



n-Refined Neutrosophic Vector Spaces

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Abstract

This paper introduces the concept of n-refined neutrosophic vector spaces as a generalization of neutrosophic vector spaces, and it studies elementary properties of them. Also, this work discusses some corresponding concepts such as weak/strong n-refined neutrosophic vector spaces, and n-refined neutrosophic homomorphisms.

Keywords: n-Refined weak neutrosophic vector space, n-Refined strong neutrosophic vector space, n-Refined neutrosophic homomorphism.

1. Introduction

Neutrosophy as a part of philosophy founded by F. Smarandache to study origin, nature, and indeterminacies became a strong tool in studying algebraic concepts. Neutrosophic algebraic structures were defined and studied such as neutrosophic modules, and neutrosophic vector spaces, etc. See [1,2,3,4,5,6,7,8,9]. In 2013 Smarandache introduced a perfect idea, when he extended the neutrosophic set to refined [n-valued] neutrosophic set, i.e. the truth value T is refined/split into types of sub-truths such as $(T_1, T_2, \dots)$ similarly indeterminacy I is refined/split into types of sub-indeterminacies $(I_1, I_2, \dots)$ and the falsehood F is refined/split into sub-falsehood $(F_1, F_2, \dots)$ [10,11]. Refined neutrosophic algebraic structures were studied such as refined neutrosophic rings, refined neutrosophic modules, and n-refined neutrosophic rings [4,12].

In this article authors try to define n-refined neutrosophic vector spaces, subspaces, and homomorphisms and to present some of their elementary properties.

For our purpose we use multiplication operation (defined in [12]) between indeterminacies $I_1, I_2, \dots, I_n$ as follows:

$$I_m I_s = I_{\min(m,s)}.$$

This work is a continuation of the study on the n-refined neutrosophic structures that began in [12].

2. Preliminaries

Definition 2.1: [12]

Let $(R, +, \cdot)$ be a ring and $I_k; 1 \leq k \leq n$ be n indeterminacies. We define $R_n(I) = \{a_0 + a_1I + \dots + a_nI_n; a_i \in R\}$ to be an n -refined neutrosophic ring.

Definition 4.3: [12]

(a) Let $R_n(I)$ be an n -refined neutrosophic ring and $P = \sum_{i=0}^n P_i I_i = \{a_0 + a_1I + \dots + a_nI_n; a_i \in P_i\}$, where P_i is a subset of R , we define P to be an AH-subring if P_i is a subring of R for all i . AHS-subring is defined by the condition $P_i = P_j$ for all i, j .

(b) P is an AH-ideal if P_i are two-side ideals of R for all i , the AHS-ideal is defined by the condition $P_i = P_j$ for all i, j .

(c) The AH-ideal P is said to be null if $P_i = R$ or $P_i = \{0\}$ for all i .

Definition 2.3 : [5]

Let $(V, +, \cdot)$ be a vector space over the field K ; then $(V(I), +, \cdot)$ is called a weak neutrosophic vector space over the field K , and it is called a strong neutrosophic vector space if it is a vector space over the neutrosophic field $K(I)$.

Definition 2.4 : [5]

Let $V(I)$ be a strong neutrosophic vector space over the neutrosophic field $K(I)$ and $W(I)$ be a non empty set of $V(I)$ then $W(I)$ is called a strong neutrosophic subspace if $W(I)$ is itself a strong neutrosophic vector space.

Definition 2.6 : [5]

Let $U(I), W(I)$ be two strong neutrosophic subspaces of $V(I)$ and let $f: V(I) \rightarrow W(I)$, we say that f is a neutrosophic vector space homomorphism if

(a) $f(I) = I$,

(b) f is a vector space homomorphism.

We define the kernel of f by $\text{Ker}(f) = \{x \in V(I); f(x) = 0_{W(I)}\}$.

Definition 2.7 : [5]

Let $v_1, v_2, \dots, v_s \in V(I)$ and $x \in V(I)$; we say that x is a linear combination of $\{v_i; i = 1, \dots, s\}$ if

$x = a_1v_1 + \dots + a_sv_s$ such that $a_i \in K(I)$.

The set $\{v_i; i = 1, \dots, s\}$ is called linearly independent if $a_1v_1 + \dots + a_sv_s = 0$ implies $a_i = 0$ for all i .

3. Main concepts and results

Definition 3.1:

Let $(K, +, \cdot)$ be a field, we say that $K_n(I) = K + KI_1 + \dots + KI_n = \{a_0 + a_1I_1 + \dots + a_nI_n; a_i \in K\}$ is an n -refined neutrosophic field.

It is clear that $K_n(I)$ is an n -refined neutrosophic field, but not a field in the classical meaning.

