Induced generalized interval neutrosophic Shapley hybrid operators and their application in multi-attribute decision making

Peide Liu*, Guolin Tang, Weilong Liu
School of Management Science and Engineering, Shandong University of Finance and Economics,
Jinan Shandong 250014, China
Corresponding author. Tel.: +86 531 82222188. E-mail: Peide.liu@gmail.com

Abstract: With respect to the interval neutrosophic multi-attribute decision-making (MADM) problems, the MADM method based on some interval neutrosophic aggregation operators is developed. Firstly, the induced generalized interval neutrosophic hybrid arithmetic averaging (IGINHAA) operator and the induced generalized interval neutrosophic hybrid geometric mean (IGINHGM) operator are proposed, which can weight all the input arguments and their ordered positions. Further, regarding the situation where the input elements are interdependent, the induced generalized interval neutrosophic Shapley hybrid arithmetic averaging (IGINSHAA) operator and the induced generalized interval neutrosophic Shapley hybrid geometric mean (IGINSHGM) operator are proposed, which are extensions of IGINHAA and IGINHGM operators, respectively, and some properties of these given operators are investigated. Furthermore, the interval neutrosophic cross entropy, which is an extension of single valued neutrosophic cross entropy, is defined, and the models based on the interval neutrosophic cross entropy and generalized Shapley function are respectively constructed to determine the optimal fuzzy measures on the attribute set and on the ordered set. Finally, an approach to interval neutrosophic MADM with interactive conditions and incomplete known weight information is proposed based on these given operators, and a practical example is shown to verify the practicality and feasibility of the new approach.

Keywords: multi-attribute decision making; interval neutrosophic set; aggregation operator; cross entropy; generalized Shapley function.

1. Introduction

Since Neutrosophic set (NS), which is a generalization of fuzzy set [43], vague set [10], intuitionistic fuzzy set [1], tautological set [25] and so on, was introduced by Smarandache [25], it has been applied in many different areas, such as medical diagnosis [41-42], imaging processing [22], pattern recognition [19], and decision making problems [8,17-18]. It should be noted that the words "neutrosophy" and "neutrosophic" were introduced by Smarandache [25]. Etymologically, "neutro-sophy" (noun) means knowledge of neutral thought while "neutrosophic" (adjective), means having the nature of, or having the characteristic of Neutrosophy. Because NS consists of three completely independent parts, which are truth-membership degree, indeterminacy-membership degree and falsity-membership degree, it is very suitable to capture the incomplete, indeterminate and inconsistent information. Now NS has attracted more and more attentions [2,6,21,28,37,44]. Wang et al. [31] defined the single valued neutrosophic set (SVNS), as a subclass of the NS. Majumdar et al. [14] introduced normalized Hamming distance and normalized Euclidian distance between two SVNSs, and further gave the similarity between two SVNSs and the entropy of a SVNS. Ye [40] defined the single valued neutrosophic weighted cross entropy and applied it to the MADM problem. Pătraşcu [20] gave two types of neutrosophic entropy as extensions of Kaufman's formula and Kosko's formula, respectively. Biswas et al. [3] developed a modified GRA method for the single valued neutrosophic MADM problems with completely unknown weight information, and entropy of the SVNS is used to obtain the weight information. Biswas et al. [4] developed an extended TOPSIS method for the neutrosophic MADM problems. In fact, in some complex decision environment, it is insufficient to express the truth-membership degree, indeterminacy-membership degree and falsity membership degree by crisp values. Wang et al. [30] utilized interval numbers to denote these three parts, and defined the interval neutrosophic set (INS) as an instance of NS. Chi and Liu [7] developed an extended TOPSIS method to solve MADM problems with interval neutrosophic information, and the maximization deviation method is used to determine the attribute weights.

The neutrosophic aggregation operators are an important tool to process the neutrosophic MADM problems. Ye [36] proposed the single valued neutrosophic weighted geometric averaging (SVNWGA) operator and applied it to solve single valued neutrosophic decision-making problems. Liu et al. [13] proposed the single valued neutrosophic normalized weighted Bonferroni mean (SVNNWBM) operator which has the reducibility, idempotency and commutativity, and utilized it to handle the MADM problems with correlated attributes. Zhang et al. [46] defined the operational rules and comparison rules of INSs, and proposed two interval neutrosophic aggregation operators, including interval neutrosophic number weighted averaging aggregation (INNWAA) operator and interval neutrosophic number weighted geometric aggregation (INNWGA) operator to handle the interval neutrosophic MADM problems. Ye [39] proposed an approach to the MADM problems based on interval neutrosophic number ordered weighted averaging (INNOWA) operator and the interval neutrosophic number ordered weighted geometric (INNOWG) operator. There are some differences between aforementioned interval neutrosophic aggregation operations. INNWAA and INNWGA operators only weight the input arguments themselves, but ignore their ordered positions. INNOWAA and INNOWGA operators only take into account the ordered positions of the interval neutrosophic arguments, but not consider the arguments themselves. In addition, these above interval neutrosophic number aggregation operators cannot reflect the correlations among the given arguments.

In the INNOWAA and INNOWGA operators, we can see that the reordering process depends on the values of the interval neutrosophic arguments and these two operators could not consider the importance of the aggregated arguments. However, in some real applications, this reordering may not meet our interests. Sometimes, the decision makers may want to order the arguments based on some other associated variables instead of the aggregated arguments. To overcome these shortcomings, the induced generalized interval neutrosophic hybrid arithmetic averaging (IGINHAA) operator and the induced generalized interval neutrosophic hybrid geometric mean (IGINHGM) operator are proposed in this paper. In addition, the aforementioned interval neutrosophic number operators only consider the addition of the importance of individual input arguments. However, in real decision making, the arguments (i.e., attributes) are often not independent. To address these situations, the induced generalized interval neutrosophic Shapley hybrid arithmetic averaging (IGINSHAA) operator and induced generalized interval neutrosophic Shapley hybrid geometric mean (IGINSHGM) operator are proposed in this paper. These two operators cannot only consider the importance of input arguments and their ordered positions, but also overall take into account the correlations among them and their ordered positions. Furthermore, due to time pressure, lacks of knowledge or data, and the expert’s limited expertise, the weight information in the MADM problems is usually incompletely known or completely unknown. Models based on the cross entropy of the INNs are constructed to determine the attribute weight information. As a very important goal, a method of interval neutrosophic MADM with interactive criteria and incompletely known weight information is developed.

The remainder of this paper is constructed as follows. In Sect. 2, some basics in regard to INS are introduced, and the cross entropy of INSs is defined. In Sect. 3, we propose IGINHAA and IGINHGM operators and investigate their several proprieties. In Sect. 4, we review some basic concepts, such as the fuzzy measure and generalized Shapley function, and propose IGINSHAA and IGINSHGM operators. In Sect. 5, the models based on cross entropy of INSs are constructed to obtain the attribute weight information, and a method for the neutrosophic MADM is developed. In Sect. 6, a practical example is shown to verify the effectiveness of the developed method. In Sect. 7, the conclusions are summarized.

2. Preliminaries

2.1. Interval neutrosophic set (INS)

Definition 1 [46]. Suppose X is a universe of discourse, with a generic element in X represented by x. An INS $\hat{A}$ in X is expressed by

$$\begin{aligned}\hat{A} &= \left\{\left\langle x, T_{\hat{A}}(x), I_{\hat{A}}(x), F_{\hat{A}}(x)\right\rangle \mid x \in X\right\} \\ &= \left\{ x, \left\langle \left[T^L{}_{\hat{A}}(x), T^U{}_{\hat{A}}(x)\right], \left[I^L{}_{\hat{A}}(x), I^U{}_{\hat{A}}(x)\right], \left[F^L{}_{\hat{A}}(x), F^U{}_{\hat{A}}(x)\right]\right\rangle \mid x \in X \right\}\end{aligned} \tag{1}$$

where the functions $T_{\hat{A}}(x), I_{\hat{A}}(x)$ and $F_{\hat{A}}(x)$ are the truth-membership function, indeterminacy-membership function and falsity-membership function, respectively, and satisfy $T_{\hat{A}} = \left[T^L{}_{\hat{A}}, T^U{}_{\hat{A}}\right] \subseteq [0,1]$, $I_{\hat{A}} = \left[I^L{}_{\hat{A}}, I^U{}_{\hat{A}}\right] \subseteq [0,1]$, $\mathrm{F}_{\hat{A}} = \left[\mathrm{F}^L{}_{\hat{A}}, \mathrm{F}^U{}_{\hat{A}}\right] \subseteq [0,1]$ and $0 \le T^U{}_{\hat{A}} + I^U{}_{\hat{A}} + F^U{}_{\hat{A}} \le 3$.

For convenience, we call $\hat{a} = \left\langle T(\hat{a}), I(\hat{a}), F(\hat{a})\right\rangle = \left\langle \left[T^L(\hat{a}), T^U(\hat{a})\right], \left[I^L(\hat{a}), I^U(\hat{a})\right], \left[F^L(\hat{a}), F^U(\hat{a})\right]\right\rangle$ an interval neutrosophic number (INN).

Let $\hat{a} = \left\langle T(\hat{a}), I(\hat{a}), F(\hat{a})\right\rangle$ and $\hat{b} = \left\langle T(\hat{b}), I(\hat{b}), F(\hat{b})\right\rangle$ be any two INNs, the operational laws of them are shown as follows [46].

