Truth-Value Based Interval Neutrosophic Sets

Haibin Wang, Yan-Qing Zhang, *Member, IEEE,* Rajshekhar Sunderraman

Abstract—**Neutrosophic set is a part of neutrosophy which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophic set is a powerful general formal framework that has been recently proposed. However, neutrosophic set needs to be specified from a technical point of view. To this effect, we define the set-theoretic operators on an instance of neutrosophic set, we call it truth-value based interval neutrosophic set. We provide various properties of truth-value based interval neutrosophic sets, which are connected to the operations and relations over truth-value based interval neutrosophic sets.**

Index Terms—**Neutrosophic set, interval neutrosophic set, truth-value based interval neutrosophic set, set-theoretic operator**

I. Introduction

ZADEH introduced the concept of fuzzy sets in 1965 [6]. Since then fuzzy sets and fuzzy logic have been applied in many real applications to handle uncertainty. The traditional fuzzy set uses one real value $\mu_A(x) \in [0,1]$ to represent the grade of membership of fuzzy set A defined on universe X. Sometimes $\mu_A(x)$ itself is uncertain and hard to be defined by a crisp value. So the concept of interval valued fuzzy sets was proposed [4] to capture the uncertainty of grade of membership. Interval valued fuzzy set uses an interval value $[\mu_A^L(x), \mu_A^U(x)]$ with $0 \le \mu_A^L(x) \le \mu_A^U(x) \le 1$ to represent the grade of membership of fuzzy set A. In some applications such as expert system, belief system and information fusion, we should consider not only the truth-membership supported by the evident but also the falsity-membership against by the evident. The concept of truth-membership and falsity-membership is beyond the scope of fuzzy sets and interval valued fuzzy sets. In 1986, Atanassov introduced the intuitionistic fuzzy sets [1] which are a generalization of fuzzy sets and provably equivalent to interval valued fuzzy sets. The intuitionistic fuzzy sets consider both truth-membership $t_A(x)$ and falsity-membership $f_A(x)$, with $t_A(x), f_A(x) \in [0,1]$ and $0 \le t_A(x)+f_A(x) \le 1$. Later on, intuitionistic fuzzy sets were extended to the interval valued intuitionistic fuzzy sets [2]. The interval valued intuitionistic fuzzy set uses a pair of intervals $[t^-,t^+], 0 \le t^- \le t^+ \le 1$ and $[f^-,f^+], 0 \le f^- \le f^+ \le 1$ with $t^+ + f^+ \le 1$ to describe the degree of true belief and false belief, respectively. Because of the restriction that $t^+ + f^+ \le 1$, intuitionistic fuzzy sets and interval valued intuitionistic fuzzy sets can only handle incomplete information not the indeterminate information and inconsistent information which exists commonly in real situations. For example, when we ask the opinion of an expert about certain statement, he or she may that the possibility that the statement is true is between 0.5 and 0.7, and the statement is false is between 0.2 and 0.4, and the degree that he or she is not sure is between 0.1 and 0.3. Here is another example, suppose there are 10 voters during a voting process. In time t1, three vote "yes", two vote "no" and five are undecided, using neutrosophic notation, it can be expressed as x(0.3,0.5,0.2); in time t2, three vote "yes", two vote "no", two give up and three are undecided, it then can be expressed as x(0.3,0.3,0.2). Intuitionistic fuzzy sets cannot handle such situation.

In intuitionistic fuzzy sets, indeterminacy is $1-t_A(x)-f_A(x)$ by default. In neutrosophic set, indeterminacy is quantified explicitly and truth-membership, indeterminacy-membership and falsity-membership are independent. This assumption is very important in a lot of situations such as information fusion when we try to combine the data from different sensors. Neutrosophy was introduced by Smarandache in 1995. "It is a branch of philosophy which studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra" [5]. The neutrosophic set is a powerful general formal framework which generalizes the concept of the classic set, fuzzy set [6], interval valued fuzzy set [4], intuitionistic fuzzy set [1], etc. A neutrosophic set A defined on universe U. $x = x(T,I,F) \in A$ with T, I and F being the real standard or non-standard subsets of $]0^-,1^+[$. T is the degree of truth-membership function in the set A, I is the indeterminacy-membership function in the set A and F is the falsity-membership function in the set A.

