



New Operators Using Neutrosophic δ -Open Set

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Abstract

In this paper, we introduce some new operators called neutrosophic δ frontier, neutrosophic δ border and neutrosophic δ exterior with the help of neutrosophic δ -open sets in neutrosophic topological space. Also, we discuss the important properties of them and the relations between them.

Keywords: neutrosophic open set, neutrosophic δ -open set, neutrosophic δ frontier, neutrosophic δ border, neutrosophic δ exterior

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1 Introduction

In 1965, the idea of fuzzy set (briefly, fs) gives a degree of membership function was first introduced by Zadeh.¹⁹ In 1968, the concept of fuzzy topological space (briefly, fts) was introduced by Chang.⁴ In 1983, the next stage of fuzzy set was developed by Atanassov³ which gives a degree of membership and a degree of non-membership functions named as intuitionistic fuzzy set (briefly, IFS). In 1997, Coker⁵ introduced the concept of intuitionistic fuzzy topological space (briefly, $IFts$) in intuitionistic fuzzy set. In 2005, the concept of neutrosophic crisp set and neutrosophic set (briefly, $N_s s$) was investigated by Smaradache.^{9,14,15} After the introduction of neutrosophic set, there are many fields of mathematics and various applications.^{1,7,8,13} In 2012, Salama and Alblawi¹⁰ defined neutrosophic topological space (briefly, $N_s ts$) and many of its applications in.^{11,12} The neutrosophic closed sets and neutrosophic continuous functions were introduced by Salama et al.¹² in 2014. Saha¹⁶ defined δ -open sets in topological spaces. Vadivel et al. in^{17,18} introduced δ -open sets and their maps in a neutrosophic topological space. P. Iswarya and K. Bageerathi,⁶ studied neutrosophic frontier and semi-frontier in neutrosophic topological spaces.

In this paper we introduce neutrosophic δ frontier, neutrosophic δ border and neutrosophic δ exterior and discuss their properties in $N_s ts$'s.

2 Preliminaries

Definition 2.1. ¹⁰ Let Y be a non-empty set. A neutrosophic set (briefly, $N_s s$) L is an object having the form $L = \{\langle y, \mu_L(y), \sigma_L(y), \nu_L(y) \rangle : y \in Y\}$ where $\mu_L \rightarrow [0, 1]$ denote the degree of membership function, $\sigma_L \rightarrow [0, 1]$ denote the degree of indeterminacy function and $\nu_L \rightarrow [0, 1]$ denote the degree of non-membership function respectively of each element $y \in Y$ to the set L and $0 \leq \mu_L(y) + \sigma_L(y) + \nu_L(y) \leq 3$ for each $y \in Y$.

Remark 2.2. ¹⁰ A $N_s s$ $L = \{\langle y, \mu_L(y), \sigma_L(y), \nu_L(y) \rangle : y \in Y\}$ can be identified to an ordered triple $\langle y, \mu_L(y), \sigma_L(y), \nu_L(y) \rangle$ in $[0, 1]$ on Y .

Definition 2.3. ¹⁰ Let Y be a non-empty set and the $N_s s$'s L and M in the form $L = \{\langle y, \mu_L(y), \sigma_L(y), \nu_L(y) \rangle : y \in Y\}$, $M = \{\langle y, \mu_M(y), \sigma_M(y), \nu_M(y) \rangle : y \in Y\}$, then

- (i) $0_N = \langle y, 0, 0, 1 \rangle$ and $1_N = \langle y, 1, 1, 0 \rangle$,
- (ii) $L \subseteq M$ iff $\mu_L(y) \leq \mu_M(y)$, $\sigma_L(y) \leq \sigma_M(y)$ & $\nu_L(y) \geq \nu_M(y) : y \in Y$,

- (iii) $L = M$ iff $L \subseteq M$ and $M \subseteq L$,
- (iv) $1_N - L = \{\langle y, \nu_L(y), 1 - \sigma_L(y), \mu_L(y) \rangle : y \in Y\} = L^c$,
- (v) $L \cup M = \{\langle y, \max(\mu_L(y), \mu_M(y)), \max(\sigma_L(y), \sigma_M(y)), \min(\nu_L(y), \nu_M(y)) \rangle : y \in Y\}$,
- (vi) $L \cap M = \{\langle y, \min(\mu_L(y), \mu_M(y)), \min(\sigma_L(y), \sigma_M(y)), \max(\nu_L(y), \nu_M(y)) \rangle : y \in Y\}$.

Definition 2.4.¹⁰ A neutrosophic topology (briefly, $N_s t$) on a non-empty set Y is a family Ψ_N of neutrosophic subsets of Y satisfying

- (i) $0_N, 1_N \in \Psi_N$.
- (ii) $L_1 \cap L_2 \in \Psi_N$ for any $L_1, L_2 \in \Psi_N$.
- (iii) $\bigcup L_x \in \Psi_N, \forall L_x : x \in X \subseteq \Psi_N$.

Then (Y, Ψ_N) is called a neutrosophic topological space (briefly, $N_s ts$) in Y . The Ψ_N elements are called neutrosophic open sets (briefly, $N_s os$) in Y . A $N_s s$ C is called a neutrosophic closed sets (briefly, $N_s cs$) iff its complement C^c is $N_s os$.

Definition 2.5.¹⁰ Let (Y, Ψ_N) be $N_s ts$ on Y and L be an $N_s s$ on Y , then the neutrosophic interior of L (briefly, $N_s int(L)$) and the neutrosophic closure of L (briefly, $N_s cl(L)$) are defined as

$$N_s int(L) = \bigcup \{I : I \subseteq L \text{ and } I \text{ is a } N_s os \text{ in } Y\}$$

$$N_s cl(L) = \bigcap \{J : L \subseteq J \text{ and } J \text{ is a } N_s cs \text{ in } Y\}.$$

Definition 2.6.² Let (Y, Ψ_N) be $N_s ts$ on Y and L be an $N_s s$ on Y . Then L is said to be a neutrosophic regular open set (briefly, $N_s ros$) if $L = N_s int(N_s cl(L))$.

The complement of a $N_s ros$ is called a neutrosophic regular closed set (briefly, $N_s rcs$) in Y .

Definition 2.7.¹⁷ A set K is said to be a neutrosophic

- (i) δ interior of G (briefly, $N_s \delta int(K)$) is defined by $N_s \delta int(K) = \bigcup \{B : B \subseteq K \text{ and } B \text{ is a } N_s ros \text{ in } Y\}$.
- (ii) δ closure of K (briefly, $N_s \delta cl(K)$) is defined by $N_s \delta cl(K) = \bigcap \{J : K \subseteq J \text{ and } J \text{ is a } N_s rcs \text{ in } Y\}$.

Definition 2.8.¹⁷ A set L is said to be a neutrosophic δ -open set (briefly, $N_s \delta os$) if $L = N_s \delta int(L)$.

The complement of an $N_s \delta os$ is called a neutrosophic δ closed set (briefly, $N_s \delta cs$) in Y .