1) $$\hat{a} \oplus \hat{b} = \left\langle \begin{array}{l}[T^L(\hat{a}) + T^L(\hat{b}) - T^L(\hat{a})T^L(\hat{b}), T^U(\hat{a}) + T^U(\hat{b}) - T^U(\hat{a})T^U(\hat{b})], \\ [I^L(\hat{a})I^L(\hat{a}), I^U(\hat{a})I^U(\hat{b})], [F^L(\hat{a})F^L(\hat{b}), F^U(\hat{a})F^U(\hat{b})]\end{array}\right\rangle \tag{2}$$

2) $$\hat{a} \otimes \hat{b} = \left\langle \begin{array}{l}[T^L(\hat{a})T^L(\hat{b}), T^U(\hat{a})T^U(\hat{b})], \\ [I^L(\hat{a}) + I^L(\hat{a}) - I^L(\hat{a})I^L(\hat{a}), I^U(\hat{a}) + I^U(\hat{b}) - I^U(\hat{a})I^U(\hat{b})], \\ [F^L(\hat{a}) + F^L(\hat{b}) - F^L(\hat{a})F^L(\hat{b}), F^U(\hat{a}) + F^U(\hat{b}) - F^U(\hat{a})F^U(\hat{b})]\end{array}\right\rangle \tag{3}$$

3) $$\lambda\hat{a} = \left\langle \begin{array}{l}[1 - (1 - T^L(\hat{a}))^\lambda, 1 - (1 - T^U(\hat{a}))^\lambda], \\ [(I^L(\hat{a}))^\lambda, (I^U(\hat{a}))^\lambda], [(F^L(\hat{a}))^\lambda, (F^U(\hat{a}))^\lambda]\end{array}\right\rangle \quad \lambda > 0 \tag{4}$$

4) $$(\hat{a})^\lambda = \left\langle \begin{array}{l}[(T^L(\hat{a}))^\lambda, (T^U(\hat{a}))^\lambda], \\ [1 - (1 - I^L(\hat{a}))^\lambda, 1 - (1 - I^U(\hat{a}))^\lambda], [1 - (1 - F^L(\hat{a}))^\lambda, 1 - (1 - F^U(\hat{a}))^\lambda]\end{array}\right\rangle \quad \lambda > 0 \tag{5}$$

Definition 2 [30]. Suppose $\hat{a} = \left\langle T(\hat{a}), I(\hat{a}), F(\hat{a})\right\rangle = \left\langle \left[T^L(\hat{a}), T^U(\hat{a})\right], \left[I^L(\hat{a}), I^U(\hat{a})\right], \left[F^L(\hat{a}), F^U(\hat{a})\right]\right\rangle$ is an INN, the complement of $\hat{a}$ is defined as follows.

$$\hat{a}^c = \left\langle T^c(\hat{a}), I^c(\hat{a}), F^c(\hat{a}) \right\rangle = \left\langle \left[F^L(\hat{a}), F^U(\hat{a})\right], \left[1 - I^U(\hat{a}), 1 - I^L(\hat{a})\right], \left[T^L(\hat{a}), T^U(\hat{a})\right] \right\rangle$$

Definition 3 [5]. Let $\hat{a} = \left\langle T(\hat{a}), I(\hat{a}), F(\hat{a}) \right\rangle$ and $\hat{b} = \left\langle T(\hat{b}), I(\hat{b}), F(\hat{b}) \right\rangle$ be any two INNs, then the Euclidean distance between $\hat{a}$ and $\hat{b}$ is defined as follows.

$$d(\hat{a}, \hat{b}) = \sqrt{\frac{1}{6}\begin{pmatrix} |T^L(\hat{a}) - T^L(\hat{b})|^2 + |T^U(\hat{a}) - T^U(\hat{b})|^2 + |I^L(\hat{a}) - I^L(\hat{b})|^2 + \\ |I^U(\hat{a}) - I^U(\hat{b})|^2 + |F^L(\hat{a}) - F^L(\hat{b})|^2 + |F^U(\hat{a}) - F^U(\hat{b})|^2 \end{pmatrix}} \tag{6}$$

Definition 4 [46]. Suppose $\hat{a} = \left\langle (T(\hat{a}), I(\hat{a}), F(\hat{a})) \right\rangle$ is an INN, and then

$$S(\hat{a}) = [T^L(\hat{a}) + 1 - I^U(\hat{a}) + 1 - F^U(\hat{a}), T^U(\hat{a}) + 1 - I^L(\hat{a}) + 1 - F^L(\hat{a})]$$

$$\mathrm{A}(\hat{a}) = [\min\{T^L(\hat{a}) - F^L(\hat{a}), T^U(\hat{a}) - F^U(\hat{a})\}, \max\{T^L(\hat{a}) - F^L(\hat{a}), T^U(\hat{a}) - F^U(\hat{a})\}]$$

$$\mathrm{C}(\hat{a}) = [T^L(\hat{a}), T^U(\hat{a})]$$

where $S(\hat{a})$, $A(\hat{a})$ and $C(\hat{a})$ represent the score function, accuracy function and certainty function of INN $\hat{a}$, respectively.

For an INN $\hat{a}$, the bigger the truth-membership $T(\hat{a})$ degree is, the smaller the indeterminacy-membership $I(\hat{a})$ degree and the falsity-membership $F(\hat{a})$ degree are, and the greater the INN is. As for the accuracy function, the bigger difference between truth-membership $T(\hat{a})$ and falsity-membership $F(\hat{a})$ is, the surer the statement is. For the certainty function, the bigger thetruth-membership $T(\hat{a})$ is, the greater the corresponding INN is.

Based on the score function, accuracy function and certainty function of INNs, the comparison rules of INNs are shown as follows.

Definition 5 [46]. Let $\hat{a}$ and $\hat{b}$ be any two INNs, the comparison rules are provided as follows.

1) if $S(\hat{a}) \geq S(\hat{b}) > \frac{1}{2}$, then $\hat{a} \succ \hat{b}$.

2) if $S(\hat{a}) \geq S(\hat{b}) = \frac{1}{2}$ and $A(\hat{a}) \geq A(\hat{b}) > \frac{1}{2}$, then $\hat{a} \succ \hat{b}$.

3) if $S(\hat{a}) \geq S(\hat{b}) = \frac{1}{2}$, $A(\hat{a}) \geq A(\hat{b}) = \frac{1}{2}$ and $C(\hat{a}) \geq C(\hat{b}) > \frac{1}{2}$, then $\hat{a} \succ \hat{b}$.

4) if $S(\hat{a}) \geq S(\hat{b}) = \frac{1}{2}$, $A(\hat{a}) \geq A(\hat{b}) = \frac{1}{2}$ and $C(\hat{a}) \geq C(\hat{b}) = \frac{1}{2}$, then $\hat{a} = \hat{b}$.

2.2. The cross entropy of INS

Before introducing the cross entropy and discrimination information measures between two INNs, we firstly review the notions of cross entropy and discrimination information measures between two fuzzy sets.

Definition 6 [23]. Let $A = (A(x_1), A(x_2), ..., A(x_n))$ and $B = (B(x_1), B(x_2), ..., B(x_n))$ be two fuzzy sets in the universe of discourse $X = \{x_1, x_2, ..., x_n\}$. The cross entropy between A and B is shown as follows.

$$H(A,B)=\sum_{i=1}^{n}\left(A(x_i)\log_2\frac{A(x_i)}{(A(x_i)+B(x_i))/2}+(1-A(x_i))\log_2\frac{1-A(x_i)}{1-(A(x_i)+B(x_i))/2}\right) \tag{7}$$

However, the cross entropy $H(A,B)$ is not symmetric in regard to its elements; a symmetric discrimination information measure is introduced by Shang et al. [23], and sown as follows.

$$I(A,B)=H(A,B)+H(B,A) \tag{8}$$

According to the cross entropy and discrimination information measures between two fuzzy sets, the cross entropy and discrimination information measures between two interval neutrosophic sets are defined as follows.

Definition 7. Let $\hat{A}=(\hat{A}(x_1),\hat{A}(x_2),...,\hat{A}(x_n))$ and $\hat{B}=(\hat{B}(x_1),\hat{B}(x_2),...,\hat{B}(x_n))$ be two interval neutrosophic sets in the universe of discourse $X=\{x_1,x_2,...,x_n\}$. The cross entropy between $\hat{A}$ and $\hat{B}$ is defined as follows.

$$\begin{aligned}E(\hat{A},\hat{B})=&\sum_{i=1}^{n}\left[\frac{T^L{}_{\hat{A}}(x_i)+T^U{}_{\hat{A}}(x_i)}{2}\log_2\frac{T^L{}_{\hat{A}}(x_i)+T^U{}_{\hat{A}}(x_i)}{(T^L{}_{\hat{A}}(x_i)+T^U{}_{\hat{A}}(x_i)+T^L{}_{\hat{B}}(x_i)+T^U{}_{\hat{B}}(x_i))/2}\right]+\\&\sum_{i=1}^{n}\left[(1-\frac{T^L{}_{\hat{A}}(x_i)+T^U{}_{\hat{A}}(x_i)}{2})\log_2\frac{1-(T^L{}_{\hat{A}}(x_i)+T^U{}_{\hat{A}}(x_i))/2}{1-(T^L{}_{\hat{A}}(x_i)+T^U{}_{\hat{A}}(x_i)+T^L{}_{\hat{B}}(x_i)+T^U{}_{\hat{B}}(x_i))/4}\right]+\\&\sum_{i=1}^{n}\left[\frac{I^L{}_{\hat{A}}(x_i)+I^U{}_{\hat{A}}(x_i)}{2}\log_2\frac{I^L{}_{\hat{A}}(x_i)+I^U{}_{\hat{A}}(x_i)}{(I^L{}_{\hat{A}}(x_i)+I^U{}_{\hat{A}}(x_i)+I^L{}_{\hat{B}}(x_i)+I^U{}_{\hat{B}}(x_i))/2}\right]+\\&\sum_{i=1}^{n}\left[(1-\frac{I^L{}_{\hat{A}}(x_i)+I^U{}_{\hat{A}}(x_i)}{2})\log_2\frac{1-(I^L{}_{\hat{A}}(x_i)+I^U{}_{\hat{A}}(x_i))/2}{1-(I^L{}_{\hat{A}}(x_i)+I^U{}_{\hat{A}}(x_i)+I^L{}_{\hat{B}}(x_i)+I^U{}_{\hat{B}}(x_i))/4}\right]+\\&\sum_{i=1}^{n}\left[\frac{F^L{}_{\hat{A}}(x_i)+F^U{}_{\hat{A}}(x_i)}{2}\log_2\frac{F^L{}_{\hat{A}}(x_i)+F^U{}_{\hat{A}}(x_i)}{(F^L{}_{\hat{A}}(x_i)+F^U{}_{\hat{A}}(x_i)+F^L{}_{\hat{B}}(x_i)+F^U{}_{\hat{B}}(x_i))/2}\right]+\\&\sum_{i=1}^{n}\left[(1-\frac{F^L{}_{\hat{A}}(x_i)+F^U{}_{\hat{A}}(x_i)}{2})\log_2\frac{1-(F^L{}_{\hat{A}}(x_i)+F^U{}_{\hat{A}}(x_i))/2}{1-(F^L{}_{\hat{A}}(x_i)+F^U{}_{\hat{A}}(x_i)+F^L{}_{\hat{B}}(x_i)+F^U{}_{\hat{B}}(x_i))/4}\right]\end{aligned} \tag{9}$$

where $\hat{A}=\left\{\left\langle x,T_{\hat{A}}(x),I_{\hat{A}}(x),F_{\hat{A}}(x)\right\rangle \mid x\in X\right\}=\left\{x,\left\langle\left[T^L{}_{\hat{A}}(x),T^U{}_{\hat{A}}(x)\right],\left[I^L{}_{\hat{A}}(x),I^U{}_{\hat{A}}(x)\right],\left[F^L{}_{\hat{A}}(x),F^U{}_{\hat{A}}(x)\right]\right\rangle \mid x\in X\right\}$,

$\hat{B}=\left\{\left\langle x,T_{\hat{B}}(x),I_{\hat{B}}(x),F_{\hat{B}}(x)\right\rangle \mid x\in X\right\}=\left\{x,\left\langle\left[T^L{}_{\hat{B}}(x),T^U{}_{\hat{B}}(x)\right],\left[I^L{}_{\hat{B}}(x),I^U{}_{\hat{B}}(x)\right],\left[F^L{}_{\hat{B}}(x),F^U{}_{\hat{B}}(x)\right]\right\rangle \mid x\in X\right\}$.

