MBJ-neutrosophic filters of equality algebras

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Abstract

In this paper, we introduce the notion of MBJ-neutrosophic sub-algebra and MBJ-neutrosophic filter on equality algebras and investigate some equivalence definitions, properties and the relation between them. Also, by using the notion of MBJ-neutrosophic filter, we introduce a congruence relation on equality algebra and show that the quotient is an equality algebra.

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1   Introduction

Non-classical logic has become a considerable formal tool for computer science and artificial intelligence to deal with fuzzy information and uncertainty information. Many-valued logic, a great extension and development of classical logic, has always been a crucial direction in non-classical logic. A crucial question for every many-valued logic is, what should be structure of its truth values. It is generally accepted that in fuzzy logic, it should be a residuated lattice, possibly fulfilling some additional properties. On the basis of that, we may now distinguish various kinds of formal fuzzy logics. Most important among them seem to be BL-, MTL-, IMTL- logics. The answer to the above question is positive and the fuzzy type theory (FTT) has indeed been introduced in [20]. However, the basic connective in FTT is a fuzzy equality since it is developed as a generalization of the elegant classical formal system originated by Henkin (see [7]). So Novák in [21] introduced a special algebra that is called EQ-algebra and that reflects directly the syntax of FTT. Viewing the

axioms of EQ-algebras with a purely algebraic eye it appears that unlike in the case of residuated lattices where the adjointness condition ties product with implication, the product in EQ-algebras is quite loosely related to the other connectives. For instance, a moments reflection shows that one can replace the product of an EQ-algebra by any other binary operation which is smaller or equal than the original product (viewed as a two-place function) and still obtains an EQ-algebra. However, the huge freedom in choosing the product might prohibit to find deep related algebraic results, hence our aim was to find something similar to EQ-algebras but without a product: an axiomatic treatment of equality/equivalence. Becuase of that Jeni in [10] introduced a new structure, called equality algebras. It has two connectives, a meet operation and an equivalence, and a constant 1.

Neutrosophic set theory was introduced by Smarandache in 1998 ([23]). Neutrosophic sets are a new mathematical tool for dealing with uncertainties which are free from many difficulties that have troubled the usual theoretical approaches. Research works on neutrosophic set theory for many applications such as information fussion, probability theory, control theory, decision making, measurement theory, etc. Kandasamy and Smarandache introduced the concept of neutrosophic algebraic structures ([15, 16]). Since then many researchers worked in this area and lots of literatures had been produced about the theory of neutrosophic set. The notion of MBJ-neutrosophic sets is introduced as another generalization of neutrosophic set, it is applied to BCK/BCI-algebras. Mohseni et al.[19] introduced the concept of MBJ-neutrosophic subalgebras in BCK/BCI-algebras, and investigated related properties. They gave a characterization of MBJ-neutrosophic subalgebra, and established a new MBJ-neutrosophic subalgebra by using an MBJ-neutrosophic subalgebra of a BCI-algebra. They considered the homomorphic inverse image of MBJ-neutrosophic subalgebra, and discussed translation of MBJ-neutrosophic subalgebra.

In this paper, the notion of MBJ-neutrosophic filter on equality algebras is introduced and investigated some equivalence definitions and some properties of it. Also, by using the notion of MBJ-neutrosophic filter, a congruence relation on equality algebra is introduced.

2   Preliminaries

In this section, we give some basic definitions and results of equality algebras which will be used in this paper.

Definition 2.1. [9] *An algebraic structure* $(X, \wedge, \sim, 1)$ *is called equality algebra, if for any* $a, b, c \in X$, *it satisfies the following conditions.*

(E1) $(X, \wedge, 1)$ *is a commutative idempotent integral monoid,*

(E2) *the operation "$\sim$" is commutative,*

(E3) $a \sim a = 1$,

(E4) $a \sim 1 = a$,

(E5) *if* $a \leq b \leq c$, *then* $a \sim c \leq b \sim c$ *and* $a \sim c \leq a \sim b$,

(E6) $a \sim b \leq (a \wedge c) \sim (b \wedge c)$,

(E7) $a \sim b \leq (a \sim c) \sim (b \sim c)$,

where $a \leq b$ *if and only if* $a \wedge b = a$.

In an equality algebra $X = (X, \wedge, \sim, 1)$, we define two operations "$\rightarrow$" and "$\leftrightarrow$" on X as follows:

$$a \rightarrow b := a \sim (a \wedge b) \quad \text{and} \quad a \leftrightarrow b := (a \rightarrow b) \wedge (b \rightarrow a).$$

Proposition 2.2. [9] *Let $X := (X, \wedge, \sim, 1)$ be an equality algebra. Then for any $a, b, c \in X$, the following assertions are valid:*
(i) $a \rightarrow b = 1$ if and only if $a \leq b$,
(ii) $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c)$,
(iii) $1 \rightarrow a = a$, $a \rightarrow 1 = 1$ and $a \rightarrow a = 1$,
(iv) $a \leq b \rightarrow c$ if and only if $b \leq a \rightarrow c$,
(v) $a \leq b \rightarrow a$,
(vi) $a \leq (a \rightarrow b) \rightarrow b$,
(vii) $a \rightarrow b \leq (b \rightarrow c) \rightarrow (a \rightarrow c)$,
(viii) If $b \leq a$, then $a \leftrightarrow b = a \rightarrow b = a \sim b$,
(ix) $a \sim b \leq a$ if and only if $b \leq a \rightarrow b$,
(x) If $a \leq b$, then $b \rightarrow c \leq a \rightarrow c$ and $c \rightarrow a \leq c \rightarrow b$.

An equality algebra $X := (X, \wedge, \sim, 1)$ is said to be *bounded* if there exists an element $0 \in X$ such that $0 \leq a$, for all $a \in X$. In a bounded equality algebra $X := (X, \wedge, \sim, 1)$, we define the negation "$\neg$" on X by $\neg a = a \rightarrow 0 = a \sim 0$ for all $a \in X$.

A subset F of X is called a *deductive system* (or *filter*) of X (see [10]) if for any $a, b \in X$, it satisfies:
(F_1) $1 \in F$,
(F_2) if $a \in F$ and $a \leq b$, then $b \in F$,
(F_3) if $a \in F$ and $a \sim b \in F$, then $b \in F$.
The set of all deductive systems of X is denoted by $\mathcal{DS}(X)$.

Lemma 2.3. [8] *Let X be an equality algebra. A subset F of X is called a deductive system of X if and only if it satisfies (F_1) and for any $a, b \in X$,*
(F) *if $a \in F$ and $a \rightarrow b \in F$, then $b \in F$.*

Definition 2.4. [27] *An equality algebra $X = (X, \wedge, \sim, 1)$ is called a commutative if for any $a, b \in X$, $(a \rightarrow b) \rightarrow b = (b \rightarrow a) \rightarrow a$.*

Definition 2.5. [1] *An equality algebra $X = (X, \wedge, \sim, 1)$ is called a positive implicative if for any $a, b, c \in X$, $a \rightarrow (b \rightarrow c) = (a \rightarrow b) \rightarrow (a \rightarrow c)$.*