Proposition 2.9.¹⁷ The Neutrosophic δ -interior operator satisfies

- (i) $N_s \delta int(K) \subseteq K$.
- (ii) $K \subseteq M \Rightarrow N_s \delta int(K) \subseteq N_s \delta int(M)$.
- (iii) $N_s \delta int(K \cap M) = N_s \delta int(K) \cap N_s \delta int(M)$.
- (iv) $N_s \delta int(K)$ is the greatest $N_s \delta os$ containing K .
- (v) $N_s \delta int(K) = K$ iff K is an $N_s \delta os$.
- (vi) $N_s \delta int(N_s \delta int(K)) = N_s \delta int(K)$.
- (vii) $(1_{N_s} - N_s \delta int(K)) = N_s \delta cl(1_{N_s} - K)$.

Proposition 2.10.¹⁷ The Neutrosophic δ -closure operator satisfies

- (i) $K \subseteq N_s \delta cl(K)$.
- (ii) $K \subseteq M \Rightarrow N_s \delta cl(K) \subseteq N_s \delta cl(M)$.
- (iii) $N_s \delta cl(K \cup M) = N_s \delta cl(K) \cup N_s \delta cl(M)$.
- (iv) $N_s \delta cl(K)$ is the smallest $N_s \delta cs \subseteq K$.

- (v) $N_s\delta cl(K) = K$ iff K is an $N_s\delta c$ set.
- (vi) $N_s\delta cl(N_s\delta cl(K)) = N_s\delta cl(K)$.
- (vii) $(1_{N_s} - N_s\delta cl(K)) = N_s\delta int(1_{N_s} - K)$.
- (viii) $y \in N_s\delta cl(K)$ iff $K \cap C \neq \emptyset$ for every $N_s\delta os$ C containing y .
- (ix) $N_s\delta cl(S \cap T) \subseteq N_s\delta cl(S) \cap N_s\delta cl(T)$.

Proposition 2.11. ¹⁷ The statements are hold for $N_s ts$. Every $N_s\delta os$ (resp. $N_s\delta cs$) is a $N_s os$ (resp. $N_s cs$). But not converse.

3 Neutrosophic δ frontier

In this section, we introduce neutrosophic δ frontier and discuss their properties in neutrosophic topological spaces.

Definition 3.1. Let (Y, Ψ_N) be a $N_s ts$. Let A be a neutrosophic subset of Y . Then the neutrosophic (resp. δ) frontier of a neutrosophic subset A were denoted by $N_s Fr(A)$ (resp. $N_s\delta Fr(A)$) and were defined by $N_s Fr(A) = N_s cl(A) \cap N_s cl(A^c)$ (resp. $N_s\delta Fr(A) = N_s\delta cl(A) \cap N_s\delta cl(A^c)$).

Example 3.2. Let $Y = \{l, m, n\}$ and define N_s 's Y_1, Y_2 & Y_3 in Y are

$$Y_1 = \langle Y, (\frac{\mu_l}{0.2}, \frac{\mu_m}{0.3}, \frac{\mu_n}{0.4}), (\frac{\sigma_l}{0.5}, \frac{\sigma_m}{0.5}, \frac{\sigma_n}{0.5}), (\frac{\nu_l}{0.8}, \frac{\nu_m}{0.7}, \frac{\nu_n}{0.6}) \rangle,$$

$$Y_2 = \langle Y, (\frac{\mu_l}{0.1}, \frac{\mu_m}{0.1}, \frac{\mu_n}{0.4}), (\frac{\sigma_l}{0.5}, \frac{\sigma_m}{0.5}, \frac{\sigma_n}{0.5}), (\frac{\nu_l}{0.9}, \frac{\nu_m}{0.9}, \frac{\nu_n}{0.6}) \rangle.$$

Then we have $\tau_N = \{0_N, Y_1, Y_2, 1_N\}$. Let $A = \langle Y, (\frac{\mu_l}{0.2}, \frac{\mu_m}{0.4}, \frac{\mu_n}{0.4}), (\frac{\sigma_l}{0.5}, \frac{\sigma_m}{0.5}, \frac{\sigma_n}{0.5}), (\frac{\nu_l}{0.8}, \frac{\nu_m}{0.6}, \frac{\nu_n}{0.6}) \rangle$, then

- (i) $N_s Fr(A) = \langle Y, (\frac{\mu_l}{0.8}, \frac{\mu_m}{0.7}, \frac{\mu_n}{0.6}), (\frac{\sigma_l}{0.5}, \frac{\sigma_m}{0.5}, \frac{\sigma_n}{0.5}), (\frac{\nu_l}{0.2}, \frac{\nu_m}{0.3}, \frac{\nu_n}{0.4}) \rangle$.
- (ii) $N_s\delta Fr(A) = \langle Y, (\frac{\mu_l}{0.8}, \frac{\mu_m}{0.7}, \frac{\mu_n}{0.6}), (\frac{\sigma_l}{0.5}, \frac{\sigma_m}{0.5}, \frac{\sigma_n}{0.5}), (\frac{\nu_l}{0.2}, \frac{\nu_m}{0.3}, \frac{\nu_n}{0.4}) \rangle$.

Remark 3.3. For a neutrosophic subset A of Y , $N_s Fr(A)$ (resp. $N_s\delta Fr(A)$) is $N_s c$ (resp. $N_s\delta c$).

Theorem 3.4. For a neutrosophic subset A in $N_s ts (Y, \Psi_N)$,

- (i) $N_s Fr(A) = N_s Fr(A^c)$.
- (ii) $N_s\delta Fr(A) = N_s\delta Fr(A^c)$.

Proof. (i) Let A be a neutrosophic subset in $N_s ts (Y, \Psi_N)$. Then by Definition 3.1 $N_s Fr(A) = N_s cl(A) \cap N_s cl(A^c) = N_s cl(A^c) \cap N_s cl(A) = N_s cl(A^c) \cap (N_s cl(A^c))^c$. Again by Definition 3.1 this is equal to $N_s Fr(A^c)$. Hence $N_s Fr(A) = N_s Fr(A^c)$.

(ii) Let A be a neutrosophic subset in $N_s ts (Y, \Psi_N)$. Then by Definition 3.1 $N_s\delta Fr(A) = N_s\delta cl(A) \cap N_s\delta cl(A^c) = N_s\delta cl(A^c) \cap N_s\delta cl(A) = N_s\delta cl(A^c) \cap (N_s\delta cl(A^c))^c$. Again by Definition 3.1 this is equal to $N_s\delta Fr(A^c)$. Hence $N_s\delta Fr(A) = N_s\delta Fr(A^c)$. \square

Theorem 3.5. Let A be a neutrosophic subset in $N_s ts (Y, \Psi_N)$. Then

- (i) $N_s Fr(A) = N_s cl(A) - N_s int(A)$.
- (ii) $N_s\delta Fr(A) = N_s\delta cl(A) - N_s\delta int(A)$.

Proof. (i) Let A be a neutrosophic subset in $N_s ts (Y, \Psi_N)$. By Theorem 2.10 (vii), $(N_s cl(A^c))^c = N_s int(A)$ and by Definition 3.1, $N_s Fr(A) = N_s cl(A) \cap (N_s cl(A^c))^c = N_s cl(A) \cap (N_s int(A))^c$. By using $A - B = A \cap B^c$, $N_s Fr(A) = N_s cl(A) - N_s int(A)$. Hence $N_s Fr(A) = N_s cl(A) - N_s int(A)$.

