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ORIGINAL ARTICLE

bi-Strong Smarandache *BL*-algebras

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Abstract In this paper, we introduce the notion of *bi*-Smarandache *BL*-algebra, *bi*-weak Smarandache *BL*-algebra, *bi*-*Q*-Smarandache ideal and *bi*-*Q*-Smarandache implicative filter, we obtain some related results and construct quotient of *bi*-Smarandache *BL*-algebras via MV-algebras (or briefly *bi*-Smarandache quotient *BL*-algebra) and prove some theorems. Finally, the notion of *bi*-strong Smarandache *BL*-algebra is presented and relationship between *bi*-strong Smarandache *BL*-algebra and *bi*-Smarandache *BL*-algebra are studied.

Keywords *bi*-Smarandache *BL*-algebra · *bi*-weak Smarandache *BL*-algebra · *bi*-*Q*-Smarandache ideal · *bi*-implicative filter · *n*-Smarandache strong structure

1. Introduction

A Smarandache structure on a set A means a weak structure W on A such that there exists a proper subset B of A which is embedded with a strong structure S. In [9], W. B. Vasantha Kandasamy studied the concept of Smarandache groupoids, subgroupoids, ideal of groupoids and strong Bol groupoids and obtained many interesting results about them. Smarandache semigroups are very important for the study of congruences, and it was studied by R. Padilla [7]. It will be very interesting to study the Smarandache structure in this algebraic structures.

Processing of the certain information, especially inferences based on certain information is based on classical two-valued logic. Due to strict and complete logical foundation (classical logic), making inference levels. thus, it is natural and necessary in an attempt to establish some rational logic system as the logical foundation for uncertain information processing. It is evident that this kind of logic cannot be

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two-valued logic itself but might form a certain extension of two-valued logic. Various kinds of non-classical logic systems have therefore been extensively researched in order to construct natural and efficient inference systems to deal with uncertainty. *BL*-algebra have been invented by P. Hajek [5] in order to provide an algebraic proof of the completeness theorem of "Basic Logic" (*BL*, for short) arising from the continuous triangular norms, familiar in the fuzzy logic framework. The language of propositional Hajek basic logic [5] contains the binary connectives $\odot$ and $\rightarrow$ and the constant $\overline{0}$. Axioms of *BL* are:

(A_1) $(\phi \rightarrow \chi) \rightarrow ((\chi \rightarrow \psi) \rightarrow (\phi \rightarrow \psi))$;
(A_2) $(\phi \odot \chi) \rightarrow \phi$;
(A_3) $(\phi \odot \chi) \rightarrow (\chi \odot \phi)$;
(A_4) $(\phi \odot (\phi \rightarrow \chi)) \rightarrow (\chi \odot (\chi \rightarrow \phi))$;
(A_{5a}) $(\phi \rightarrow (\chi \rightarrow \psi)) \rightarrow ((\phi \odot \chi) \rightarrow \psi))$;
(A_{5b}) $((\phi \odot \chi) \rightarrow \psi) \rightarrow (\phi \rightarrow (\chi \rightarrow \psi))$;
(A_6) $((\phi \rightarrow \chi) \rightarrow \psi) \rightarrow (((\chi \rightarrow \phi) \rightarrow \psi) \rightarrow \psi)$;
(A_7) $\overline{0} \rightarrow \omega$.

MV-algebras were originally introduced by Chang in order to give an algebraic counterpart of the Lukasiewicz many valued logic. This structure is directly obtained from Lukasiewicz logic, in the sense that the operations coincide with the basic logical connectives [4]. Lukasiewicz logic is an axiomatic extension of *BL*-logic and consequently, *MV*-algebras are particular class of *BL*-algebras.

It is clear that any *MV*-algebra is a *BL*-algebra. An *MV*-algebras is a weaker structure than *BL*-algebra, thus we can consider in any *BL*-algebra a weaker structure as *MV*-algebra.

The authors introduced the notion of *bi*-*BL*-algebra, *bi*-filter, *bi*-deductive system and *bi*-Boolean center of a *bi*-*BL*-algebra. They have also presented classes of *bi*-*BL*-algebras and we stated relation between *bi*-filters and quotient *bi*-*BL*-algebra [1].

A. Borumand Saeid et al introduced the notion of Smarandache *BL*-algebra and dealt with Smarandache ideal structures in Smarandache *BL*-algebra. They constructed the quotient of Smarandache *BL*-algebra via *MV*-algebras (or briefly Smarandache quotient *BL*-algebras) and proved that this quotient is a *BL*-algebra [2].

In this paper, we introduce the notion of *bi*-Smarandache *BL*-algebra, *bi*-Strong Smarandache *BL*-algebra and investigate relationship between *bi*-Smarandache *BL*-algebra and *bi*-Strong Smarandache *BL*-algebra. We deal with *bi*-Smarandache ideal structures in *bi*-Smarandache *BL*-algebra. We introduce the notions of *bi*-weak Smarandache *BL*-algebra and *bi*-Smarandache (implicative) ideals in *bi*-*BL*-algebra, we construct the quotient of *bi*-Smarandache *BL*-algebra via *MV*-algebras and we prove that this quotient is a *bi*-*BL*-algebra.

2. Preliminaries

Definition 1 [5] *A BL-algebra is an algebra* $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ *with four binary operations* $\wedge, \vee, \odot, \rightarrow$ *and two constants* $0, 1$ *such that:*
(BL1) $(L, \wedge, \vee, \rightarrow, 0, 1)$ *is a bounded lattice,*
(BL2) $(L, \odot, 1)$ *is a commutative monoid,*

(BL3) $\odot$ *and* $\rightarrow$ *form an adjoint pair i.e,* $a \odot b \leq c$ *if and only if* $a \leq b \rightarrow c$,
(BL4) $a \wedge b = a \odot (a \rightarrow b)$,
(BL5) $(a \rightarrow b) \vee (b \rightarrow a) = 1$,
for all $a, b, c \in L$.

A BL-algebra L is called an MV-algebra if $x^{**} = x$, for all $x \in L$, where $x^* = x \rightarrow 0$.

Lemma 1 [5] *In each BL-algebra L, the following relations hold, for all* $x, y, z \in L$:
(1) $x \odot (x \rightarrow y) \leq y$,
(2) $x \leq (y \rightarrow (x \odot y))$,
(3) $x \leq y$ *if and only if* $x \rightarrow y = 1$,
(4) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(5) *If* $x \leq y$, *then* $y \rightarrow z \leq x \rightarrow z$ *and* $z \rightarrow x \leq z \rightarrow y$,
(6) $y \leq (y \rightarrow x) \rightarrow x$,
(7) $y \rightarrow x \leq (z \rightarrow y) \rightarrow (z \rightarrow x)$,
(8) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$,
(9) $x \vee y = [(x \rightarrow y) \rightarrow y] \wedge [(y \rightarrow x) \rightarrow x]$.