The neutrosophic set generalizes the above-mentioned sets from philosophical point of view. From scientific or engineering point of view, the neutrosophic set and set-theoretic operators need to be specified. Otherwise, it will be difficult to apply in the real applications. In this paper, we define the set-theoretic operators on an instance of neutrosophic set called truth-value based interval neutrosophic set.

Manuscript received March 21, 2005.

Haibin Wang is with the Department of Computer Science, Georgia State University, Atlanta, GA 30302 USA (phone: 404-368-0193; fax: 404-463-9912; e-mail: hwang17@student.gsu.edu).

Yan-Qing Zhang is with the Department of Computer Science, Georgia State University, Atlanta, GA 30302 USA (e-mail: yzhang@cs.gsu.edu).

Rajshekhar Sunderraman is with the Department of Computer Science, Georgia State University, Atlanta, GA 30302 USA (e-mail: raj@cs.gsu.edu).

II. NEUTROSOPHIC SET

This section gives a brief overview of concepts of the neutrosophic set defined in [5]. Here, we use different notations to express the same meaning. Let S_1 and S_2 be two real standard or non-standard subsets, then $S_1+S_2 = \{x|x = s_1+s_2, s_1 \in S_1$ and $s_2 \in S_2\}$, $\{1^+\}+S_2 = \{x|x = 1^+ + s_2, s_2 \in S_2\}$, $S_1-S_2 = \{x|x=s_1-s_2, s_1 \in S_1$ and $s_2 \in S_2\}$, $\{1^+\}-S_2 = \{x|x = 1^+ - s_2, s_2 \in S_2\}$, $S_1 \times S_2 = \{x|x = s_1 \times s_2, s_1 \in S_1$ and $s_2 \in S_2\}$.

Definition 1 (Neutrosophic Set) Let X be a space of points (objects), with a generic element in X denoted by x. A neutrosophic set A in X is characterized by a truth-membership function TA, an indeterminacy-membership function IA and falsity-membership function F_A. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]0^-,1^+[$. That is

$$T_A: X \to]0^-,1^+[, \quad (1)$$
$$I_A: X \to]0^-,1^+[, \quad (2)$$
$$F_A: X \to]0^-,1^+[\quad (3)$$

There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0^- \le \sup T_A(x)+\sup I_A(x)+\sup F_A(x) \le 3^+$.

Definition 2 (Complement) The complement of a neutrosophic set A is denoted by c(A) and is defined by

$$T_c(A)(x) = \{1^+\} - T_A(x), \quad (4)$$
$$Ic(A)(x) = \{1^+\} - I_A(x), \quad (5)$$
$$F_c(A)(x) = \{1^+\} - F_A(x), \quad (6)$$

for all x in X.

Definition 3 (Containment) A neutrosophic set A is contained in the other neutrosophic set B, $A \subseteq B$, if and only if

$$\inf T_A(x) \le \inf T_B(x),\ \sup T_A(x) \le \sup T_B(x), \quad (7)$$
$$\inf F_A(x) \ge \inf F_B(x),\ \sup F_B(x) \ge \sup F_B(x). \quad (8)$$

Definition 4 (Union) The union of two neutrosophic sets A and B is a neutrosophic set C, written as $C = A \cup B$, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of A and B by

$$T_C(x) = T_A(x)+T_B(x)-T_A(x)\times T_B(x), \quad (9)$$
$$I_C(x) = I_A(x)+I_B(x)-I_A(x)\times I_B(x), \quad (10)$$
$$F_C(x) = F_A(x)+F_B(x)-F_A(x)\times F_B(x), \quad (11)$$

for all x in X.

Definition 5 (Intersection) The intersection of two neutrosophic sets A and B is a neutrosophic set C, written as $C = A \cap B$, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of A and B by

$$T_C(x) = T_A(x)\times T_B(x), \quad (12)$$
$$I_C(x) = I_A(x)\times I_B(x), \quad (13)$$
$$F_C(x) = F_A(x)\times F_B(x), \quad (14)$$

for all x in X.