Example 3.2 :

Let $K = Q$ be the field of rationals. The corresponding 3-refined neutrosophic field is

$$Q_3(I) = \{a + bI_1 + cI_2 + dI_3; a, b, c, d \in Q\}.$$

Definition 3.3 :

Let $(V, +, \cdot)$ be a vector space over the field K . Then we say that $V_n(I) = V + VI_1 + \dots + VI_n = \{x_0 + x_1I_1 + \dots + x_nI_n; x_i \in V\}$ is a weak n -refined neutrosophic vector space over the field K . Elements of $V_n(I)$ are called n -refined neutrosophic vectors, elements of K are called scalars.

If we take scalars from the n -refined neutrosophic field $K_n(I)$, we say that $V_n(I)$ is a strong n -refined neutrosophic vector space over the n -refined neutrosophic field $K_n(I)$. Elements of $K_n(I)$ are called n -refined neutrosophic scalars.

Remark 3.4:

If we take $n=1$ we get the classical neutrosophic vector space.

Addition on $V_n(I)$ is defined as:

$$\sum_{i=0}^n a_i I_i + \sum_{i=0}^n b_i I_i = \sum_{i=0}^n (a_i + b_i) I_i.$$

Multiplication by a scalar $m \in K$ is defined as:

$$m \cdot \sum_{i=0}^n a_i I_i = \sum_{i=0}^n (m \cdot a_i) I_i.$$

Multiplication by an n -refined neutrosophic scalar $m = \sum_{i=0}^n m_i I_i \in K_n(I)$ is defined as:

$$(\sum_{i=0}^n m_i I_i) \cdot (\sum_{i=0}^n a_i I_i) = \sum_{i,j=0}^n (m_i \cdot a_j) I_i I_j,$$

where $a_i \in V, m_i \in K, I_i I_j = I_{\min(i,j)}$.

Theorem 3.5 :

Let $(V, +, \cdot)$ be a vector space over the field K . Then a weak n -refined neutrosophic vector space $V_n(I)$ is a vector space over the field K . A strong n -refined neutrosophic vector space is not a vector space but a module over the n -refined neutrosophic field $K_n(I)$.

Proof:

It is similar to that of Theorem 2.3 in [5].

Example 3.6:

Let $V = Z_2$ be the finite vector space of integers modulo 2 over itself:

(a) The corresponding weak 2-refined neutrosophic vector space over the field Z_2 is

$$V_n(I) = \{0, 1, I_1, I_2, I_1 + I_2, 1 + I_1 + I_2, 1 + I_1, 1 + I_2\}.$$

Definition 3.7:

Let $V_n(I)$ be a weak n -refined neutrosophic vector space over the field K ; a nonempty subset $W_n(I)$ is called a weak n -refined neutrosophic subspace of $V_n(I)$ if $W_n(I)$ is a subspace of $V_n(I)$ itself.

Definition 3.8:

Let $V_n(I)$ be a strong n -refined neutrosophic vector space over the n -refined neutrosophic field $K_n(I)$; a nonempty subset $W_n(I)$ is called a strong n -refined neutrosophic subspace of $V_n(I)$ if $W_n(I)$ is a submodule of $V_n(I)$ itself.

Theorem 3.9:

Let $V_n(I)$ be a weak n -refined neutrosophic vector space over the field K , $W_n(I)$ be a nonempty subset of $V_n(I)$. Then $W_n(I)$ is a weak n -refined neutrosophic subspace if and only if:

$$x + y \in W_n(I), m \cdot x \in W_n(I) \text{ for all } x, y \in W_n(I), m \in K.$$

Proof:

It holds directly from the condition of subspace.

Theorem 3.10:

Let $V_n(I)$ be a strong n -refined neutrosophic vector space over an n -refined neutrosophic field $K_n(I)$, $W_n(I)$ be a nonempty subset of $V_n(I)$. Then $W_n(I)$ is a strong n -refined neutrosophic subspace if and only if:

$$x + y \in W_n(I), m \cdot x \in W_n(I) \text{ for all } x, y \in W_n(I), m \in K_n(I).$$

Proof:

It holds directly from the condition of submodule.

Example 3.11:

Let $V = R^2$ be a vector space over the field R , $W = \langle (0,1) \rangle$ is a subspace of V , $R_2^2(I) = \{(a,b) + (m,s)I_1 + (k,t)I_2; a,b,m,s,k,t \in R\}$ is the corresponding weak/strong 2-refined neutrosophic vector space.

$W_2(I) = \{a_0 + a_1I_1 + a_2I_2\} = \{(0,x) + (0,y)I_1 + (0,z)I_2; x,y,z \in R\}$ is a weak 2-refined neutrosophic subspace of the weak 2-refined neutrosophic vector space $R_2^2(I)$ over the field R .