(ii) Let A be a neutrosophic subset in $N_s ts (Y, \Psi_N)$. By Theorem 2.10 (vii), $(N_s\delta cl(A^c))^c = N_s\delta int(A)$ and by Definition 3.1, $N_s\delta Fr(A) = N_s\delta cl(A) \cap (N_s\delta cl(A^c))^c = N_s\delta cl(A) \cap (N_s\delta int(A))^c$. By using $A - B = A \cap B^c$, $N_s\delta Fr(A) = N_s\delta cl(A) - N_s\delta int(A)$. Hence $N_s\delta Fr(A) = N_s\delta cl(A) - N_s\delta int(A)$. \square

Theorem 3.6. A neutrosophic subset A is $N_s c$ (resp. $N_s\delta c$) set in Y if and only if $N_s Fr(A) \subseteq A$ (resp. $N_s\delta Fr(A) \subseteq A$).

Proof. Let A be a $N_s\delta c$ set in the $N_s ts$ (Y, Ψ_N) . Then by Definition 3.1, $N_s\delta Fr(A) = N_s\delta cl(A) \cap N_s\delta cl(A^c) \subseteq N_s\delta cl(A)$. By using Theorem 2.10 (v), $N_s\delta cl(A) = A$. Hence $N_s\delta Fr(A) \subseteq A$, if A is $N_s\delta c$ in Y .

Conversely, Assume that, $N_s\delta Fr(A) \subseteq A$. Then $N_s\delta cl(A) - N_s\delta int(A) \subseteq A$. Since $N_s\delta int(A) \subseteq A$, we conclude that $N_s\delta cl(A) = A$ and hence A is $N_s\delta c$.

The proof of the other is similar. \square

Theorem 3.7. If A is a $N_s o$ (resp. $N_s\delta o$) set in Y , then $N_s Fr(A) \subseteq A^c$ (resp. $N_s\delta Fr(A) \subseteq A^c$).

Proof. Let A be a $N_s\delta o$ set in the $N_s ts$ (Y, Ψ_N) . By Definition 2.8, A^c is $N_s\delta c$ set in Y . By Theorem 3.6, $N_s\delta Fr(A^c) \subseteq A^c$ and by Theorem 3.6, we get $N_s\delta Fr(A) \subseteq A^c$.

The proof of the other is similar. \square

Theorem 3.8. Let $A \subseteq B$ and B be any $N_s c$ (resp. $N_s\delta c$) set in Y . Then $N_s Fr(A) \subseteq B$ (resp. $N_s\delta Fr(A) \subseteq B$).

Proof. By Theorem 2.10 (ii), $A \subseteq B$, $N_s\delta cl(A) \subseteq N_s\delta cl(B)$. By Definition 3.1, $N_s\delta Fr(A) = N_s\delta cl(A) \cap N_s\delta cl(A^c) \subseteq N_s\delta cl(B) \cap N_s\delta cl(A^c) \subseteq N_s\delta cl(B)$. Then by Proposition 2.10 (v), this is equal to B . Hence $N_s\delta Fr(A) \subseteq B$.

The proof of the other is similar. \square

Theorem 3.9. Let A be a neutrosophic subset in the $N_s ts$ (Y, Ψ_N) . Then $(N_s Fr(A))^c = N_s int(A) \cup N_s int(A^c)$ (resp. $(N_s\delta Fr(A))^c = N_s\delta int(A) \cup N_s\delta int(A^c)$).

Proof. Let A be a neutrosophic subset in the $N_s ts$ (Y, Ψ_N) . Then by Definition 3.1, $(N_s\delta Fr(A))^c = (N_s\delta cl(A) \cap N_s\delta cl(A^c))^c = ((N_s\delta cl(A))^c \cup (N_s\delta cl(A^c))^c)$. By Theorem 2.10 (vii), which is equal to $N_s\delta int(A^c) \cup N_s\delta int(A)$. Hence $(N_s\delta Fr(A))^c = N_s\delta int(A) \cup N_s\delta int(A^c)$.

The proof of the other is similar. \square

Theorem 3.10. For a neutrosophic subset A in the $N_s ts$ (Y, Ψ_N) , then $N_s\delta Fr(A) \subseteq N_s Fr(A)$.

Proof. Let A be a neutrosophic subset in the $N_s ts$ (Y, Ψ_N) . Then by Proposition 2.11, $N_s\delta cl(A) \supseteq N_s cl(A)$ and $N_s cl(A^c) \subseteq N_s\delta cl(A^c)$. By Definition 3.1, $N_s\delta Fr(A) = N_s\delta cl(A) \cap N_s\delta cl(A^c) \subseteq N_s cl(A) \cap N_s cl(A^c)$, this is equal to $N_s Fr(A)$. Hence $N_s\delta Fr(A) \subseteq N_s Fr(A)$. \square

Theorem 3.11. For a neutrosophic subset A in the $N_s ts$ (Y, Ψ_N) , $N_s cl(N_s Fr(A)) \subseteq N_s Fr(A)$ (resp. $N_s\delta cl(N_s\delta Fr(A)) \subseteq N_s\delta Fr(A)$).

Proof. Let A be the neutrosophic subset in the $N_s ts$ (Y, Ψ_N) . Then by Definition 3.1, $N_s\delta cl(N_s\delta Fr(A)) = N_s\delta cl(N_s\delta cl(A) \cap (N_s\delta cl(A^c))) \subseteq (N_s\delta cl(N_s\delta cl(A))) \cap (N_s\delta cl(N_s\delta cl(A^c)))$. By Theorem 2.10 (vi), $N_s\delta cl(N_s\delta Fr(A)) = N_s\delta cl(A) \cap N_s\delta cl(A^c)$. By Definition 3.1, this is equal to $N_s\delta Fr(A)$.

The proof of the other is similar. \square

Theorem 3.12. For a neutrosophic subset A in the $N_s ts$ (Y, Ψ_N) , $N_s Fr(N_s int(A)) \subseteq N_s Fr(A)$ (resp. $N_s\delta Fr(N_s\delta int(A)) \subseteq N_s\delta Fr(A)$).

Proof. Let A be the neutrosophic subset in the $N_s ts$ (Y, Ψ_N) . Then

$$\begin{aligned} N_s\delta Fr(N_s\delta int(A)) &= N_s\delta cl(N_s int(A)) \cap (N_s\delta cl(N_s\delta int(A))^c) [\text{by Definition 3.1}] \\ &= N_s\delta cl(N_s\delta int(A)) \cap (N_s\delta cl(N_s\delta cl(A^c))) [\text{by Theorem 2.9 (vii)}] \\ &= N_s\delta cl(N_s\delta int(A)) \cap (N_s\delta cl(A^c)) [\text{by Theorem 2.10 (vi)}] \\ &\subseteq N_s\delta cl(A) \cap N_s\delta cl(A^c) [\text{by Theorem 2.9 (i)}] \\ &= N_s\delta Fr(A) [\text{by Definition 3.1}]. \end{aligned}$$

Hence $N_s\delta Fr(N_s\delta int(A)) \subseteq (N_s\delta Fr(A))$.

The proof of the other is similar. \square

Theorem 3.13. For a neutrosophic subset A in the $N_s ts$ (Y, Ψ_N) , then $N_s Fr(N_s cl(A)) \subseteq N_s Fr(A)$ (resp. $N_s\delta Fr(N_s\delta cl(A)) \subseteq N_s\delta Fr(A)$).

