\alpha(\varsigma_\eta))^2 + ({}^1\mathfrak{S}_\alpha(\varsigma_\eta))^2 + ({}^1\mathfrak{Y}_\alpha(\varsigma_\eta))^2}\sqrt{({}^2\mathfrak{A}_\alpha(\varsigma_\eta))^2 + ({}^2\mathfrak{S}_\alpha(\varsigma_\eta))^2 + ({}^2\mathfrak{Y}_\alpha(\varsigma_\eta))^2}}.$$

$\mathfrak{C}^1_{MPNS}$ satisfies the following properties,

(1) $0 \leq \mathfrak{C}^1_{MPNS} \leq 1$,
(2) $\mathfrak{C}^1_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2}) = \mathfrak{C}^1_{MPNS}(\mathcal{M}_{\mathfrak{N}_2}, \mathcal{M}_{\mathfrak{N}_1})$,
(3) $\mathfrak{C}^1_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2}) = 1$ if $\mathcal{M}_{\mathfrak{N}_1} = \mathcal{M}_{\mathfrak{N}_2}$,
(4) If $\mathcal{M}_{\mathfrak{N}_1} \subseteq \mathcal{M}_{\mathfrak{N}_2} \subseteq \mathcal{M}_{\mathfrak{N}_3}$ then $\mathfrak{C}^1_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_3}) \leq \mathfrak{C}^1_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2})$ and $\mathfrak{C}^1_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_3}) \leq \mathfrak{C}^1_{MPNS}(\mathcal{M}_{\mathfrak{N}_2}, \mathcal{M}_{\mathfrak{N}_3})$. The proof of these properties can be easily done by using the above definition.

Definition 3.18 (*Set theoretic similarity measure of MPNSs*) We define the set theoretic similarity measure for *m*-polar neutrosophic sets based on set theoretic viewpoint [40]. Suppose that $\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2} \in \mathfrak{mpn}(\mathcal{M}_{\mathfrak{N}})$, in $\mathcal{Q} = \{\varsigma_1, \varsigma_2, \ldots, \varsigma_l\}$. A set theoretic similarity measure between $\mathcal{M}_{\mathfrak{N}_1}$ $\mathcal{M}_{\mathfrak{N}_2}$ is given as

$$\mathfrak{C}^2_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2}) = \frac{1}{ml}\sum_{\eta=1}^{l}\sum_{\alpha=1}^{m} \frac{{}^1\mathfrak{A}_\alpha(\varsigma_\eta)^2\mathfrak{A}_\alpha(\varsigma_\eta) + {}^1\mathfrak{S}_\alpha(\varsigma_\eta)^2\mathfrak{S}_\alpha(\varsigma_\eta) + {}^1\mathfrak{Y}_\alpha(\varsigma_\eta)^2\mathfrak{Y}_\alpha(\varsigma_\eta)}{\max[({}^1\mathfrak{A}_\alpha(\varsigma_\eta))^2 + ({}^1\mathfrak{S}_\alpha(\varsigma_\eta))^2 + ({}^1\mathfrak{Y}_\alpha(\varsigma_\eta))^2, ({}^2\mathfrak{A}_\alpha(\varsigma_\eta))^2 + ({}^2\mathfrak{S}_\alpha(\varsigma_\eta))^2 + ({}^2\mathfrak{Y}_\alpha(\varsigma_\eta))^2]}.$$

$\mathfrak{C}^2_{MPNS}$ satisfies the following properties,

(1) $0 \leq \mathfrak{C}^2_{MPNS} \leq 1$,
(2) $\mathfrak{C}^2_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2}) = \mathfrak{C}^2_{MPNS}(\mathcal{M}_{\mathfrak{N}_2}, \mathcal{M}_{\mathfrak{N}_1})$,
(3) $\mathfrak{C}^2_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2}) = 1$ if $\mathcal{M}_{\mathfrak{N}_1} = \mathcal{M}_{\mathfrak{N}_2}$,
(4) If $\mathcal{M}_{\mathfrak{N}_1} \subseteq \mathcal{M}_{\mathfrak{N}_2} \subseteq \mathcal{M}_{\mathfrak{N}_3}$ then $\mathfrak{C}^2_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_3}) \leq \mathfrak{C}^2_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2})$ and $\mathfrak{C}^2_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_3}) \leq \mathfrak{C}_{\mathfrak{MPNS}}(\mathcal{M}_{\mathfrak{N}}, \mathcal{M}_{\mathfrak{N}})$. The proof of these properties can be easily done by using the above definition.

4 Multi-criteria Decision-Making Under *m*-Polar Neutrosophic Data

Multi-criteria decision-making (MCDM) is a process to find an optimal alternative that has the highest degree of satisfaction from a set of feasible alternatives characterized by multiple criteria, and these kinds of MCDM problems arise in many real-world situations. In this section, we discuss two applications of medical diagnosis and clustering analysis of students with the help of *m*-polar fuzzy neutrosophic data. We present three novel algorithms for multi-criteria decision-making (MCDM) with linguistic information based on the MPNTS and MPFNSs for medical diagnosis and clustering analysis.

In each algorithm, we use *m*-polar neutrosophic input data. Firstly, we collect input information for every algorithm in the form of linguistic variables and then convert them into *m*-polar neutrosophic numbers (MPNNs) by using fuzzy logics. When our data set is covered into proposed *m*-polar neutrosophic numeric values, then we apply each algorithm one by one. At last, we get better results for medical diagnosis and clustering analysis.

4.1 MCDM for Medical Diagnosis

In this part of our manuscript, we establish two different techniques based on MPNTS and on similarity measures to investigate the disease with *m*-polar neutrosophic information.

Proposed Technique of Algorithm 1

Algorithm 1 (Algorithm for m-polar neutrosophic topological space)

Input:

Step 1: Input the set $\mathfrak{P}$ for a patient according to his doctor, corresponding to the "m" number of symptoms appearing to the patient. All the input data leads to those "p" diseases which will be possible outcome according to the appearing symptoms in the form of m-polar neutrosophic set.

Step 2: Input the sets $\Im_\xi; \xi = 1,2,...,z$, for "p" diseases $\eth_\delta; \delta = 1,2,...,p$, according to "z" number of experts, corresponding to the "m" number of symptoms in the form of m-polar neutrosophic sets (MPNSs).

Calculations:

Step 3: Construct m-polar neutrosophic topological space (MPNTS) $\mathcal{T}_{\mathcal{M}_\mathfrak{N}}$ using MPNSs $\Im_\xi; \xi = 1,2,...,z$ given by "z" number of experts.

Step 4: Find interior $\mathfrak{P}^o$ of $\mathfrak{P}$ by using Definition 3.6 under the constructed $\mathcal{T}_{\mathcal{M}_\mathfrak{N}}$. $\mathfrak{P}^o$ shows the actual condition of the patient according to the "z" number of experts and give better decision to diagnosis.

Step 5: Calculate scores of each disease corresponding to "m" number of symptoms by using Definition 2.6.

Output:

Step 6: We rank the alternative (disease) on the basis of score values according to the Definition 2.7.

Step 7: Alternative (disease) with the higher score has the maximum rank according to the given numerical example. This implies that patient is suffering from that disease.