According to Shannon's inequality [11], we can easily prove that $E(\hat{A},\hat{B})\geq 0$, and $E(\hat{A},\hat{B})=0$ if and only if $T_{\hat{A}}(x)=T_{\hat{B}}(x)$, $I_{\hat{A}}(x)=I_{\hat{B}}(x)$, and $F_{\hat{A}}(x)=F_{\hat{B}}(x)$ for any $x_i\in X$. In addition, we can easily prove that $E(\hat{A}^c,\hat{B}^c)=E(\hat{A},\hat{B})$, where $\hat{A}^c$ and $\hat{B}^c$ are the complement of INSs $\hat{A}$ and $\hat{B}$, respectively.

$E(\hat{A},\hat{B})$ denotes the degree of discrimination of $\hat{A}$ form $\hat{B}$, which can also be named a discrimination for INSs. Since $E(\hat{A},\hat{B})$ is not symmetric, a modified symmetric discrimination information measures for INSs is defined as follows.

$$D(\hat{A},\hat{B})=E(\hat{A},\hat{B})+E(\hat{B},\hat{A}) \tag{10}$$

The smaller $D(\hat{A},\hat{B})$ is, the smaller the difference between $\hat{A}$ and $\hat{B}$ is.

3. Induced generalized interval neutrosophic hybrid aggregation operators based on the additive measures

In this section, we will propose the IGINHAA and IGINHGM operators, which can weight the interval neutrosophic arguments and their ordered positions by the induced variables.

3.1. IGHAA operator

The induced generalized hybrid averaging (IGHA) operator [16] is a generalization of induced ordered weighted averaging (IOWA) operator [35] and the hybrid weighted averaging (HWA) operator [34]. The HWA operator weights both the input arguments and their ordered positions. However, the HWA operator has not the boundedness, idempotency. Further, Lin et al. [12] proposed a new hybrid weighted arithmetical averaging (HWAA) operator with the boundedness and idempotency.

Definition 8 [12]. A HWAA operator of dimension n is a mapping HWAA: $R^n \to R$ that has an associated weight vector $w=(w_1,w_2,...,w_n)^T$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, denoted by

$$HWAA(a_1,a_2,...,a_n)=\frac{\sum_{j=1}^{n} w_j \omega_{(j)} a_{(j)}}{\sum_{j=1}^{n} w_j \omega_{(j)}} \tag{11}$$

where $a_{(j)}$ is the j th largest value of $a_i\ (i=1,2,...,n)$, and $\omega=(\omega_1,\omega_2,...,\omega_n)^T$ is the weight vector of $a_i\ (i=1,2,...,n)$ with $\omega_j \in [0,1]$ and $\sum_{j=1}^{n}\omega_j = 1$.

Moreover, Merigo proposed the induced generalized hybrid averaging (IGHA) operator [16], which is a generalization of HWA operator by utilizing generalized means [9] and order inducing variables. However, the IGHA operator has not the properties, such as boundedness and idempotency. Based on HWAA operator, Meng et al. [15] proposed an induced generalized hybrid arithmetical averaging (IGHAA) operator which has the boundedness and idempotency.

Definition 9 [15]. Suppose $u_i (i=1,2,...,n)$ is a set of order-inducing variables, and γ is a parameter with $\gamma \in (0,+\infty)$. An IGHAA operator of dimension n is a mapping IGHAA: $R^n \to R$ on the set of the second arguments of two tuples $<u_1,a_1>,<u_2,a_2>, \ ...,<u_n,a_n>$. Such that

$$IGHAA(<u_1,a_1>,<u_2,a_2>,...,<u_n,a_n>)=\left(\frac{\bigoplus_{j=1}^{n} w_j \omega^{\gamma}{}_{(j)} a^{\gamma}{}_{(j)}}{\sum_{j=1}^{n} w_j \omega^{\gamma}{}_{(j)}}\right)^{\frac{1}{\gamma}} \tag{12}$$

where $a_{(j)}$ is the a_i value of the IGHAA pair $<u_i,a_i>$ having the j th largest value of u_i, $\omega=(\omega_1,\omega_2,...,\omega_n)^T$ is the weight vector of $a_i\ (i=1,2,...,n)$ such that $\omega_j \in [0,1]$, $\sum_{j=1}^{n}\omega_j = 1$, and

$w=(w_1,...,w_n)^T$ is a weight vector on the ordered set $N=\{1,2,...,n\}$ such that $w_j\in[0,1]$, $\sum_{j=1}^{n}w_j=1$.

3.2. IGINHAA operator

The IGHAA operator fails to aggregate interval neutrosophic information, in this section; we will extend it to INNs, and propose the IGINHAA operator.

Definition 10. Suppose Ω is the set of all INNs, $\hat{a}_i=\langle T(\hat{a}_i), I(\hat{a}_i), F(\hat{a}_i)\rangle\ (i=1,2,...,n)$ is a collection of INNs, $u_i\,(i=1,2,...,n)$ is a set of order-inducing variables, and γ is a parameter with $\gamma\in(0,+\infty)$. An IGINHAA operator of dimension n is a mapping IGINHAA: $\Omega^n\to\Omega$ on the set of second components of 2-tuple arguments $<u_1,\hat{a}_1>,<u_2,\hat{a}_2>,...<u_n,\hat{a}_n>$. Such that,

$$IGINHAA(<u_1,\hat{a}_1>,<u_2,\hat{a}_2>,...,<u_n,\hat{a}_n>)=\left(\frac{\overset{n}{\underset{j=1}{\oplus}}w_j\omega^{\gamma}_{(j)}\hat{a}^{\gamma}_{(j)}}{\sum_{j=1}^{n}w_j\omega^{\gamma}_{(j)}}\right)^{\frac{1}{\gamma}} \tag{13}$$

where $\hat{a}_{(j)}$ is the $\hat{a}_i$ value of the IGINHAA pair $<u_i,\hat{a}_i>$ having the jth largest value of u_i, $\omega=(\omega_1,\omega_2,...,\omega_n)^T$ is the weight vector of $\hat{a}_i\ (i=1,2,...,n)$ such that $\omega_j\in[0,1]$ and $\sum_{j=1}^{n}\omega_j=1$, and $w=(w_1,...,w_n)^T$ is a weight vector on the ordered set $N=\{1,...,n\}$ such that $w_j\in[0,1]$, $\sum_{j=1}^{n}w_j=1$.

However, if there is a tie between $<u_i,\hat{a}_i>$ and $<u_j,\hat{a}_j>$ regarding order-inducing variables such that $u_i=u_j$, in this case, we replace second components of 2-tuple arguments $<u_i,\hat{a}_i>$ and $<u_j,\hat{a}_j>$ by their generalized mean $((\hat{a}_i^{\gamma}\oplus\hat{a}_j^{\gamma})/2)^{1/\gamma}$. If k order-inducing variables are equal, we replace these by k replicas of their generalized mean.

Theorem 1. With the operations of INNs, IGINHAA operator (13) can be transformed into the following form.

$$\begin{aligned}&IGINHAA(<u_1,\hat{a}_1>,<u_2,\hat{a}_2>,...,<u_n,\hat{a}_n>)\\
&=\left\langle\left[\left(1-\prod_{j=1}^{n}(1-(T^L(\hat{a}_{(j)}))^{\gamma})^{\frac{w_j\omega^{\gamma}_{(j)}}{\sum_{j=1}^{n}w_j\omega^{\gamma}_{(j)}}}\right)^{\frac{1}{\gamma}},\left(1-\prod_{j=1}^{n}(1-(T^U(\hat{a}_{(j)}))^{\gamma})^{\frac{w_j\omega^{\gamma}_{(j)}}{\sum_{j=1}^{n}w_j\omega^{\gamma}_{(j)}}}\right)^{\frac{1}{\gamma}}\right],\right.\\
&\left[1-\left(1-\prod_{j=1}^{n}(1-(1-I^L(\hat{a}_{(j)}))^{\gamma})^{\frac{w_j\omega^{\gamma}_{(j)}}{\sum_{j=1}^{n}w_j\omega^{\gamma}_{(j)}}}\right)^{\frac{1}{\gamma}},1-\left(1-\prod_{j=1}^{n}(1-(1-I^U(\hat{a}_{(j)}))^{\gamma})^{\frac{w_j\omega^{\gamma}_{(j)}}{\sum_{j=1}^{n}w_j\omega^{\gamma}_{(j)}}}\right)^{\frac{1}{\gamma}}\right],\\
&\left.\left[1-\left(1-\prod_{j=1}^{n}(1-(1-F^L(\hat{a}_{(j)}))^{\gamma})^{\frac{w_j\omega^{\gamma}_{(j)}}{\sum_{j=1}^{n}w_j\omega^{\gamma}_{(j)}}}\right)^{\frac{1}{\gamma}},1-\left(1-\prod_{j=1}^{n}(1-(1-F^U(\hat{a}_{(j)}))^{\gamma})^{\frac{w_j\omega^{\gamma}_{(j)}}{\sum_{j=1}^{n}w_j\omega^{\gamma}_{(j)}}}\right)^{\frac{1}{\gamma}}\right]\right\rangle\end{aligned} \tag{14}$$

Whose aggregated valued is also an INN.

Proof.