Proof. Let A be a neutrosophic subset in the $N_s ts$ (Y, Ψ_N) . Then

$$\begin{aligned} N_s \delta Fr(N_s \delta cl(A)) &= N_s \delta cl(N_s \delta cl(A)) \cap (N_s \delta cl(N_s \delta cl(A))^c) [\text{by Definition 3.1}] \\ &= N_s \delta cl(A) \cap (N_s \delta cl(N_s \delta int(A^c))) [\text{by Theorems 2.10 (vii) and 2.10 (ii) \& (vi)}] \\ &\subseteq N_s \delta cl(A) \cap N_s \delta cl(A^c) [\text{by Theorem 2.9 (i)}] \\ &= N_s \delta Fr(A) [\text{by Definition 3.1}] \end{aligned}$$

Hence $N_s \delta Fr(N_s \delta cl(A)) \subseteq N_s \delta Fr(A)$.

The proof of the other is similar. \square

Theorem 3.14. Let A be a neutrosophic subset in the $N_s ts$ (Y, Ψ_N) . Then $N_s int(A) \subseteq A - N_s Fr(A)$ (resp. $N_s \delta int(A) \subseteq A - N_s \delta Fr(A)$).

Proof. Let A be a neutrosophic subset in the $N_s ts$ (Y, Ψ_N) . Now by Definition 3.1,

$$\begin{aligned} A - N_s \delta Fr(A) &= A \cap (N_s \delta Fr(A))^c \\ &= A \cap [N_s \delta cl(A) \cap N_s \delta cl(A^c)]^c \\ &= A \cap [N_s \delta int(A^c) \cup N_s \delta int(A)] \\ &= [A \cap N_s \delta int(A^c)] \cup [A \cap N_s \delta int(A)] \\ &= [A \cap N_s \delta int(A^c)] \cup N_s \delta int(A) \supseteq N_s \delta int(A) \end{aligned}$$

Hence $N_s \delta int(A) \subseteq A - N_s \delta Fr(A)$.

The proof of the other is similar. \square

Remark 3.15. In general topology, the following conditions are hold:

- (i) $N_s Fr(A) \cap N_s int(A) = 0_N$ (resp. $N_s \delta Fr(A) \cap N_s \delta int(A) = 0_N$),
- (ii) $N_s int(A) \cup N_s Fr(A) = N_s cl(A)$ (resp. $N_s \delta int(A) \cup N_s \delta Fr(A) = N_s \delta cl(A)$),
- (iii) $N_s int(A) \cup N_s int(A^c) \cup N_s Fr(A) = 1_N$ (resp. $N_s \delta int(A) \cup N_s \delta int(A^c) \cup N_s \delta Fr(A) = 1_N$).

But the neutrosophic topology, we give counter-examples to show that the condition of neutrosophic subset of the above remark may not be hold in general.

Example 3.16. In Example 3.2, Let $A = \langle Y, (\frac{\mu_l}{0.2}, \frac{\mu_m}{0.4}, \frac{\mu_n}{0.4}), (\frac{\sigma_l}{0.5}, \frac{\sigma_m}{0.5}, \frac{\sigma_n}{0.5}), (\frac{\nu_l}{0.8}, \frac{\nu_m}{0.6}, \frac{\nu_n}{0.6}) \rangle$, then

$$\begin{aligned} \text{(i)} \quad N_s \delta Fr(A) &= \langle Y, (\frac{\mu_l}{0.8}, \frac{\mu_m}{0.7}, \frac{\mu_n}{0.6}), (\frac{\sigma_l}{0.5}, \frac{\sigma_m}{0.5}, \frac{\sigma_n}{0.5}), (\frac{\nu_l}{0.2}, \frac{\nu_m}{0.3}, \frac{\nu_n}{0.4}) \rangle, \\ N_s \delta int(A) &= \langle Y, (\frac{\mu_l}{0.2}, \frac{\mu_m}{0.3}, \frac{\mu_n}{0.4}), (\frac{\sigma_l}{0.5}, \frac{\sigma_m}{0.5}, \frac{\sigma_n}{0.5}), (\frac{\nu_l}{0.8}, \frac{\nu_m}{0.7}, \frac{\nu_n}{0.6}) \rangle, \\ N_s \delta Fr(A) \cap N_s \delta int(A) &= \langle Y, (\frac{\mu_l}{0.2}, \frac{\mu_m}{0.3}, \frac{\mu_n}{0.4}), (\frac{\sigma_l}{0.5}, \frac{\sigma_m}{0.5}, \frac{\sigma_n}{0.5}), (\frac{\nu_l}{0.8}, \frac{\nu_m}{0.7}, \frac{\nu_n}{0.6}) \rangle. \\ \implies N_s \delta Fr(A) \cap N_s \delta int(A) &\neq 0_N. \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad N_s \delta int(A) &= \langle Y, (\frac{\mu_l}{0.2}, \frac{\mu_m}{0.3}, \frac{\mu_n}{0.4}), (\frac{\sigma_l}{0.5}, \frac{\sigma_m}{0.5}, \frac{\sigma_n}{0.5}), (\frac{\nu_l}{0.8}, \frac{\nu_m}{0.7}, \frac{\nu_n}{0.6}) \rangle, \\ N_s \delta int(A^c) &= \langle Y, (\frac{\mu_l}{0.2}, \frac{\mu_m}{0.3}, \frac{\mu_n}{0.4}), (\frac{\sigma_l}{0.5}, \frac{\sigma_m}{0.5}, \frac{\sigma_n}{0.5}), (\frac{\nu_l}{0.8}, \frac{\nu_m}{0.7}, \frac{\nu_n}{0.6}) \rangle, \\ N_s \delta Fr(A) &= \langle Y, (\frac{\mu_l}{0.8}, \frac{\mu_m}{0.7}, \frac{\mu_n}{0.6}), (\frac{\sigma_l}{0.5}, \frac{\sigma_m}{0.5}, \frac{\sigma_n}{0.5}), (\frac{\nu_l}{0.2}, \frac{\nu_m}{0.3}, \frac{\nu_n}{0.4}) \rangle. \\ \implies N_s \delta int(A) \cup N_s \delta int(A^c) \cup N_s \delta Fr(A) &\neq 1_N. \end{aligned}$$

Example 3.17. In Example 3.2, Let $A = \langle Y, (\frac{\mu_l}{0.2}, \frac{\mu_m}{0.4}, \frac{\mu_n}{0.7}), (\frac{\sigma_l}{0.5}, \frac{\sigma_m}{0.5}, \frac{\sigma_n}{0.5}), (\frac{\nu_l}{0.8}, \frac{\nu_m}{0.6}, \frac{\nu_n}{0.3}) \rangle$, then

$$\begin{aligned} N_s \delta int(A) &= \langle Y, (\frac{\mu_l}{0.2}, \frac{\mu_m}{0.3}, \frac{\mu_n}{0.4}), (\frac{\sigma_l}{0.5}, \frac{\sigma_m}{0.5}, \frac{\sigma_n}{0.5}), (\frac{\nu_l}{0.8}, \frac{\nu_m}{0.7}, \frac{\nu_n}{0.6}) \rangle, \\ N_s \delta Fr(A) &= \langle Y, (\frac{\mu_l}{0.8}, \frac{\mu_m}{0.7}, \frac{\mu_n}{0.6}), (\frac{\sigma_l}{0.5}, \frac{\sigma_m}{0.5}, \frac{\sigma_n}{0.5}), (\frac{\nu_l}{0.2}, \frac{\nu_m}{0.3}, \frac{\nu_n}{0.4}) \rangle, \\ N_s \delta cl(A) &= 1_N. \\ \implies N_s \delta int(A) \cup N_s \delta Fr(A) &\neq N_s \delta cl(A) \end{aligned}$$

Theorem 3.18. Let A and B be neutrosophic subsets in the $N_s ts$ (Y, Ψ_N) . Then $N_s Fr(A \cup B) \subseteq N_s Fr(A) \cup N_s Fr(B)$ (resp. $N_s \delta Fr(A \cup B) \subseteq N_s \delta Fr(A) \cup N_s \delta Fr(B)$).