In this algorithm, rating of each criteria according to the corresponding alternative is constructed by using m-polar neutrosophic information for MCDM and given in input matrix (can be taken in tabular form by using m-polar neutrosophic numbers) as

$$\mathfrak{P} = [\Im^\alpha_{\eta\xi}]_{r\times s} = [\langle \mathfrak{A}^\alpha_{\eta\xi}, \mathfrak{S}^\alpha_{\eta\xi}, \mathfrak{Y}^\alpha_{\eta\xi}\rangle]_{r\times s}; \alpha = 1,2,3,\ldots,m$$
$$\mathfrak{P} = [\Im^\alpha_{\eta\xi}]_{r\times s} = \begin{pmatrix} (\langle \mathfrak{A}^\alpha_{11}, \mathfrak{S}^\alpha_{11}, \mathfrak{Y}^\alpha_{11}\rangle) & (\langle \mathfrak{A}^\alpha_{12}, \mathfrak{S}^\alpha_{12}, \mathfrak{Y}^\alpha_{12}\rangle) & \cdots & (\langle \mathfrak{A}^\alpha_{1s}, \mathfrak{S}^\alpha_{1s}, \mathfrak{Y}^\alpha_{1s}\rangle) \\ (\langle \mathfrak{A}^\alpha_{21}, \mathfrak{S}^\alpha_{21}, \mathfrak{Y}^\alpha_{21}\rangle) & (\langle \mathfrak{A}^\alpha_{22}, \mathfrak{S}^\alpha_{22}, \mathfrak{Y}^\alpha_{22}\rangle) & \cdots & (\langle \mathfrak{A}^\alpha_{2s}, \mathfrak{S}^\alpha_{2s}, \mathfrak{Y}^\alpha_{2s}\rangle) \\ \vdots & \vdots & \ddots & \vdots \\ (\langle \mathfrak{A}^\alpha_{r1}, \mathfrak{S}^\alpha_{r1}, \mathfrak{Y}^\alpha_{r1}\rangle) & (\langle \mathfrak{A}^\alpha_{r2}, \mathfrak{S}^\alpha_{r2}, \mathfrak{Y}^\alpha_{r2}\rangle) & \cdots & (\langle \mathfrak{A}^\alpha_{rs}, \mathfrak{S}^\alpha_{rs}, \mathfrak{Y}^\alpha_{rs}\rangle) \end{pmatrix} \tag{1}$$

In matrix $\mathfrak{P}$, the entries $\mathfrak{A}^\alpha_{\eta\xi}$, $\mathfrak{S}^\alpha_{\eta\xi}$, and $\mathfrak{Y}^\alpha_{\eta\xi}$ represent truth, indeterminacy, and falsity membership grades for alternative $\eth_\eta$ corresponding to the criteria $\mathfrak{C}_\xi$, where $\eta = 1,2,3,\ldots,r; \xi = 1,2,3,\ldots,s$. These grades satisfies the following properties under MPN environment.

(1) $0 \le \mathfrak{A}^\alpha_{\eta\xi} \le 1; 0 \le \mathfrak{S}^\alpha_{\eta\xi} \le 1; 0 \le \mathfrak{Y}^\alpha_{\eta\xi} \le 1$.

(2) $0 \le \mathfrak{A}^\alpha_{\eta\xi} + \mathfrak{S}^\alpha_{\eta\xi} + \mathfrak{Y}^\alpha_{\eta\xi} \le 3$, for $\eta = 1,2,3,\ldots,r; \xi = 1,2,3,\ldots,s; \alpha = 1,2,3\ldots,m$.

The rating of each criteria corresponding to the alternative for m-triplets is illustrated in this work. The input decision matrices $\Im_\xi; \xi = 1,2,3,\ldots,z$ for z number of experts can be written by using m-polar neutrosophic data same as Equation 2. Then we construct an m-polar neutrosophic topological space $\mathcal{T}_{\mathcal{MN}}$ by using experts data $\Im_\xi; \xi = 1,2,3,\ldots,z$. Find interior $\mathfrak{P}^o$ of MPN-matrix $\mathfrak{P}$ under the constructed $\mathcal{T}_{\mathcal{M}_\mathfrak{N}}$. Then we calculate score values of all the alternatives in $\mathfrak{P}^o$. We rank these fuzzy values and choose alternative having maximum fuzzy value as an optimal decision. The step-wise description of this proposed technique is given as Algorithm 1.

4.1.1 Proposed Technique of Algorithm 2:

In this algorithm, rating of each criteria according to the corresponding alternative is constructed by using m-polar neutrosophic information for MCDM and given in input matrix (can be taken in tabular form by using m-polar neutrosophic numbers) as

$$\mathfrak{P} = [\Im^\alpha_{\eta\xi}]_{r\times s} = [\langle \mathfrak{A}^\alpha_{\eta\xi}, \mathfrak{S}^\alpha_{\eta\xi}, \mathfrak{Y}^\alpha_{\eta\xi}\rangle]_{r\times s}; \alpha = 1,2,3,\ldots,m$$
$$\mathfrak{P} = [\Im^\alpha_{\eta\xi}]_{r\times s} = \begin{pmatrix} (\langle \mathfrak{A}^\alpha_{11}, \mathfrak{S}^\alpha_{11}, \mathfrak{Y}^\alpha_{11}\rangle) & (\langle \mathfrak{A}^\alpha_{12}, \mathfrak{S}^\alpha_{12}, \mathfrak{Y}^\alpha_{12}\rangle) & \cdots & (\langle \mathfrak{A}^\alpha_{1s}, \mathfrak{S}^\alpha_{1s}, \mathfrak{Y}^\alpha_{1s}\rangle) \\ (\langle \mathfrak{A}^\alpha_{21}, \mathfrak{S}^\alpha_{21}, \mathfrak{Y}^\alpha_{21}\rangle) & (\langle \mathfrak{A}^\alpha_{22}, \mathfrak{S}^\alpha_{22}, \mathfrak{Y}^\alpha_{22}\rangle) & \cdots & (\langle \mathfrak{A}^\alpha_{2s}, \mathfrak{S}^\alpha_{2s}, \mathfrak{Y}^\alpha_{2s}\rangle) \\ \vdots & \vdots & \ddots & \vdots \\ (\langle \mathfrak{A}^\alpha_{r1}, \mathfrak{S}^\alpha_{r1}, \mathfrak{Y}^\alpha_{r1}\rangle) & (\langle \mathfrak{A}^\alpha_{r2}, \mathfrak{S}^\alpha_{r2}, \mathfrak{Y}^\alpha_{r2}\rangle) & \cdots & (\langle \mathfrak{A}^\alpha_{rs}, \mathfrak{S}^\alpha_{rs}, \mathfrak{Y}^\alpha_{rs}\rangle) \end{pmatrix} \tag{2}$$

In matrix $\mathfrak{P}$, the entries $\mathfrak{A}^\alpha_{\eta\xi}$, $\mathfrak{S}^\alpha_{\eta\xi}$, and $\mathfrak{Y}^\alpha_{\eta\xi}$ represents truth, indeterminacy, and falsity membership grades for alternative $\eth_\eta$ corresponding to the criteria $\mathfrak{C}_\xi$, where $\eta = 1,2,3,\ldots,r; \xi = 1,2,3,\ldots,s$. These grades satisfies the following properties under MPN environment.

(1) $0 \le \mathfrak{A}^\alpha_{\eta\xi} \le 1; 0 \le \mathfrak{S}^\alpha_{\eta\xi} \le 1; 0 \le \mathfrak{Y}^\alpha_{\eta\xi} \le 1$.

(2) $0 \le \mathfrak{A}^\alpha_{\eta\xi} + \mathfrak{S}^\alpha_{\eta\xi} + \mathfrak{Y}^\alpha_{\eta\xi} \le 3$, for $\eta = 1,2,3,\ldots,r; \xi = 1,2,3,\ldots,s; \alpha = 1,2,3\ldots,m$.

The rating of each criteria corresponding to the alternative for m-triplets is illustrated in this work. The input decision matrices $\Im_{\xi}; \xi = 1, 2, 3, \ldots, z$ for z number of experts can be written by using m-polar neutrosophic data same as Equation 2. We calculate cosine similarity measure and set theoretic similarity measure between $\Im_{\xi}; \xi = 1, 2, 3, \ldots, z$ and $\mathfrak{P}$. We choose the m-polar neutrosophic sets from $\Im_{\xi}; \xi = 1, 2, 3, \ldots, z$ having highest cosine similarity measure and highest set theoretic similarity measure. Then we calculate score values of all the alternatives in the selected sets from $\Im_{\xi}; \xi = 1, 2, 3, \ldots, z$. We rank these fuzzy values and choose alternative having maximum fuzzy value as an optimal decision. The step-wise description of this proposed technique is given as Algorithm 2.

Algorithm 2 (Algorithm for m-polar neutrosophic sets using similarity measures)

Input:

Step 1: Input the set $\mathfrak{P}$ for a patient according to his doctor, corresponding to the "m" number of symptoms appearing to the patient. All the input data leads to those "p" diseases which will be possible outcome according to the appearing symptoms in the form of m-polar neutrosophic set.