III. TRUTH-VALUE BASED INTERVAL NEUTROSOPHIC SET

In this section, we present the notion of the truth-value based interval neutrosophic set. The truth-value based interval neutrosophic set is an instance of neutrosophic set which can be used in real scientific and engineering applications.

Definition 6 (Truth-value Based Interval Neutrosophic Set) Let X be a space of points (objects), with a generic element in X denoted by x. A truth-value based interval neutrosophic set A in X is characterized by truth-membership function T_A, indeterminacy-membership function I_A and falsity-membership function F_A. For each point x in X, $T_A(x), I_A(x), F_A(x) \subseteq [0,1]$.

When X is continuous, a truth-value based interval neutrosophic set A can be written as

$$A = \int_X \langle T(x), I(x), F(x)\rangle / x, x \in X \quad (15)$$

When X is discrete, a truth-value based interval neutrosophic set A can be written as

$$A = \sum_{i=1}^{n} \langle T(xi), I(xi), F(xi)\rangle / xi, xi \in X \quad (16)$$

Consider parameters such as capability, trustworthiness and price of semantic Web services. These parameters are commonly used to define quality of service of semantic Web services.

Example 1 Assume that $X=\{x_1,x_2,x_3\}$. x_1 is capability, x_2 is trustworthiness and x_3 is price. The values of x_1, x_2 and x_3 are subset of [0,1]. They are obtained from the questionnaire of some domain experts, their option could be degree of good, degree of indeterminacy and degree of poor. A is a truth-value based interval neutrosophic set of X defined by

$A = <[0.2,0.4],[0.3,0.5],[0.3,0.5]>/x_1 + <[0.5,0.7],[0,0.2],[0.2,0.3]>/x_2 + <[0.6,0.8],[0.2,0.3],[0.2,0.3]>/x_3$.

B is a truth-value based interval neutrosophic set of X defined by

$B = <[0.5,0.7],[0.1,0.3],[0.1,0.3]>/x_1 + <[0.2,0.3],[0.2,0.4],[0.5,0.8]>/x_2 + <[0.4,0.6],[0,0.1],[0.3,0.4]>/x_3$.

Definition 8 A truth-value based interval neutrosophic set A is ***empty*** if and only if its $\inf T_A(x) = \sup T_A(x)=0$, $\inf I_A(x)=\sup I_A(x)=1$ and $\inf F_A(x)=\sup T_A(x)=0$, for all x in X.

Definition 9 (Truth function) Let g be the function mapping indeterminacy-membership to the subinterval of [0,1], that is

$$g: I \to \text{subinterval of } [0,1]. \quad (17)$$

The purpose of truth function is to extract the true information implicitly represented by the indeterminacy component. For different applications, we could choose different appropriate truth functions. Here we give one example.

Definition 10 (Truth function g1) Let g_1 be the truth function mapping indeterminacy-membership of truth-value based interval neutrosophic set A to the subinterval of [0,1], such that

$$\inf g_1(I_A(x)) = a*(\inf I_A(x)/2) + b*(1-\sup I_A(x)/2), \quad (18)$$
$$\sup g_1(I_A(x)) = a*(\sup I_A(x)/2) + b*(1-\inf I_A(x)/2), \quad (19)$$
$a,b > 0$, $a+b < 1$, $x \in X$.

We now present the set-theoretic operators on the truth-value based interval neutrosopic sets.

Definition 11 Let A and B be two truth-value based interval neutrosophic sets defined on X. ***A(x) ≤ B(x)*** if and only if

$$\inf T_A(x) \le \inf T_B(x),\ \sup T_A(x) \le \sup T_B(x), \quad (20)$$
$$\inf g_1(I_A(x)) \le \inf g_1(I_B(x)),\ \sup g_1(I_A(x)) \le \sup g_1(I_B(x)) \quad (21)$$
$$\inf F_A(x) \ge \inf F_B(x),\ \sup F_A(x) \ge \sup F_B(x). \quad (22)$$

Definition 12 (Containment) A truth-value based interval neutrosophic set A is contained in the other truth-value based interval neutrosophic set B, $A \subseteq B$, if and only if $A(x) \le B(x)$, for all x in X.