Definition 2 [5] *Let L be a BL-algebra. Then subset I of L is called an ideal of L if following conditions hold:*
(I_1) $0 \in I$,
(I_2) $x \in I$ *and* $(x^* \rightarrow y^*)^* \in I$ *imply* $y \in I$ *for all* $x, y \in L$.

Definition 3 [5] *An MV-algebra is an algebra* $Q = (Q, \oplus, ^*, 0)$ *of type (2,1,0) satisfying the following equations:*
(MV_1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(MV_2) $x \oplus y = y \oplus x$;
(MV_3) $x \oplus 0 = x$;
(MV_4) $x^{**} = x$;
(MV_5) $x \oplus 0^* = 0^*$;
(MV_6) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,
for all $x, y, z \in Q$.

From now on, $L = (L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL-algebra and $Q = (Q, \oplus, ^*, 0)$ is an MV-algebra unless otherwise specified.

Definition 4 [1] *A nonempty set* $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ *with four binary operations and two constants is said to be a bi-BL-algebra if* $L = L_1 \cup L_2$, *where* L_1 *and* L_2 *are proper subsets of L and*

i. $(L_1, \wedge, \vee, \odot, \rightarrow, 0, 1)$ *is a non-trivial BL-algebra,*

ii. $(L_2, \wedge, \vee, \odot, \rightarrow, 0, 1)$ *is a non-trivial BL-algebra.*

Definition 5 [1] *If L is a bi-BL-algebra and also a BL-algebra, then we say that L is a super BL-algebra.*

Definition 6 [1] *Let $L = L_1 \cup L_2$ be a bi-BL-algebra. We say the subset $F = F_1 \cup F_2$ of L is a bi-filter of L if F_i is a filter of L_i, where $i = 1, 2$ respectively.*

Example 1 Let $L_1 = \{0, a, b, c, d, 1\}$ and $L_2 = \{0, d, e, 1\}$. Define $\odot$ and $\rightarrow$ as follow:

L_1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	c	c	d	a
b	0	c	b	c	d	b
c	0	c	c	c	d	c
d	0	d	d	d	0	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	b	b	d	1
b	0	a	1	a	d	1
c	0	1	1	1	d	1
d	d	1	1	1	1	1
1	0	a	b	c	d	1

L_2

$\odot$	0	d	e	1
0	0	0	0	0
d	0	0	d	d
e	0	d	e	e
1	0	d	e	1

$\rightarrow$	0	d	e	1
0	1	1	1	1
d	d	1	1	1
e	0	d	1	1
1	0	d	e	1

For L, whose tables are the following:

L

$\odot$	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	a	c	c	d	e	a
b	0	c	b	c	d	b	b
c	0	c	c	c	d	e	c
d	0	d	d	d	0	d	d
e	0	e	b	e	d	e	e
1	0	a	b	c	d	e	1

$\rightarrow$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	0	1	b	b	d	e	1
b	0	a	1	a	d	d	1
c	0	1	1	1	d	e	1
d	d	1	1	1	1	1	1
e	0	d	b	d	d	1	1
1	0	a	b	c	d	e	1

Consider $F_1 = \{a, b, c, 1\}$ and $F_2 = \{e, 1\}$. Then $F = F_1 \cup F_2 = \{a, b, c, e, 1\}$ is a *bi*-filter of L.

Theorem 1 [1] *Let $F = F_1 \cup F_2$ be a bi-filter of a bi-BL-algebra $L = L_1 \cup L_2$ such that F_i is a filter of L_i, where $i = 1, 2$. Then $\frac{\mathcal{L}}{\mathcal{F}} := \frac{L_1}{F_1} \cup \frac{L_2}{F_2}$ is a bi-BL-algebra, where $\frac{L_i}{F_i} = \{[x]_{F_i} | x \in L_i\}$ and $[x]_{F_i} = \{y \in L_i | x \rightarrow y \in F_i\ ,\ y \rightarrow x \in F_i\}$, where $x \in L_i$ and $i = 1, 2$.*

Definition 7 [2] *A Smarandache BL-algebra is defined to be a BL-algebra L in which there exists a proper subset Q of A such that:*
(S_1) $0, 1 \in Q$ and $|Q| > 2$,
(S_2) Q is an MV-algebra under the operations of L.

Remark 1 *If $|Q| = 2$, i.e., $Q = \{0, 1\}$, then every BL-algebra is a Smarandache BL-algebra.*

In the following, Q is a nontrivial MV-algebra under operations in L and also $|Q| > 2$.

Definition 8 [2] *A nonempty subset I of L is called Smarandache ideal of L related to Q (or briefly Q-Smarandache ideal of A) if it satisfies:*
(c_1) If $x \in I$, $y \in Q$ and $y \leq x$, then $y \in I$.
(c_2) If $x, y \in I$, then $x \oplus y \in I$.

Theorem 2 [2] *If I is an ideal of L, then I is a Q-Smarandache ideal of L.*

Definition 9 [2] *A nonempty subset F of L is called Smarandache implicative filter of L relative to Q (or briefly Q-Smarandache implicative filter of L) if it satisfies:*

(F_1) $1 \in F$.

(F_2) If $x \in F$, $y \in Q$ and $x \to y \in F$, then $y \in F$.

In the following example, we show that every Q-Smarandache implicative filter of L is not a filter of L.

Example 2 Let $L = \{0, a, b, c, d, 1\}$. Define $\odot$ and $\to$ as follow:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	c	c	d	a
b	0	c	b	c	d	b
c	0	c	c	c	d	c
d	0	d	d	d	0	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	b	b	d	1
b	0	a	1	a	d	1
c	0	1	1	1	d	1
d	d	1	1	1	1	1
1	0	a	b	c	d	1

$(L, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra. $Q = \{0, d, 1\}$ is the only MV-algebra which is properly contained in L, which the following tables:

Q

$\oplus$	0	d	1
0	0	d	1
d	d	d	1
1	1	1	1

$*$	0	d	1
	1	d	0

Therefore L is a Smarandache BL-algebra. Consider $F = \{d, 1\}$, then F is a Q-Smarandache implicative filter of L, but not a filter of L since $d \leq c$ and $c \notin F$.

Remark 2 [2] *Let F be a Q-Smarandache implicative filter of L. Then $F \neq \phi$.*

Definition 10 [2] *A Q-Smarandache ideal M of L is called maximal Q-Smarandache ideal if only if the following conditions hold:*

(M_1) M is a proper Q-Smarandache ideal.

(M_2) For every Q-Smarandache ideal I such that $M \subseteq I$, we have either $M = I$ or $I = L$.