By an *interval number* we mean a closed subinterval $\tilde{a} = [a^-, a^+]$ of $I = [0, 1]$, where $0 \leq a^- \leq a^+ \leq 1$. Denote by $[I]$ the set of all interval numbers. Let us define what is known as *refined minimum* (briefly, rmin) and *refined maximum* (briefly, rmax) of two elements in $[I]$. We also define the symbols "$\succeq$", "$\preceq$", "$=$" in case of two elements in $[I]$. Consider two interval numbers $\tilde{a}_1 := [a_1^-, a_1^+]$ and $\tilde{a}_2 := [a_2^-, a_2^+]$. Then

$$\text{rmin}\{\tilde{a}_1, \tilde{a}_2\} = \left[\min\{a_1^-, a_2^-\}, \min\{a_1^+, a_2^+\}\right],$$
$$\text{rmax}\{\tilde{a}_1, \tilde{a}_2\} = \left[\max\{a_1^-, a_2^-\}, \max\{a_1^+, a_2^+\}\right],$$
$$\tilde{a}_1 \succeq \tilde{a}_2 \Leftrightarrow a_1^- \geq a_2^-, \ a_1^+ \geq a_2^+,$$

and similarly we may have $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 = \tilde{a}_2$. To say $\tilde{a}_1 \succ \tilde{a}_2$ (resp. $\tilde{a}_1 \prec \tilde{a}_2$) we mean $\tilde{a}_1 \succeq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$ (resp. $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$). Let $\tilde{a}_i \in [I]$ where $i \in \Lambda$. We define

$$\text{rinf}_{i \in \Lambda} \tilde{a}_i = \left[\inf_{i \in \Lambda} a_i^-, \inf_{i \in \Lambda} a_i^+\right] \quad \text{and} \quad \text{rsup}_{i \in \Lambda} \tilde{a}_i = \left[\sup_{i \in \Lambda} a_i^-, \sup_{i \in \Lambda} a_i^+\right].$$

Let X be a nonempty set. A function $A : X \to [I]$ is called an *interval-valued fuzzy set* (briefly, an *IVF set*) in X. Let $[I]^X$ stand for the set of all IVF sets in X. For every $A \in [I]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the *degree* of membership of an element x to A, where $A^- : X \to I$ and $A^+ : X \to I$ are fuzzy sets in X which are called a *lower fuzzy set* and an *upper fuzzy set* in X, respectively. For simplicity, we denote $A = [A^-, A^+]$.

Let X be a non-empty set. A *neutrosophic set* (NS) in X (see [24]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x) \rangle \mid x \in X\},$$

where $A_T : X \to [0, 1]$ is a *truth membership function*, $A_I : X \to [0, 1]$ is an *indeterminate membership function*, and $A_F : X \to [0, 1]$ is a *false membership function*.

Let X be a non-empty set. By an *MBJ-neutrosophic set* in X (see [19]), we mean a structure of the form:

$$\mathcal{A} := \{\langle x; M_A(x), \tilde{B}_A(x), J_A(x) \rangle \mid x \in X\},$$

where M_A and J_A are fuzzy sets in X, which are called a truth membership function and a false membership function, respectively, and $\tilde{B}_A$ is an IVF set in X which is called an *indeterminate interval-valued membership function*.

We use the symbol $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ for the MBJ-neutrosophic set

$$\mathcal{A} := \{\langle x; M_A(x), \tilde{B}_A(x), J_A(x) \rangle \mid x \in X\}.$$

Note. From now on, we let $X = (X, \wedge, \sim, 1)$ or X as an equality algebra unless otherwise state.

3 MBJ-neutrosophic sub-algberas and filters

In this section we define the notion of MBJ-neutrosophic sub-algebra and MBJ-neutrosophic filter of X and investigate some equivalence definitions and some propoerties of them.
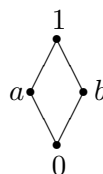
Definition 3.1. *An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is called an MBJ-neutrosophic sub-algebra of X if it satisfies*

$$(\forall x, y, z \in X) \begin{pmatrix} M_A(x \sim y) \geq \min\{M_A(x), M_A(y)\} \\ \tilde{B}_A(x \sim y) \succeq \mathrm{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\} \\ J_A(x \sim y) \leq \max\{J_A(x), J_A(y)\} \end{pmatrix}, \tag{3.1}$$

and

$$(\forall x, y, z \in X) \begin{pmatrix} M_A(x \wedge y) \geq \min\{M_A(x), M_A(y)\} \\ \tilde{B}_A(x \wedge y) \succeq \mathrm{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\} \\ J_A(x \wedge y) \leq \max\{J_A(x), J_A(y)\} \end{pmatrix}. \tag{3.2}$$

Example 3.2. *Let $X = \{0, a, b, 1\}$ be a set with the following Hasse diagram.*

*Then $(X, \wedge, 1)$ is a commutative idempotent integral monoid. We define a binary operation $\sim$
on X by Table 1.*

$\sim$	0	a	b	1
0	1	b	a	0
a	b	1	0	a
b	a	0	1	b
1	0	a	b	1

Table 1: Cayley table for the implication "$\sim$"

Then $\mathcal{E} = (X, \wedge, \sim, 1)$ is an equality algebra. Let define $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X as Table 2.

X	M_A	$\tilde{B}_A$	J_A
0	0.4	$[0.4, 0.5]$	0.5
a	0.5	$[0.3, 0.45]$	0.5
b	0.4	$[0.3, 0.45]$	0.5
1	0.7	$[0.7, 0.8]$	0.2

Table 2: Cayley table for the implication "$\mathcal{A}$"

*By routine calculation, we can see that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic sub-algebra
of X.*

Proposition 3.3. *If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic sub-algebra of X, then $M_A(1) \geq
M_A(x), \tilde{B}_A(1) \succeq \tilde{B}_A(x)$ and $J_A(1) \leq J_A(x)$, for all $x \in X$.*

Proof. For any $x \in X$, we get

$$M_A(1) = M_A(x \sim x) \geq \min\{M_A(x), M_A(x)\} = M_A(x),$$
$$\tilde{B}_A(1) = \tilde{B}_A(x \sim x) \succeq \mathrm{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x)\} = \tilde{B}_A(x),$$

and

$$J_A(1) = J_A(x \sim x) \leq \max\{J_A(x), J_A(x)\} = J_A(x).$$

$\square$

In the following example we show that the converse of above proposition may not be true, in
general.

Example 3.4. *Let X be the equality algebra as Example 3.2. Define $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X as
Table 3.*

X	M_A	$\tilde{B}_A$	J_A
0	0.4	$[0.2, 0.3]$	0.5
a	0.5	$[0.4, 0.5]$	0.5
b	0.4	$[0.5, 0.6]$	0.5
1	0.7	$[0.7, 0.8]$	0.2

Table 3: Cayley table for the implication "$\mathcal{A}$"

It is clear that for all $x \in X$, $M_A(1) \geq M_A(x)$, $\tilde{B}_A(1) \succeq \tilde{B}_A(x)$ and $J_A(1) \leq J_A(x)$. But $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is not an MBJ-neutrosophic sub-algebra of X, because

$$[0.2, 0.3] = \tilde{B}_A(0) = \tilde{B}_A(a \wedge b) \nsucceq \mathrm{rmin}\{\tilde{B}_A(a), \tilde{B}_A(b)\} = \tilde{B}_A(a) = [0.4, 0.5].$$

Theorem 3.5. *Let* $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ *be an MBJ-neutrosophic sub-algebra of* X. *Then the following staements hold:*

$$(\forall x, y \in X) \left(\begin{array}{c} M_A(x \to y) \geq \min\{M_A(x), M_A(y)\} \\ \tilde{B}_A(x \to y) \succeq \mathrm{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\} \\ J_A(x \to y) \leq \max\{J_A(x), J_A(y)\} \end{array} \right). \tag{3.3}$$

Proof. Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic sub-algebra of X and $x, y \in X$. Then by Definition 3.1, we have

$$M_A(x \to y) = M_A(x \sim (x \wedge y)) \geq \min\{M_A(x), M_A(x \wedge y)\} \geq \min\{M_A(x), M_A(y)\},$$
$$\tilde{B}_A(x \to y) = \tilde{B}_A(x \sim (x \wedge y)) \succeq \mathrm{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x \wedge y)\} \succeq \mathrm{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\},$$
$$J_A(x \to y) = J_A(x \sim (x \wedge y)) \leq \max\{J_A(x), J_A(x \wedge y)\} \leq \max\{J_A(x), J_A(y)\}.$$

Hence (3.3) holds. $\square$

In the following example we show that the converse of Theorem 3.5 may not be true, in general.