Step 2: Input the sets $\Im_{\xi}; \xi = 1, 2, ..., z$, for "p" diseases $\eth_{\delta}; \delta = 1, 2, ..., p$, according to "z" number of experts, corresponding to the "m" number of symptoms in the form of m-polar neutrosophic sets (MPNSs).

Calculations:

Step 3: calculate cosine similarity measure using Definition 3.17 between $\Im_{\xi}; \xi = 1, 2, ..., z$ and $\mathfrak{P}$.

Step 3': calculate set theoretic similarity measure using Definition 3.18 between $\Im_{\xi}; \xi = 1, 2, ..., z$ and $\mathfrak{P}$.

Step 4: Choose the MPNS from $\Im_{\xi}; \xi = 1, 2, ..., z$ having highest cosine similarity measure with $\mathfrak{P}$. That $\Im_{\xi}$ gives the best decision for diagnosis of patient.

Step 4': Choose the MPNS from $\Im_{\xi}; \xi = 1, 2, ..., z$ having highest set theoretic similarity measure with $\mathfrak{P}$. That $\Im_{\xi}$ gives the best decision for diagnosis of patient.

Step 5: Calculate scores of each disease $\eth_{\delta}$ of selected $\Im_{\xi}$ after finding cosine and set theoretic similarity measures corresponding to "m" number of symptoms by using Definition 2.6. From this method we get two different results (rankings) according to two different similarity measures.

Output:

Step 6: We rank the alternative (disease) on the basis of score values according to the Definition 2.7.

Step 7: Alternative (disease) with the higher score has the maximum rank according to the given numerical example. This implies that patient is suffering from that disease.

The flow chart diagram of proposed algorithms can be seen in Fig. 2.

4.1.2 Numerical example

Suppose that a patient is facing some health issues and the symptoms are temperature, headache, fatigue, loss of appetite, stomach pain, inadequate immune system, muscle, and joint pain. According to the doctor's opinion, all these symptoms lead to the following diseases Tuberculosis, Hepatitis C, and Typhoid fever. Let us consider the set $\mathcal{Q} = \{\eth_1, \eth_2, \eth_3\}$ of the alternatives consisting of three diseases and the set $\mathfrak{Z} = \{\mathcal{J}, \mathcal{J}, \mathcal{J}, \mathcal{J}\}$ of symptoms, where

$\eth_1 =$ Tuberculosis, $\eth_2 =$ Hepatitis C, $\eth_3 =$ Typhoid fever,
$\mathcal{J}_1 =$ Fever, $\mathcal{J}_2 =$ Poor immune system
$\mathcal{J}_3 =$ Muscle and joint pain, fatigue,
$\mathcal{J}_4 =$ Unintentional weight loss, loss of appetite

We input the data of patient according to his doctor in the form of 4-polar neutrosophic set for each disease corresponding to every symptom. In this data, the numeric values corresponding to each symptom show that how many chances he have to be suffered from the considered disease. In Table 5 for disease $\eth_1$ =Tuberculosis, the first triplet $\langle 0.635, 0.115, 0.114 \rangle$ shows that according to his symptom "$\mathcal{J}_1$ =fever" patient has 63, 5% truth chances, 11.5% indeterminacy, and 11.4% falsity chances to have tuberculosis. Similarly, we can observe all values of patient according to his symptoms for all diseases.

We consider that we have "z=3" highly qualified experts, then according to these experts the data of each disease corresponding to each symptom is given in tabular form of 4-polar neutrosophic sets as Tables 6, 7, and 8. Each $\Im_{\xi}; \xi = 1, 2, 3$ representing the data of each disease corresponding to each symptom according to 3 experts. This means that for expert $\Im_1$ and disease $\eth_1$ =tuberculosis

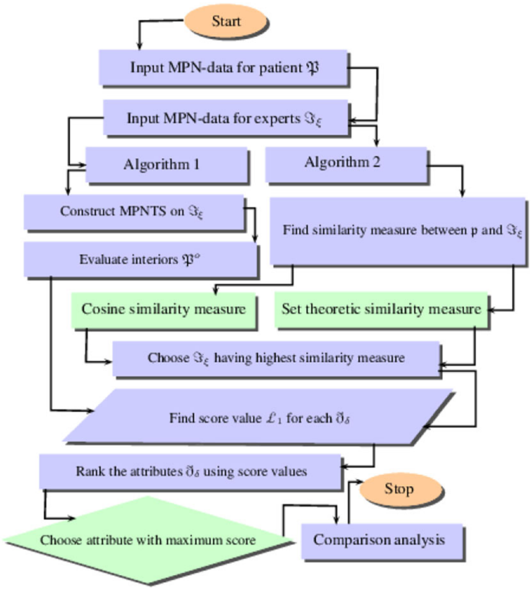

Fig. 2 Flowchart diagram of proposed algorithm 1 and algorithm 2

the first triplet $\langle 0.511, 0.311, 0.213\rangle$ shows that according to symptom "$\mathcal{J}_1 =$ fever" there are 63, 5% truth chances, 11.5% indeterminacy, and 11.4% falsity chances to have tuberculosis. On the same pattern, we can observe all values of diseases according to the corresponding symptoms for each expert.

4.1.3 Solution by using Algorithm 1

Now we construct 4-polar neutrosophic topological space $\mathcal{T}_{\mathcal{M}_{\mathfrak{N}}}$ on $\mathfrak{I}_\xi; \xi = 1, 2, 3$ given as $\mathcal{T}_{\mathcal{MN}} = \{\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3, {}^0\mathcal{M}_{\mathfrak{N}}, {}^1\mathcal{M}_{\mathfrak{N}}\}$. We find the interior $\mathfrak{P}^o$ of $\mathfrak{P}$ by using Definition 3.6 under the 4PNTS $\mathcal{T}_{\mathcal{M}_{\mathfrak{N}}}$. Thus $\mathfrak{P}^o = {}^0\mathcal{M}_{\mathfrak{N}} \cup \mathfrak{I}_1 \cup \mathfrak{I}_2 = \mathfrak{I}_2$. Now we use Definition 2.6 on $\mathfrak{I}_2$ to find scores of all the diseases $\eth_\delta, \delta = 1, 2, 3$.

$$\begin{aligned} £_1(\mathfrak{I}_{2\eth_1}) = & \frac{1}{2 \times 4}(4 + (0.611 - 2(0.213) - 0.118) \\ & + (0.711 - 2(0.321) - 0.118) \\ & + (0.412 - 2(0.511) - 0.611) \\ & + (0.813 - 2(0.211) - 0.341)) = 0.3558. \end{aligned}$$

Similarly, we can find $£_1(\mathfrak{I}_{2\eth_2}) = 0.662$ and $£_1(\mathfrak{I}_{2\eth_3}) = 0.3691$. By Definition 2.7 we can write that $\eth_2 \succ \eth_3 \succ \eth_1$. Hence, patient is suffering from Hepatitis C. Graphically results can be seen as Fig. 3.