Proof. Let A and B be neutrosophic subsets in the $Nsts(Y, \Psi_N)$. Then

$$\begin{aligned}
 N_s \delta Fr(A \cup B) &= N_s \delta cl(A \cup B) \cap N_s \delta cl(A \cup B)^c [\text{by Definition 3.1}] \\
 &= N_s \delta cl(A \cup B) \cap N_s \delta cl(A^c \cap B^c) \\
 &\subseteq (N_s \delta cl(A) \cup N_s \delta cl(B) \cap ((N_s \delta cl(A^c))) \cap (N_s \delta cl(B^c))) [\text{by Theorem 2.10 (iii) \& (ix)}] \\
 &= [(N_s \delta cl(A) \cup (N_s \delta cl(B)) \cap (N_s \delta cl(A^c)))] \cap [(N_s \delta cl(A) \cup (N_s \delta cl(B)) \cap (N_s \delta cl(B^c)))] \\
 &= [(N_s \delta cl(A) \cap N_s \delta cl(A^c)) \cup ((N_s \delta cl(B) \cap (N_s \delta cl(A^c))))] \cap [(N_s \delta cl(A) \cap (N_s \delta cl(B^c))) \\
 &\quad \cup ((N_s \delta cl(B) \cap (N_s \delta cl(B^c))))] \\
 &= [N_s \delta Fr(A) \cup (N_s \delta cl(B)) \cap (N_s \delta cl(A^c))] \cap [(N_s \delta cl(A) \cap (N_s \delta cl(B^c))) \cup (N_s \delta Fr(B))] \\
 &\quad [\text{by Definition 3.1}] \\
 &= (N_s \delta Fr(A) \cup N_s \delta Fr(B)) \cap [(N_s \delta cl(B) \cap (N_s \delta cl(A^c))) \cup ((N_s \delta cl(A) \cap N_s \delta cl(B^c)))] \\
 &\subseteq N_s \delta Fr(A) \cup N_s \delta Fr(B).
 \end{aligned}$$

Hence, $N_s \delta Fr(A \cup B) \subseteq N_s \delta Fr(A) \cup N_s \delta Fr(B)$.

The proof of the other is similar. \square

Note 1. The following example shows that

- (i) $N_s Fr(A \cap B) \not\subseteq N_s Fr(A) \cup N_s Fr(B)$ and $N_s Fr(A) \cap N_s Fr(B) \not\subseteq N_s Fr(A \cap B)$.
- (ii) $N_s \delta Fr(A \cap B) \not\subseteq N_s \delta Fr(A) \cup N_s \delta Fr(B)$ and $N_s \delta Fr(A) \cap N_s \delta Fr(B) \not\subseteq N_s \delta Fr(A \cap B)$.

Theorem 3.19. For any neutrosophic subsets A and B in the $Nsts(Y, \Psi_N)$, $N_s Fr(A \cap B) \subseteq (N_s Fr(A) \cap (N_s \delta cl(B))) \cup (N_s Fr(B) \cap N_s \delta cl(A))$ (resp. $N_s \delta Fr(A \cap B) \subseteq (N_s \delta Fr(A) \cap (N_s \delta cl(B))) \cup (N_s \delta Fr(B) \cap N_s \delta cl(A))$).

Proof. Let A and B be neutrosophic subsets in the $Nsts(Y, \Psi_N)$. Then

$$\begin{aligned}
 N_s \delta Fr(A \cap B) &= N_s \delta cl(A \cap B) \cap (N_s \delta cl(A \cap B))^c [\text{by Definition 3.1}] \\
 &= N_s \delta cl(A \cap B) \cap (N_s \delta cl(A^c \cup B^c)) \\
 &\subseteq (N_s \delta cl(A) \cap N_s \delta cl(B)) \cap (N_s \delta cl(A^c) \cup N_s \delta cl(B^c)) [\text{by Theorem 2.10 (iii) \& (ix)}] \\
 &= [(N_s \delta cl(A) \cap N_s \delta cl(B)) \cap N_s \delta cl(A^c)] \cup [(N_s \delta cl(A) \cap N_s \delta cl(B)) \cap N_s \delta cl(B^c)] \\
 &= (N_s \delta Fr(A) \cap N_s \delta cl(B)) \cup (N_s \delta Fr(B) \cap N_s \delta cl(A)) [\text{by Definition 3.1}].
 \end{aligned}$$

Hence $N_s \delta Fr(A \cap B) \subseteq ((N_s \delta Fr(A) \cap (N_s \delta cl(B))) \cup (N_s \delta Fr(B) \cap (N_s \delta cl(A))))$.

The proof of the other is similar. \square

Corollary 3.20. For any neutrosophic subsets A and B in the $Nsts(Y, \Psi_N)$, $N_s Fr(A \cap B) \subseteq N_s Fr(A) \cup N_s Fr(B)$ (resp. $N_s \delta Fr(A \cap B) \subseteq N_s \delta Fr(A) \cup N_s \delta Fr(B)$).

Proof. Let A and B be neutrosophic subsets in the $Nsts(Y, \Psi_N)$. Then

$$\begin{aligned}
 N_s \delta Fr(A \cap B) &= N_s \delta cl(A \cap B) \cap (N_s \delta cl(A \cap B))^c [\text{by Definition 3.1}] \\
 &= N_s \delta cl(A \cap B) \cap (N_s \delta cl(A^c \cup B^c)) \\
 &\subseteq (N_s \delta cl(A) \cap N_s \delta cl(B)) \cap (N_s \delta cl(A^c) \cup N_s \delta cl(B^c)) [\text{by Theorem 2.10 (iii) \& (ix)}] \\
 &= (N_s \delta cl(A) \cap N_s \delta cl(B)) \cap (N_s \delta cl(A^c) \cup (N_s \delta cl(A) \cap N_s \delta cl(B)) \cap (N_s \delta cl(B^c))) \\
 &= (N_s \delta Fr(A) \cap N_s \delta cl(B)) \cup (N_s \delta cl(A) \cap N_s \delta Fr(B)) [\text{by Definition 3.1}] \\
 &\subseteq N_s \delta Fr(A) \cup N_s \delta Fr(B).
 \end{aligned}$$

Hence $N_s \delta Fr(A \cap B) \subseteq N_s \delta Fr(A) \cup N_s \delta Fr(B)$.