Example 3.6. *Let* X *be the equality algebra as Example 3.2. It is clear that the table of implication is as Table 4:*

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Table 4: Cayley table for the implication "$\to$"

Define $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ *in* X *as Table 5.*

X	M_A	$\tilde{B}_A$	J_A
0	0.3	$[0.4, 0.5]$	0.5
a	0.5	$[0.3, 0.45]$	0.5
b	0.4	$[0.3, 0.45]$	0.5
1	0.7	$[0.7, 0.8]$	0.2

Table 5: Cayley table for the implication "$\mathcal{A}$"

Routine calculation show that the condition (3.3) *holds but* $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ *is not an MBJ-neutrosophic sub-algebra of* X, *because*

$$0.3 = M_A(0) = M_A(a \wedge b) \ngeq \min\{M_A(a), M_A(b)\} = M_A(b) = 0.4.$$

Theorem 3.7. *Let X be a linearly order equality algebra and $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X which satisfies in condition (3.3). Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic sub-algebra in X.*

Proof. Let $x, y \in X$ and $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X. Since X is a linearly order equality algebra, we get $x \leq y$ or $y \leq x$. Suppose $x \leq y$. Then by Proposition 2.2(viii) and (E2), we have $y \to x = y \sim x = x \sim y$. Thus

$$M_A(x \sim y) = M_A(y \to x) \geq \min\{M_A(x), M_A(y)\},$$
$$\tilde{B}_A(x \sim y) = \tilde{B}_A(y \to x) \succeq \mathrm{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\},$$
$$J_A(x \sim y) = J_A(y \to x) \leq \max\{J_A(x), J_A(y)\}.$$

Moreover, since $x \leq y$, we have $x \wedge y = x$ and it is clear that

$$M_A(x \wedge y) \geq \min\{M_A(x), M_A(y)\},$$
$$\tilde{B}_A(x \wedge y) \succeq \mathrm{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\},$$
$$J_A(x \wedge y) \leq \max\{J_A(x), J_A(y)\}.$$

Therefore, $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic sub-algebra in X. $\qquad\square$

In the following we introduce the notion of MBJ-neutrosophic filters of equality algebras.

Definition 3.8. *An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is called an MBJ-neutrosophic filter of X if it satisfies*

$$(\forall x \in X) \left(M_A(x) \leq M_A(1),\ \tilde{B}_A(x) \preceq \tilde{B}_A(1),\ J_A(x) \geq J_A(1) \right), \tag{3.4}$$

$$(\forall x, y, z \in X) \left(\begin{array}{l} M_A(y) \geq \min\{M_A(x), M_A(x \sim y)\} \\ \tilde{B}_A(y) \succeq \mathrm{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x \sim y)\} \\ J_A(y) \leq \max\{J_A(x), J_A(x \sim y)\} \end{array} \right), \tag{3.5}$$

and

$$(\forall x, y \in X) \left(x \leq y \ \Rightarrow \left\{ \begin{array}{l} M_A(y) \geq M_A(x) \\ \tilde{B}_A(y) \succeq \tilde{B}_A(x) \\ J_A(y) \leq J_A(x) \end{array} \right. \right). \tag{3.6}$$

Example 3.9. *Let X be the equality algebra as Example 3.2. Define $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X as Table 6.*

X	M_A	$\tilde{B}_A$	J_A
0	0.4	$[0.3, 0.4]$	0.6
a	0.5	$[0.4, 0.5]$	0.4
b	0.4	$[0.4, 0.5]$	0.5
1	0.7	$[0.7, 0.8]$	0.2

Table 6: Cayley table for the implication "$\mathcal{A}$"

Then routine calculations show that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic filter in X.

Now, in the follow, we discuss characterizations of an MBJ-neutrosophic filter;

Theorem 3.10. *Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X. Then $\mathcal{A}$ is an MBJ-neutrosophic filter of X if and only if it satisfies (3.4) and the following condition:*

$$(\forall x, y \in X) \left(\begin{array}{l} M_A(y) \geq \min\{M_A(x), M_A(x \to y)\} \\ \tilde{B}_A(y) \succeq \operatorname{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x \to y)\} \\ J_A(y) \leq \max\{J_A(x), J_A(x \to y)\} \end{array} \right). \tag{3.7}$$

Proof. Assume that $\mathcal{A}$ is an MBJ-neutrosophic filter of X. It is clear that the condition (3.4) holds. Let $x, y \in X$. By Definition 3.8, for all $x, y \in X$, we get

$$\min\{M_A(x), M_A(x \to y)\} = \min\{M_A(x), M_A(x \sim (x \wedge y))\} \leq M_A(x \wedge y) \leq M_A(y),$$
$$\operatorname{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x \to y)\} = \operatorname{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x \sim (x \wedge y))\} \preceq \tilde{B}_A(x \wedge y) \preceq \tilde{B}_A(y),$$
$$\max\{J_A(x), J_A(x \to y)\} = \max\{J_A(x), J_A(x \sim (x \wedge y))\} \geq J_A(x \wedge y) \geq J_A(y).$$

Hence, for all $x, y \in X$, we have

$$M_A(y) \geq \min\{M_A(x), M_A(x \to y)\},$$
$$\tilde{B}_A(y) \succeq \operatorname{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x \to y)\},$$
$$J_A(y) \leq \max\{J_A(x), J_A(x \to y)\}.$$

Conversely, suppose that $\mathcal{A}$ satisfies in (3.4) and (3.7). Let $x, y \in X$. If $x \leq y$, then $x \to y = 1$ and by (3.4) and (3.7), we get

$$M_A(x) = \min\{M_A(x), M_A(1)\} = \min\{M_A(x), M_A(x \to y)\} \leq M_A(y),$$
$$\tilde{B}_A(x) = \operatorname{rmin}\{\tilde{B}_A(x), \tilde{B}_A(1)\} = \operatorname{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x \to y)\} \preceq \tilde{B}_A(y),$$
$$J_A(x) = \max\{J_A(x), J_A(1)\} = \max\{J_A(x), J_A(x \to y)\} \geq J_A(y),$$

and so for all $x, y \in X$, $M_A(y) \geq M_A(x), \tilde{B}_A(y) \succeq \tilde{B}_A(x)$ and $J_A(y) \leq J_A(x)$. By (3.7) and by Proposition 2.2(ix), we have $M_A(x \sim y) \leq M_A(x \to y), \tilde{B}_A(x \sim y) \preceq \tilde{B}_A(x \to y)$ and $J_A(x \sim y) \geq J_A(x \to y)$, and so

$$\min\{M_A(x), M_A(x \sim y)\} \leq \min\{M_A(x), M_A(x \to y)\} \leq M_A(y),$$
$$\operatorname{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x \sim y)\} \preceq \operatorname{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x \to y)\} \preceq \tilde{B}_A(y),$$
$$\max\{J_A(x), J_A(x \sim y)\} \geq \max\{J_A(x), J_A(x \to y)\} \geq J_A(y).$$

Therefore, $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic filter of X. $\qquad\square$

In the following example we show that every MBJ-neutrosophic filter in X is not an MBJ-neutrosophic sub-algebra, in general.