Firstly, we prove Eq. (14) holds.

$$IGINHAA(<\mu_1,\hat{a}_1>,<\mu_2,\hat{a}_2>,\ldots,<\mu_n,\hat{a}_n>)$$

$$=\left(\frac{\bigoplus_{j=1}^{n} w_j\omega^{\gamma}_{(j)}\left\langle \begin{array}{l}\left[(T^{L}_{(\hat{a}_{(j)})})^{\gamma},(T^{U}_{(\hat{a}_{(j)})})^{\gamma}\right],\left[1-(1-I^{L}_{(\hat{a}_{(j)})})^{\gamma},1-(1-I^{U}_{(\hat{a}_{(j)})})^{\gamma}\right],\\ \left[1-(1-F^{L}_{(\hat{a}_{(j)})})^{\gamma},1-(1-F^{U}_{(\hat{a}_{(j)})})^{\gamma}\right]\end{array}\right\rangle}{\sum_{j=1}^{n} w_j\omega^{\gamma}_{(j)}}\right)^{\frac{1}{\gamma}}$$

$$=\left(\bigoplus_{j=1}^{n}\left\langle\begin{array}{l}\left[1-(1-(T^{L}_{(\hat{a}_{(j)})})^{\gamma})^{\frac{w_j\omega^{\gamma}_{(j)}}{\sum_{j=1}^{n}w_j\omega^{\gamma}_{(j)}}},1-(1-(T^{U}_{(\hat{a}_{(j)})})^{\gamma})^{\frac{w_j\omega^{\gamma}_{(j)}}{\sum_{j=1}^{n}w_j\omega^{\gamma}_{(j)}}}\right],\\ \left[(1-(1-I^{L}_{(\hat{a}_{(j)})})^{\gamma})^{\frac{w_j\omega^{\gamma}_{(j)}}{\sum_{j=1}^{n}w_j\omega^{\gamma}_{(j)}}},(1-(1-I^{U}_{(\hat{a}_{(j)})})^{\gamma})^{\frac{w_j\omega^{\gamma}_{(j)}}{\sum_{j=1}^{n}w_j\omega^{\gamma}_{(j)}}}\right],\\ \left[(1-(1-F^{L}_{(\hat{a}_{(j)})})^{\gamma})^{\frac{w_j\omega^{\gamma}_{(j)}}{\sum_{j=1}^{n}w_j\omega^{\gamma}_{(j)}}},(1-(1-F^{U}_{(\hat{a}_{(j)})})^{\gamma})^{\frac{w_j\omega^{\gamma}_{(j)}}{\sum_{j=1}^{n}w_j\omega^{\gamma}_{(j)}}}\right]\end{array}\right\rangle\right)^{\frac{1}{\gamma}}$$

$$IGINHAA(<u_1,\hat{a}_1>,<u_2,\hat{a}_2>,\ldots,<u_n,\hat{a}_n>)$$

$$=\left\langle\begin{array}{l}\left[\left(1-\prod_{j=1}^{n}(1-(T^{L}(\hat{a}_{(j)}))^{\gamma})^{\frac{w_j\omega^{\gamma}_{(j)}}{\sum_{j=1}^{n}w_j\omega^{\gamma}_{(j)}}}\right)^{\frac{1}{\gamma}},\left(1-\prod_{j=1}^{n}(1-(T^{U}(\hat{a}_{(j)}))^{\gamma})^{\frac{w_j\omega^{\gamma}_{(j)}}{\sum_{j=1}^{n}w_j\omega^{\gamma}_{(j)}}}\right)^{\frac{1}{\gamma}}\right],\\ \left[1-\left(1-\prod_{j=1}^{n}(1-(1-I^{L}(\hat{a}_{(j)}))^{\gamma})^{\frac{w_j\omega^{\gamma}_{(j)}}{\sum_{j=1}^{n}w_j\omega^{\gamma}_{(j)}}}\right)^{\frac{1}{\gamma}},1-\left(1-\prod_{j=1}^{n}(1-(1-I^{U}(\hat{a}_{(j)}))^{\gamma})^{\frac{w_j\omega^{\gamma}_{(j)}}{\sum_{j=1}^{n}w_j\omega^{\gamma}_{(j)}}}\right)^{\frac{1}{\gamma}}\right],\\ \left[1-\left(1-\prod_{j=1}^{n}(1-(1-F^{L}(\hat{a}_{(j)}))^{\gamma})^{\frac{w_j\omega^{\gamma}_{(j)}}{\sum_{j=1}^{n}w_j\omega^{\gamma}_{(j)}}}\right)^{\frac{1}{\gamma}},1-\left(1-\prod_{j=1}^{n}(1-(1-F^{U}(\hat{a}_{(j)}))^{\gamma})^{\frac{w_j\omega^{\gamma}_{(j)}}{\sum_{j=1}^{n}w_j\omega^{\gamma}_{(j)}}}\right)^{\frac{1}{\gamma}}\right]\end{array}\right\rangle$$

Next, we prove Eq. (14) is an INN. It is easy to prove the following inequalities.

$$\left[\left(1-\prod_{i=1}^{n}\left(1-(T^{L}_{(\hat{a}_{(j)})})^{\gamma}\right)^{\frac{w_j\omega^{\gamma}_{(j)}}{\sum_{j=1}^{n}w_j\omega^{\gamma}_{(j)}}}\right)^{\frac{1}{\gamma}},\left(1-\prod_{i=1}^{n}\left(1-(T^{U}_{(\hat{a}_{(j)})})^{\gamma}\right)^{\frac{w_j\omega^{\gamma}_{(j)}}{\sum_{j=1}^{n}w_j\omega^{\gamma}_{(j)}}}\right)^{\frac{1}{\gamma}}\right]\subseteq[0,1]$$

$$\left[1-\left(1-\prod_{i=1}^{n}\left(1-(1-I^{L}_{(\hat{a}_{(j)})})^{\gamma}\right)^{\frac{w_j\omega^{\gamma}_{(j)}}{\sum_{j=1}^{n}w_j\omega^{\gamma}_{(j)}}}\right)^{\frac{1}{\gamma}},1-\left(1-\prod_{i=1}^{n}\left(1-(1-I^{U}_{(\hat{a}_{(j)})})^{\gamma}\right)^{\frac{w_j\omega^{\gamma}_{(j)}}{\sum_{j=1}^{n}w_j\omega^{\gamma}_{(j)}}}\right)^{\frac{1}{\gamma}}\right]\subseteq[0,1]$$

$$\left[\left(1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-F^{L}(\hat{a}_{(j)})\right)^{\gamma}\right)^{\frac{w_j\omega^{\gamma}_{(j)}}{\sum_{j=1}^{n} w_j\omega^{\gamma}_{(j)}}}\right)^{\frac{1}{\gamma}}\right),\left(1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-F^{U}(\hat{a}_{(j)})\right)^{\gamma}\right)^{\frac{w_j\omega^{\gamma}_{(j)}}{\sum_{j=1}^{n} w_j\omega^{\gamma}_{(j)}}}\right)^{\frac{1}{\gamma}}\right)\right] \subseteq [0,1]$$

Thus, it is an INN.

Remark 1. If $\gamma = 1$, then the IGINHAA operator reduces to the induced interval neutrosophic hybrid arithmetic averaging (IINHAA) operator.

$$IINHAA(<u_1,\hat{a}_1>,<u_2,\hat{a}_2>,...,<u_n,\hat{a}_n>) = \frac{\oplus_{j=1}^{n} w_j\omega_{(j)}\hat{a}_{(j)}}{\sum_{j=1}^{n} w_j\omega_{(j)}} \tag{15}$$

Furthermore, If $w_j = \frac{1}{n} (j = 1,2,...,n)$, then we get the interval neutrosophic number weighted averaging aggregation (INNWAA) operator [46].

Remark 2. If $\gamma = 2$, then the IGINHAA operator reduces to the induced interval neutrosophic hybrid quadratic averaging (IINHQA) operator.

$$IINHQA(<u_1,\hat{a}_1>,<u_2,\hat{a}_2>,...,<u_n,\hat{a}_n>) = \left(\frac{\oplus_{j=1}^{n} w_j\omega^2_{(j)}\hat{a}^2_{(j)}}{\sum_{j=1}^{n} w_j\omega^2_{(j)}}\right)^{\frac{1}{2}} \tag{16}$$

From these remarks, we can know that IGINHAA operator is the generalized form of IINHAA, INNWAA and IINHQA operators.

Based on the operational laws of INNs, we shall prove that the IGINHAA operator has the following properties.

Theorem 2 (Idempotency). Suppose $\hat{a}_i = a = \langle T(a), I(a), F(a)\rangle\ (i = 1,2,...,n)$, then we have $IGINHAA(<u_1,\hat{a}_1>,<u_2,\hat{a}_2>,...,<u_n,\hat{a}_n>) = \hat{a}$.

Theorem 3 (Commutativity). Suppose $(\hat{a}'_1,\hat{a}'_2,...,\hat{a}'_n)$ is any permutation of $(\hat{a}_1,\hat{a}_2,...,\hat{a}_n)$, then we get $IGINHNAA(<u_1,\hat{a}_1>,<u_2,\hat{a}_2>,...,<u_n,\hat{a}_n>) = IGINHNAA(<u'_1,\hat{a}'_1>,<u'_2,\hat{a}'_2>,...,<u'_n,\hat{a}'_n>)$.

Theorem 4 (Monotonicity). Suppose $A = (\hat{a}_1,\hat{a}_2,...,\hat{a}_n)$ and $\tilde{B} = (\hat{b}_1,\hat{b}_2,...,\hat{b}_n)$ are two collections of INNs, $\hat{a}_i = \langle T(\hat{a}_i), I(\hat{a}_i), F(\hat{a}_i)\rangle\ (i = 1,2,...,n)$, $\hat{b}_i = \langle T(\hat{b}_i), I(\hat{b}_i), F(\hat{b}_i)\rangle\ (i = 1,2,...,n)$, and (.) is a permutation on $\tilde{A}$ and $\tilde{B}$ respectively, such that $T^L(\hat{a}_{(i)}) \le T^L(\hat{b}_{(i)}), T^U(\hat{a}_{(i)}) \le T^U(\hat{b}_{(i)})$, $I^L(\hat{a}_{(i)}) \ge I^L(\hat{b}_{(i)})$, $I^U(\hat{a}_{(i)}) \ge I^U(\hat{b}_{(i)})$, $F^L(\hat{a}_{(i)}) \ge F^L(\hat{b}_{(i)}), F^U(\hat{a}_{(i)}) \ge F^U(\hat{b}_{(i)})$, then we have

$$IGINHAA(<u_1,\hat{a}_1>,<u_2,\hat{a}_2>,...,<u_n,\hat{a}_n>) \le IGINHAA(<u_1,\hat{b}_1>,<u_2,\hat{b}_2>,...,<u_n,\hat{b}_n>)$$

Theorem 5 (Boundedness). Suppose $\hat{a}_i = \langle T(\hat{a}_i), I(\hat{a}_i), F(\hat{a}_i)\rangle\ (i = 1,2,...,n)$ is a collection of INNs, and,

$$\hat{a}^- = \langle [\max(T^L(\hat{a}_i)), \max(T^U(\hat{a}_i))], [\min(I^L(\hat{a}_i)), \min(I^U(\hat{a}_i))], [\min(F^L(\hat{a}_i)), \min(F^U(\hat{a}_i))]\rangle$$

$$\hat{a}^+ = \langle [\min(T^L(\hat{a}_i)), \min(T^U(\hat{a}_i))], [\max(I^L(\hat{a}_i)), \max(I^U(\hat{a}_i))], [\max(F^L(\hat{a}_i)), \max(F^U(\hat{a}_i))]\rangle$$

then,

$$\hat{a}^- \leq IGINHAA(<u_1, \hat{a}_1>, <u_2, \hat{a}_2>, \ldots, <u_n, \hat{a}_n>) \leq \hat{a}^+ .$$

Limited to the space, the proofs of the properties of IGINHAA operator are omitted in here.

3.3. IGINHGM operator

In terms of the IGINHAA operator and geometric mean, we define an IGINHGM operator.