The proof of the other is similar. \square

Theorem 3.21. For any neutrosophic subset A in the $Nsts(Y, \Psi_N)$,

- (i) (a) $N_s Fr(N_s Fr(A)) \subseteq N_s Fr(A)$,
 (b) $N_s Fr(N_s Fr(N_s Fr(A))) \subseteq N_s Fr(N_s \delta Fr(A))$.
- (ii) (a) $N_s \delta Fr(N_s \delta Fr(A)) \subseteq N_s \delta Fr(A)$,

$$(b) N_s \delta Fr(N_s \delta Fr(N_s \delta Fr(A))) \subseteq N_s \delta Fr(N_s \delta Fr(A)).$$

Proof. (ii) (a) Let A be a neutrosophic subset in the $N_s ts$ (Y, Ψ_N). Then

$$\begin{aligned} N_s \delta Fr(N_s \delta Fr(A)) &= N_s \delta cl(N_s \delta Fr(A)) \cap N_s \delta cl(N_s \delta Fr(A)^c) [\text{by Definition 3.1}] \\ &= N_s \delta cl(N_s \delta cl(A) \cap (N_s \delta cl(A^c)) \cap (N_s \delta cl(N_s \delta cl(A)) \cap (N_s \delta cl(A^c))^c)) \\ &\quad [\text{by Definition 3.1}] \\ &\subseteq (N_s \delta cl(N_s \delta cl(A)) \cap (N_s \delta cl(N_s \delta cl(A^c))) \cap (N_s \delta cl(N_s \delta int(A^c))) \cup (N_s \delta int(A))) \\ &\quad [\text{by Theorem 2.10 (vi) \& (ix)}] \\ &= (N_s \delta cl(A) \cap (N_s \delta cl(A^c)) \cap (N_s \delta cl(N_s \delta int(A) \cup N_s \delta int(A^c)))) \\ &\quad [\text{by Theorem 2.10 (vi)}] \\ &\subseteq N_s \delta cl(A) \cap N_s \delta cl(A^c) \\ &= N_s \delta Fr(A) [\text{by Definition 3.1}]. \end{aligned}$$

Therefore $N_s \delta Fr(N_s \delta Fr(A)) \subseteq N_s \delta Fr(A)$.

(b) Again, $N_s \delta Fr(N_s \delta Fr(N_s \delta Fr(A))) \subseteq N_s \delta Fr(N_s \delta Fr(A))$.

The proof of the other is similar. □

4 Neutrosophic δ border and neutrosophic δ exterior

In this section, we introduce the neutrosophic δ border, neutrosophic δ exterior using neutrosophic δ open sets and their properties are discussed in $N_s ts$'s.

Definition 4.1. Let A be a neutrosophic subset of $N_s ts$ (Y, Ψ_N). Then the set $N_s Br(A) = A \cap N_s int(A)$ (resp. $N_s \delta Br(A) = A \cap N_s \delta int(A)$) is called the neutrosophic (resp. δ) border of A (briefly, $N_s Br(A)$ (resp. $N_s \delta Br(A)$)).

Example 4.2. In Example 3.2, Let $A = \langle Y, (\frac{\mu_l}{0.2}, \frac{\mu_m}{0.4}, \frac{\mu_n}{0.4}), (\frac{\sigma_l}{0.5}, \frac{\sigma_m}{0.5}, \frac{\sigma_n}{0.5}), (\frac{\nu_l}{0.8}, \frac{\nu_m}{0.6}, \frac{\nu_n}{0.6}) \rangle$, then

$$\begin{aligned} (i) N_s int(A) &= \langle Y, (\frac{\mu_l}{0.2}, \frac{\mu_m}{0.3}, \frac{\mu_n}{0.4}), (\frac{\sigma_l}{0.5}, \frac{\sigma_m}{0.5}, \frac{\sigma_n}{0.5}), (\frac{\nu_l}{0.8}, \frac{\nu_m}{0.7}, \frac{\nu_n}{0.6}) \rangle. \\ A \cap N_s int(A) &= \langle Y, (\frac{\mu_l}{0.2}, \frac{\mu_m}{0.3}, \frac{\mu_n}{0.4}), (\frac{\sigma_l}{0.5}, \frac{\sigma_m}{0.5}, \frac{\sigma_n}{0.5}), (\frac{\nu_l}{0.8}, \frac{\nu_m}{0.7}, \frac{\nu_n}{0.6}) \rangle. \\ \implies N_s Br(A) &= \langle Y, (\frac{\mu_l}{0.2}, \frac{\mu_m}{0.3}, \frac{\mu_n}{0.4}), (\frac{\sigma_l}{0.5}, \frac{\sigma_m}{0.5}, \frac{\sigma_n}{0.5}), (\frac{\nu_l}{0.8}, \frac{\nu_m}{0.7}, \frac{\nu_n}{0.6}) \rangle. \\ (ii) N_s \delta int(A) &= \langle Y, (\frac{\mu_l}{0.2}, \frac{\mu_m}{0.3}, \frac{\mu_n}{0.4}), (\frac{\sigma_l}{0.5}, \frac{\sigma_m}{0.5}, \frac{\sigma_n}{0.5}), (\frac{\nu_l}{0.8}, \frac{\nu_m}{0.7}, \frac{\nu_n}{0.6}) \rangle. \\ A \cap N_s \delta int(A) &= \langle Y, (\frac{\mu_l}{0.2}, \frac{\mu_m}{0.3}, \frac{\mu_n}{0.4}), (\frac{\sigma_l}{0.5}, \frac{\sigma_m}{0.5}, \frac{\sigma_n}{0.5}), (\frac{\nu_l}{0.8}, \frac{\nu_m}{0.7}, \frac{\nu_n}{0.6}) \rangle. \\ \implies N_s \delta Br(A) &= \langle Y, (\frac{\mu_l}{0.2}, \frac{\mu_m}{0.3}, \frac{\mu_n}{0.4}), (\frac{\sigma_l}{0.5}, \frac{\sigma_m}{0.5}, \frac{\sigma_n}{0.5}), (\frac{\nu_l}{0.8}, \frac{\nu_m}{0.7}, \frac{\nu_n}{0.6}) \rangle. \end{aligned}$$

Theorem 4.3. If a subset A of Y is $N_s c$ (resp. $N_s \delta c$), then $N_s Br(A) = N_s Fr(A)$ (resp. $N_s \delta Br(A) = N_s \delta Fr(A)$).

Proof. Let A be a $N_s \delta c$ subset of Y . Then by Theorem 2.10 (vii), $N_s \delta cl(A) = A$. Now, $N_s \delta Fr(A) = N_s \delta cl(A) - N_s \delta int(A) = A - N_s \delta int(A) = N_s \delta Br(A)$.

The proof of the other is similar. □

Theorem 4.4. For a neutrosophic subset A of Y , $A = N_s int(A) \cup N_s Br(A)$ (resp. $A = N_s \delta int(A) \cup N_s \delta Br(A)$).

Proof. Let $x_{(\alpha, \beta, \gamma)} \in A$. If $x_{(\alpha, \beta, \gamma)} \in N_s \delta int(A)$, then the result is obvious. If $x_{(\alpha, \beta, \gamma)} \notin N_s \delta int(A)$, then by the definition of $N_s \delta Br(A)$, $x_{(\alpha, \beta, \gamma)} \in N_s \delta Br(A)$. Hence $x_{(\alpha, \beta, \gamma)} \in N_s \delta int(A) \cup N_s \delta Br(A)$ and so $A \subseteq N_s \delta int(A) \cup N_s \delta Br(A)$. On the other hand, since $N_s \delta int(A) \subseteq A$ and $N_s \delta Br(A) \subseteq A$, we have $N_s \delta int(A) \cup N_s \delta Br(A) \subseteq A$.