Theorem 3 [2] *The relation $\sim_Q$ on a Smarandache BL-algebra L which is defined by*

$$x \sim_Q y \quad \Leftrightarrow \quad (x \to y \in Q, y \to x \in Q)$$

is a congruence relation.

Definition 11 [2] *Let L be a BL-algebra and Q be an MV-algebra. Then $\frac{L}{Q} = \{[x] | x \in L\}$ and $[x] = \{y \in L | x \sim_Q y\}$ are quotient algebra via the congruence relation $\sim_Q$ (or briefly Smarandache quotient BL-algebra).*

We defined on $\frac{L}{Q}$:

$[x] \oplus [y] = [x \oplus y]$, $[x]^* = [x^*]$, $[x] \to [y] = [x \to y]$, $[x] \odot [y] = [x \odot y]$,
$[x] \wedge [y] = [x \wedge y]$, $[x] \vee [y] = [x \vee y]$, $[0] = \frac{0}{Q}$, $[1] = \frac{1}{Q}$.

For convenience, let $x * y = x \odot y^*$.

Definition 12 [2] *A Q-Smarandache ideal I of L is called a Smarandache implicative ideal of L related to Q (or briefly Q-Smarandache implicative ideal of L), if it satisfies: if $(x * y) * z \in I$ and $y * z \in I$ imply $x * z \in I$ for all $x, y, z \in Q$.*

3. *bi*-Smarandache *BL*-algebra

Definition 13 *A bi-smarandache BL-algebra $L = (L, \wedge, \vee, \odot, \to, 0, 1)$ is a nonempty set with four binary operations $\wedge, \vee, \odot, \to$ and two constants $0, 1$ such that $L = L_1 \cup L_2$, where L_1 and L_2 are proper subset of L and*

i. $(L_1, \wedge, \vee, \odot, \to, 0, 1)$ *is a Smarandache BL-algebra,*

ii. $(L_2, \wedge, \vee, \odot, \to, 0, 1)$ *is a Smarandache BL-algebra.*

Example 3 Let $L_1 = \{0, a, b, c, d, n\}$ and $L_2 = \{n, e, f, 1\}$. With the following tables:

L_1

$\odot$	0	a	b	c	d	n
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
n	0	a	b	c	d	n

$\to$	0	a	b	c	d	n
0	n	n	n	n	n	n
a	d	n	d	n	d	n
b	c	c	n	n	n	n
c	b	c	d	n	d	n
d	a	a	c	c	n	n
n	0	a	b	c	d	n

L_2

$\odot$	n	e	f	1
n	n	n	n	n
e	n	n	e	e
f	n	e	f	f
1	n	e	f	1

$\to$	n	e	f	1
n	1	1	1	1
e	e	1	1	1
f	n	e	1	1
1	n	e	f	1

For L, whose tables are the following:

L

$\odot$	0	a	b	c	d	n	e	f	1
0	0	0	0	0	0	0	0	0	0
a	0	a	0	a	0	a	a	a	a
b	0	0	0	0	b	b	b	b	b
c	0	a	0	a	b	c	c	c	c
d	0	0	b	b	d	d	d	d	d
n	0	a	b	c	d	n	n	n	n
e	0	a	b	c	d	n	n	e	e
f	0	a	b	c	d	n	e	f	f
1	0	a	b	c	d	n	e	f	1

$\to$	0	a	b	c	d	n	e	f	1
0	1	1	1	1	1	1	1	1	1
a	d	1	d	1	d	1	1	1	1
b	c	c	1	1	1	1	1	1	1
c	b	c	d	1	d	1	1	1	1
d	a	a	c	c	1	1	1	1	1
n	0	a	b	c	d	1	1	1	1
e	0	a	b	c	d	e	1	1	1
f	0	a	b	c	d	n	e	1	1
1	0	a	b	c	d	n	e	f	1

$(L, \wedge, \vee, \odot, \to, 0, 1)$ is a *bi*-*BL*-algebra. $Q_1 = \{0, a, d, n\}$ and $Q_2 = \{n, e, 1\}$ are *MV*-algebras which are properly contained in L_1 and L_2, respectively, with the following tables:

Q_1

$\oplus$	0	a	d	n
0	0	a	d	n
a	a	a	n	n
d	d	n	d	n
n	n	n	n	n

$*$	0	a	d	n
	n	d	a	0

Q_2

$\oplus$	n	e	1
n	n	e	1
e	e	1	1
1	1	1	1

$*$	n	e	1
	1	e	n

Then L_1 and L_2 are Smarandache *BL*-algebras. Therefore L is a *bi*-smarandache *BL*-algebra.

Example 4 Consider *bi*-*BL*-algebra $\mathcal{D}_{2\times 2,2}$, with the support set $D_{2\times 2,2} = L_{2\times 2} \cup L_2 =$

$\{0, a, b, c\} \cup \{c, 1\} = \{0, a, b, c, 1\}$ and the following tables:

$\mathcal{L}_{2\times 2}$

$\odot$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

$\to$	0	a	b	c
0	c	c	c	c
a	b	c	b	c
b	a	a	c	c
c	0	a	b	c

$\mathcal{L}_2$

$\odot$	c	1
c	c	c
1	c	1

$\to$	c	1
c	1	1
1	c	1

$Q_1 = \{0, c\}$ and $Q_2 = \{c, 1\}$ are the only MV-algebras which are properly contained in $\mathcal{L}_{2\times 2}$ and $\mathcal{L}_2$, respectively, with the following tables:

Q_1

$\oplus$	0	c
0	0	c
c	c	c

$*$	0	c
	c	0

Q_2

$\oplus$	c	1
c	c	1
1	1	1

$*$	c	1
	1	c

Therefore $\mathcal{L}_{2\times 2}$ and $\mathcal{L}_2$ are not Smarandache BL-algebras. Thus $\mathcal{D}_{2\times 2,2}$ is not a *bi*-smarandache BL-algebra.

In the following example, we show that every Smarandache BL-algebra is not a *bi*-Smarandache BL-algebra.