Example 3.11. *Let X be the equality algebra as Example 3.2 and $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic filter of X as Example 3.9. It is clear that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is not an MBJ-neutrosophic sub-algebra of X, because*

$$[0.3, 0.4] = \tilde{B}_A(0) = \tilde{B}_A(a \sim b) \nsucceq \operatorname{rmin}\{\tilde{B}_A(a), \tilde{B}_A(b)\} = \tilde{B}_A(a) = [0.4, 0.5].$$

Theorem 3.12. *In a linearly order equality algebra, every MBJ-neutrosophic filter is an MBJ-neutrosophic sub-algebra.*

Proof. Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic filter of X. Since for all $x, y \in X$, we have $y \leq x \to y$, then by (3.6), $M_A(y) \leq M_A(x \to y), \tilde{B}_A(y) \preceq \tilde{B}_A(x \to y)$ and $J_A(y) \geq J_A(x \to y)$. It follows from the assumption that for all $x, y \in X$

$$M_A(x \to y) \geq M_A(y) \geq \min\{M_A(x), M_A(x \to y)\} \geq \min\{M_A(x), M_A(y)\},$$
$$\tilde{B}_A(x \to y) \succeq \tilde{B}_A(y) \succeq \text{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x \to y)\} \succeq \text{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\},$$
$$J_A(x \to y) \leq J_A(y) \leq \max\{J_A(x), J_A(x \to y)\} =\leq \max\{J_A(x), J_A(y)\}.$$

Hence by Theorem 3.7, $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic sub-algebra of X. □

Theorem 3.13. *An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is an MBJ-neutrosophic filter of X if and only if it satisfies:*

$$(\forall x, y, z \in X) \left(x \leq y \to z \Rightarrow \left\{ \begin{array}{l} M_A(z) \geq \min\{M_A(x), M_A(y)\} \\ \tilde{B}_A(z) \succeq \text{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\} \\ J_A(z) \leq \max\{J_A(x), J_A(y)\} \end{array} \right. \right). \qquad (3.8)$$

Proof. Suppose $\mathcal{A}$ is an MBJ-neutrosophic filter of X. Let $x, y, z \in X$ such that $x \leq y \to z$. Then $M_A(x) \leq M_A(y \to z), \tilde{B}_A(x) \preceq \tilde{B}_A(y \to z)$ and $J_A(x) \geq J_A(y \to z)$. Thus

$$\min\{M_A(y), M_A(x)\} \leq \min\{M_A(y), M_A(y \to z)\} \leq M_A(z),$$
$$\text{rmin}\{\tilde{B}_A(y), \tilde{B}_A(x)\} \preceq \text{rmin}\{\tilde{B}_A(y), \tilde{B}_A(y \to z)\} \preceq \tilde{B}_A(z),$$
$$\max\{J_A(y), J_A(x)\} \geq \max\{J_A(y), J_A(y \to z)\} \geq J_A(z).$$

Conversely, suppose that $\mathcal{A}$ satisfies in (3.8). Since $x \leq 1$ and for any $x \in X, 1 = x \to 1$, we have $x \leq x \to 1$. Then by (3.8) we have,

$$M_A(1) \geq \min\{M_A(x), M_A(x)\} = M_A(x),$$
$$\tilde{B}_A(1) \succeq \text{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x)\} = \tilde{B}_A(x),$$
$$J_A(1) \leq \max\{J_A(x), J_A(x)\} = J_A(x).$$

Since for any $x, y \in X, x \to y \leq x \to y$, we have

$$\min\{M_A(x), M_A(x \to y)\} \leq M_A(y),$$
$$\text{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x \to y)\} \preceq \tilde{B}_A(y),$$
$$\max\{J_A(x), J_A(x \to y)\} \geq J_A(y).$$

Hence by Theorem 3.10, $\mathcal{A}$ is an MBJ-neutrosophic filter of X. □

Proposition 3.14. *Let X be a positive implicative equality algebra and $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set of X. Then for any $x, y, z \in X$, the following statements are equivalent:*
(i) $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic filter of X,
(ii)

$$\min\{M_A(x \to (y \to z)), M_A(x \to y)\} \leq M_A(x \to z),$$
$$\text{rmin}\{\tilde{B}_A(x \to (y \to z)), \tilde{B}_A(x \to y)\} \preceq \tilde{B}_A(x \to z),$$
$$\max\{J_A(x \to (y \to z)), J_A(x \to y)\} \geq J_A(x \to z),$$

(iii)

$$\min\{M_A(z \to (y \to (y \to x))), M_A(z)\} \leq M_A(y \to x),$$
$$\text{rmin}\{\tilde{B}_A(z \to (y \to (y \to x))), \tilde{B}_A(z)\} \preceq \tilde{B}_A(y \to x),$$
$$\max\{J_A(z \to (y \to (y \to x))), J_A(z)\} \geq J_A(y \to x).$$

Proof. $(i \Rightarrow iii)$ Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic filter of X. Since X is a positive implicative equality algebra, we have

$$\min\{M_A(z \to (y \to (y \to x))), M_A(z)\} \leq M_A(y \to (y \to x)) = M_A(y \to x),$$
$$\mathrm{rmin}\{\tilde{B}_A(z \to (y \to (y \to x))), \tilde{B}_A(z)\} \preceq \tilde{B}_A(y \to (y \to x)) = \tilde{B}_A(y \to x),$$
$$\max\{J_A(z \to (y \to (y \to x))), J_A(z)\} \geq J_A(y \to (y \to x)) = J_A(y \to x).$$

$(iii \Rightarrow ii)$ Since X is a positive implicative equality algebra, we have

$$\min\{M_A(x \to (y \to z)), M_A(x \to y)\} = \min\{M_A((x \to y) \to (x \to z)), M_A(x \to y)\}$$
$$= \min\{M_A((x \to y) \to (x \to (x \to z))), M_A(x \to y)\} \leq M_A(x \to z),$$
$$\mathrm{rmin}\{\tilde{B}_A(x \to (y \to z)), \tilde{B}_A(x \to y)\} = \min\{\tilde{B}_A((x \to y) \to (x \to z)), \tilde{B}_A(x \to y)\}$$
$$= \min\{\tilde{B}_A((x \to y) \to (x \to (x \to z))), \tilde{B}_A(x \to y)\} \preceq \tilde{B}_A(x \to z),$$
$$\max\{J_A(x \to (y \to z)), J_A(x \to y)\} = \max\{J_A((x \to y) \to (x \to z)), J_A(x \to y)\}$$
$$= \max\{J_A((x \to y) \to (x \to (x \to z))), J_A(x \to y)\} \geq J_A(x \to z).$$

$(ii \Rightarrow i)$ It is enough to let $x = 1$. $\hfill\square$

Proposition 3.15. *Let $\mathcal{A}$ be an MBJ-neutrosophic filter of X. Then for any $x, y, z, s, t \in X$ the following statements hold:*

(1) If $M_A(x \to y) = M_A(1)$, then $M_A(x) \leq M_A(y)$.

(2) If $\tilde{B}_A(x \to y) = \tilde{B}_A(1)$, then $\tilde{B}_A(x) \preceq \tilde{B}_A(y)$.

(3) If $J_A(x \to y) = J_A(1)$, then $J_A(x) \geq J_A(y)$.

(4) $\min\{M_A(x), M_A(x \to y)\} \leq M_A(x \wedge y)$,

(5) $\mathrm{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x \to y)\} \preceq \tilde{B}_A(x \wedge y)$,

(6) $\max\{J_A(x), J_A(x \to y)\} \geq J_A(x \wedge y)$,

(7) $\min\{M_A(x), M_A(y)\} = M_A(x \wedge y)$,

(8) $\mathrm{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\} = \tilde{B}_A(x \wedge y)$,

(9) $\max\{J_A(x), J_A(y)\} = J_A(x \wedge y)$,

(10) $\min\{M_A(x \sim y), M_A(y \sim z)\} \leq M_A(x \sim z)$,

(11) $\mathrm{rmin}\{\tilde{B}_A(x \sim y), \tilde{B}_A(y \sim z)\} \preceq \tilde{B}_A(x \sim z)$,

(12) $\max\{J_A(x \sim y), J_A(y \sim z)\} \geq J_A(x \sim z)$,

(13) $\min\{M_A(x \to y), M_A(y \to z)\} \leq M_A(x \to z)$,

(14) $\mathrm{rmin}\{\tilde{B}_A(x \to y), \tilde{B}_A(y \to z)\} \preceq \tilde{B}_A(x \to z)$,

(15) $\max\{J_A(x \to y), J_A(y \to z)\} \geq J_A(x \to z)$,

(16) If $y \leq x$, then $M_A(y) = \min\{M_A(x), M_A(x \to y)\}$.