The proof of the other is similar. □

Theorem 4.5. For a neutrosophic subset A of Y , $N_s int(A) \cap N_s Br(A) = 0_N$ (resp. $N_s \delta int(A) \cap N_s \delta Br(A) = 0_N$).

Proof. Suppose $N_s \delta int(A) \cap N_s \delta Br(A) \neq 0_N$. Let $x_{(\alpha, \beta, \gamma)} \in N_s \delta int(A) \cap N_s \delta Br(A)$. Then $x_{(\alpha, \beta, \gamma)} \in N_s \delta int(A)$ and $x_{(\alpha, \beta, \gamma)} \in N_s \delta Br(A)$. Since $N_s \delta Br(A) = A - N_s \delta int(A)$, then $x_{(\alpha, \beta, \gamma)} \in A$. But $x_{(\alpha, \beta, \gamma)} \in N_s \delta int(A)$, $x_{(\alpha, \beta, \gamma)} \in A$. There is a contradiction. Hence $N_s \delta int(A) \cap N_s \delta Br(A) = 0_N$.

The proof of the other is similar. □

Theorem 4.6. For a neutrosophic subset A of Y , A is a $N_s o$ (resp. $N_s \delta o$) set if and only if $N_s Br(A) = 0_N$ (resp. $N_s \delta Br(A) = 0_N$).

Proof. Necessity: Suppose A is $N_s \delta o$. Then by Theorem 2.9 (v), $N_s \delta int(A) = A$. Now, $N_s \delta Br(A) = A - N_s \delta int(A) = A - A = 0_N$.

Sufficiency: Suppose $N_s \delta Br(A) = 0_N$. This implies, $A - N_s \delta int(A) = 0_N$. Therefore $A = N_s \delta int(A)$ and hence A is $N_s \delta o$.

The proof of the other is similar. \square

Corollary 4.7. For a $N_s ts$, $N_s Br(0_N) = 0_N$ and $N_s Br(1_N) = 0_N$ (resp. $N_s \delta Br(0_N) = 0_N$ and $N_s \delta Br(1_N) = 0_N$).

Proof. Since 0_N and 1_N are $N_s \delta o$, by Theorem 4.6, $N_s \delta Br(0_N) = 0_N$ and $N_s \delta Br(1_N) = 0_N$.

The proof of the other is similar. \square

Theorem 4.8. For a neutrosophic subset A of Y , $N_s Br(N_s int(A)) = 0_N$ (resp. $N_s \delta Br(N_s \delta int(A)) = 0_N$).

Proof. By the definition of $N_s \delta$ border, $N_s \delta Br(N_s \delta int(A)) = N_s \delta int(A) - N_s \delta int(N_s \delta int(A))$. By Theorem 2.9 (vi), $N_s \delta int(N_s \delta int(A)) = N_s \delta int(A)$ and hence $N_s \delta Br(N_s \delta int(A)) = 0_N$.

The proof of the other is similar. \square

Theorem 4.9. For a neutrosophic subset A of Y , $N_s int(N_s Br(A)) = 0_N$ (resp. $N_s \delta int(N_s \delta Br(A)) = 0_N$).

Proof. Let $x_{(\alpha, \beta, \gamma)} \in N_s \delta int(N_s \delta Br(A))$. Since $N_s \delta Br(A) \subseteq A$, by Theorem 2.9 (i), $N_s \delta int(N_s \delta Br(A)) \subseteq N_s \delta int(A)$. Hence $x_{(\alpha, \beta, \gamma)} \in N_s \delta int(A)$. Since $N_s \delta int(N_s \delta Br(A)) \subseteq N_s \delta Br(A)$, $x_{(\alpha, \beta, \gamma)} \in N_s \delta Br(A)$. Therefore $x_{(\alpha, \beta, \gamma)} \in N_s \delta int(A) \cap N_s \delta Br(A)$, $x_{(\alpha, \beta, \gamma)} = 0_N$.

The proof of the other is similar. \square

Theorem 4.10. For a neutrosophic subset A of Y , $N_s Br(N_s Br(A)) = N_s Br(A)$ (resp. $N_s \delta Br(N_s \delta Br(A)) = N_s \delta Br(A)$).

Proof. By the definition of $N_s \delta$ border, $N_s \delta Br(N_s \delta Br(A)) = N_s \delta Br(A) - N_s \delta int(N_s \delta Br(A))$. By Theorem 4.9 $N_s \delta int(N_s \delta Br(A)) = 0_N$ and hence $N_s \delta Br(N_s \delta Br(A)) = N_s \delta Br(A)$.

The proof of the other is similar. \square

Theorem 4.11. For a subset A of a space Y , the following statements are equivalent

- (i) A is $N_s o$ (resp. $N_s \delta o$).
- (ii) $A = N_s int(A)$ (resp. $A = N_s \delta int(A)$).
- (iii) $N_s Br(A) = 0_N$ (resp. $N_s \delta Br(A) = 0_N$).

Proof. (i) \rightarrow (ii) Obvious from Theorem 2.9.

(ii) \rightarrow (iii). Suppose that $A = N_s \delta int(A)$. Then by Definition, $N_s \delta Br(A) = N_s \delta int(A) - N_s \delta int(A) = 0_N$.

(iii) \rightarrow (i). Let $N_s \delta Br(A) = 0_N$. Then by Definition 4.1, $A - N_s \delta int(A) = 0_N$ and hence $A = N_s \delta int(A)$.

The proof of the other is similar. \square

Theorem 4.12. Let A be a neutrosophic subset of Y . Then, $N_s Br(A) = A \cap N_s cl(A^c)$ (resp. $N_s \delta Br(A) = A \cap N_s \delta cl(A^c)$).

Proof. Since $N_s \delta Br(A) = A - N_s \delta int(A)$ and by Theorem 2.10, $N_s \delta Br(A) = A - (N_s \delta cl(A^c))^c = A \cap (N_s \delta cl(A^c))^c = A \cap N_s \delta cl(A^c)$.

The proof of the other is similar. \square

Theorem 4.13. For a neutrosophic subset A of Y , $N_s Br(A) \subseteq N_s Fr(A)$ (resp. $N_s \delta Br(A) \subseteq N_s \delta Fr(A)$).

Proof. Since $A \subseteq N_s \delta cl(A)$, $A - N_s \delta int(A) \subseteq N_s \delta cl(A) - N_s \delta int(A)$. That implies, $N_s \delta Br(A) \subseteq N_s \delta Fr(A)$.

The proof of the other is similar. \square

Definition 4.14. Let A be a neutrosophic subset of a $N_s ts$ (Y, Ψ_N) . The neutrosophic (resp. δ) interior of A^c is called the neutrosophic (resp. δ) exterior of A and it is denoted by $N_s Ext(A)$ (resp. $N_s \delta Ext(A)$). That is, $N_s Ext(A) = N_s int(A^c)$ (resp. $N_s \delta Ext(A) = N_s \delta int(A^c)$).