Table 5 4-Polar neutrosophic data of patient $\mathfrak{P}$

$\mathfrak{P}$	4-polar neutrosophic sets
$\eth_1$	$(\langle 0.635, 0.115, 0.114\rangle, \langle 0.813, 0.239, 0.115\rangle, \langle 0.513, 0.431, 0.513\rangle \langle 0.911, 0.119, 0.238\rangle)$
$\eth_2$	$(\langle 0.739, 0.119, 0.115\rangle, \langle 0.923, 0.111, 0.108\rangle, \langle 0.889, 0.108, 0.117\rangle, \langle 0.835, 0.113, 0.218\rangle)$
$\eth_3$	$(\langle 0.919, 0.113, 0.122\rangle, \langle 0.818, 0.112, 0.211\rangle, \langle 0.611, 0.513, 0.618\rangle, \langle 0.713, 0.218, 0.319\rangle)$

Table 6 4-polar neutrosophic data for expert $\mathfrak{I}_1$

$\mathfrak{I}_1$	4-polar neutrosophic sets
$\eth_1$	$(\langle 0.511, 0.311, 0.213\rangle, \langle 0.631, 0.431, 0.211\rangle, \langle 0.328, 0.611, 0.782\rangle \langle 0.713, 0.348, 0.411\rangle)$
$\eth_2$	$(\langle 0.638, 0.324, 0.237\rangle, \langle 0.816, 0.118, 0.119\rangle, \langle 0.717, 0.115, 0.218\rangle, \langle 0.719, 0.222, 0.249\rangle)$
$\eth_3$	$(\langle 0.889, 0.212, 0.213\rangle, \langle 0.699, 0.189, 0.232\rangle, \langle 0.413, 0.718, 0.818\rangle, \langle 0.518, 0.421, 0.518\rangle)$

Table 7 4-polar neutrosophic data for expert $\mathfrak{I}_2$

$\mathfrak{I}_2$	4-polar neutrosophic sets
$\eth_1$	$(\langle 0.611, 0.213, 0.118\rangle, \langle 0.711, 0.321, 0.118\rangle, \langle 0.412, 0.511, 0.611\rangle \langle 0.813, 0.211, 0.341\rangle)$
$\eth_2$	$(\langle 0.718, 0.211, 0.117\rangle, \langle 0.916, 0.113, 0.112\rangle, \langle 0.817, 0.113, 0.211\rangle, \langle 0.815, 0.211, 0.234\rangle)$
$\eth_3$	$(\langle 0.918, 0.116, 0.132\rangle, \langle 0.713, 0.116, 0.213\rangle, \langle 0.511, 0.611, 0.713\rangle, \langle 0.613, 0.321, 0.416\rangle)$

Table 8 4-polar neutrosophic data for expert $\mathfrak{I}_3$

$\mathfrak{I}_3$	4-polar neutrosophic sets
$\eth_1$	$(\langle 0.711, 0.118, 0.108\rangle, \langle 0.811, 0.213, 0.108\rangle, \langle 0.512, 0.421, 0.521\rangle \langle 0.815, 0.118, 0.213\rangle)$
$\eth_2$	$(\langle 0.723, 0.119, 0.111\rangle, \langle 0.928, 0.112, 0.110\rangle, \langle 0.888, 0.111, 0.119\rangle, \langle 0.889, 0.181, 0.201\rangle)$
$\eth_3$	$(\langle 0.929, 0.115, 0.128\rangle, \langle 0.813, 0.112, 0.211\rangle, \langle 0.611, 0.511, 0.613\rangle, \langle 0.718, 0.213, 0.325\rangle)$

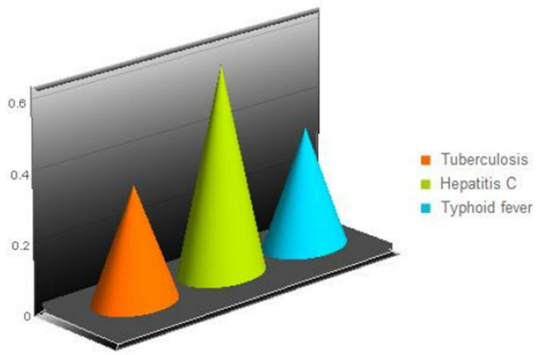

Fig. 3 Ranking of alternatives under MPNTS

4.1.4 Solution by using Algorithm 2

Now by using Tables 5, 6, 7, and 8, we find cosine similarity measures between $(\mathfrak{I}_1, \mathfrak{P}), (\mathfrak{I}_2, \mathfrak{P})$ and $(\mathfrak{I}_3, \mathfrak{P})$ by using Definition 3.17 given as

$$\mathfrak{C}^1_{MPNS}(\mathfrak{I}_2, \mathfrak{P}) = \frac{1}{3\times 4}\Bigg(\frac{(0.611)(0.635)+(0.213)(0.115)+(0.118)(0.114)}{\sqrt{(0.611)^2+(0.213)^2+(0.118)^2}\sqrt{(0.635)^2+(0.115)^2+(0.114)^2}} + \frac{(0.711)(0.813)+(0.321)(0.329)+(0.118)(0.115)}{\sqrt{(0.711)^2+(0.321)^2+(0.118)^2}\sqrt{(0.813)^2+(0.329)^2+(0.115)^2}} + \cdots + \frac{(0.613)(0.713)+(0.321)(0.218)+(0.416)(0.319)}{\sqrt{(0.613)^2+(0.321)^2+(0.416)^2}\sqrt{(0.713)^2+(0.218)^2+(0.319)^2}}\Bigg).$$

$\mathfrak{C}^1_{MPNS}(\mathfrak{I}_2, \mathfrak{P}) = \frac{11.89053}{12} = 0.990878$. Similarly, we can find similarity between other MPNSs given as $\mathfrak{C}^1_{MPNS}(\mathfrak{I}_1, \mathfrak{P}) = \frac{11.50807}{12} = 0.95900$, $\mathfrak{C}^1_{MPNS}(\mathfrak{I}_3, \mathfrak{P}) = \frac{11.996}{12} = 0.99966$. This shows that $\mathfrak{C}^1_{MPNS}(\mathfrak{I}_3, \mathfrak{P}) \succ \mathfrak{C}^1_{MPNS}(\mathfrak{I}_2, \mathfrak{P}) \succ \mathfrak{C}^1_{MPNS}(\mathfrak{I}_1, \mathfrak{P})$. From this ranking it is clear to see that opinion of expert $\mathfrak{I}_3$ is most related and similar to the condition of patient $\mathfrak{P}$. So, we select $\mathfrak{I}_3$ and calculate score values of all diseases $\eth_\delta; \delta = 1, 2, 3$ by using Definition 2.6. This implies that $\pounds_1(\mathfrak{I}_{3\eth_1}) = 0.5198$, $\pounds_1(\mathfrak{I}_{3\eth_2}) = 0.7301$, $\pounds_1(\mathfrak{I}_{3\eth_3}) = 0.4977$. By Definition 2.7 we can write that $\eth_2 \succ \eth_1 \succ \eth_3$. Hence patient is suffering from Hepatitis C.

Now, we use set theoretic similarity measure $\mathfrak{C}^2_{MPNS}$ to find similarity between $(\mathfrak{I}_1, \mathfrak{P}), (\mathfrak{I}_2, \mathfrak{P})$ and $(\mathfrak{I}_3, \mathfrak{P})$ by using Definition 3.18 given as

$$\mathfrak{C}^2_{MPNS}(\mathfrak{I}_2, \mathfrak{P}) = \frac{1}{3\times 4}\Bigg(\frac{(0.611)(0.635)+(0.213)(0.115)+(0.118)(0.114)}{\max((0.611)^2+(0.213)^2+(0.118)^2, (0.635)^2+(0.115)^2+(0.114)^2)} + \frac{(0.711)(0.813)+(0.321)(0.329)+(0.118)(0.115)}{\max((0.711)^2+(0.321)^2+(0.118)^2, (0.813)^2+(0.329)^2+(0.115)^2)} + \cdots + \frac{(0.613)(0.713)+(0.321)(0.218)+(0.416)(0.319)}{\max((0.613)^2+(0.321)^2+(0.416)^2, (0.713)^2+(0.218)^2+(0.319)^2)}\Bigg).$$

$\mathfrak{C}^2_{MPNS}(\mathfrak{I}_2, \mathfrak{P}) = \frac{10.44972}{12} = 0.87081$. Similarly, we can find similarity between other MPNSs given as $\mathfrak{C}^2_{MPNS}(\mathfrak{I}_1, \mathfrak{P}) = \frac{10.51971}{12} = 0.87664$, $\mathfrak{C}^2_{MPNS}(\mathfrak{I}_3, \mathfrak{P}) = \frac{11.2283}{12} = 0.9355$. This shows that $\mathfrak{C}^2_{MPNS}(\mathfrak{I}_3, \mathfrak{P}) \succ \mathfrak{C}^2_{MPNS}(\mathfrak{I}_1, \mathfrak{P}) \succ \mathfrak{C}^2_{MPNS}$ $(\mathfrak{I}_2, \mathfrak{P})$. From this ranking it is clear to see that opinion of expert $\mathfrak{I}_3$ is most related and similar to the condition of patient $\mathfrak{P}$. So, we select $\mathfrak{I}_3$ and calculate score values of all diseases $\eth_\delta; \delta = 1, 2, 3$ by using Definition 2.6. This implies that $\pounds_1(\mathfrak{I}_{3\eth_1}) = 0.5198$, $\pounds_1(\mathfrak{I}_{3\eth_2}) = 0.7301$, $\pounds_1(\mathfrak{I}_{3\eth_3}) = 0.4977$. By Definition 2.7 we can write that $\eth_2 \succ \eth_1 \succ \eth_3$. Hence patient is suffering from Hepatitis C. Graphically results can be seen as Fig 4.