(17) If $y \leq x$, then $\tilde{B}_A(y) = \text{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x \rightarrow y)\}$.

(18) If $y \leq x$, then $J_A(y) = \max\{J_A(x), J_A(x \rightarrow y)\}$.

Proof. (1), (2) and (3) by Theorem 3.10 and assumptions $M_A(x \rightarrow y) = M_A(1), \tilde{B}_A(x \rightarrow y) = \tilde{B}_A(1)$ and $J_A(x \rightarrow y) = J_A(1)$, we have

$$M_A(x) = \min\{M_A(x), M_A(1)\} = \min\{M_A(x), M_A(x \rightarrow y)\} \leq M_A(y),$$
$$\tilde{B}_A(x) = \text{rmin}\{\tilde{B}_A(x), \tilde{B}_A(1)\} = \text{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x \rightarrow y)\} \preceq \tilde{B}_A(y),$$
$$J_A(x) = \max\{J_A(x), J_A(1)\} = \max\{J_A(x), J_A(x \rightarrow y)\} \geq J_A(y).$$

(4), (5) and (6) by Definition 3.8, we have

$$\min\{M_A(x), M_A(x \rightarrow y)\} = \min\{M_A(x), M_A(x \sim (x \wedge y))\} \leq M_A(x \wedge y),$$
$$\text{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x \rightarrow y)\} = \text{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x \sim (x \wedge y))\} \preceq \tilde{B}_A(x \wedge y),$$
$$\max\{J_A(x), J_A(x \rightarrow y)\} = \max\{J_A(x), J_A(x \sim (x \wedge y))\} \geq J_A(x \wedge y).$$

(7), (8) and (9) this is clear that $M_A(x \wedge y) \leq \min\{M_A(x), M_A(y)\}, \tilde{B}_A(x \wedge y) \preceq \text{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\}$ and $J_A(x \wedge y) \geq \max\{J_A(x), J_A(y)\}$. Conversly, by Proposition 2.2(v) and items (4),(5) and (6), we obtain

$$\min\{M_A(x), M_A(y)\} \leq \min\{M_A(x), M_A(x \rightarrow y)\} \leq M_A(x \wedge y),$$
$$\text{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\} \preceq \text{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x \rightarrow y)\} \preceq \tilde{B}_A(x \wedge y),$$
$$\max\{J_A(x), J_A(y)\} \geq \max\{J_A(x), J_A(x \rightarrow y)\} \geq J_A(x \wedge y).$$

(10), (11) and (12) since $x \sim y \leq (y \sim z) \sim (x \sim z)$, by Definition 2.1 (E7) and Definition 3.8, we get

$$\min\{M_A(x \sim y), M_A(y \sim z)\} \leq \min\{M_A((y \sim z) \sim (x \sim z)), M_A(y \sim z)\} \leq M_A(x \sim z),$$
$$\text{rmin}\{\tilde{B}_A(x \sim y), \tilde{B}_A(y \sim z)\} \preceq \text{rmin}\{\tilde{B}_A((y \sim z) \sim (x \sim z)), \tilde{B}_A(y \sim z)\} \preceq \tilde{B}_A(x \sim z),$$
$$\max\{J_A(x \sim y), J_A(y \sim z)\} \geq \max\{J_A((y \sim z) \sim (x \sim z)), J_A(y \sim z)\} \geq J_A(x \sim z).$$

(13), (14) and (15) by Proposition 2.2(vii), we have $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$, then by Definition 3.8 and Theorem 3.10 we get

$$\min\{M_A(x \rightarrow y), M_A(y \rightarrow z)\} \leq \min\{M_A((y \rightarrow z) \rightarrow (x \rightarrow z)), M_A(y \rightarrow z)\} \leq M_A(x \rightarrow z),$$
$$\text{rmin}\{\tilde{B}_A(x \rightarrow y), \tilde{B}_A(y \rightarrow z)\} \preceq \text{rmin}\{\tilde{B}_A((y \rightarrow z) \rightarrow (x \rightarrow z)), \tilde{B}_A(y \rightarrow z)\} \preceq \tilde{B}_A(x \rightarrow z),$$
$$\max\{J_A(x \rightarrow y), J_A(y \rightarrow z)\} \geq \max\{J_A((y \rightarrow z) \rightarrow (x \rightarrow z)), J_A(y \rightarrow z)\} \geq J_A(x \rightarrow z).$$

(16), (17) and (18) since $y \leq x$, $y \leq x \rightarrow y$ and $\mathcal{A}$ is an MBJ-neutrosophic filter of X, we have

$$M_A(y) \leq \min\{M_A(x \rightarrow y), M_A(x)\} \leq M_A(y),$$
$$\tilde{B}_A(y) \preceq \text{rmin}\{\tilde{B}_A(x \rightarrow y), \tilde{B}_A(x)\} \preceq \tilde{B}_A(y),$$
$$J_A(y) \geq \max\{J_A(x \rightarrow y), J_A(x)\} \geq J_A(y).$$

$\square$

Proposition 3.16. *Let $\{F_i\}_{i \in \Lambda}$ be a family of MBJ-neutrosophic filters of X. Then infinite intersection ($\bigwedge_{i \in \Lambda}$) of F_i is an MBJ-neutrosophic filter of X.*

Proof. Suppose that for any $x, y, z \in X, x \leq y \rightarrow z$. Since F_i are an MBJ-neutrosophic filters of X, by Theorem 3.13 for all $x, y, z \in X$, we have

$$M_A(z) \geq \min\{M_A(x), M_A(y)\},$$
$$\tilde{B}_A(z) \succeq \mathrm{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\},$$
$$J_A(z) \leq \max\{J_A(x), J_A(y)\},$$

Thus,

$$\bigwedge_{i \in \Lambda} M_A^i(z) = \inf_{i \in \Lambda} M_A^i(z) \geq \inf_{i \in \Lambda}(\min\{M_A^i(x), M_A^i(y)\}) = \min\{\inf_{i \in \Lambda} M_A^i(x), \inf_{i \in \Lambda} M_A^i(y)\}$$

$$= \min\{\bigwedge_{i \in \Lambda} M_A^i(x), \bigwedge_{i \in \Lambda} M_A^i(y)\},$$

$$\bigwedge_{i \in \Lambda} \tilde{B}_A^i(z) = \mathrm{rinf}_{i \in \Lambda} \tilde{B}_A^i(z) \succeq \mathrm{rinf}_{i \in \Lambda}(\mathrm{rmin}\{\tilde{B}_A^i(x), \tilde{B}_A^i(y)\}) = \mathrm{rmin}\{\mathrm{rinf}_{i \in \Lambda} \tilde{B}_A^i(x), \mathrm{rinf}_{i \in \Lambda} \tilde{B}_A^i(y)\}$$

$$= \min\{\bigwedge_{i \in \Lambda} \tilde{B}_A^i(x), \bigwedge_{i \in \Lambda} \tilde{B}_A^i(y)\},$$

$$\bigwedge_{i \in \Lambda} J_A^i(z) = \inf_{i \in \Lambda} J_A^i(z) \leq \inf_{i \in \Lambda}(\max\{J_A^i(x), J_A^i(y)\}) = \max\{\inf_{i \in \Lambda} J_A^i(x), \inf_{i \in \Lambda} J_A^i(y)\}$$

$$= \max\{\bigwedge_{i \in \Lambda} J_A^i(x), \bigwedge_{i \in \Lambda} J_A^i(y)\}.$$

Therefore, by Theorem 3.13, infinite intersection of F_i is an MBJ-neutrosophic filter of X. $\qquad\square$

Let X and Y be two equality algebras. $f : X \rightarrow Y$ is called an *equality homomorphism* if for any $x, y \in X$, we have

$$f(x \sim y) = f(x) \sim f(y), \quad \text{and} \quad f(x \wedge y) = f(x) \wedge f(y).$$

It is clear that $f(x \rightarrow y) = f(x) \rightarrow f(y)$ and if X and Y are bounded, then $f(0) = 0$.