$W_2(I) = \{a_0 + a_1I_1 + a_2I_2\} = \{(0,x) + (0,y)I_1 + (0,z)I_2; x,y,z \in R\}$ is a strong 2-refined neutrosophic subspace of the strong 2-refined neutrosophic vector space $R_2^2(I)$ over the n -refined neutrosophic field $R_2(I)$.

Definition 3.12:

Let $V_n(I)$ be a weak n -refined neutrosophic vector space over the field K , x be an arbitrary element of $V_n(I)$, we say that x is a linear combination of $\{x_1, x_2, \dots, x_m\} \subseteq V_n(I)$, or $x = a_1x_1 + a_2x_2 + \dots + a_mx_m$: $a_i \in K, x_i \in V_n(I)$.

Example 3.13:

Consider the weak 2-refined neutrosophic vector space in Example 3.11,

$x = (0,2) + (1,3)I \in R_2^2(I)$, $x = 2(0,1) + 1(1,0)I_1 + 3(0,1)I_2$, i.e x is a linear combination of the set $\{(0,1), (1,0)I_1, (0,1)I_2\}$ over the field R .

Definition 3.14:

Let $V_n(I)$ be a strong n -refined neutrosophic vector space over an n -refined neutrosophic field $K_n(I)$, x be an arbitrary element of $V_n(I)$, we say that x is a linear combination of $\{x_1, x_2, \dots, x_m\} \subseteq V_n(I)$ is $x = a_1x_1 + a_2x_2 + \dots + a_mx_m$: $a_i \in K_n(I), x_i \in V_n(I)$.

Example 3.15:

Consider the strong 2-refined neutrosophic vector space $R_2^2(I) = \{(a,b) + (m,s)I_1 + (k,t)I_2; a,b,m,s,k,t \in R\}$ over the 2-refined neutrosophic field $R_2(I)$,

$x = (0,2) + (3,3)I_1 + (-1,0)I_2 = (2 + I_1) \cdot (0,1) + (1 + I_2) \cdot (1,1)I_1 + (I_1 - I_2) \cdot (1,0)I_2$, hence x is a linear combination of the set $\{(0,1), (1,1)I_1, (1,0)I_2\}$ over the 2-refined neutrosophic field $R_2(I)$.

Definition 3.16:

Let $X = \{x_1, \dots, x_m\}$ be a subset of a weak n -refined neutrosophic vector space $V_n(I)$ over the field K , X is a weak linearly independent set if $\sum_{i=1}^m a_i x_i = 0$ implies $a_i = 0$; $a_i \in K$.

Definition 3.17:

Let $X = \{x_1, \dots, x_m\}$ be a subset of a strong n -refined neutrosophic vector space $V_n(I)$ over the n -refined neutrosophic field $K_n(I)$, X is a weak linearly independent set if $\sum_{i=1}^m a_i x_i = 0$ implies $a_i = 0$; $a_i \in K_n(I)$.

Definition 3.18:

Let $V_n(I), W_n(I)$ be two strong n -refined neutrosophic vector space over the n -refined neutrosophic field $K_n(I)$, let $f: V_n(I) \rightarrow W_n(I)$ be a well defined map. It is called a strong n -refined neutrosophic homomorphism if:

$$f(a \cdot x + b \cdot y) = a \cdot f(x) + b \cdot f(y) \text{ for all } x, y \in V_n(I), a, b \in K_n(I).$$

A weak n -refined neutrosophic homomorphism can be defined as the same.

We can understand the strong n -refined homomorphism as a module homomorphism, weak n -refined neutrosophic homomorphism can be understood as a vector space homomorphism.

Remark:

The previous definition of n -refined homomorphism between two strong/weak n -refined vector spaces is a classical homomorphism between two modules/spaces. We can not add a similar condition to the concept of neutrosophic homomorphism ($f(I_i) = I_i$), since I_i is not supposed to be an element of $V_n(I)$ if V has more than one dimension for example. According to our definition, $\text{Ker}(f)$ will be a subspace (which is different from classical neutrosophic vector space case) since f was defined as a classical homomorphism without any additional condition.

Definition 3.19:

Let $f: V_n(I) \rightarrow W_n(I)$ be a weak/strong n -refined neutrosophic homomorphism, we define:

$$(a) \text{Ker}(f) = \{x \in V_n(I); f(x) = 0\}.$$

(b) $Im(f) = \{y \in U_n(I); \exists x \in V_n(I) \text{ and } y = f(x)\}$.