Example 4.15. In Example 3.2, Let $A = \langle Y, (\frac{\mu_l}{0.2}, \frac{\mu_m}{0.4}, \frac{\mu_n}{0.4}), (\frac{\sigma_l}{0.5}, \frac{\sigma_m}{0.5}, \frac{\sigma_n}{0.5}), (\frac{\nu_l}{0.8}, \frac{\nu_m}{0.6}, \frac{\nu_n}{0.6}) \rangle$, then

$$\begin{aligned} \text{(i)} \quad N_s \text{int}(A^c) &= \langle Y, (\frac{\mu_l}{0.2}, \frac{\mu_m}{0.3}, \frac{\mu_n}{0.4}), (\frac{\sigma_l}{0.5}, \frac{\sigma_m}{0.5}, \frac{\sigma_n}{0.5}), (\frac{\nu_l}{0.8}, \frac{\nu_m}{0.7}, \frac{\nu_n}{0.6}) \rangle. \\ &\implies N_s \text{Ext}(A) = \langle Y, (\frac{\mu_l}{0.2}, \frac{\mu_m}{0.3}, \frac{\mu_n}{0.4}), (\frac{\sigma_l}{0.5}, \frac{\sigma_m}{0.5}, \frac{\sigma_n}{0.5}), (\frac{\nu_l}{0.8}, \frac{\nu_m}{0.7}, \frac{\nu_n}{0.6}) \rangle. \\ \text{(ii)} \quad N_s \delta \text{int}(A^c) &= \langle Y, (\frac{\mu_l}{0.2}, \frac{\mu_m}{0.3}, \frac{\mu_n}{0.4}), (\frac{\sigma_l}{0.5}, \frac{\sigma_m}{0.5}, \frac{\sigma_n}{0.5}), (\frac{\nu_l}{0.8}, \frac{\nu_m}{0.7}, \frac{\nu_n}{0.6}) \rangle. \\ &\implies N_s \delta \text{Ext}(A) = \langle Y, (\frac{\mu_l}{0.2}, \frac{\mu_m}{0.3}, \frac{\mu_n}{0.4}), (\frac{\sigma_l}{0.5}, \frac{\sigma_m}{0.5}, \frac{\sigma_n}{0.5}), (\frac{\nu_l}{0.8}, \frac{\nu_m}{0.7}, \frac{\nu_n}{0.6}) \rangle. \end{aligned}$$

Theorem 4.16. For a neutrosophic subset A of Y , $N_s \text{Ext}(A) = (N_s \text{cl}(A))^c$ (resp. $N_s \delta \text{Ext}(A) = (N_s \delta \text{cl}(A))^c$).

Proof. We know that, $1_N - N_s \delta \text{cl}(A) = N_s \delta \text{int}(A^c)$, then $N_s \delta \text{Ext}(A) = N_s \delta \text{int}(A^c) = (N_s \delta \text{cl}(A))^c$.

The proof of the other is similar. \square

Theorem 4.17. For a neutrosophic subset A of Y , $N_s \text{Ext}(N_s \text{Ext}(A)) = N_s \text{int}(N_s \text{cl}(A)) \supseteq N_s \text{int}(A)$ (resp. $N_s \delta \text{Ext}(N_s \delta \text{Ext}(A)) = N_s \delta \text{int}(N_s \delta \text{cl}(A)) \supseteq N_s \delta \text{int}(A)$).

Proof. Now, $N_s \delta \text{Ext}(N_s \delta \text{Ext}(A)) = N_s \delta \text{Ext}(N_s \delta \text{int}(A^c)) = N_s \delta \text{int}((N_s \delta \text{int}(A^c))^c) = N_s \delta \text{int}(N_s \delta \text{cl}(A)) \supseteq N_s \delta \text{int}(A)$.

The proof of the other is similar. \square

Theorem 4.18. For a neutrosophic subset A of Y , If $A \subseteq B$, then $N_s \text{Ext}(B) \subseteq N_s \text{Ext}(A)$ (resp. $N_s \delta \text{Ext}(B) \subseteq N_s \delta \text{Ext}(A)$).

Proof. Suppose $A \subseteq B$. Now, $N_s \delta \text{Ext}(B) = N_s \delta \text{int}(B^c) \subseteq N_s \delta \text{int}(A^c) = N_s \delta \text{Ext}(A)$.

The proof of the other is similar. \square

Theorem 4.19. For a neutrosophic subset A of Y , $N_s \text{Ext}(1_N) = 0_N$ and $N_s \text{Ext}(0_N) = 1_N$ (resp. $N_s \delta \text{Ext}(1_N) = 0_N$ and $N_s \delta \text{Ext}(0_N) = 1_N$).

Proof. Now, $N_s \delta \text{Ext}(1_N) = N_s \delta \text{int}((1_N)^c) = N_s \delta \text{int}(0_N)$ and $N_s \delta \text{Ext}(0_N) = N_s \delta \text{int}((0_N)^c) = N_s \delta \text{int}(1_N)$. Since 0_N and 1_N are $N_s \delta$ sets, then $N_s \delta \text{int}(0_N) = 0_N$ and $N_s \delta \text{int}(1_N) = 1_N$. Hence $N_s \delta \text{Ext}(0_N) = 1_N$ and $N_s \delta \text{Ext}(1_N) = 0_N$.

The proof of the other is similar. \square

Theorem 4.20. For a neutrosophic subset A of Y , $N_s \text{Ext}(A) = N_s \text{Ext}((N_s \text{Ext}(A))^c)$ (resp. $N_s \delta \text{Ext}(A) = N_s \delta \text{Ext}((N_s \delta \text{Ext}(A))^c)$).

Proof. Now, $N_s \delta \text{Ext}((N_s \delta \text{Ext}(A))^c) = N_s \delta \text{Ext}((N_s \delta \text{int}(A^c))^c) = N_s \delta \text{int}((((N_s \delta \text{int}(A^c))^c))^c) = N_s \delta \text{int}(N_s \delta \text{int}(A^c)) = N_s \delta \text{int}(A^c) = N_s \delta \text{Ext}(A)$.

The proof of the other is similar. \square

Theorem 4.21. For a sub sets A and B of Y , the followings are valid.

- (i) $N_s \text{Ext}(A \cup B) \subseteq N_s \text{Ext}(A) \cap N_s \text{Ext}(B)$ (resp. $N_s \delta \text{Ext}(A \cup B) \subseteq N_s \delta \text{Ext}(A) \cap N_s \delta \text{Ext}(B)$).
- (ii) $N_s \text{Ext}(A \cap B) \supseteq N_s \text{Ext}(A) \cup N_s \text{Ext}(B)$ (resp. $N_s \delta \text{Ext}(A \cap B) \supseteq N_s \delta \text{Ext}(A) \cup N_s \delta \text{Ext}(B)$).

Proof. (i) $N_s \delta \text{Ext}(A \cup B) = N_s \delta \text{int}((A \cup B)^c) = N_s \delta \text{int}((A^c) \cap (B^c)) \subseteq N_s \delta \text{cl}(A^c) \cap N_s \delta \text{cl}(B^c) = N_s \delta \text{Ext}(A) \cap N_s \delta \text{Ext}(B)$.

(ii) $N_s \delta \text{Ext}(A \cap B) = N_s \delta \text{int}((A \cap B)^c) = N_s \delta \text{int}((A^c) \cup (B^c)) \supseteq N_s \delta \text{cl}(A^c) \cup N_s \delta \text{cl}(B^c) = N_s \delta \text{Ext}(A) \cup N_s \delta \text{Ext}(B)$.

The proof of the other is similar. \square

5 Conclusions

So far, we have studied some new operators called neutrosophic δ frontier respective border and exterior with the help of neutrosophic δ -open sets in neutrosophic topological space. Also, we discussed the important properties of them and the relations between them. This can be extended to some weaker forms of neutrosophic open sets, in future.

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