Notation. Let X and Y be two equality algebras and $f : X \rightarrow Y$ be an equality homomorphism. For any MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in Y, we define a new MBJ-neutrosophic set $\mathcal{A}^f = (M_A^f, \tilde{B}_A^f, J_A^f)$ in X by $M_A^f(x) = M_A(f(x))$, $\tilde{B}_A^f(x) = \tilde{B}_A(f(x))$ and $J_A^f(x) = J_A(f(x))$, for any $x \in X$.

Theorem 3.17. *Let $f : X \rightarrow Y$ be an equality homomorphism.*
(i) If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic filter of Y, then $\mathcal{A}^f = (M_A^f, \tilde{B}_A^f, J_A^f)$ is an MBJ-neutrosophic filter of X.
(ii) If f is an equality epimorphism and $\mathcal{A}^f = (M_A^f, \tilde{B}_A^f, J_A^f)$ is an MBJ-neutrosophic filter of X, then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic filter of Y.

Proof. (i) Let $x, y \in X$. Since $f(x) \leq 1 = f(1)$ and $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic filter of Y, we have

$$M_A^f(x) = M_A(f(x)) \leq M_A(f(1)) = M_A^f(1),$$
$$\tilde{B}_A^f(x) = \tilde{B}_A(f(x)) \preceq \tilde{B}_A(f(1)) = \tilde{B}_A^f(1),$$
$$J_A^f(x) = J_A(f(x)) \geq J_A(f(1)) = J_A^f(1).$$

Moreover, since f is an equality homomotphism, we get

$$\min\{M_A^f(x), M_A^f(x \to y)\} = \min\{M_A(f(x \to y)), M_A(f(x))\} = \min\{M_A(f(x) \to f(y)), M_A(f(x))\}$$
$$\leq M_A(f(y)) = M_A^f(y),$$

$$\min\{\tilde{B}_A^f(x), \tilde{B}_A^f(x \to y)\} = \min\{\tilde{B}_A(f(x \to y)), \tilde{B}_A(f(x))\} = \min\{\tilde{B}_A(f(x) \to f(y)), \tilde{B}_A(f(x))\}$$
$$\preceq \tilde{B}_A(f(y)) = \tilde{B}_A^f(y),$$

$$\max\{M_A^f(x), M_A^f(x \to y)\} = \max\{M_A(f(x \to y)), M_A(f(x))\} = \max\{M_A(f(x) \to f(y)), M_A(f(x))\}$$
$$\geq M_A(f(y)) = M_A^f(y),$$

Therefore $\mathcal{A}^f = (M_A^f, \tilde{B}_A^f, J_A^f)$ is an MBJ-neutrosophic filter of X.

(ii) Let $y \in Y$. Since f is an equality epimorphism and $\mathcal{A}^f = (M_A^f, \tilde{B}_A^f, J_A^f)$ is an MBJ-neutrosophic filter of X, there exists $x \in X$ such that $f(x) = y$ and

$$M_A(y) = M_A(f(x)) = M_A^f(x) \leq M_A^f(1) = M_A(f(1)) = M_A(1),$$
$$\tilde{B}_A(y) = \tilde{B}_A(f(x)) = \tilde{B}_A^f(x) \preceq \tilde{B}_A^f(1) = \tilde{B}_A(f(1)) = \tilde{B}_A(1),$$
$$J_A(y) = J_A(f(x)) = J_A^f(x) \geq J_A^f(1) = J_A(f(1)) = J_A(1).$$

Now, suppose $x, y \in Y$. Then there exist $a, b \in X$ such that $f(a) = x$ and $f(b) = y$. Thus

$$M_A(y) = M_A(f(b)) = M_A^f(b) \geq \min\{M_A^f(a), M_A^f(a \to b)\} = \min\{M_A(f(a \to b)), M_A(f(a))\}$$
$$= \min\{M_A(f(a) \to f(b)), M_A(f(a))\} = \min\{M_A(x \to y), M_A(x)\}.$$

$$\tilde{B}_A(y) = \tilde{B}_A(f(b)) = \tilde{B}_A^f(b) \preceq \min\{\tilde{B}_A^f(a), \tilde{B}_A^f(a \to b)\} = \min\{\tilde{B}_A(f(a \to b)), \tilde{B}_A(f(a))\}$$
$$= \min\{\tilde{B}_A(f(a) \to f(b)), \tilde{B}_A(f(a))\} = \min\{\tilde{B}_A(x \to y), \tilde{B}_A(x)\}.$$

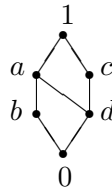$$J_A(y) = J_A(f(b)) = J_A^f(b) \leq \min\{J_A^f(a), J_A^f(a \to b)\} = \min\{J_A(f(a \to b)), J_A(f(a))\}$$
$$= \min\{J_A(f(a) \to f(b)), J_A(f(a))\} = \min\{J_A(x \to y), J_A(x)\}.$$

Therefore, $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic filter of Y. □

Define $K(x, y) = \{z \in X \mid x \leq y \to z\}$ which is called *an upper set of x and y*. It is easy to see that $1, x, y \in K(x, y)$, for any $x, y \in X$.

In the following example we can see that every $K(x, y)$ is not a filter of X in general.

Example 3.18. *Let $X = \{0, a, b, c, d, 1\}$ be a set with the following Hasse diagram.*

Then $(X, \wedge, 1)$ is a meet semilattice with top element 1. Define an operation $\sim$ on X by Table 7.

~	0	a	b	c	d	1
0	1	d	c	b	a	0
a	d	1	a	d	c	a
b	c	a	1	0	d	b
c	b	d	0	1	a	c
d	a	c	d	a	1	d
1	0	a	b	c	d	1

Table 7: Cayley table for the binary operation "~"

Thus $\mathcal{E} = (X, \wedge, \sim, 1)$ is an equality algebra, and the implication "$\rightarrow$" is given by Table 8.

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

Table 8: Cayley table for the implication "$\rightarrow$"

Hence, $K(c, a) = \{a, c, d, 1\}$ is not a filter of X, because $d \rightarrow b = a \in K(c, a)$ and $d \in K(c, a)$, but $b \notin K(c, a)$.

Given an MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X, we consider the following sets,

$$U(M_A; t) := \{x \in X | M_A(x) \geq t\}, \tag{3.9}$$

$$U(\tilde{B}_A; [\delta_1, \delta_2]) := \{x \in X | \tilde{B}_A(x) \succeq [\delta_1, \delta_2]\}, \tag{3.10}$$

$$L(J_A; s) := \{x \in X | J_A(x) \leq s\}. \tag{3.11}$$

Where $t, s \in [0, 1] = I$ and $[\delta_1, \delta_2] \in [I]$.