Definition 11. Suppose Ω is the set of all INNs, $\hat{a}_i = \langle T(\hat{a}_i), I(\hat{a}_i), F(\hat{a}_i) \rangle$ $(i = 1,2,..., n)$ is a collection of INNs, $u_i (i = 1,2,..., n)$ is a set of order-inducing variables, and γ is a parameter with $\gamma \in (0,+\infty)$. An IGINHGM operator of dimension n is a mapping IGINHGM: $\Omega^n \to \Omega$ on the set of second components of 2-tuple arguments $< u_1, \hat{a}_1 >, < u_2, \hat{a}_2 >, ... < u_n, \hat{a}_n >$. Such that,

$$IGINHGM(< u_1, \hat{a}_1 >, < u_2, \hat{a}_2 >, ..., < u_n, \hat{a}_n >) = \frac{1}{\gamma}\left(\mathop{\otimes}_{i=1}^{n} (\gamma \hat{a}_{(j)})^{\frac{w_j \omega_{(j)}}{\sum_{j=1}^{n} w_j \omega_{(j)}}} \right) \tag{17}$$

where $\hat{a}_{(j)}$ is the $\hat{a}_i$ value of the IGINHGM pair $< u_i, \hat{a}_i >$ having the j th largest value of μ_i, $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the weight vector of $\hat{a}_i$ $(i = 1,2,..., n)$ such that $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$, and $w = (w_1, w_2, ..., w_n)^T$ is a weight vector on the ordered set $N = \{1,2,..., n\}$ such that $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

However, if there is a tie between $< u_i, \hat{a}_i >$ and $< u_j, \hat{a}_j >$ regarding order-inducing variables such that $u_i = u_j$, in this case, we replace second components of 2-tuple arguments $< u_i, \hat{a}_i >$ and $< u_j, \hat{a}_j >$ by their generalized geometric mean $\frac{1}{\gamma}((\gamma \hat{a}_i)^{\frac{1}{2}} \otimes (\gamma \hat{a}_i)^{\frac{1}{2}})$. If k items are tied, we replace these by k replicas of their generalized geometric mean.

Theorem 6. With the operations of INNs, IGINHGM operator (17) can be transformed into the following form.

$$IGINHGM(<u_1,\hat{a}_1>,<u_2,\hat{a}_2>,...,<u_n,\hat{a}_n>)$$
$$=\left\langle \begin{array}{l}\left[1-\left(1-\prod_{j=1}^{n}(1-(1-T^L(\hat{a}_{(j)}))^{\gamma})^{\frac{w_j\omega_{(j)}}{\sum_{j=1}^{n}w_j\omega_{(j)}}}\right)^{\frac{1}{\gamma}},1-\left(1-\prod_{j=1}^{n}(1-(1-T^U(\hat{a}_{(j)}))^{\gamma})^{\frac{w_j\omega_{(j)}}{\sum_{j=1}^{n}w_j\omega_{(j)}}}\right)^{\frac{1}{\gamma}}\right],\\ \left[\left(1-\prod_{j=1}^{n}(1-(I^L(\hat{a}_{(j)}))^{\gamma})^{\frac{w_j\omega_{(j)}}{\sum_{j=1}^{n}w_j\omega_{(j)}}}\right)^{\frac{1}{\gamma}},\left(1-\prod_{j=1}^{n}(1-(I^U(\hat{a}_{(j)}))^{\gamma})^{\frac{w_j\omega_{(j)}}{\sum_{j=1}^{n}w_j\omega_{(j)}}}\right)^{\frac{1}{\gamma}}\right],\\ \left[\left(1-\prod_{j=1}^{n}(1-(F^L(\hat{a}_{(j)}))^{\gamma})^{\frac{w_j\omega_{(j)}}{\sum_{j=1}^{n}w_j\omega_{(j)}}}\right)^{\frac{1}{\gamma}},\left(1-\prod_{j=1}^{n}(1-(F^U(\hat{a}_{(j)}))^{\gamma})^{\frac{w_j\omega_{(j)}}{\sum_{j=1}^{n}w_j\omega_{(j)}}}\right)^{\frac{1}{\gamma}}\right]\end{array}\right\rangle \quad (18)$$

Whose aggregated valued is also an INN.

Similar to the proof of the Theorem 1, it is easy to get the result.

Remark 3. If $\gamma = 1$, then the IGINHGM operator reduces to the induced interval neutrosophic hybrid geometric mean (IINHGM) operator.

$$IINHGM(<u_1,\hat{a}_1>,<u_2,\hat{a}_2>,...,<u_n,\hat{a}_n>)=\bigotimes_{i=1}^{n}(\hat{a}_{(j)})^{\frac{w_j\omega_{(j)}}{\sum_{j=1}^{n}w_j\omega_{(j)}}} \quad (19)$$

Furthermore, if $w_j = \frac{1}{n}(j = 1,2,...,n)$, then we get the interval neutrosophic number weighted geometric aggregation (INNWGA) operator [46].

Remark 4. If $\gamma = 2$, then the IGINHGM operator reduces to the induced interval neutrosophic hybrid quadratic geometric (IINHQG) operator.

$$IINHQG(<u_1,\hat{a}_1>,<u_2,\hat{a}_2>,...,<u_n,\hat{a}_n>)=\frac{1}{2}\left(\bigotimes_{i=1}^{n}(2\cdot\hat{a}_{(j)})^{\frac{w_j\omega_{(j)}}{\sum_{j=1}^{n}w_j\omega_{(j)}}}\right) \quad (20)$$

From these remarks, we can know that IGINHGM operator is the generalized form of IINHGM, INNWGA and IINHQG operators.

Similar to the proofs of the properties of IGINHAA operator, we can prove that the IGINHGM operator also has idempotency, commutativity, monotonicity and boundedness. It is omitted here.

4. The induced generalized interval neutrosophic hybrid aggregation operators based on the Shapley function

Although the IGINHAA and IGINHGM operators can weight the interval neutrosophic arguments and their ordered positions, they are under the implicit assumption that the aggregated interval neutrosophic arguments are independent. However, in some situations, especially in some decision making, the input elements maybe exist some correlations, these two operators cannot process this condition. To address this situation, we shall propose two new aggregation interval neutrosohphic operators, named IGINSHAA and IGINSHGM operators, which not only weight the interval neutrosophic arguments and their ordered positions, but also consider the interactions among them and among their ordered positions.

4.1. Fuzzy measures and the generalized Shapley function

Definition 12 [27]. A fuzzy measure on a finite set $\mathrm{N}=\{1,2,..., \mathrm{n}\}$ is a set function $\mu: P(N)\rightarrow[0,1]$. It satisfies the following conditions:

(1) Boundary: $\mu(\phi)=0,\ \mu(\mathrm{N})=1$,

(2) Monotonicity: if $F, G\in P(N)$ and $F\subseteq G$, then $\mu(F)\leq\mu(G)$,

where $P(N)$ is the power set of N .

In MADM problems, $\mu(F)$ can be regarded as the importance the of the attribute set F .

The generalized Shapley function was provided by Shapley [24], which is shown as follows.

$$\varphi_i(\mu, N)=\sum_{F\subseteq N\setminus i}\frac{(n-f-1)!\,f!}{n!}(\mu(i\cup f)-\mu(f))\quad\forall i\in N \tag{21}$$

where μ is a fuzzy measure on $N=\{1,2,...,n\}$, n and f represent the cardinalities of N and F , respectively. From the generalized Shapley function, we know that it is an expectation value of the overall interaction between the element $i\in N$ and every combination in $N\setminus i$. From the Eq. (21) and the Definition 12, it is easy to know that $\{\varphi_i(\mu, N)\}_{i\in N}$ is a weight vector, because $\varphi_i(\mu, N)\geq 0$ for any element $i\in N$, and $\sum_{i=1}^{n}\varphi_i(\mu, N)=1$. It should be noted that if there are no interactions between elements, then the Shapley values are equal to their own importance.

4.2. IGINSHAA operator

Definition 13. Suppose Ω is the set of all INNs, $\hat{a}_i=\langle T(\hat{a}_i), I(\hat{a}_i), F(\hat{a}_i)\rangle\ (i=1,2,...,n)$ is a collection of INNs, $u_i(i=1,2,...,n)$ is a set of order-inducing variables, and γ is a parameter with $\gamma\in(0,+\infty)$. An IGINSHAA operator of dimension n is a mapping IGINSHAA: $\Omega^n\rightarrow\Omega$ on the set of second components of 2-tuple arguments $<u_1,\hat{a}_1>,<u_2,\hat{a}_2>,...<u_n,\hat{a}_n>$. Such that,

$$IGINSHAA(<\mu_1,\hat{a}_1>,<\mu_2,\hat{a}_2>,...,<\mu_n,\hat{a}_n>)=\left(\frac{\oplus_{j=1}^{n}\varphi_j(\mu, N)\varphi^{\gamma}{}_{(j)}(\nu, A)\hat{a}^{\gamma}{}_{(j)}}{\sum_{j=1}^{n}\varphi_j(\mu, N)\varphi^{\gamma}{}_{(j)}(\nu, A)}\right)^{\frac{1}{\gamma}} \tag{22}$$

where $\hat{a}_{(j)}$ is the $\hat{a}_i$ value of the IGINSHAA pair $<u_i,\hat{a}_i>$ having the j th largest value of u_i , $\varphi_j(\mu, N)$ is the Shapley value in regard to the associated fuzzy measure μ on $N=\{1,2,...,n\}$ for j th ordered positions, and $\varphi_{(j)}(\nu, A)$ is the Shapley value in regard to the fuzzy measure υ on $A=\{\hat{a}_1,\hat{a}_2,...,\hat{a}_n\}$ for $\hat{a}_{(j)}$ $(j=1,2,...,n)$.

However, if there is a tie between $<u_i,\hat{a}_i>$ and $<u_j,\hat{a}_j>$ with respect to order-inducing variables such that $u_i=u_j$, in this case, we replace second components of 2-tuple arguments $<u_i,\hat{a}_i>$ and $<u_j,\hat{a}_j>$ by their generalized mean $((\hat{a}_i{}^{\gamma}\oplus\hat{a}_j{}^{\gamma})/2)^{1/\gamma}$. If k items are tied, we replace these by k replicas of their generalized mean.