Example 5 Let $L_1 = \{0, a, c, 1\}$ and $L_2 = \{0, b, c, d, 1\}$. With the following tables:

L_1

$\odot$	0	a	c	1
0	0	0	0	0
a	0	a	c	a
c	0	c	c	c
1	0	a	c	1

$\to$	0	a	c	1
0	1	1	1	1
a	0	1	c	1
c	0	1	1	1
1	0	a	c	1

L_2

$\odot$	0	b	c	d	1
0	0	0	0	0	0
b	0	b	c	d	b
c	0	c	c	d	c
d	0	d	d	0	d
1	0	b	c	d	1

$\rightarrow$	0	b	c	d	1
0	1	1	1	1	1
b	0	1	c	d	1
c	0	1	1	d	1
d	d	1	1	1	1
1	0	b	c	d	1

For L, whose tables are the following:

L

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	c	c	d	a
b	0	c	b	c	d	b
c	0	c	c	c	d	c
d	0	d	d	d	0	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	b	b	d	1
b	0	a	1	a	d	1
c	0	1	1	1	d	1
d	d	1	1	1	1	1
1	0	a	b	c	d	1

L is BL-algebra such that L is super BL-algebra. $Q_1 = \{0, 1\}$ and $Q_2 = \{0, d, 1\}$ are the only MV-algebras which are properly contained in L_1 and L_2, respectively. Therefore L is not a *bi*-Smarandache BL-algebra, but $Q = \{0, d, 1\}$ is the only MV-algebras which are properly contained in L, which the following tables:

Q

$\oplus$	0	d	1
0	0	d	1
d	d	1	1
1	1	1	1

$*$	0	d	1
	1	d	0

Therefore L is a Smarandache BL-algebras.

Definition 14 *Let $L = L_1 \cup L_2$ be a bi-BL-algebra. If only one of L_1 or L_2 is a Smarandache BL-algebra, then we call L a bi-weak smarandache BL-algebra.*

Example 6 In Example 5, L_2 is a Smarandache BL-algebra and L_1 is not a Smarandache BL-algebra. Thus $L = L_1 \cup L_2$ is a *bi*-weak Smarandache BL-algebra.

Theorem 4 *All bi-Smarandache BL-algebras are bi-weak Smarandache BL-algebras and not conversely.*

Example 7 $\mathcal{H}_{2,2\times 2} = \mathcal{L}_2 \cup \mathcal{L}_{2\times 2}$ is a super BL-algebra. $\mathcal{L}_2$ and $\mathcal{L}_{2\times 2}$ are not Smarandache BL-algebras, thus $\mathcal{H}_{2,2\times 2}$ is not a *bi*-weak Smarandache BL-algebra.

Example 8 In Example 3, L is a *bi*-weak Smarandache BL-algebra (by Theorem 4), but L is not a super BL-algebra.

Theorem 5 *Let $L = L_1 \cup L_2$ be a super BL-algebra and bi-Smarandache BL-algebra. Then L is a Smarandache BL-algebra.*

Proof Let $L = (L_1 \cup L_2, \wedge, \vee, \odot, \rightarrow, 0, 1)$ be a super BL-algebra and *bi*-Smarandache BL-algebra. Then there exist MV-algebras Q_1 and Q_2 of L_1 and L_2, respectively, and we have $0 \in Q_1$ or $0 \in Q_2$. Let $0 \in Q_1$. Now we consider the following cases:

1) If $1 \in Q_1$, then Q_1 is an MV-algebra which is contained in L. Thus L is a Smarandache BL-algebra.

2) If $1 \notin Q_1$, since Q_1 is an MV-algebra of L_1, thus we have the greatest element $g \in L_1$ such that $0^* = g$ and $g^* = 0$. Consider $Q = (Q_1 - \{g\}) \cup \{1\}$. Now we verify that $(Q, \oplus, ^*, 0)$ is an MV-algebra.

Let $x, y \in Q$. Then we have the following cases:

1) Let $x, y \in Q_1 - \{g\}$ and $x, y \neq 1$. Then $x \oplus y \in Q_1$. If $x \oplus y \neq g$, then $x \oplus y \in Q$, now if $x \oplus y = g$, then we replace g with 1. Thus $x \oplus y = 1 \in Q$.

2) Let $x \in Q_1 - \{g\}$ and $y = 1$. Then $x \oplus y = x \oplus 1 = 1 \in Q$.

3) Let $x, y = 1$. Then $x \oplus y = 1 \oplus 1 = 1 \in Q$.

Thus Q is close respect to $\oplus$. And since Q_1 is an MV-algebra, thus for any $x \in Q_1 - \{0\}$, we have $x^{**} = x$ and consider $0^* = 1$ and $1^* = 0$. Therefore Q is close respect to *.

Now we verify that Q satisfy in definition of MV-algebra.
Let $x, y, z \in Q = (Q_1 - \{g\}) \cup \{1\}$. Then we have the following cases:

1) Let $x, y, z \in Q = (Q_1 - \{g\}) - \{1\}$. Since Q_1 is an MV-algebra, thus x, y, z satisfy in definition of MV-algebra (i.e., conditions $((MV_1)$ to $(MV_6))$.

2) Let $x, y, z = 1$. It is clear that x, y, z satisfy in definition of MV-algebra.

3) Let $x = 1$ and $y, z \in (Q_1 - \{g\}) - \{1\}$. In this case, we consider two cases:
(a) If $y \oplus z = g$, then we replace g with 1, i.e., $y \oplus z = 1$ and
(b) If $y \oplus z \neq g$, thus $y \oplus z \in Q_1 - \{g\} \subseteq Q$.

Now we verify conditions (MV_1) to (MV_6).
(MV_1) In Case (a), $x \oplus (y \oplus z) = 1 \oplus 1 = 1$ and $(x \oplus y) \oplus z = (1 \oplus y) \oplus z = 1 \oplus z = 1$.
In Case (b), $1 \oplus (y \oplus z) = (1 \oplus y) \oplus z = 1$. Thus $x \oplus (y \oplus z) = (x \oplus y) \oplus z$.
(MV_2) $x \oplus y = 1 \oplus y = 1 = y \oplus 1 = y \oplus x$.
(MV_3) $x \oplus 0 = 1 \oplus 0 = 1 = x$.
(MV_4) $x^{**} = 1^{**} = 0^* = 1 = x$.
(MV_5) $x \oplus 0^* = 1 \oplus 1 = 1 = x$.
(MV_6) $(x^* \oplus y)^* \oplus y = (1^* \oplus y)^* \oplus y = y^* \oplus y = 1$, since $y \in Q_1 - \{g\}$ and Q_1 is an MV-algebra, and $(y^* \oplus x)^* \oplus x = y \oplus 1 = 1$. Thus $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

4) Let $y = 1$ and $x, z \in (Q_1 - \{g\}) - \{1\}$. In this case, we consider two cases:
(a) If $x \oplus z = g$, then we replace g with 1, i.e., $x \oplus z = 1$ and
(b) If $x \oplus z \neq g$, thus $x \oplus z \in Q_1 - \{g\} \subseteq Q$. This case is similar to Case 3).

5) Let $z = 1$ and $x, y \in (Q_1 - \{g\}) - \{1\}$. In this case, we consider two cases:
(a) If $x \oplus y = g$, then we replace g with 1, i.e., $x \oplus y = 1$ and
(b) If $x \oplus y \neq g$, thus $x \oplus y \in Q_1 - \{g\} \subseteq Q$. This case is similar to Case 3).