Example 3.19. *Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic as Example 3.2. Suppose $t = 0.5$, $s = 0.4$ and $[\delta_1, \delta_2] = [0.4, 0.5]$. Then it is clear that*

$$U(M_A; 0.5) := \{x \in X | M_A(x) \geq 0.5\} = \{a, 1\},$$

$$U(\tilde{B}_A; [0.4, 0.5]) := \{x \in X | \tilde{B}_A(x) \succeq [0.4, 0.5]\} = \{0, 1\},$$

$$L(J_A; 0.4) := \{x \in X | J_A(x) \leq 0.4\} = \{1\}.$$

Theorem 3.20. *An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is an MBJ-neutrosophic filter of X if and only if for any $t, s \in [0, 1]$ and $[\delta_1, \delta_2] \in [I]$, the non-empty sets $U(M_A; t), U(\tilde{B}_A; [\delta_1, \delta_2])$ and $L(J_A; s)$ are filters of X.*

Proof. Suppose $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic filter of X. Let $t, s \in [0, 1]$ and $[\delta_1, \delta_2] \in [I]$ such that $U(M_A; t), U(\tilde{B}_A; [\delta_1, \delta_2])$ and $L(J_A; s)$ are non-empty sets. Obviously,

$1 \in U(M_A; t), 1 \in U(\tilde{B}_A; [\delta_1, \delta_2])$ and $1 \in L(J_A; s)$. For any $x, y, a, b, u, v \in X$, if $x, x \rightarrow y \in U(M_A; t), a, a \rightarrow b \in U(\tilde{B}_A; [\delta_1, \delta_2])$ and $u, u \rightarrow v \in L(J_A; s)$, then

$$M_A(y) \geq \min\{M_A(x), M_A(x \rightarrow y)\} \geq \min\{t, t\} = t,$$
$$\tilde{B}_A(b) \succeq \text{rmin}\{\tilde{B}_A(a), \tilde{B}_A(a \rightarrow b)\} \succeq \text{rmin}\{[\delta_1, \delta_2], [\delta_1, \delta_2]\} = [\delta_1, \delta_2],$$
$$J_A(v) \leq \max\{J_A(u), J_A(u \rightarrow v)\} \leq \max\{s, s\} = s,$$

and so $y \in U(M_A; t), b \in U(\tilde{B}_A; [\delta_1, \delta_2])$ and $v \in L(J_A; s)$. Therefore, $U(M_A; t), U(\tilde{B}_A; [\delta_1, \delta_2])$ and $L(J_A; s)$ are filters of X.

Conversely, suppose that for any $t, s \in [0, 1]$ and $[\delta_1, \delta_2] \in [I]$, the non-empty sets $U(M_A; t)$, $L(J_A; s)$ and $U(\tilde{B}_A; [\delta_1, \delta_2])$ are filters of X. Let $x \in U(M_A; t) \cap L(J_A; s) \cap U(\tilde{B}_A; [\delta_1, \delta_2])$, for some $x \in X$. If $M_A(1) < M_A(x), \tilde{B}_A(1) \prec \tilde{B}_A(x)$ and $J_A(1) > J_A(x)$, then $1 \notin U(M_A; t), 1 \notin U(\tilde{B}_A; [\delta_1, \delta_2])$ and $1 \notin L(J_A; s)$, which is a contradiction. Hence $M_A(x) \leq M_A(1), \tilde{B}_A(x) \preceq \tilde{B}_A(1), J_A(x) \geq J_A(1)$ for all $x \in X$. Suppose for some $x_0, y_0 \in X$, $x_0, x_0 \rightarrow y_0 \in U(M_A; t_0)$ but $y_0 \notin U(M_A; t_0)$ for $t_0 := \min\{M_A(x_0), M_A(x_0 \rightarrow y_0)\}$. This is a contradiction and thus $M_A(y) \geq \min\{M_A(x), M_A(x \rightarrow y)\}$ for all $x, y \in X$. Similarly, we can show that for all $x, y \in X$, $J_A(y) \leq \max\{J_A(x), J_A(x \rightarrow y)\}$. Suppose for some $x_0, y_0 \in X$, $\tilde{B}_A(y_0) \prec \text{rmin}\{\tilde{B}_A(x_0), \tilde{B}_A(x_0 \rightarrow y_0)\}$. Let $\tilde{B}_A(x_0 \rightarrow y_0) = [\eta_1, \eta_2], \tilde{B}_A(x_0) = [\eta_3, \eta_4]$ and $\tilde{B}_A(y_0) = [\delta_1, \delta_2]$. Then

$$[\delta_1, \delta_2] \prec \text{rmin}\{[\eta_1, \eta_2], [\eta_3, \eta_4]\} = [\min\{\eta_1, \eta_3\}, \min\{\eta_2, \eta_4\}],$$

and so $\delta_1 < \min\{\eta_1, \eta_3\}$ and $\delta_2 < \min\{\eta_2, \eta_4\}$. Taking

$$[\gamma_1, \gamma_2] := \frac{1}{2}(\tilde{B}_A(y_0) + \text{rmin}\{\tilde{B}_A(x_0 \rightarrow y_0), \tilde{B}_A(x_0)\}),$$

implies that

$$[\gamma_1, \gamma_2] = \frac{1}{2}([\delta_1, \delta_2] + \text{rmin}\{[\eta_1, \eta_2], [\eta_3, \eta_4]\}) = \frac{1}{2}([\delta_1, \delta_2] + [\min\{\eta_1, \eta_3\}, \min\{\eta_2, \eta_4\}])$$
$$= [\frac{1}{2}(\delta_1 + \min\{\eta_1, \eta_3\}), \frac{1}{2}(\delta_2 + \min\{\eta_2, \eta_4\})].$$

It follows that
$$\min\{\eta_1, \eta_3\} > \gamma_1 = \frac{1}{2}(\delta_1 + \min\{\eta_1, \eta_3\}) > \delta_1,$$

and
$$\min\{\eta_2, \eta_4\} > \gamma_2 = \frac{1}{2}(\delta_2 + \min\{\eta_2, \eta_4\}) > \delta_2.$$

Hence
$$[\min\{\eta_1, \eta_3\}, \min\{\eta_2, \eta_4\}] \succ [\gamma_1, \gamma_2] \succ [\delta_1, \delta_2] = \tilde{B}_A(y_0),$$

and so $y_0 \notin U(\tilde{B}_A; [\gamma_1, \gamma_2])$. On the other hand,

$$\tilde{B}_A(x_0 \rightarrow y_0) = [\eta_1, \eta_2] \succeq [\min\{\eta_1, \eta_3\}, \min\{\eta_2, \eta_4\}] \succ [\gamma_1, \gamma_2],$$

and
$$\tilde{B}_A(x_0) = [\eta_3, \eta_4] \succeq [\min\{\eta_1, \eta_3\}, \min\{\eta_2, \eta_4\}] \succ [\gamma_1, \gamma_2],$$

so $x_0 \rightarrow y_0, x_0 \in U(\tilde{B}_A; [\gamma_1, \gamma_2])$, which is a contradiction and so $\tilde{B}_A(y) \succeq \text{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x \rightarrow y)\}$ for all $x, y \in X$. Therefore, $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic filter of X. $\qquad\square$

Proposition 3.21. *Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic filter of X such that $x, y \in X$, $t, s \in [0,1]$ and $[\delta_1, \delta_2] \in [I]$. If $x, y \in U(M_A; t) \cap U(\tilde{B}_A; [\delta_1, \delta_2]) \cap L(J_A; s)$, then $K(x, y) \subseteq U(M_A; t) \cap U(\tilde{B}_A; [\delta_1, \delta_2]) \cap L(J_A; s)$.*

Proof. Let $x, y \in U(M_A; t) \cap U(\tilde{B}_A; [\delta_1, \delta_2]) \cap L(J_A; s)$, $t, s \in [0,1]$, $[\delta_1, \delta_2] \in [I]$ and $z \in K(x, y)$. Then $x \leq y \to z$. Since $x, y \in U(M_A; t) \cap U(\tilde{B}_A; [\delta_1, \delta_2]) \cap L(J_A; s)$ and $\mathcal{A}$ is an MBJ-neutrosophic filter of X, we have $M_A(x), M_A(y) \geq t$. By Theorem 3.13, we get $M_A(z) \geq \min\{M_A(x), M_A(y)\} \geq t$, and so $z \in U(M_A; t)$. By the similar way, we can see that $z \in U(\tilde{B}_A; [\delta_1, \delta_2]) \cap L(J_A; s)$. Hence $K(x, y) \subseteq U(M_A; t) \cap U(\tilde{B}_A; [\delta_1, \delta_2]) \cap L(J_A; s)$. $\qquad\square$

Corollary 3.22. *Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic filter of X such that $t, s \in [0,1]$ and $[\delta_1, \delta_2] \in [I]$. If $U(M_A; t) \cap U(\tilde{B}_A; [\delta_1, \delta_2]) \cap L(J_A; s) \neq \emptyset$, then*

$$U(M_A; t) = \bigcup_{x,y \in U(M_A; t)} K(x, y), \ U(\tilde{B}_A; [\delta_1, \delta_2]) = \bigcup_{x,y \in U(\tilde{B}_A; t)} K(x, y), \ L(J_A; s) = \bigcup_{x,y \in U(J_A; t)} K(x, y).$$

Proof. The proof is straightforward. $\qquad\square$

Theorem 3.23. *Given an filter F of X, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X that defined by*

$$M_A(x) = \begin{cases} t & \text{if } x \in F, \\ 0 & \text{otherwise,} \end{cases} \quad \tilde{B}_A(x) = \begin{cases} [\delta_1, \delta_2] & \text{if } x \in F, \\ [0,0] & \text{otherwise,} \end{cases} \quad J_A(x) = \begin{cases} s & \text{if } x \in F, \\ 1 & \text{otherwise,} \end{cases}$$
(3.12)

Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic filter of X such that $U(M_A; t) = U(\tilde{B}_A; [\delta_1, \delta_2]) = L(J_A; s) = F$.