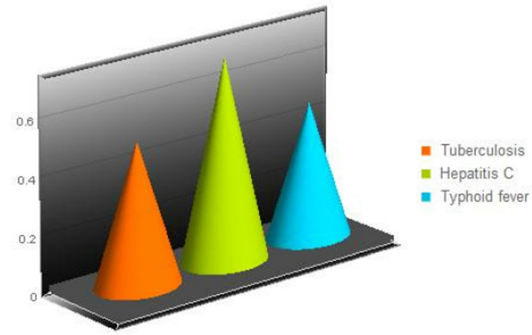

Fig. 4 Ranking of attributes under similarity measures

4.1.5 Discussion and Comparison Analysis:

In this section, we discuss advantages validity, simplicity, flexibility, and superiority of our proposed approach and algorithms. We also give a brief comparison analysis of proposed method with existing approaches.

Advantages of Proposed Approach

Now we discuss some advantages of the proposed techniques based on MPNSs.

(i) Validity of the Method

The suggested method is valid and suitable for all types of input data. we present two novel algorithms in this manuscript one for MPNTS and other for similarity measures. We introduced two similarity measures between MPNSs. It is interesting to note that both algorithms and both formulas of similarity gives the same result (see Table 9). In this work, both algorithms have their own importance and can be used according to the requirement of decision-maker. Both algorithms are valid and give best decision in multi-criteria decision-making (MCDM) problems.

(ii) Simplicity and Flexibility Dealing with Different Criteria

In MCDM problems, we experience different types of criteria and input data according to the given situations. The proposed algorithms are simple and easy to understand which can be applied easily on whatever type of alternatives and measures. Both algorithms are flexible and easily altered according to the different situations, inputs, and outputs. There is a slightly difference between the ranking of the proposed approaches because different formulae have

Table 9 Score values for optimal choice under both algorithms

Algorithm	Method	$\eth_1$	$\eth_2$	$\eth_3$	Ranking of alternatives
Algorithm1	m-Polar neutrosophic topological space	0.3558	0.622	0.3691	$\eth_2 \succ \eth_3 \succ \eth_1$
Algorithm2	Cosine similarity on m-polar neutrosophic sets	0.5198	0.7301	0.4977	$\eth_2 \succ \eth_1 \succ \eth_3$
Algorithm2	Set theoretic similarity on m-polar neutrosophic sets	0.5198	0.7301	0.4977	$\eth_2 \succ \eth_1 \succ \eth_3$

Table 10 Comparison of proposed algorithms with some existing approaches

Methods	Similarity measures on sets	Ranking of alternatives
Wei [37]	Picture fuzzy set	$\eth_2 \succ \eth_1 \succ \eth_3$
Xu and Chen [39, 40]	Intuitionistic fuzzy set and correlation measures	$\eth_2 \succ \eth_1 \succ \eth_3$
Ye [45]	Correlation coefficient of neutrosophic set	$\eth_2 \succ \eth_1 \succ \eth_3$
Ye [47]	Intuitionistic fuzzy set	$\eth_2 \succ \eth_3 \succ \eth_1$
Li and Cheng [17]	Intuitionistic fuzzy set	$\eth_2 \succ \eth_3 \succ \eth_1$
Lin [18]	Hesitant fuzzy linguistic information	$\eth_2 \succ \eth_1 \succ \eth_3$
Wei [38]	Interval-valued intuitionistic fuzzy set	$\eth_2 \succ \eth_3 \succ \eth_1$
Proposed algorithm1	m-Polar neutrosophic topological space	$\eth_2 \succ \eth_3 \succ \eth_1$
Proposed algorithm2	Cosine similarity on m-polar neutrosophic sets	$\eth_2 \succ \eth_1 \succ \eth_3$
Proposed algorithm2	Set theooretic similarity on m-polar neutrosophic sets	$\eth_2 \succ \eth_1 \succ \eth_3$

different ordering strategies, so they can afford the slightly different effect according to their deliberations.

(iii) Superiority of Proposed Method

From all above discussion, we observe that our proposed models of m-polar neutrosophic set and m-polar neutrosophic topological space are superior to existing approaches including fuzzy neutrosophic sets, m-polar intuitionistic fuzzy sets, interval-valued m-polar fuzzy sets and m-polar fuzzy sets. Moreover, many hybrid structures of fuzzy sets become the special cases of m-polar neutrosophic set with the addition of some suitable conditions (see Fig. 1). So our proposed approach is valid, flexible, simple, and superior to other hybrid structures of fuzzy sets.

Comparison Analysis

(1) In our proposed method, we define m-polar neutrosophic topological space and two algorithms based on MPN input data. The impressive point of this model is that we can use it for mathematical modeling at a large scale or "m" numbers of criteria with its truth, falsity, and indeterminacy part. These m-degrees basically show the corresponding properties or any set criteria about the alternatives. As in giving numerical example, we use $m = 4$ to analyze the data for four symptoms appearing to the patient. The value of "m" can be taken as large as possible, which is not possible for other approaches. Moreover, many hybrid structures of fuzzy set become the special cases of m-polar neutrosophic set with the addition of some suitable conditions (see Fig. 1).

(2) Table 10 as given above listing the results of the comparison in the final ranking of top 3 alternatives (diseases). As it could be observed in the comparison Table 10, the best selection made by the proposed methods is comparable to already established methods which is expressive in itself and approves the reliability and validity of the proposed method. Now the question turns out that why we need to specify a novel algorithm based on this novel structure? There are many arguments which show that proposed operator is more suitable than other existing methods. As we know that intuitionistic fuzzy sets, picture fuzzy sets, fuzzy sets, hesitant fuzzy sets, neutrosophic sets, and other existing hybrid structures of fuzzy sets have some limitations and not able to present full information about the situation. But our proposed model of m-polar neutrosophic set is most suitable for MCDM methods and deals with multi-criteria having truth, indeterminacy, and falsity values. Due to the addition of neutrosophic nature in multi-polarity, these three grades go independent of each other and give a lot of information about the multiple criteria for the alternatives.

(3) The similarity measures for other existing hybrid structures of fuzzy set become special cases of similarity measures of *m*-polar neutrosophic set. So, this model is more generalized and can easily deal with the problems involving intuitionistic, neutrosophy, hesitant, picture, and fuzziness of alternatives. The constructed topological space on MPNS becomes superior to existing topological spaces and easily deals with the problems in MCDM methods.

4.2 Clustering Analysis in Multi-criteria Decision-Making

We introduce a novel clustering algorithm under *m*-polar neutrosophic environment to solve multi-criteria decision-making problem. Before this, we revise some basic concepts.