6) Let $x, y = 1$ and $z \in (Q_1 - \{g\}) - \{1\}$. It is clear that x, y, z satisfy in definition of MV-algebra.

7) Let $x, z = 1$ and $y \in (Q_1 - \{g\}) - \{1\}$. It is clear that x, y, z satisfy in definition of MV-algebra.

8) Let $y, z = 1$ and $x \in (Q_1 - \{g\}) - \{1\}$. It is clear that x, y, z satisfy in definition of MV-algebra.

Therefore $(Q, \oplus, ^*, 0)$ is an MV-algebra which is properly contained in L. Thus L is a Smarandache BL-algebra.

Example 9 Let $L_1 = \{0, e, f, g\}$ and $L_2 = \{g, a, b, c, d, 1\}$. With the following tables:

L_1

$\odot$	0	e	f	g
0	0	0	0	0
e	0	0	e	e
f	0	e	f	f
g	0	e	f	g

$\to$	0	e	f	g
0	g	g	g	g
e	e	g	g	g
f	0	e	g	g
g	0	e	f	g

L_2

$\odot$	g	a	b	c	d	1
g	g	g	g	g	g	g
a	g	a	g	a	g	a
b	g	g	b	b	b	b
c	g	a	b	c	b	c
d	g	g	b	b	d	d
1	g	a	b	c	d	1

$\to$	g	a	b	c	d	1
g	1	1	1	1	1	1
a	d	1	d	1	d	1
b	a	a	1	1	1	1
c	g	a	d	1	d	1
d	a	a	c	c	1	1
1	g	a	b	c	d	1

For $L = L_1 \cup L_2$, whose tables are the following:

L

$\odot$	0	e	f	g	a	b	c	d	1
0	0	0	0	0	0	0	0	0	0
e	0	0	e	e	e	e	e	e	e
f	0	e	f	f	f	f	f	f	f
g	0	e	f	g	g	g	g	g	g
a	0	e	f	g	a	g	a	g	a
b	0	e	f	g	g	b	b	b	b
c	0	e	f	g	a	b	c	b	c
d	0	e	f	g	g	b	b	d	d
1	0	e	f	g	a	b	c	d	1

$\to$	0	a	b	c	d	n	e	f	1
0	1	1	1	1	1	1	1	1	1
e	e	1	1	1	1	1	1	1	1
f	0	e	1	1	1	1	1	1	1
g	0	e	f	1	1	1	1	1	1
a	0	e	f	d	1	d	1	d	1
b	0	e	f	a	a	1	1	1	1
c	0	e	f	g	a	d	1	d	1
d	0	e	f	a	a	c	c	1	1
1	0	e	f	g	a	b	c	d	1

Then L is super BL-algebra. $Q_1 = \{0, e, g\}$ and $Q_2 = \{g, a, d, 1\}$ are MV-algebras which are properly contained in L_1 and L_2, respectively, with the following tables:

Q_1

$\oplus$	0	e	g
0	0	e	g
e	e	g	g
g	g	g	g

$*$	0	e	g
	g	e	0

Q_2

$\oplus$	g	a	d	1
g	g	a	d	1
a	a	a	a	1
d	d	a	d	1
1	1	1	1	1

$*$	g	a	d	1
	1	d	a	g

Therefore L_1 and L_2 are Smarandache BL-algebras. Thus L is a *bi*-Smarandache BL-algebra. Also $\acute{Q} = \{0, e, 1\}$ is the only MV-algebra which is properly contained in L, with the following tables:

$\acute{Q}$

$\oplus$	0	e	1
0	0	e	1
e	e	1	1
1	1	1	1

$*$	0	e	1
	1	e	0

Therefore L is a Smarandache BL-algebra.

From now on, $(Q_i, \oplus, ^*, 0)$ is an MV-algebra unless otherwise specified.

Definition 15 *Let $L = L_1 \cup L_2$ be a bi-BL-algebra. A nonempty subset I of L is called bi-Smarandache ideal of L related to Q (or briefly bi-Q-Smarandache ideal of L), where $Q = Q_1 \cup Q_2$ if $I = I_1 \cup I_2$ such that I_1 and I_2 are Q_1-Smarandache ideal of L_1 and Q_2-Smarandache ideal of L_2, respectively.*

Example 10 In Example 3, we consider $I_1 = \{0, a\}$ and $I_2 = \{n, e, 1\}$. I_1 is a Q_1-Smarandache ideal of L_1 and I_2 is a Q_2-Smarandache ideal of L_2. Thus $I = I_1 \cup I_2 = \{0, a, n, e, 1\}$ is a *bi*-Q-Smarandache ideal of L, where $Q = Q_1 \cup Q_2 = \{0, a, d, n, e, 1\}$.

Theorem 6 *Let $L = L_1 \cup L_2$ be a bi-BL-algebra and $I = I_1 \cup I_2$ be a bi-ideal of L. Then I is a bi-Q-Smarandache ideal of L.*

Proof Let $I = I_1 \cup I_2$ be a *bi*-ideal of $L = L_1 \cup L_2$. Then I_1 is an ideal of L_1 and I_2 is an ideal of L_2, hence by Theorem 2, I_1 is a Q_1-Smarandache ideal of L_1 and I_2 is a Q_2-Smarandache ideal of L_2. Thus $I = I_1 \cup I_2$ is a *bi*-Q-Smarandache ideal of L, where $Q = Q_1 \cup Q_2$.

In the following example, we show that the converse of Theorem 6 is not true.

Example 11 In Example 3, consider $I_1 = \{0, a, d, n\}$. It is clear that I_1 is a Q_1-Smarandache ideal but not an ideal of L_1. Since $d \in I_1$, $(d^* \to b^*)^* = n^* = 0 \in I_1$

but $b \notin I_1$ and $I_2 = \{n, e, 1\}$ is a Q_2-Smarandache ideal but not an ideal of L_2. Since $n \in I_2$, $(n^* \to f^*)^* = n^* = 1 \in I_2$ but $f \notin I_2$. Thus $I = I_1 \cup I_2 = \{0, a, d, n, e, 1\}$ is not a *bi*-ideal of L.

Definition 16 *Let $L = L_1 \cup L_2$ be a bi-BL-algebra. A bi-Q-Smarandache ideal $I = I_1 \cup I_2$ of $L = L_1 \cup L_2$ is called a bi-Smarandache implicative ideal of L related to $Q = Q_1 \cup Q_2$ (or briefly bi-Q-Smarandache implicative ideal of L) if I_1 and I_2 are Q_1-Smarandache implicative ideal of L_1 and Q_2-Smarandache implicative ideal of L_2, respectively.*

Example 12 In Example 3, $I_1 = \{0, a\}$ is a Q_1-Smarandache implicative ideal of L_1 and $I_2 = \{n, e, 1\}$ is a Q_2-Smarandache implicative ideal of L_2. Thus $I = I_1 \cup I_2 = \{0, a, n, e, 1\}$ is a *bi*-Q-Smarandache implicative ideal of L, where $Q = Q_1 \cup Q_2 = \{0, a, d, n, e, 1\}$.