Theorem 7. With the operations of INNs, IGINSHAA operator (22) can be transformed into the following form.

$$IGINSHAA\ (<u_1,\hat{a}_1>,<u_2,\hat{a}_2>,...,<u_n,\hat{a}_n>)$$
$$=\left\langle \left[\left(1-\prod_{j=1}^{n}(1-(T^L(\hat{a}_{(j)}))^{\gamma})^{\frac{\varphi_j(\mu,N)\varphi^{\gamma}_{(j)}(\mu,A)}{\sum_{j=1}^{n}\varphi_j(\mu,N)\varphi^{\gamma}_{(j)}(\mu,A)}}\right)^{\frac{1}{\gamma}},\left(1-\prod_{j=1}^{n}(1-(T^U(\hat{a}_{(j)}))^{\gamma})^{\frac{\varphi_j(\mu,N)\varphi^{\gamma}_{(j)}(\mu,A)}{\sum_{j=1}^{n}\varphi_j(\mu,N)\varphi^{\gamma}_{(j)}(\mu,A)}}\right)^{\frac{1}{\gamma}}\right],\right.$$
$$\left[1-\left(1-\prod_{j=1}^{n}(1-(1-I^L(\hat{a}_{(j)}))^{\gamma})^{\frac{\varphi_j(\mu,N)\varphi^{\gamma}_{(j)}(\mu,A)}{\sum_{j=1}^{n}\varphi_j(\mu,N)\varphi^{\gamma}_{(j)}(\mu,A)}}\right)^{\frac{1}{\gamma}},1-\left(1-\prod_{j=1}^{n}(1-(1-I^U(\hat{a}_{(j)}))^{\gamma})^{\frac{\varphi_j(\mu,N)\varphi^{\gamma}_{(j)}(\mu,A)}{\sum_{j=1}^{n}\varphi_j(\mu,N)\varphi^{\gamma}_{(j)}(\mu,A)}}\right)^{\frac{1}{\gamma}}\right], \tag{23}$$
$$\left.\left[1-\left(1-\prod_{j=1}^{n}(1-(1-F^L(\hat{a}_{(j)}))^{\gamma})^{\frac{\varphi_j(\mu,N)\varphi^{\gamma}_{(j)}(\mu,A)}{\sum_{j=1}^{n}\varphi_j(\mu,N)\varphi^{\gamma}_{(j)}(\mu,A)}}\right)^{\frac{1}{\gamma}},1-\left(1-\prod_{j=1}^{n}(1-(1-F^U(\hat{a}_{(j)}))^{\gamma})^{\frac{\varphi_j(\mu,N)\varphi^{\gamma}_{(j)}(\mu,A)}{\sum_{j=1}^{n}\varphi_j(\mu,N)\varphi^{\gamma}_{(j)}(\mu,A)}}\right)^{\frac{1}{\gamma}}\right]\right\rangle$$

Whose aggregated valued is also an INN.

Similar to the proof of the Theorem 1, it is easy to get the result.

Remark 5. If the fuzzy measures μ and ν are both additive, then the IGINSHAA operator degenerates to the IGINHAA operator.

Remark 6. If $\gamma=1$, then the IGINSHAA operator reduces to the induced interval neutrosophic Shapley hybrid arithmetic averaging (IINSHAA) operator.

$$IINSHAA(<\mu_1,\hat{a}_1>,<\mu_2,\hat{a}_2>,...,<\mu_n,\hat{a}_n>)=\frac{\oplus_{j=1}^{n}\varphi_j(\mu,N)\varphi_{(j)}(\nu,A)\hat{a}_{(j)}}{\sum_{j=1}^{n}\varphi_j(\mu,N)\varphi_{(j)}(\nu,A)} \tag{24}$$

Remark 7. If $\gamma=2$, then the IGINSHAA operator reduces to the induced interval neutrosophic Shapley hybrid quadratic averaging (IINSHQA) operator.

$$IGINSHAA(<\mu_1,\hat{a}_1>,<\mu_2,\hat{a}_2>,...,<\mu_n,\hat{a}_n>)=\left(\frac{\oplus_{j=1}^{n}\varphi_j(\mu,N)\varphi^2{}_{(j)}(\nu,A)\hat{a}^2{}_{(j)}}{\sum_{j=1}^{n}\varphi_j(\mu,N)\varphi^2{}_{(j)}(\nu,A)}\right)^{\frac{1}{2}} \tag{25}$$

Similar to the proofs of the properties of IGINHAA operator, we can see that the IGINSHAA operator has properties, such as idempotency, commutativity, monotonicity and boundedness.

4.3. IGINSHGM operator

Definition 14. Suppose Ω is the set of all INNs, $\hat{a}_i=\langle T(\hat{a}_i),I(\hat{a}_i),F(\hat{a}_i)\rangle\ (i=1,2,...,n)$ is a collection of INNs, $u_i\,(i=1,2,...,n)$ is a set of order-inducing variables, and γ is a parameter with $\gamma\in(0,+\infty)$. An IGINSHGM operator of dimension n is a mapping IGINSHGM: $\Omega^n\to\Omega$ on the set of second components of 2-tuple arguments $<u_1,\hat{a}_1>,<u_2,\hat{a}_2>,...<u_n,\hat{a}_n>$. Such that,

$$IGINSHGM(<u_1,\hat{a}_1>,<u_2,\hat{a}_2>,...,<u_n,\hat{a}_n>)=\frac{1}{\gamma}\left(\underset{i=1}{\overset{n}{\otimes}}(\gamma\hat{a}_{(j)})^{\frac{\varphi_j(\mu,N)\varphi_{(j)}(\nu,A)}{\sum_{j=1}^{n}\varphi_j(\mu,N)\varphi_{(j)}(\nu,A)}}\right) \tag{26}$$

where $\hat{a}_{(j)}$ is the $\hat{a}_i$ value of the IGINSHGM pair $<u_i,\hat{a}_i>$ having the j th largest value of u_i, $\varphi_j(\mu,N)$ is the Shapley value in regard to the associated fuzzy measure μ on $N=\{1,2,...,n\}$ for j th ordered positions, and $\varphi_{(j)}(\nu,A)$ is the Shapley value in regard to the fuzzy measure υ on $A=\{\hat{a}_1,\hat{a}_2,...,\hat{a}_n\}$ for $\hat{a}_{(j)}$ $(j=1,2,...,n)$.

However, if there is a tie between $<u_i,\hat{a}_i>$ and $<u_j,\hat{a}_j>$ with respect to order-inducing variables such that $u_i=u_j$, in this case, we replace second components of 2-tuple arguments $<u_i,\hat{a}_i>$ and $<u_j,\hat{a}_j>$ by their generalized geometric mean $\frac{1}{\gamma}((\gamma\hat{a}_i)^{\frac{1}{2}}\otimes(\gamma\hat{a}_i)^{\frac{1}{2}})$. If k items are tied, we replace these by k replicas of their generalized geometric mean.

Theorem 8. With the operations of INNs, IGINSHGM operator (26) can be transformed into the following form.

$$\begin{aligned}
&IGINSHGM\ (<\mu_1,\hat{a}_1>,<\mu_2,\hat{a}_2>,...,<\mu_n,\hat{a}_n>)\\
&=\Bigg\langle\left[1-\left(1-\prod_{j=1}^{n}(1-(1-T^L(\hat{a}_{(j)}))^{\gamma})^{\frac{\varphi_{j(\mu,N)}\varphi_{(j)}(\nu,A)}{\sum_{j=1}^{n}\varphi_{j(\mu,N)}\varphi_{(j)}(\nu,A)}}\right)^{\frac{1}{\gamma}},1-\left(1-\prod_{j=1}^{n}(1-(1-T^U(\hat{a}_{(j)}))^{\gamma})^{\frac{\varphi_{j(\mu,N)}\varphi_{(j)}(\nu,A)}{\sum_{j=1}^{n}\varphi_{j(\mu,N)}\varphi_{(j)}(\nu,A)}}\right)^{\frac{1}{\gamma}}\right],\\
&\left[\left(1-\prod_{j=1}^{n}(1-(I^L(\hat{a}_{(j)}))^{\gamma})^{\frac{\varphi_{j(\mu,N)}\varphi_{(j)}(\nu,A)}{\sum_{j=1}^{n}\varphi_{j(\mu,N)}\varphi_{(j)}(\nu,A)}}\right)^{\frac{1}{\gamma}},\left(1-\prod_{j=1}^{n}(1-(I^U(\hat{a}_{(j)}))^{\gamma})^{\frac{\varphi_{j(\mu,N)}\varphi_{(j)}(\nu,A)}{\sum_{j=1}^{n}\varphi_{j(\mu,N)}\varphi_{(j)}(\nu,A)}}\right)^{\frac{1}{\gamma}}\right],\\
&\left[\left(1-\prod_{j=1}^{n}(1-(F^L(\hat{a}_{(j)}))^{\gamma})^{\frac{\varphi_{j(\mu,N)}\varphi_{(j)}(\nu,A)}{\sum_{j=1}^{n}\varphi_{j(\mu,N)}\varphi_{(j)}(\nu,A)}}\right)^{\frac{1}{\gamma}},\left(1-\prod_{j=1}^{n}(1-(F^U(\hat{a}_{(j)}))^{\gamma})^{\frac{\varphi_{j(\mu,N)}\varphi_{(j)}(\nu,A)}{\sum_{j=1}^{n}\varphi_{j(\mu,N)}\varphi_{(j)}(\nu,A)}}\right)^{\frac{1}{\gamma}}\right]\Bigg\rangle
\end{aligned} \tag{27}$$

Whose aggregated valued is also an INN.

Similar to the proof of the Theorem 1, it is easy to get the result.

Remark 8. If μ and ν are both additive, then the IGINSHGM operator degenerates to the IGINHGM operator.

Remark 9. If $\gamma=1$, then the IGINSHGM operator reduces to the induced interval neutrosophic Shapley hybrid geometric mean (IINSHGM) operator.

$$IGINSHGM(<u_1,\hat{a}_1>,<u_2,\hat{a}_2>,...,<u_n,\hat{a}_n>)=\bigotimes_{i=1}^{n}(\hat{a}_{(j)})^{\frac{\varphi_j(\mu,N)\varphi_{(j)}(\nu,A)}{\sum_{j=1}^{n}\varphi_j(\mu,N)\varphi_{(j)}(\nu,A)}} \tag{28}$$

Remark 10. If $\gamma=2$, then the IGINSHGM operator degenerates to the induced interval neutrosophic Shapley hybrid quadratic geometric (IINSHQG) operator.

$$IINSHQG(<u_1,\hat{a}_1>,<u_2,\hat{a}_2>,...,<u_n,\hat{a}_n>)=\frac{1}{2}\left(\bigotimes_{i=1}^{n}(2\cdot\hat{a}_{(j)})^{\frac{\varphi_j(\mu,N)\varphi_{(j)}(\nu,A)}{\sum_{j=1}^{n}\varphi_j(\mu,N)\varphi_{(j)}(\nu,A)}}\right) \tag{29}$$

From the above analysis, we can know that IGINSHGM operator is the generalized form of IGINHGM, IINSHGM and IINSHQG operators.

Similar to the IGINSHAA operator, we can prove that the IGINSHGM operator has idempotency, commutativity, monotonicity and boundedness. It is omitted here.

5. An approach to MADM under interval neutrosophic environment

For a MADM problem with interval neutrosophic information, in which the criteria are interactive, suppose $A=\{A_1,A_2,...,A_m\}$ is a set of m candidate alternatives, $C=\{C_1,C_2,...,C_n\}$ is a set of n attributes. Assume $\mathrm{R}=[r_{ij}]_{m\times n}$ is a decision matrix, where $r_{ij}=<[T^L{}_{ij},T^U{}_{ij}],\ [I^L{}_{ij},I^U{}_{ij}],\ [F^L{}_{ij},F^U{}_{ij}]>$ $(i=1,2,...,m;\ j=1,2,...,n)$ is the attribute value expressed by the interval neutrosophic information for alternative A_i with respect to attribute C_j, satisfying $[T^L{}_{ij},T^U{}_{ij}]\subseteq[0,1]$, $[I^L{}_{ij},I^U{}_{ij}]\subseteq[0,1]$, $[F^L{}_{ij},F^U{}_{ij}]\subseteq[0,1]$, and $0\le T^U{}_{ij}+I^U{}_{ij}+F^U{}_{ij}\le 3$.