Definition 4.1 [41] Let $\mathcal{M}_{\mathfrak{N}_\zeta}$ be "q" *m*-polar neutrosophic sets (MPNSs), then $\mathcal{G} = (g_{\beta\zeta})_{q\times q}$ is said to be similarity matrix, where $g_{\beta\zeta} = \mathfrak{C}(\mathcal{M}_{\mathfrak{N}_\beta}, \mathcal{M}_{\mathfrak{N}_\zeta})$ represents the similarity measure of MPNSs $\mathcal{M}_{\mathfrak{N}_\beta}$ and $\mathcal{M}_{\mathfrak{N}_\zeta}$ and satisfy the following:

(1) $0 \leq g_{\beta\zeta} \leq 1; \beta, \zeta = 1, 2, 3, \ldots, q,$
(2) $g_{\beta\beta} = 1; \beta = 1, 2, 3, \ldots, q,$
(3) $g_{\beta\zeta} = g_{\zeta\beta}; \beta, \zeta = 1, 2, 3, \ldots, q.$

Definition 4.2 [41] Let $\mathcal{G} = (g_{\beta\zeta})_{q\times q}$ be the similarity matrix. Then $\mathcal{G}^2 = \mathcal{G} \circ \mathcal{G} = (\overline{g_{\beta\zeta}})_{q\times q}$ is said to be a composition matrix of $\mathcal{G}$, where

$$\overline{g_{\beta\zeta}} = \max_{\delta}\{\min\{g_{\beta\delta}, g_{\delta\zeta}\}\}; \quad \beta, \zeta = 1, 2, 3, \ldots, q$$

Theorem 4.3 [41] *Let $\mathcal{G} = (g_{\beta\zeta})_{q\times q}$ be a similarity matrix, then after a finite compositions ($\mathcal{G} \to \mathcal{G}^2 \to \mathcal{G}^4 \to \cdots \to \mathcal{G}^{2^\delta} \to \cdots$), $\exists$ a positive integer δ such that $\mathcal{G}^{2^\delta} = \mathcal{G}^{2^{(\delta+1)}}$. $\mathcal{G}^{2^\delta}$ is an equivalence similarity matrix.*

Definition 4.4 [41] Let $\mathcal{G} = (g_{\beta\zeta})_{q\times q}$ be an equivalence similarity matrix. Then $\mathcal{G}_\eth = (g^{\eth}_{\beta\zeta})_{q\times q}$ is said to be ð-cutting matrix of $\mathcal{G}$, where

$$g^{\eth}_{\beta\zeta} = \begin{cases} 0 & \text{if} g_{\beta\zeta} < \eth \\ 1 & \text{if} g_{\beta\zeta} \geq \eth \end{cases}$$

$\beta, \zeta = 1, 2, 3, \ldots, q$ and ð is confidence level with $\eth \in [0, 1]$.

Now, we use these basic ideas for the construction of a novel clustering algorithm based on MPNSs given as algorithm 3. In the constructed numerical example of clustering analysis, we discuss algorithm 3 with more detail and clarity.

Algorithm 3 (Algorithm for clustering analysis using m-polar neutrosophic sets)

Input:

Step 1: Let $\{\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2}, ..., \mathcal{M}_{\mathfrak{N}_q}\}$ be an assembling of MPNSs over $\mathcal{Q}$ and $\{\eta_1, \eta_2, ..., \eta_r\}$ be the collection of attributes. Input MPN-data in tabular form to see the relationship between sets and attributes.

Calculations:

Step 2: Construct similarity matrix $\mathcal{G} = (g_{\beta\zeta})_{q\times q}$, where $g_{\beta\zeta} = \mathfrak{C}(\mathcal{M}_{\mathfrak{N}_\beta}, \mathcal{M}_{\mathfrak{N}_\zeta})$ and can be calculated as

$$\mathfrak{C}(\mathcal{M}_{\mathfrak{N}_\beta}, \mathcal{M}_{\mathfrak{N}_\zeta}) = 1 - \frac{1}{3m}\sum_{\alpha=1}^{m}\sum_{t=1}^{r}\wp_t(|^{\beta t}\mathfrak{A}_\alpha - {}^{\zeta t}\mathfrak{A}_\alpha| + |^{\beta t}\mathfrak{S}_\alpha - {}^{\zeta t}\mathfrak{S}_\alpha| + |^{\beta t}\mathfrak{Y}_\alpha - {}^{\zeta t}\mathfrak{Y}_\alpha|).$$

Step 3: Find $\mathcal{G}^2$ and check whether the similarity matrix satisfy $\mathcal{G}^2 \subseteq \mathcal{G}$. If it does not hold, then find the equivalence similarity matrix $\mathcal{G}^{2^\delta}$:

$$(\mathcal{G} \to \mathcal{G}^2 \to \mathcal{G}^4 \to ... \to \mathcal{G}^{2^\delta} \to ...) \quad \text{until,} \quad \mathcal{G}^{2^\delta} = \mathcal{G}^{2^{(\delta+1)}}.$$

Step 4: Find confidence level ð and construct a ð-cutting matrix $\mathcal{G}_\eth = (g^{\eth}_{\beta\zeta})_{q\times q}$ by using Definition 4.4.

Output:

Step 5: Classify the MPNSs by using the following argument:
If all the members of βth line (column) in $\mathcal{G}_\eth$ are same as the corresponding elements of ζth line (column) in $\mathcal{G}_\eth$, then MPNSs $\mathcal{M}_{\mathfrak{N}_\beta}$ and $\mathcal{M}_{\mathfrak{N}_\zeta}$ are of the same type, otherwise not.

Table 11 Characteristics of decision variables

Decision variables	Characteristics for 2-polar neutrosophic soft set
Intellectually curious	$\langle \text{creative}, \text{originality} \rangle$
Obedient and punctual	$\langle \text{hard} - \text{working}, \text{honest} \rangle$
Experience	$\langle \text{high}, \text{mediumhigh} \rangle$

Table 12 Linguistic terms for rating criteria for weight vector

Linguistic terms (LTs)	Fuzzy numbers
Good/G	$0.60 \leq x \leq 1$
Medium good/MG	$0.20 \leq x < 0.60$
Medium/M	$0.10 \leq x < 0.20$
Medium bad/MB	$0.05 \leq x < 0.10$
Bad/B	$0 \leq x < 0.05$

4.2.1 Numerical Example

Suppose that $\mathcal{Q} = \{\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2}, \mathcal{M}_{\mathfrak{N}_3}, \mathcal{M}_{\mathfrak{N}_4}, \mathcal{M}_{\mathfrak{N}_5}, \mathcal{M}_{\mathfrak{N}_6}, \mathcal{M}_{\mathfrak{N}_7}\}$ be the collection of seven students. They take an admission in a Science project learning academy for the preparation of a national competition on Science projects. Every student is evaluated on the basis of some important educational parameters, which are set according to the experts of that academy. To get fair assessment of these students, the evaluation committee establish the set of decision variables given as $\mathfrak{Z} = \{\eta_1, \eta_2, \eta_3\}$, where

$$\eta_1 = \text{Intellectually curious}, \eta_2 = \text{Obedient and punctual}, \eta_3 = \text{Experience}$$

Experts need to categorize the students according to these parameters and create their clustering corresponding to different sections of that academy. We subdivide these parameters into further criteria given as

- "Intellectually curious" student may be creative and give his original ideas.
- "Obedient and punctual" may be hard-working and honest.
- "Experience" means that some students have high or medium high experience.

In tabular form, this information can be seen as Table 11.

Some linguistic terms are defined to convert verbal description of experts about $\mathfrak{Z}$ into mathematical language given in Table 12.

Experts select the weight vector "$\wp$" for the strength of established decision variables as $\wp = (0.60, 0.25, 0.15)^T$. To clarify the differences of the opinion of experts and to cover the input data, we construct 2-polar neutrosophic sets given in Table 13. The flow chart diagram of proposed algorithm is given in Fig. 5.

Now, we calculate similarity measure $\mathfrak{C}$ between elements of Table 13 and construct similarity matrix.