Example 13 Let $L_1 = \{0, a, b, c, d, e, f, g, n\}$ and $L_2 = \{n, h, i, 1\}$. With the following tables:

L_1

$\odot$	0	a	b	c	d	e	f	g	n
0	0	0	0	0	0	0	0	0	0
a	0	0	a	0	0	a	0	0	a
b	0	a	b	0	a	b	0	a	b
c	0	0	0	0	0	0	c	c	c
d	0	0	a	0	0	a	c	c	d
e	0	a	b	0	a	b	c	d	e
f	0	0	0	c	c	c	f	f	f
g	0	0	a	c	c	d	f	f	g
n	0	a	b	c	d	e	f	g	n

$\to$	0	a	b	c	d	e	f	g	n
0	n	n	n	n	n	n	n	n	n
a	g	n	n	g	n	n	g	n	n
b	f	g	n	f	g	n	f	g	n
c	e	e	e	n	n	n	n	n	n
d	d	e	e	g	n	n	g	n	n
e	c	d	e	f	g	n	f	g	n
f	b	b	b	e	e	e	n	n	n
g	a	b	b	d	e	e	g	n	n
n	0	a	b	c	d	e	f	g	n

L_2

$\odot$	n	h	i	1
n	n	n	n	n
h	n	n	h	h
i	n	h	i	i
1	n	h	i	1

$\to$	n	h	i	1
n	1	1	1	1
h	h	1	1	1
i	n	h	1	1
1	n	h	i	1

For $L = L_1 \cup L_2$, whose tables are the following:

⊙	0	a	b	c	d	e	f	g	n	h	i	1
0	0	0	0	0	0	0	0	0	0	0	0	0
a	0	0	a	0	0	a	0	0	a	a	a	a
b	0	a	b	0	a	b	0	a	b	b	b	b
c	0	0	0	0	0	0	c	c	c	c	c	c
d	0	0	a	0	0	a	c	c	d	d	d	d
e	0	a	b	0	a	b	c	d	e	e	e	e
f	0	0	0	c	c	c	f	f	f	f	f	f
g	0	0	a	c	c	d	f	f	g	g	g	g
n	0	a	b	c	d	e	f	g	n	n	n	n
h	0	a	b	c	d	e	f	g	n	n	h	h
n	0	a	b	c	d	e	f	g	n	h	i	i
n	0	a	b	c	d	e	f	g	n	h	i	1

→	0	a	b	c	d	e	f	g	n	h	i	1
0	1	1	1	1	1	1	1	1	1	1	1	1
a	g	1	1	g	1	1	g	1	1	1	1	1
b	f	g	1	f	g	1	f	g	1	1	1	1
c	e	e	e	1	1	1	1	1	1	1	1	1
d	d	e	e	g	1	1	g	1	1	1	1	1
e	c	d	e	f	g	1	f	g	1	1	1	1
f	b	b	b	e	e	e	1	1	1	1	1	1
g	a	b	b	d	e	e	g	1	1	1	1	1
n	0	a	b	c	d	e	f	g	1	1	1	1
h	0	a	b	c	d	e	f	g	h	1	1	1
n	0	a	b	c	d	e	f	g	n	h	1	1
1	0	a	b	c	d	e	f	g	n	h	i	1

Then $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a *bi-BL*-algebra. $Q_1 = \{0, b, f, c, e, n\}$ and $Q_2 = \{n, h, 1\}$ are MV-algebras which are properly contained in L_1 and L_2, respectively, with the following tables:

Q_1

⊕	0	b	c	e	f	n
0	0	b	c	e	f	n
b	b	b	e	e	n	n
c	c	e	f	n	f	1
e	e	e	n	n	n	n
f	f	n	f	n	f	n
n	n	n	n	n	n	n

∗	0	b	c	e	f	n
	n	f	e	c	b	0

Q_2

⊕	n	h	1
n	n	h	1
h	h	1	1
1	1	1	1

∗	n	h	1
	1	h	n

Therefore L is a *bi*-Smarandache BL-algebra. Then $I_1 = \{0, b\}$ is Q_1-Smarandache ideal of L_1, but not a Q_1-Smarandache implicative ideal of L_1. Since $(f * c) * e = (f \odot e) \odot c = 0 \in I_1$ and $c * e = c \odot c = 0 \in I_1$, but $f * e = f \odot c = c \notin I_1$. $I_2 = \{n, h, 1\}$ is a Q_2-Smarandache implicative ideal of L_2. Thus $I = I_1 \cup I_2$ is a *bi*-Q-Smarandache ideal of $L = L_1 \cup L_2$, but not a *bi*-Q-Smarandache implicative ideal of L.

Definition 17 *Let $L = L_1 \cup L_2$ be a bi-BL-algebra. A nonempty subset F of L is called bi-Smarandache implicative filter of L related to Q, where $Q = Q_1 \cup Q_2$ (or* *briefly bi-Q-Smarandache implicative filter of L), if $F = F_1 \cup F_2$ such that F_1 and F_2 are Q_1-Smarandache implicative filters of L_1 and Q_2-Smarandache implicative filter of L_2, respectively.*

Example 14 In Example 3, $F_1 = \{d, n\}$ is a Q_1-Smarandache implicative filter of L_1 and $F_2 = \{f, 1\}$ is a Q_2-Smarandache implicative filter of L_2. Thus $F = F_1 \cup F_2 = \{d, n, f, 1\}$ is a *bi-Q*-Smarandache implicative filter of L, where $Q = Q_1 \cup Q_2$.

Remark 3 *Let F be a bi-Q-Smarandache implicative filter of L. Then $F \neq \phi$ and F is not a bi-Smarandache BL-algebra since $0 \notin F$.*

Proposition 1 *Each filter of a BL-algebra is a Q-Smarandache implicative filter and not conversely.*

Proof Let F be a filter of a BL-algebra L. Then $1 \in F$. Now let $x \in F$, $y \in Q$ and $x \to y \in F$. Since $Q \subseteq L$, then $y \in L$, thus $y \in F$. Therefore F is a Q-Smarandache implicative filter.

Consider BL-algebra $\mathcal{L}_{3\times 2}$, with the following tables:

$\mathcal{L}_{3\times 2}$

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

$Q = \{0, a, d, 1\}$ is an MV-algebra which is properly contained in $\mathcal{L}_{3\times 2}$, with the following tables:

Q

$\oplus$	0	a	d	1
0	0	a	d	1
a	a	a	1	1
d	d	1	d	1
1	1	1	1	1

$*$	0	a	d	1
	1	d	a	0

Therefore $\mathcal{L}_{3\times 2}$ is Smarandache BL-algebra. Then $F = \{a, 1\}$ is a Q-Smarandache implicative filter of $\mathcal{L}_{3\times 2}$, but not a filter of $\mathcal{L}_{3\times 2}$, since $a \leq c$ and $a \in F$, but $c \notin F$.