Definition 13 Two truth-value based interval neutrosophic sets A and B are equal, written as A=B, if and only if $A \subseteq B$ and $B \subseteq A$.

Definition 14 (Complement) The complement of a truth-value based interval neutrosophic set A is denoted by A' and is defined by

$$T_{A'}(x) = F_A(x), \quad (23)$$
$$\inf g_1(I_{A'}(x)) = 1 - \sup g_1(I_A(x)), \quad (24)$$
$$\sup g_1(I_{A'}(x)) = 1 - \inf g_1(I_A(x)), \quad (25)$$
$$F_{A'}(x) = T_A(x), \quad (26)$$
for all x in X.

Definition 15 (Intersection) The intersection of two truth-value based interval neutrosophic sets A and B is a truth-value based interval neutrosophic set C, written as $C = A \cap B$, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of A and B by

$$\inf T_C(x) = \min(\inf T_A(x), \inf T_B(x)), \quad (27)$$
$$\sup T_C(x) = \min(\sup T_A(x), \sup T_B(x)), \quad (28)$$
$$\inf g_1(I_C(x)) = \min(\inf g_1(I_A(x)), \inf g_1(I_B(x))), \quad (29)$$
$$\sup g_1(I_C(x)) = \min(\sup g_1(I_A(x)), \sup g_1(I_B(x))), \quad (30)$$
$$\inf F_C(x) = \max(\inf F_A(x), \inf F_B(x)), \quad (31)$$
$$\sup F_C(x) = \max(\sup F_A(x), \sup F_B(x)), \quad (32)$$
for all x in X.

Theorem 1 $A \cap B$ is the largest truth-value based interval neutrosophic set contained in both A and B.
Proof: It is easily verified from the definition of containment and intersection of truth-value based interval neutrosophic sets.

Definition 16 (Union) The union of two truth-value based interval neutrosophic sets A and B is a truth-value based interval neutrosophic set C, written as $C = A \cup B$, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of A and B by

$$\inf T_C(x) = \max(\inf T_A(x), \inf T_B(x)), \quad (33)$$
$$\sup T_C(x) = \max(\sup T_A(x), \sup T_B(x)), \quad (34)$$
$$\inf g_1(I_C(x)) = \max(\inf g_1(I_A(x)), \inf g_1(I_B(x))), \quad (35)$$
$$\sup g_1(IC(x)) = \max(\sup g_1(I_A(x)), \sup g_1(I_B(x))), \quad (36)$$
$$\inf F_C(x) = \min(\inf F_A(x), \inf F_B(x)), \quad (37)$$
$$\sup F_C(x) = \min(\sup F_A(x), \sup F_B(x)), \quad (38)$$
for all x in X.

Theorem 2 $A \cup B$ is the smallest truth-value based interval neutrosophic set containing both A and B.
Proof: It is straightforward from the definition of containment and union of truth-value based interval neutrosophic sets.

Theorem 3 $A \subseteq B \Leftrightarrow B' \subseteq A'$.
Proof: It is direct from the definition of containment and complement of truth-value based interval neutrosophic sets.

Now, we will define two operators: truth-favorite(Δ) and false-favorite (∇) to remove the indeterminacy in the truth-value based interval neutrosophic sets and transform it into interval valued intuitionistic fuzzy sets or interval valued paraconsistent sets. These two operators are unique on truth-valued interval neutrosophic sets.

Definition 17 (Truth-favorite) The truth-favorite of truth-value based interval neutrosophic set A is a truth-value based interval neutrosophic set B, written as $B = \Delta A$, whose truth-membership and falsity-membership functions are related to those of A by

$$\inf T_B(x) = \min(\inf T_A(x)+\inf g_1(I_A(x)),1), \quad (39)$$
$$\sup T_B(x) = \min(\sup T_A(x)+\sup g_1(I_A(x)),1), \quad (40)$$
$$\inf I_B(x) = 0, \quad (41)$$
$$\sup I_B(x) = 0, \quad (42)$$
$$\inf F_B(x) = \inf F_A(x), \quad (43)$$
$$\sup F_B(x) = \sup F_A(x), \quad (44)$$
for all x in X.