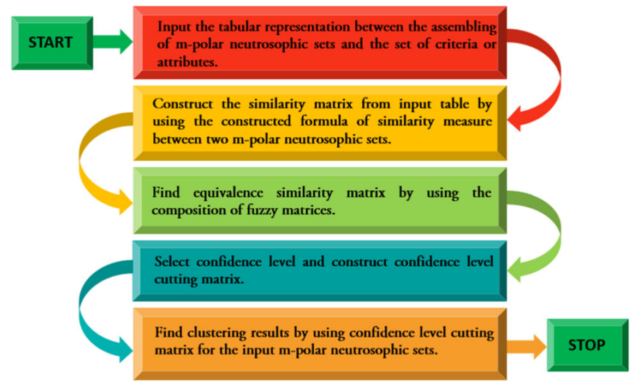

Fig. 5 Flow chart diagram of proposed algorithm 3 for clustering

Table 13 2-Polar neutrosophic input table

Students	η_1	η_2	η_3
$\mathcal{M}_{\mathfrak{N}_1}$	$(\langle 0.81, 0.21, 0.11\rangle, \langle 0.89, 0.23, 0.38\rangle)$	$(\langle 0.78, 0.32, 0.17\rangle, \langle 0.83, 0.21, 0.11\rangle)$	$(\langle 0.61, 0.42, 0.31\rangle, \langle 0.71, 0.31, 0.41\rangle)$
$\mathcal{M}_{\mathfrak{N}_2}$	$(\langle 0.73, 0.23, 0.18\rangle, \langle 0.79, 0.21, 0.31\rangle)$	$(\langle 0.79, 0.23, 0.14\rangle, \langle 0.81, 0.31, 0.21\rangle)$	$(\langle 0.83, 0.31, 0.18\rangle, \langle 0.73, 0.41, 0.37\rangle)$
$\mathcal{M}_{\mathfrak{N}_3}$	$(\langle 0.91, 0.11, 0.15\rangle, \langle 0.86, 0.31, 0.24\rangle)$	$(\langle 0.83, 0.21, 0.43\rangle, \langle 0.89, 0.21, 0.41\rangle)$	$(\langle 0.72, 0.43, 0.39\rangle, \langle 0.69, 0.41, 0.43\rangle)$
$\mathcal{M}_{\mathfrak{N}_4}$	$(\langle 0.74, 0.31, 0.44\rangle, \langle 0.79, 0.37, 0.28\rangle)$	$(\langle 0.79, 0.28, 0.32\rangle, \langle 0.73, 0.41, 0.28\rangle)$	$(\langle 0.81, 0.31, 0.21\rangle, \langle 0.83, 0.19, 0.22\rangle)$
$\mathcal{M}_{\mathfrak{N}_5}$	$(\langle 0.93, 0.11, 0.18\rangle, \langle 0.91, 0.12, 0.15\rangle)$	$(\langle 0.91, 0.21, 0.31\rangle, \langle 0.89, 0.15, 0.19\rangle)$	$(\langle 0.89, 0.21, 0.23\rangle, \langle 0.87, 0.23, 0.24\rangle)$
$\mathcal{M}_{\mathfrak{N}_6}$	$(\langle 0.78, 0.21, 0.37\rangle, \langle 0.75, 0.21, 0.41\rangle)$	$(\langle 0.82, 0.31, 0.34\rangle, \langle 0.79, 0.25, 0.42\rangle)$	$(\langle 0.88, 0.28, 0.23\rangle, \langle 0.75, 0.21, 0.15\rangle)$
$\mathcal{M}_{\mathfrak{N}_7}$	$(\langle 0.79, 0.28, 0.15\rangle, \langle 0.83, 0.15, 0.19\rangle)$	$(\langle 0.86, 0.23, 0.31\rangle, \langle 0.87, 0.13, 0.31\rangle)$	$(\langle 0.89, 0.31, 0.24\rangle, \langle 0.79, 0.28, 0.24\rangle)$

$$\mathcal{G}=\begin{pmatrix} 0.1000 & 0.9339 & 0.9100 & 0.8670 & 0.8863 & 0.9055 & 0.9092 \\ 0.9339 & 0.1000 & 0.8860 & 0.9130 & 0.8903 & 0.9207 & 0.9388 \\ 0.9100 & 0.8860 & 0.1000 & 0.8634 & 0.9145 & 0.8771 & 0.9100 \\ 0.8670 & 0.9130 & 0.8634 & 0.1000 & 0.8535 & 0.9204 & 0.8973 \\ 0.8863 & 0.8903 & 0.9145 & 0.8535 & 0.1000 & 0.8701 & 0.9354 \\ 0.9055 & 0.9207 & 0.8771 & 0.9204 & 0.8701 & 0.1000 & 0.9085 \\ 0.9092 & 0.9388 & 0.9100 & 0.8973 & 0.9354 & 0.9085 & 0.1000 \end{pmatrix}$$

$$\mathcal{G}^2=\begin{pmatrix} 0.1000 & 0.9339 & 0.9100 & 0.9130 & 0.9100 & 0.9207 & 0.9339 \\ 0.9339 & 0.1000 & 0.9100 & 0.9204 & 0.9354 & 0.9207 & 0.9388 \\ 0.9100 & 0.9100 & 0.1000 & 0.8973 & 0.9145 & 0.9100 & 0.9145 \\ 0.9130 & 0.9204 & 0.8973 & 0.1000 & 0.8973 & 0.9204 & 0.9130 \\ 0.9100 & 0.9354 & 0.9145 & 0.8973 & 0.1000 & 0.9085 & 0.9354 \\ 0.9207 & 0.9207 & 0.9100 & 0.9204 & 0.9085 & 0.1000 & 0.9207 \\ 0.9339 & 0.9388 & 0.9145 & 0.9130 & 0.9354 & 0.9207 & 0.1000 \end{pmatrix}$$

As $\mathcal{G}^2 \nsubseteq \mathcal{G}$, so we move towards the further calculations.

$$\mathcal{G}^4=\begin{pmatrix} 0.1000 & 0.9339 & 0.9145 & 0.9204 & 0.9339 & 0.9207 & 0.9339 \\ 0.9339 & 0.1000 & 0.9145 & 0.9204 & 0.9354 & 0.9207 & 0.9388 \\ 0.9145 & 0.9145 & 0.1000 & 0.9130 & 0.9145 & 0.9145 & 0.9145 \\ 0.9204 & 0.9204 & 0.9130 & 0.1000 & 0.9204 & 0.9204 & 0.9204 \\ 0.9339 & 0.9354 & 0.9145 & 0.9204 & 0.1000 & 0.9207 & 0.9354 \\ 0.9207 & 0.9207 & 0.9145 & 0.9204 & 0.9207 & 0.1000 & 0.9388 \\ 0.9339 & 0.9388 & 0.9145 & 0.9204 & 0.9354 & 0.9388 & 0.1000 \end{pmatrix}$$

$$\mathcal{G}^8=\begin{pmatrix} 0.1000 & 0.9339 & 0.9145 & 0.9204 & 0.9339 & 0.9339 & 0.9339 \\ 0.9339 & 0.1000 & 0.9145 & 0.9204 & 0.9354 & 0.9339 & 0.9388 \\ 0.9145 & 0.9145 & 0.1000 & 0.9145 & 0.9145 & 0.9145 & 0.9145 \\ 0.9204 & 0.9204 & 0.9145 & 0.1000 & 0.9204 & 0.9204 & 0.9204 \\ 0.9339 & 0.9354 & 0.9145 & 0.9204 & 0.1000 & 0.9354 & 0.9354 \\ 0.9339 & 0.9339 & 0.9145 & 0.9204 & 0.9354 & 0.1000 & 0.9388 \\ 0.9339 & 0.9388 & 0.9145 & 0.9204 & 0.9354 & 0.9388 & 0.1000 \end{pmatrix}$$

$$\mathcal{G}^{16}=\begin{pmatrix} 0.1000 & 0.9339 & 0.9145 & 0.9204 & 0.9339 & 0.9339 & 0.9339 \\ 0.9339 & 0.1000 & 0.9145 & 0.9204 & 0.9354 & 0.9339 & 0.9388 \\ 0.9145 & 0.9145 & 0.1000 & 0.9145 & 0.9145 & 0.9145 & 0.9145 \\ 0.9204 & 0.9204 & 0.9145 & 0.1000 & 0.9204 & 0.9204 & 0.9204 \\ 0.9339 & 0.9354 & 0.9145 & 0.9204 & 0.1000 & 0.9354 & 0.9354 \\ 0.9339 & 0.9388 & 0.9145 & 0.9204 & 0.9354 & 0.1000 & 0.9388 \\ 0.9339 & 0.9388 & 0.9145 & 0.9204 & 0.9354 & 0.9388 & 0.1000 \end{pmatrix}$$

$$\mathcal{G}^{32}=\begin{pmatrix} 0.1000 & 0.9339 & 0.9145 & 0.9204 & 0.9339 & 0.9339 & 0.9339 \\ 0.9339 & 0.1000 & 0.9145 & 0.9204 & 0.9354 & 0.9339 & 0.9388 \\ 0.9145 & 0.9145 & 0.1000 & 0.9145 & 0.9145 & 0.9145 & 0.9145 \\ 0.9204 & 0.9204 & 0.9145 & 0.1000 & 0.9204 & 0.9204 & 0.9204 \\ 0.9339 & 0.9354 & 0.9145 & 0.9204 & 0.1000 & 0.9354 & 0.9354 \\ 0.9339 & 0.9388 & 0.9145 & 0.9204 & 0.9354 & 0.1000 & 0.9388 \\ 0.9339 & 0.9388 & 0.9145 & 0.9204 & 0.9354 & 0.9388 & 0.1000 \end{pmatrix}$$