Proposition 2 *Each bi-filter of a bi-BL-algebra is a bi-Q-Smarandache implicative-filter and not conversely.*

Definition 18 *Let $L = L_1 \cup L_2$ be a bi-Smarandache BL-algebra. A bi-Q-Smarandache ideal $M = M_1 \cup M_2$ of L is called bi-maximal-Q-Smarandache ideal, where $Q = Q_1 \cup Q_2$ if only if the following conditions hold:*

(M_1) *M_i is a proper Q_i-Smarandache ideal.*

(M_2) *For every Q_i-Smarandache ideal I_i such that $M_i \subseteq I_i$, we have either $M_i = I_i$ or $I_i = L_i$,*

where $i = 1, 2$.

Example 15 In Example 3, $I_1 = \{0, a, c, d, n\}$ is maximal Q_1-Smarandache ideal of L_1 and $I_2 = \{n, e, 1\}$ is maximal Q_2-Smarandache ideal of L_2. Thus $I = I_1 \cup I_2 = \{0, a, c, d, n, e, 1\}$ is a *bi*-maximal-Q-Smarandache ideal of L, where $Q = Q_1 \cup Q_2$.

Definition 19 *Let $L = L_1 \cup L_2$ be a bi-Smarandache BL-algebra. Then there exist MV-algebras Q_1 and Q_2 which are properly contained in L_1 and L_2, respectively. Then $\frac{L_i}{Q_i} = \{[x]_{Q_i} | x \in L_i\}$ and $[x]_{Q_i} = \{y \in L_i | x \sim_{Q_i} y\} = \{y \in L_i | x \to y \in Q_i, y \to x \in Q_i\}$ are quotient algebras via the congruence relations $\sim_{Q_i}$, where $i = 1, 2$ (or briefly bi-Smarandache quotient BL-algebra).*

We defined on $\frac{L_i}{Q_i}$:
$[x]_{Q_i} \oplus [y]_{Q_i} = [x \oplus y]_{Q_i}$, $[x]^*_{Q_i} = [x^*]_{Q_i}$, $[x]_{Q_i} \to [y]_{Q_i} = [x \to y]_{Q_i}$,
$[x]_{Q_i} \odot [y]_{Q_i} = [x \odot y]_{Q_i}$, $[x]_{Q_i} \wedge [y]_{Q_i} = [x \wedge y]_{Q_i}$, $[x]_{Q_i} \vee [y]_{Q_i} = [x \vee y]_{Q_i}$,
$[0]_{Q_i} = \frac{0}{Q_i}$, $[1]_{Q_i} = \frac{1}{Q_i}$, *where $i = 1, 2$.*
Then $\frac{\mathcal{L}}{\mathcal{Q}} := \frac{L_1}{Q_1} \cup \frac{L_2}{Q_2}$.

Example 16 In Example 3, consider $L_1 = \{0, a, b, c, d, n\}$, $L_2 = \{n, e, f, 1\}$, $Q_1 = \{0, a, d, n\}$ and $Q_2 = \{n, e, 1\}$, then $\frac{L_1}{Q_1} = \{[0]_{Q_1}, [a]_{Q_1}, [b]_{Q_1}, [c]_{Q_1}, [d]_{Q_1}, [n]_{Q_1}\}$ and $\frac{L_2}{Q_2} = \{[n]_{Q_2}, [e]_{Q_2}, [f]_{Q_2}, [1]_{Q_2}\}$ such that $[0]_{Q_1} = [a]_{Q_1} = [d]_{Q_1} = [n]_{Q_1} = \{0, a, d, n\}$ and $[b]_{Q_1} = [c]_{Q_1} = \{b, c\}$ and $[n]_{Q_2} = [e]_{Q_2} = [f]_{Q_2} = [1]_{Q_2} = \{n, e, f, 1\}$.

Thus $\frac{\mathcal{L}}{\mathcal{Q}} = \{[0]_{Q_1}, [b]_{Q_1}, [1]_{Q_2}\}$.

Example 17 In Example 9, consider $L_1 = \{0, e, f, g\}$, $L_2 = \{g, a, b, c, d, 1\}$, $Q_1 = \{0, e, g\}$ and $Q_2 = \{g, a, d, 1\}$, then in $\frac{L_1}{Q_1}$, we have $[0]_{Q_1} = [e]_{Q_1} = [f]_{Q_1} = [g]_{Q_1}$, thus $\frac{L_1}{Q_1} = \{[0]_{Q_1}\}$ and in $\frac{L_2}{Q_2}$, we have $[g]_{Q_2} = [a]_{Q_2} = [b]_{Q_2} = [c]_{Q_2} = [d]_{Q_2} = [1]_{Q_2}$, thus $\frac{L_2}{Q_2} = \{[g]_{Q_2}\}$. Therefore $\frac{\mathcal{L}}{\mathcal{Q}} = \{[0]_{Q_1}, [g]_{Q_2}\}$.

But in $\frac{L}{\acute{Q}}$, we have $[0]_{\acute{Q}} = [e]_{\acute{Q}} = [g]_{\acute{Q}} = [a]_{\acute{Q}} = [b]_{\acute{Q}} = [c]_{\acute{Q}} = [d]_{\acute{Q}} = [1]_{\acute{Q}}$, then $\frac{L}{\acute{Q}} = \{[0]_{\acute{Q}}\}$. Thus $\frac{\mathcal{L}}{\mathcal{Q}} \neq \frac{L}{\acute{Q}}$.

4. *bi*-Strong Smarandache *BL*-algebra

Definition 20 *Let $L = (L, \wedge, \vee, \odot, \to, 0, 1)$ be a BL-algebra. If there exists a chain of proper subsets*

$$P_{n-1} < P_{n-2} < \cdots < P_2 < P_1 < L,$$

where " $<$ " means"included in" whose corresponding structure verify the inverse chain

$$W_{n-1} > W_{n-2} > \cdots > W_2 > W_1 > L,$$

where " $>$ " signifies strictly strong (i.e., structure satisfying more axioms). Then we call $L = (L, \wedge, \vee, \odot, \to, 0, 1)$ a strong Smarandache BL-algebra of rank n.