5.1. The models based on cross entropy for obtaining the optimal fuzzy measures

Due to time pressure, lack of knowledge, the expert's limited expertise about the complex problems, the attribute weights are usually incompletely known [33]. Because the cross entropy is a very important method to obtain the weight vector [32,38,45], the models based on the cross entropy are respectively established to obtain the optimal fuzzy measures on the attribute set and on the ordered set.

Here, the Shapley values can overall reflects the correlations between criteria, and it is regarded as the criteria weights.

Firstly, we use the approach proposed by Zhang et al. [45] to obtain the optimal fuzzy measure ν on the attribute set C. Suppose $R^+=(r_1{}^+,r_2{}^+,...,r_n{}^+)$ and $R^-=(r_1{}^-,r_2{}^-,...,r_n{}^-)$ represent the positive and negative ideal alternatives, respectively, where

$$\begin{aligned} r_j{}^+ &= \left\langle\left[\max_{1\le i\le m}T^L{}_{ij},\max_{1\le i\le m}T^U{}_{ij}\right],\left[\min_{1\le i\le m}I^L{}_{ij},\min_{1\le i\le m}I^U{}_{ij}\right],\left[\min_{1\le i\le m}F^U{}_{ij},\min_{1\le i\le m}F^U{}_{ij}\right]\right\rangle \\ r_j{}^- &= \left\langle\left[\min_{1\le i\le m}T^L{}_{ij},\min_{1\le i\le m}T^U{}_{ij}\right],\left[\max_{1\le i\le m}I^L{}_{ij},\max_{1\le i\le m}I^U{}_{ij}\right],\left[\max_{1\le i\le m}F^U{}_{ij},\max_{1\le i\le m}F^U{}_{ij}\right]\right\rangle \end{aligned} \tag{30}$$

We can get the performance of alternative A_i with respect to attribute C_j as follows.

$$D_{ij}=\frac{D_{ij}{}^-}{D_{ij}{}^-+D_{ij}{}^+} \tag{31}$$

where D_{ij}^{-} and D_{ij}^{+} are the degrees of discriminations of A_i from positive ideal solution r_j^{+} and form negative ideal solution r_j^{-} with respect to attribute C_j, respectively.

If the criteria weights are incompletely known, according to the model established by Zhang et al. [45], we can construct the following model to obtain the optimal fuzzy measure ν on the attribute C.

$$\max \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{D_{ij}^{-}}{D_{ij}^{-}+D_{ij}^{+}} \varphi_{c_j}(\upsilon, C)$$
$$s.t.\begin{cases} \upsilon(\Phi)=0, \upsilon(C)=1 \\ \upsilon(G) \le \upsilon(F) \forall G, F \subseteq C, G \subseteq F \\ \upsilon(C_j) \in Wc_j,\ j=1,2,...,n \end{cases} \tag{32}$$

where $\varphi_{c_j}(\upsilon, C)$ is the Shapley value of the attribute C_j with respect to the fuzzy measure ν, $Wc_j (j=1,2,...,n)$ is the known weight information. If the criteria weights are completely unknown $\upsilon(c_j) \in Wc_j\ (j=1,2,...,n)$ in the above model should be omitted.

Then, we consider how to get the optimal fuzzy measure μ on the ordered set $N=\{1,2,...,n\}$. if the weight information about ordered positions is partly known, the following model can be constructed to obtain the optimal fuzzy measure μ on the ordered set N.

$$\max \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{D_{i(j)}^{-}}{D_{i(j)}^{-}+D_{i(j)}^{+}} \varphi_{j}(\mu, N)$$
$$s.t.\begin{cases} \mu(\Phi)=0, \mu(N)=1 \\ \mu(G) \le \mu(F) \forall G, F \subseteq N, G \subseteq F \\ \mu(j) \in W_j,\ j=1,2,...,n \end{cases} \tag{33}$$

where $\varphi_j(\mu, N)$ is the Shapley value of the j th position with respect to the fuzzy measure μ, and $(.)$ is a permutation on $N\}$ for each $i=1,2,...,m$ such that $\frac{D_{i(j)}^{-}}{D_{i(j)}^{-}+D_{i(j)}^{+}}$ is the j th largest value of $\frac{D_{ij}^{-}}{D_{ij}^{-}+D_{ij}^{+}}$. It should be noted that If the weight information for the ordered positions is completely unknown $\mu(j) \in W_j (j=1,2,...,n)$ in the above model should be omitted.

5.2. The decision procedure

According to above models and induced generalized interval neutrosophic Shapley hybrid operators, we propose a procedure to handle MADM problems in which the attribute values are in the form of INNs and the weight information for attributes and ordered positions are incompletely known. The steps are given as follows.

Step 1: Evaluate the alternatives with respect to the criteria to construct the interval neutrosophic matrix $\mathrm{R}=[r_{ij}]_{m\times n}$.

Step 2: Utilize Eq. (31) to calculate performance of alternative $A_i(i=1,2,...,m)$ with respect to attribute $C_j(j=1,2,...,n)$.

Step 3: Utilize model (32) to obtain the optimal fuzzy measure v on the attribute C, and calculate their Shapley values.

Step 4: Utilize model (33) to obtain the optimal fuzzy measure μ on the ordered set N, and calculate their Shapley values.

Step 5: Utilize IGINSHAA operator or IGINSHGM operator to determine comprehensive interval neutrosophic value $\tilde{r}_i$ of each alternative $A_i(i=1,2,...,,m)$.

Step 6: Utilize Definition 5 to construct the possibility matrix of the score function value.

Step 7: Rank all the alternatives $\{a_1,a_2,...,a_n\}$ by comparison rules of interval neutrosophic numbers, and select the best alternative(s).

Step 8: End.

6. An illustrative example

Let us consider the MADM problems with respect to a manufacturing company to evaluate the global suppliers based on the core competencies of suppliers (adapted from Ref. [29]). Suppose there are four suppliers whose core competencies are evaluated by the following four attributes $C=\{C_1,C_2,C_3,C_4\}$: the level of technology innovation (C_1); the control ability of flow (C_2); the ability of management (C_3); and the level of service (C_4). The evaluated value of supplier A_i $(i=1,2,3,4)$ with respect to C_j $(j=1,2,3,4)$ can be expressed by interval neutrosophic number $r_{ij}=<[T^L{}_{ij},T^U{}_{ij}],\ [I^L{}_{ij},I^U{}_{ij}],[F^L{}_{ij},F^U{}_{ij}])>$ $(i=1,2,...,m;\ \ j=1,2,...,n)$. The interval neutrosophic matrix $R=[r_{ij}]_{m\times n}$ is listed in Table 1. Assume that the importance of attributes is respectively given as $\omega_1=[0.3,0.4],\omega_2=[0.15,0.25],\omega_1=[0.2,025]$ and $\omega_4=[0.25,0.3]$. Furthermore, the importance of the ordered positions is respectively defined by $w_1=[0.1,0.2],w_2=[0.2,0.3]$, $w_3=[0.3,0.4]$ and $w_4=[0.2,0.3]$.

Table 1 interval neutrosophic decision matrix

	C_1	C_2	C_3	C_4
A_1	$\langle$[[0.3,0.5],[0.2,0.3],[0.2,0.5]$\rangle$	$\langle$[0.3,0.5],[0.1,0.3],[0.3,0.5]$\rangle$	$\langle$[0.5,0.6],[0.1,0.2],[0.1,0.3]$\rangle$	$\langle$[0.3,0.5],[0.2,0.4],[0.2,0.3]$\rangle$
A_2	$\langle$[0.4,0.5],[0.1,0.2],[0.3,0.4]$\rangle$	$\langle$[0.2,0.4],[0.1,0.2],[0.3,0.5]$\rangle$	$\langle$[0.6,0.7],[0.1,0.2],[0.1,0.2]$\rangle$	$\langle$[0.4,0.5],[0.3,0.4],[0.2,0.4]$\rangle$
A_3	$\langle$[0.3,0.5],[0.1,0.2],[0.2,0.3]$\rangle$	$\langle$[0.3,0.5],[0.1,0.2],[0.3,0.5]$\rangle$	$\langle$[0.4,0.6],[0.2,0.4],[0.2,0.3]$\rangle$	$\langle$[0.6,0.7],[0.1,0.2],[0.1,0.3]$\rangle$
A_4	$\langle$[0.4,0.6],[0.2,0.3],[0.2,0.3]$\rangle$	$\langle$[0.2,0.4],[0.2,0.3],[0.3,0.5]$\rangle$	$\langle$[0.5,0.7],[0.1,0.2],[0.2,0.3]$\rangle$	$\langle$[0.5,0.7],[0.1,0.2],[0.1,0.3]$\rangle$

6.1. The evaluation steps by IGINSHAA operator

Step 1: Utilize Eq. (31) to calculate performance of alternative A_i with respect to attribute C_j. We can get

$$\frac{D_{ij}^{-}}{D_{ij}^{-}+D_{ij}^{+}}=\begin{pmatrix} 0.0659 & 0.7709 & 0.7244 & 0.0567 \\ 0.5922 & 0.5886 & 1.0000 & 0.0230 \\ 0.8043 & 1.0000 & 0.0000 & 1.0000 \\ 0.6951 & 0.0000 & 0.6994 & 0.9759 \end{pmatrix}$$

Step 2: Utilize model (32) to obtain the optimal fuzzy measures on the attribute set C, the programming model is shown as follows.

$$\begin{aligned} \max = & -0.03055(\upsilon(c_1)-\upsilon(c_2,c_3,c_4))+0.03679(\upsilon(c_2)-\upsilon(c_1,c_3,c_4))+0.05824(\upsilon(c_3)-\upsilon(c_1,c_2,c_4)) \\ & -0.06448(\upsilon(c_4)-\upsilon(c_1,c_2,c_3))+0.00312(\upsilon(c_1,c_2)-\upsilon(c_3,c_4))+0.01385(\upsilon(c_1,c_3)-\upsilon(c_2,c_4)) \\ & -0.04752(\upsilon(c_1,c_4)-\upsilon(c_2,c_3))+2.24908 \end{aligned}$$

$$\begin{cases} \upsilon(c_1,c_2,c_3,c_4)=1\ \upsilon(\Phi)=0 \\ \upsilon(G)\le\upsilon(F)\ \forall F,G\subseteq\{c_1,c_2,c_3,c_4\}\ G\subseteq F \\ \upsilon(c_1)\in[0.3,0.4],\ \upsilon(c_2)\in[0.15,0.25],\upsilon(c_3)\in[0.2,0.25],\upsilon(c_4)\in[0.25,0.3] \end{cases}$$