Definition 18 (False-favorite) The false-favorite of truth-value based interval neutrosophic set A is a truth-value based interval neutrosophic set B, written as $B = \nabla A$, whose truth-membership and falsity-membership functions are related to those of A by

$$\inf T_B(x) = \inf T_A(x), \quad (45)$$
$$\sup T_B(x) = \sup T_A(x), \quad (46)$$
$$\inf I_B(x) = 0, \quad (47)$$
$$\sup I_B(x) = 0, \quad (48)$$
$$\inf F_B(x) = \min(\inf F_A(x)+1-\sup g_1(I_A(x)),1), \quad (49)$$
$$\sup F_B(x) = \min(\sup F_A(x)+1-\inf g_1(I_A(x)),1), \quad (50)$$
for all x in X.

Theorem 4 For every two truth-value based interval neutrosophic sets A and B:

1. $\Delta(A \cup B) \subseteq \Delta A \cup \Delta B$
2. $\Delta A \cap \Delta B \subseteq \Delta(A \cap B)$
3. $\nabla A \cup \nabla B \subseteq \nabla(A \cup B)$
4. $\nabla(A \cap B) \subseteq \nabla A \cap \nabla B$

Proof: It is straightforward from the definition of truth-favorite and false-favorite.

IV. PROPERTIES OF SET-THEORETIC OPERATORS

In this section, we will give some properties of set-theoretic operators defined on truth-value based interval neutrosophic sets as in Section 3. The proof of these properties is left for the readers.

Property 1 (Commutativity) $A \cup B = B \cup A$, $A \cap B = B \cap A$.
Property 2 (Associativity) $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$.
Property 3 (Distributivity) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
Property 4 (Idempotency) $A \cup A = A$, $A \cap A = A$.
Property 5 $A \cap \phi = \phi$, $A \cup X = X$, where $\inf T\phi = \sup T\phi$ $\inf I\phi = \sup I\phi = 0$, $\inf F\phi = \sup F\phi = 1$ and $\inf T_X = \sup T_X = \inf I_X = \sup I_X = 1$, $\inf F_X = \sup F_X = 0$.

Property 6 $A \cup \phi = A$, $A \cap X = A$, where $\inf T\phi = \sup T\phi$ $\inf I\phi = \sup I\phi = 0$, $\inf F\phi = \sup F\phi = 1$ and $\inf T_X = \sup T_X = \inf I_X = \sup I_X = 1$, $\inf F_X = \sup F_X = 0$.

Property 7 (Absorption) $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$.

Property 8 (De Morgan's Laws) $(A \cup B)' = A' \cap B'$, $A' \cap B' = A' \cup B'$.

Property 9 (Involution) $(A')' = A$.

V. CONCLUSIONS AND FUTURE WORK

In this paper, we have presented an instance of the neutrosophic set called the truth-value based interval neutrosophic set. The truth-value based interval neutrosophic set is an extension of classic set, fuzzy set, interval valued fuzzy set, intuitionistic fuzzy sets, interval valued intuitionistic fuzzy set, interval type-2 fuzzy set [3] and paraconsistent set. The notions of inclusion, complement, intersection, union have been defined on the truth-value based interval neutrosophic set. Various properties of set-theoretic operators have been given. In the future, we will create the logic system based on the truth-value based interval neutrosophic set and apply the theory to solve practical applications in areas such as information fusion, data mining, bioinformatics, and Web intelligence, etc.

ACKNOWLEDGMENT

The authors would like to thank Dr. Florentin Smarandache for his valuable suggestions.

REFERENCES

[1] K. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, pp. 87-96, 1986.
[2] K. Atanassov and G. Gargov, "Interval valued intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 31, pp. 343-349, 1989.
[3] Q. Liang and J. Mendel, "Interval type-2 fuzzy logic systems: theory and design," *IEEE Transaction on Fuzzy Systems*, vol. 8, pp。535-550， 2000
[4] I. Turksen, "Interval valued fuzzy sets based on normal forms," *Fuzzy Sets and Systems*, vol. 20, pp. 191-210, 1986.
[5] F. Smarandache, "A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability and Statistics," Phoneix:Xiquan, 2003.
[6] L. Zadeh, "Fuzzy sets," *Inform. and Control*, vol. 8, pp. 338-353, 1965.