It is clear that $\mathcal{G}^{32} = \mathcal{G}^{16} \circ \mathcal{G}^{16} = \mathcal{G}^{16}$ is an equivalence similarity matrix. Since the confidence level ð has a strong connection with the elements of the equivalence similarity matrix. For ð we construct ð-cutting matrix $\mathcal{G}_ð$. Different ð produces different $\mathcal{G}_ð$ and different clustering for the universal set $\mathcal{Q}$. For different values of ð different clustering results are given in Table 14.

The clustering effect diagram for different ð-cutting of seven students can be seen in Fig. 6. This means that by utilizing this novel algorithm experts of academy can easily classify the students corresponding to different sections of the academy according to their ability. All the clustering depend upon the parameter ð, which is confidence level and selected according to the opinions and suggestions of experts.

4.2.2 Comparison

Now, we compare our proposed method with some existing approaches and we see that our proposed approach has the following advantages.

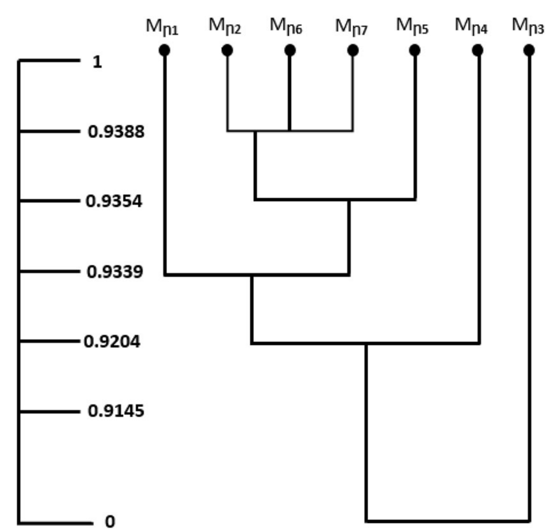

Fig. 6 The clustering effect diagram of seven students

Table 14 The clustering results of seven students

Confidence level ð	Clusters
$0.9388 < ð \leq 1$	$\{\mathcal{M}_{\mathfrak{N}_1}\}, \{\mathcal{M}_{\mathfrak{N}_2}\}, \{\mathcal{M}_{\mathfrak{N}_3}\}, \{\mathcal{M}_{\mathfrak{N}_4}\}, \{\mathcal{M}_{\mathfrak{N}_5}\}, \{\mathcal{M}_{\mathfrak{N}_6}\}, \{\mathcal{M}_{\mathfrak{N}_7}\}$
$0.9354 < ð \leq 0.9388$	$\{\mathcal{M}_{\mathfrak{N}_1}\}, \{\mathcal{M}_{\mathfrak{N}_2}, \mathcal{M}_{\mathfrak{N}_6}, \mathcal{M}_{\mathfrak{N}_7}\}, \{\mathcal{M}_{\mathfrak{N}_3}\}, \{\mathcal{M}_{\mathfrak{N}_4}\}, \{\mathcal{M}_{\mathfrak{N}_5}\}$
$0.9339 < ð \leq 0.9354$	$\{\mathcal{M}_{\mathfrak{N}_1}\}, \{\mathcal{M}_{\mathfrak{N}_2}, \mathcal{M}_{\mathfrak{N}_5}, \mathcal{M}_{\mathfrak{N}_6}, \mathcal{M}_{\mathfrak{N}_7}\}, \{\mathcal{M}_{\mathfrak{N}_3}\}, \{\mathcal{M}_{\mathfrak{N}_4}\}$
$0.9204 < ð \leq 0.9339$	$\{\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2}, \mathcal{M}_{\mathfrak{N}_5}, \mathcal{M}_{\mathfrak{N}_6}, \mathcal{M}_{\mathfrak{N}_7}\}, \{\mathcal{M}_{\mathfrak{N}_3}\}, \{\mathcal{M}_{\mathfrak{N}_4}\}$
$0.9145 < ð \leq 0.9204$	$\{\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2}, \mathcal{M}_{\mathfrak{N}_4}, \mathcal{M}_{\mathfrak{N}_5}, \mathcal{M}_{\mathfrak{N}_6}, \mathcal{M}_{\mathfrak{N}_7}\}, \{\mathcal{M}_{\mathfrak{N}_3}\}$
$0 \leq ð \leq 0.9145$	$\{\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2}, \mathcal{M}_{\mathfrak{N}_3}, \mathcal{M}_{\mathfrak{N}_4}, \mathcal{M}_{\mathfrak{N}_5}, \mathcal{M}_{\mathfrak{N}_6}, \mathcal{M}_{\mathfrak{N}_7}\}$,

Table 15 Comparison of proposed approach with the existing methodologies

Authors	Set	Truth grade	Indeterminacy grade	Falsity grade	Multi-polarity	Loss of information
Xu et al. [41]	IFS	✓	×	✓	×	×
Zhang et al. [54]	IFS	✓	×	✓	×	✓
Peng et al. [22]	PFS	✓	×	✓	×	×
Proposed approach	MPNS	✓	✓	✓	✓	×

(1) By using the methods of Xu et al. [41] and Zhang et al. [54], we cannot handle the multi-polar input data and cannot deal with the indeterminacy part of the alternatives. They used intuitionistic fuzzy sets (IFSs) for the clustering of input data. In our proposed approach, we deal our clustering with the multiple data with the truth, indeterminacy, and falsity part of the alternatives. So, our method is more efficient and deal with numerous applications having multiple data.

(2) Peng et al. [22] presented the clustering idea on Pythagorean fuzzy sets (PFSs). They increased the domain of Xu et al. [41] and Zhang et al. [54] approaches, but they cannot handle the multi-polar input data and cannot deal with the indeterminacy part of the alternatives. Our proposed method removes these restrictions and can easily handle multi-criteria decision-making problems.

(3) According to Peng et al. [22] research idea, Zhang et al. [54] produced the loss of too much information in the data during the calculation by using intuitionistic fuzzy similarity degrees. This loss effects upon the final result of clustering. Our proposed approach does not lose any input data during the calculations and produces accurate and appropriate results. This comparison is given in tabular form in Table 15.

5 Conclusion

Decision analysis has been intensively examined by numerous scholars and researchers. The techniques developed for this task mainly depend on the type of decision problem under consideration. Most of its relating issues are associated with uncertain, imprecise and multi-polar information, which cannot be tackled properly through fuzzy set. To overcome this particular deficiency of fuzzy sets, Chen et al. [5] have proposed the concept of *m*-polar fuzzy set (MPFS) in 2014, which has the capability to deal with the data having vagueness and uncertainty under multi-polar information. Neutrosophic set deals with the MCDM methods having truth, falsity, and indeterminacy grades for the corresponding alternatives. In this manuscript, we have established the idea of *m*-polar neutrosophic set (MPNS) by combining the two independent concepts of *m*-polar fuzzy set and neutrosophic set. We have established the notion of *m*-polar neutrosophic topology and defined interior, closure, exterior, and frontier in the context of MPNSs with the help of illustrations. We have presented cosine similarity measure and set theoretic similarity measure to find the similarity between MPNSs. Three novel algorithms for multi-criteria decision-making (MCDM) with linguistic information have been developed on the basis of MPNTS, similarity measures, and clustering analysis. Furthermore, we have presented advantages, simplicity, flexibility, and validity of the proposed algorithms. We have discussed and compared our results with some existing methodologies.

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Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

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