Proof. Let $x, y \in X$ and $x, x \to y \in F$. Since F is a filter of X, we have $y \in F$ and so

$$M_A(y) = t = \min\{M_A(x), M_A(x \to y)\},$$
$$\tilde{B}_A(y) = [\delta_1, \delta_2] = \operatorname{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x \to y)\},$$
$$J_A(y) = s = \max\{J_A(x), J_A(x \to y)\}.$$

If $x \to y \in F$ and $x \notin F$, then $M_A(x \to y) = t, \tilde{B}_A(x \to y) = [\delta_1, \delta_2]$ and $J_A(x \to y) = s$. Also, $M_A(x) = 0, \tilde{B}_A(x) = [0,0]$ and $J_A(x) = 1$. Hence

$$M_A(y) \geq 0 = \min\{t, 0\} = \min\{M_A(x), M_A(x \to y)\},$$
$$\tilde{B}_A(y) \succeq [0,0] = \operatorname{rmin}\{[\delta_1, \delta_2], [0,0]\} = \operatorname{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x \to y)\},$$
$$J_A(y) \leq 1 = \max\{s, 1\} = \max\{J_A(x), J_A(x \to y)\}.$$

If $x \to y, x \notin F$, then $M_A(x) = M_A(x \to y) = 0, \tilde{B}_A(x) = \tilde{B}_A(x \to y) = [0,0]$ and $J_A(x) = J_A(x \to y) = 1$. Hence

$$M_A(y) \geq 0 = \min\{0, 0\} = \min\{M_A(x), M_A(x \to y)\},$$
$$\tilde{B}_A(y) \succeq [0,0] = \operatorname{rmin}\{[0,0], [0,0]\} = \operatorname{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x \to y)\},$$
$$J_A(y) \leq 1 = \max\{1, 1\} = \max\{J_A(x), J_A(x \to y)\}.$$

Since F is a filter of X, we have $1 \in F$ and so it is clear that for all $x \in X$, $M_A(x) \leq M_A(1)$, $\tilde{B}_A(x) \preceq \tilde{B}_A(1)$, $J_A(x) \geq J_A(1)$. Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic filter of X and we have $U(M_A; t) = U(\tilde{B}_A; [\delta_1, \delta_2]) = L(J_A; s) = F$. $\qquad\square$

Theorem 3.24. *For any non-empty subset F of X, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set of X which is given in (3.12). If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic filter of X, then F is a filter of X.*

Proof. Since for any $x \in U(M_A; t) \cap L(J_A; s) \cap U(\tilde{B}_A; [\delta_1, \delta_2])$, $M_A(x) \leq M_A(1)$, $\tilde{B}_A(x) \preceq \tilde{B}_A(1)$, $J_A(x) \geq J_A(1)$, it is clear that $1 \in F$. Let $x, y \in X$ such that $x \to y \in F$ and $x \in F$. Then $M_A(x) = M_A(x \to y) = t, \tilde{B}_A(x) = \tilde{B}_A(x \to y) = [\delta_1, \delta_2]$ and $J_A(x) = J_A(x \to y) = s$. Thus

$$M_A(y) \geq \min\{M_A(x), M_A(x \to y)\} = t,$$
$$\tilde{B}_A(y) \succeq \operatorname{rmin}\{\tilde{B}_A(x), \tilde{B}_A(x \to y)\} = [\delta_1, \delta_2],$$
$$J_A(y) \leq \max\{J_A(x), J_A(x \to y)\} = s,$$

and so $M_A(y) = t$, $\tilde{B}_A(y) = [\delta_1, \delta_2]$ and $J_A(y) = s$. Hence, $y \in F$. Therefore, F is a filter of X. $\square$

Proposition 3.25. *Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic filter of X. Then the sets $X_{M_A} = \{x \in X \mid M_A(x) = M_A(1)\}$, $X_{\tilde{B}_A} = \{x \in X \mid \tilde{B}_A(x) = \tilde{B}_A(1)\}$, $X_{J_A} = \{x \in X \mid J_A(x) = J_A(1)\}$ are filters of X.*

Proof. It is clear that $1 \in X_{M_A} \cap X_{\tilde{B}_A} \cap X_{J_A}$. Let $x, x \to y \in X_{M_A}$. Then $M_A(x \to y) = M_A(x) = M_A(1)$. Since $\mathcal{A}$ is an MBJ-neutrosophic filter of X, we have

$$M_A(1) \geq M_A(y) \geq \min\{M_A(x \to y), M_A(x)\} = \min\{M_A(1), M_A(1)\} = M_A(1).$$

Hence $M_A(y) = M_A(1)$ and so $y \in X_{M_A}$. Therefore X_{M_A} is a filter of X. The proofs of other cases are similar. $\square$

   Note. Since $1 \in X_{M_A} \cap X_{\tilde{B}_A} \cap X_{J_A}$, we get $X_{M_A} \cap X_{\tilde{B}_A} \cap X_{J_A} \neq \emptyset$ and by Proposition 3.25, the sets X_{M_A}, $X_{\tilde{B}_A}$ and X_{J_A} are filters of X, by using an MBJ-neutrosophic filter of X, for any $x, y \in X$, we can define a congruence relation on X as follows:

$$x \equiv_\mathcal{A} y \;\; \text{if and only if} \;\; x \to y, y \to x \in X_{M_A} \cap X_{\tilde{B}_A} \cap X_{J_A}.$$

Corollary 3.26. *Let $\frac{X}{\equiv_\mathcal{A}} = \{\frac{x}{\equiv_\mathcal{A}} \mid x \in X\}$. Define the operations $\simeq$ and $\sqcap$ on $\frac{X}{\equiv_\mathcal{A}}$ as follows:*

$$\frac{x}{\equiv_\mathcal{A}} \sqcap \frac{y}{\equiv_\mathcal{A}} = \frac{x \wedge y}{\equiv_\mathcal{A}} \;\; and \;\; \frac{x}{\equiv_\mathcal{A}} \simeq \frac{y}{\equiv_\mathcal{A}} = \frac{x \sim y}{\equiv_\mathcal{A}}.$$

Hence $\left(\frac{X}{\equiv_\mathcal{A}}, \sqcap, \simeq, \frac{1}{\equiv_\mathcal{A}}\right)$ is an equality algebra.

4   Conclusions

In this paper, the notion of MBJ-neutrosophic sub-algebra and MBJ-neutrosophic filter on equality algebras are introduced and some equivalence definitions, properties and the relation between them are investigated. Also, by using the notion of MBJ-neutrosophic filter, a congruence relation is defined on equality algebra and we prove that the quotient structure that is made by this is an equality algebra.

   In future work, we can study MBJ-neutrosophic notion on different kinds of filters such as (positive) implicative filters, fantastic filters, prime and maximal filters of equality algebras and investigate the quotient that is made by them.

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