Solving this above model by Lingo software, we can get

$\upsilon(c_1)=\upsilon(c_1,c_2)=\upsilon(c_1,c_3)=\upsilon(c_1,c_4)=\upsilon(c_1,c_2,c_4)=\upsilon(c_1,c_3,c_4)=0.3,$

$\upsilon(c_2)=\upsilon(c_3)=\upsilon(c_4)=\upsilon(c_2,c_4)=\upsilon(c_3,c_4)=0.2,\upsilon(c_2,c_3)=\upsilon(c_{1,},c_2,c_3)=\upsilon(c_2,c_3,c_4)=\upsilon(c_1,c_2,c_3,c_4)=1$

By Eq. (21), we can obtain the following attribute Shapley values

$$\varphi_{c_1}(\nu,C)=0.0958,\varphi_{c_2}(\nu,C)=0.4208,\varphi_{c_3}(\nu,C)=0.4208,\varphi_{c_4}(\nu,C)=0.0625$$

Step 3: Utilize model (33) to obtain the optimal fuzzy measure μ on the ordered set $N=\{1,2,...,n\}$, the programming model is shown as follows.

$$\begin{aligned} \max = & 0.49925(\mu(1)-\mu(2,3,4))+0.25563(\mu(c2)-\mu(1,3,4))-0.03175(\mu(3)-\mu(1,2,4)) \\ & -0.72313(\mu(4)-\mu(1,2,3))+0.37744(\mu(1,2)-\mu(3,4))+0.23375(\mu(1,3)-\mu(2,4)) \\ & -0.11194(\mu(1,4)-\mu(2,3))+2.24908 \end{aligned}$$

$$\begin{cases} \mu(1,2,3,4)=1\ \upsilon(\Phi)=0 \\ \mu(G)\le\upsilon(G)\ \forall G,F\subseteq\{1,2,3,4\}\ G\subseteq F \\ \mu(1)\in[0.1,0.2],\ \mu(2)\in[0.2,0.3],\ \mu(3)\in[0.3,0.4],\ \mu(4)\in[0.2,0.3] \end{cases}$$

Solving this above model by Lingo software, we can obtain

$$\begin{aligned} & \mu(1)=\mu(4)=\mu(1,4)=0.2, \\ & \mu(2)=\mu(3)=\mu(1,3)=\mu(2,3)=\upsilon(2,4)=\mu(3,4)=\mu(1,3,4)=\mu(2,3,4)=0.3, \\ & \mu(1,2)=\mu(1,2,4)=\mu(1,2,3)=\mu(1,2,3,4)=1. \end{aligned}$$

By Eq. (21), we get the following position Shapley values

$$\varphi_1(\mu,N)=0.4000,\varphi_2(\mu,N)=0.4500,\varphi_3(\mu,N)=0.1000,\varphi_4(\mu,N)=0.0500$$

Step 4: Utilize the IGINSHAA operator to get the comprehensive interval neutrosophic value $\tilde{r}_i$ of each alternative $A_i(i=1,2,...,,m)$. (suppose $\lambda=1$), we can get

$$\begin{aligned} & \tilde{r}_1=<[0.4106,0.5539],[0.1024,0.2444],[0.1687,0.3834]> \\ & \tilde{r}_2=<[0.5179,0.6315],[0.1013,0.2017],[0.1452,0.2633]> \\ & \tilde{r}_3=<[0.4583,0.6073],[0.1066,0.2133],[0.1766,0.3742]> \\ & \tilde{r}_4=<[0.4757,0.6780],[0.1091,0.2104],[0.1929,0.3135]> \end{aligned}$$

Step 5: Utilize Definition 5 to construct the possibility matrix of the score function value, we can get.

$$\begin{bmatrix} 0.5 & 0.2763 & 0.4321 & 0.3581 \\ 0.7237 & 0.5 & 0.6546 & 0.5727 \\ 0.5679 & 0.3454 & 0.5 & 0.4243 \\ 0.6419 & 0.4273 & 0.5757 & 0.5 \end{bmatrix}$$

Step 6: Get the order of the alternatives, we have.

$$a_2 \succ a_4 \succ a_3 \succ a_1$$

Thus, the best alternative is a_2 .

6.2. The evaluation steps by IGINSHGM operator

Step 1-Step 3: See the above Step 1- Step 3.

Step 4: Utilize the IGINSHGM operator to get comprehensive interval neutrosophic value $\tilde{r}_i$ of each alternative $A_i(i=1,2,...,,m)$. (suppose $\lambda=1$), we can get

$$\tilde{r}_1 =<[0.3895,0.5488],[0.1036,0.2515],[0.2004,0.4045]>$$
$$\tilde{r}_2 =<[0.4658,0.6011],[0.1027,0.2028],[0.1732,0.2968]>$$
$$\tilde{r}_3 =<[0.4158,0.5882],[0.1098,0.2210],[0.2054,0.3948]>$$
$$\tilde{r}_4 =<[0.4581,0.6631],[0.1132,0.2132],[0.1996,0.3199]>$$

Step 5: Utilize Definition 5 to construct the possibility matrix of the score function value, we can get.

$$\begin{bmatrix} 0.5 & 0.3202 & 0.4520 & 0.3415 \\ 0.6789 & 0.5 & 0.6313 & 0.5102 \\ 0.5480 & 0.3687 & 0.5 & 0.3873 \\ 0.6585 & 0.4898 & 0.6127 & 0.5 \end{bmatrix}$$

Step 6: Get the order of the alternatives, we have.

$$a_2 \succ a_4 \succ a_3 \succ a_1$$

Thus, the best alternative is a_2 .

6.3 Analysis the influence of the parameter λ

In order to demonstrate the influence of the parameter λ on decision making results, we use the different values λ in IGINSHAA operator and IGINSHGM operator to rank the alternatives. The ranking results are listed in Table 2.

As we can see from Table 2, the ranking results of the alternatives are different for the different values λ in IGINSHAA and IGINSHGM operators. But the best alternative from the different values λ is the same in IGINSHAA operator. Thus, the individual or organization can properly select the favorable alternative in terms of his interest and the actual needs. In general, we can get $\lambda=1$.

Table 2 Ordering of the alternatives by utilizing the different λ in IGINSHAA and IGINSHGM operators

λ	Ranking by IGINSHAA operator	Ranking by IGINSHGM operator
$1/2$	$a_2 \succ a_4 \succ a_3 \succ a_1$	$a_4 \succ a_2 \succ a_3 \succ a_1$
1	$a_2 \succ a_4 \succ a_3 \succ a_1$	$a_2 \succ a_4 \succ a_3 \succ a_1$
2	$a_2 \succ a_4 \succ a_3 \succ a_1$	$a_2 \succ a_4 \succ a_3 \succ a_1$
5	$a_2 \succ a_4 \succ a_3 \succ a_1$	$a_4 \succ a_2 \succ a_1 \succ a_3$

10	$a_2 \succ a_3 \succ a_4 \succ a_1$	$a_4 \succ a_2 \succ a_1 \succ a_3$

6.4. Comparison with the exiting methods

In order to demonstrate the practicality and effectiveness of the developed method in this paper, we can compare the proposed method with the existing method proposed by Zhang et al. [47]. The steps are shown as follows.

Step 1: Utilize the maximization deviation method to determine the attribute weights, the model is established as follows.

$$\max = 1.0699\, w(c_1) + 0.7587\, w(c_2) + 1.1132\, w(c_3) + 1.5471 w(c_4)$$
$$\begin{cases} w(c_1) + w(c_2) + w(c_3) + w(c_4) = 1 \\ w(c_1) \in [0.3, 0.4],\ w(c_2) \in [0.15, 0.25\ ], w(c_3) \in [0.2, 0.25]\ , w(c_4) \in [0.25, 0.3] \end{cases}$$

Solving this above model by Lingo software, we obtain

$$w(c_1) = 0.3, w(c_2) = 0.15, w(c_3) = 0.25, w(c_4) = 0.3$$

Step 2: Identify the positive ideal solution and negative ideal solution, we get

$$\hat{r}^+ = (\langle[0.4,0.6],[0.1,0.2],[0.2,0.3]\rangle, \langle[0.3,0.5],[0.1,0.2],[0.3,0.5]\rangle, \langle[0.6,0.7],[0.1,0.2],[0.1,0.2]\rangle, \langle[0.6,0.7],[0.1,0.2],[0.1,0.3]\rangle)$$
$$\hat{r}^- = (\langle[0.3,0.5],[0.2,0.3],[0.3,0.5]\rangle, \langle[0.2,0.4],[0.2,0.3],[0.3,0.5]\rangle, \langle[0.4,0.6],[0.2,0.4],[0.2,0.3]\rangle, \langle[0.3,0.5],[0.3,0.4],[0.2,0.4]\rangle)$$

Step 3: Determine the distance between each alternative and the positive ideal and negative ideal solutions, respectively, we get

$$D^+ = (0.1118, 0.0818, 0.0527, 0.0595)\ \ D^- = (0.0672, 0.0808, 0.1034, 0.1114)$$

Step 4: Determine the closeness coefficient of each alternative to the ideal solution, we get

$$CC_1 = 0.3753, CC_2 = 0.4967, CC_3 = 0.6625, CC_4 = 0.6518$$

Step 5: According to the relative closeness coefficient $CC_i \quad (i = 1,2,3,4)$, get the order of the alternatives,

$$a_3 \succ a_4 \succ a_2 \succ a_1$$

So, the best alternative is the a_3.

From above analysis, we can see that the ranking results by the proposed methods in this paper are different to that obtained by the existing method. The reason may be that the existing method could not weight the input arguments' positions and capture their interrelationship, which may result in the unreasonable ranking results. However, the method proposed in this paper can effectively handle the interval neutrosophic MADM problems in which attribute and ordered position weights are incompletely known, and attributes exist correlative. So, we think the method developed in this paper is more suitable to handle this application example.

7. Conclusion

This paper developed an approach to interval neutrosophic MADM with interactive conditions and incomplete weight information. To get the comprehensive values, two types of interval neutrosophic hybrid aggregation operators are introduced. One is based on additive measures which could consider the importance of the aggregated interval neutrosophic arguments and their ordered positions, the other utilizes the Shapley function in regard to fuzzy measures which considers the importance of the aggregated interval neutrosophic arguments and their ordered positions but also captures their interrelationship. To get the weight information, the programming models based on cross entropy are established.

It should be mentioned that the fuzzy measures can well process the situations where the input arguments are correlative, but they are determined by ascertaining $2^n - 2$ parameters for n criteria. To reduce the complexity of calculating the fuzzy measures on the given set, it shall be significant to research some special kinds of fuzzy measures, such as λ-fuzzy measures, k-additive measure and p-symmetry measures.

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