Remark 4 *In above definition, W_2 can be a Boolean algebra and W_1 can be an MV-algebra.*

Example 18 Let $L = \{0, a, b, c, d, 1\}$. With the following tables:

L

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	b	b	d	0	a
b	0	b	b	0	0	b
c	0	d	0	c	d	c
d	0	0	0	d	0	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	g	a	b	c	d	1

$L = (L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL-algebra. $A = \{0, b, c, 1\}$ is an MV-algebra, $B = \{0, b, 1\}$ is a Boolean algebra and $B \subset A \subset L$. Thus L is a strong Smarandache BL-algebra of rank 3.

Proposition 3 *Every strong Smarandache BL-algebra of rank n such that $n \geq 2$, is a Smarandache BL-algebra.*

Corollary 1 *Every strong Smarandache BL-algebra of rank 2 is a Smarandache BL-algebra.*

The following example shows that the converse of Corollary 1 is not true.

Example 19 In Example 18, $A = \{0, b, c, 1\}$ is an MV-algebra which is properly contained in L. Thus L is a Smarandache BL-algebra, but L is not a strong Smarandache BL-algebra of rank 2.

Definition 21 *Let $L = L_1 \cup L_2$ be a bi-BL-algebra. If L_1 is a strong Smarandache BL-algebra of rank n_1 and L_2 is a strong Smarandache BL-algebra of rank n_2, then we call $L = L_1 \cup L_2$ a bi-strong Smarandache BL-algebra of rank n_1, n_2.*

If only one of L_1 or L_2 is a strong Smarandache BL-algebra of rank n_1 or n_2, respectively, then $L = L_1 \cup L_2$ is a bi-weak Smarandache BL-algebra.

Example 20 In Example 3, L_1 is a strong Smarandache BL-algebra of rank 3. Since $Q_1 = \{0, a, d, 1\}$ is an MV-algebra, $B_1 = \{0, d, 1\}$ is a Boolean algebra and $B_1 \subset Q_1 \subset L_1$.

L_2 is a strong Smarandache BL-algebra of rank 2. Since $Q_2 = \{n, e, 1\}$ is an MV-algebra and $Q_1 \subset L_2$. Thus $L = L_1 \cup L_2$ is a *bi*-weak Smarandache BL-algebra of rank $3, 2$.

Proposition 4 *Every bi-strong Smarandache BL-algebra of rank n_1, n_2 such that $n_1, n_2 \geq 2$, is a bi-Smarandache BL-algebra.*

Corollary 2 *Every bi-strong Smarandache BL-algebra of rank 2,2, is a bi-Smarandache BL-algebra.*

The following example shows that the converse of Corollary 2 is not true.

Example 21 In Example 3, L is a *bi*-Smarandache BL-algebra, but L is a *bi*-strong Smarandache BL-algebra of rank $3, 2$.

Now we consider case that $L = L_1 \cup L_2$ is a super BL-algebra.

Example 22 In Example 9, L_1 is a strong Smarandache BL-algebra of rank 2, since $Q_1 = \{0, e, g\}$ is an MV-algebra and $Q_1 \subset L_1$ and L_2 is a strong Smarandache BL-algebra of rank 3, since $Q_2 = \{g, a, d, 1\}$ is an MV-algebra and $B = \{g, d, 1\}$ is a Boolean algebra and $B \subset Q_2 \subset L_2$.

Thus $L = L_1 \cup L_2$ is a *bi*-strong Smarandache BL-algebra of rank $2, 3$. But in BL-algebra L, we have $Q = \{0, e, 1\}$ is the only MV-algebra which is properly contained in L and $Q \subset L$. Therefore L is a strong Smarandache BL-algebra of rank 2 (or Smarandache BL-algebra).

We show that in a strong Smarandache BL-algebra, and rank is not unique.

Example 23 Let $L = \{0, a, b, c, d, e, f, g, 1\}$. Then L is a BL-algebra with the following tables:

	$\odot$	0	a	b	c	d	e	f	g	1
	0	0	0	0	0	0	0	0	0	0
	a	0	0	a	0	0	a	0	0	a
	b	0	a	b	0	a	b	0	a	b
L	c	0	0	0	0	0	0	c	c	c
	d	0	0	a	0	0	a	c	c	d
	e	0	a	b	0	a	b	c	d	e
	f	0	0	0	c	c	c	f	f	f
	g	0	0	a	c	c	d	f	f	g
	1	0	a	b	c	d	e	f	g	1

$\rightarrow$	0	a	b	c	d	e	f	g	1
0	1	1	1	1	1	1	1	1	1
a	g	1	1	g	1	1	g	1	1
b	f	g	1	f	g	1	f	g	1
c	e	e	e	1	1	1	1	1	1
d	d	e	e	g	1	1	g	1	1
e	c	d	e	f	g	1	f	g	1
f	b	b	b	e	e	e	1	1	1
g	a	b	b	d	e	e	g	1	1
1	0	a	b	c	d	e	f	g	1

$Q_1 = \{0, d, 1\}$ is an MV-algebras which is properly contained in L, i.e., $Q_1 \subset L$. Then L is a strong Smarandache BL-algebra of rank 2.

Now we consider MV-algebra $Q_2 = \{0, b, f, c, e, 1\}$ which is properly contained in L. $B_2 = \{0, b, f, 1\}$ is a Boolean algebra which is properly contained in Q_2. Thus $B_2 \subset Q_2 \subset L$. Then L is a strong Smarandache BL-algebra of rank 3.

Theorem 7 *All bi-strong Smarandache BL-algebras of rank* n_1, n_2 *are bi-weak Smarandache BL-algebras and not conversely.*

proof By Proposition 4 and Theorem 4.

5. Conclusion

Smarandache structure occurs as a weak structure in any structure.

In the present paper, by using this notion, we have introduced the concept of *bi*-Smarandache BL-algebras and investigated some of their useful properties. We have also presented definition of strong Smarandache *BL*-algebra and *bi*-strong Smarandache *BL*-algebra and investigated relationship between strong Smarandache *BL*-algebras with Smarandache *BL*-algebras and relationship between *bi*-strong Smarandache *BL*-algebras with *bi*-Smarandache *BL*-algebras and introduced the notion of *bi*-weak Smarandache *BL*-algebras and investigated relationship between *bi*-weak Smarandache *BL*-algebras with *bi*-Smarandache *BL*-algebras and *bi*-strong Smarandache *BL*-algebras.

In our future study of *bi*-Smarandache *BL*-algebras, maybe the following topics should be considered:

(1) To get more results in *bi*-Smarandache *BL*-algebras and application;

(2) To obtain more results in *bi*-strong Smarandache *BL*-algebra and application;

(3) To have more connection to strong Smarandache *BL*-algebra and Smarandache *BL*-algebra;

(4) To grasp more connection to *bi*-strong Smarandache *BL*-algebra and *bi*-Smarandache *BL*-algebra;

(5) To have more connection of ranks *bi*-strong Smarandache *BL*-algebra together.

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