Theorem 3.20:

Let $f: V_n(I) \rightarrow U_n(I)$ be a weak n-refined neutrosophic homomorphism. Then

(a) $Ker(f)$ is a weak n-refined neutrosophic subspace of $V_n(I)$.

(b) $Im(f)$ is a weak n-refined neutrosophic subspace of $U_n(I)$.

Proof:

(a) f is a vector space homomorphism since $V_n(I), U_n(I)$ are vector spaces, hence $Ker(f)$ is a subspace of the vector space $V_n(I)$, thus $Ker(f)$ is a weak n-refined neutrosophic subspace of $V_n(I)$.

(b) It holds by similar argument.

Theorem 3.21:

Let $f: V_n(I) \rightarrow U_n(I)$ be a strong n-refined neutrosophic homomorphism. Then

(a) $Ker(f)$ is a strong n-refined neutrosophic subspace of $V_n(I)$.

(b) $Im(f)$ is a strong n-refined neutrosophic subspace of $U_n(I)$.

Proof:

(a) f is a module homomorphism since $V_n(I), U_n(I)$ are modules over the n-refined neutrosophic field $K_n(I)$, hence $Ker(f)$ is a submodule of the vector space $V_n(I)$, thus $Ker(f)$ is a strong n-refined neutrosophic subspace of $V_n(I)$.

(b) Holds by similar argument.

Example 3.22:

Let $R_2^2(I) = \{x_0 + x_1I_1 + x_2I_2; x_0, x_1, x_2 \in R^2\}$, $R_2^3(I) = \{y_0 + y_1I_1 + y_2I_2; y_0, y_1, y_2 \in R^3\}$ be two weak 2-refined neutrosophic vector space over the field R . Consider $f: R_2^2(I) \rightarrow R_2^3(I)$, where

$f[(a, b) + (m, n)I_1 + (k, s)I_2] = (a, 0, 0) + (m, 0, 0)I_1 + (k, 0, 0)I_2$, f is a weak 2-refined neutrosophic homomorphism over the field R .

$Ker(f) = \{(0, b) + (0, n)I_1 + (0, s)I_2; b, n, s \in R\}$.

$Im(f) = \{(a, 0, 0) + (m, 0, 0)I_1 + (k, 0, 0)I_2; a, m, k \in R\}$.

Example 3.23:

Let $W_2(I) = \langle (0, 0, 1)I_1 \rangle = \{q \cdot (0, 0, a)I_1; a \in R, q \in R_2(I)\}$, $U_2(I) = \langle (0, 1, 0)I_1 \rangle = \{q \cdot (0, a, 0)I_1; a \in R; q \in R_2(I)\}$ be two strong 2-refined neutrosophic subspaces of the strong 2-refined neutrosophic vector space $R_2^3(I)$ over 2-refined neutrosophic field $R_2(I)$. Define $f: W_2(I) \rightarrow U_2(I); f[q(0, 0, a)I_1] = q(0, a, 0)I_1; q \in R_2(I)$.

f is a strong 2-refined neutrosophic homomorphism:

Let $A = q_1(0, 0, a)I_1, B = q_2(0, 0, b)I_1 \in W_2(I); q_1, q_2 \in R_2(I)$, we have

$$A + B = (q_1 + q_2)(0,0,a+b)I_1, f(A+B) = (q_1 + q_2).(0,a+b,0)I_1 = f(A) + f(B).$$

Let $m = c + dI_1 + eI_2 \in R_2(I)$ be a 2-refined neutrosophic scalar, we have

$$m \cdot A = c \cdot q_1(0,0,a)I_1 + d \cdot q_1(0,0,a)I_1I_1 + e \cdot q_1(0,0,a)I_2I_1 = q_1(0,0,c \cdot a + d \cdot a + e \cdot a)I_1,$$

$f(m \cdot A) = q_1(0,c \cdot a + d \cdot a + e \cdot a,0)I_1 = m \cdot f(A)$, hence f is a strong 2-refined neutrosophic homomorphism.

$$\text{Ker}(f) = (0,0,0) + (0,0,0)I_1 + (0,0,0)I_2.$$

$$\text{Im}(f) = U_2(I).$$

Remark 3.24:

A union of two n -refined neutrosophic vector spaces $V_n(I)$ and $W_n(I)$ is not supposed to be an n -refined neutrosophic vector space, since the addition operation can not be defined. For example consider $V = R^3, W = R^2, n = 2$.

5. Conclusion

In this paper we have introduced the concept of weak/strong n -refined neutrosophic vector space. Also, some related concepts such as weak/strong n -refined neutrosophic subspace, weak/strong n -refined neutrosophic homomorphism have been presented and studied.

Future research

Authors hope that some corresponding notions will be studied in future such as weak/strong n -refined neutrosophic basis, and AH-subspaces.

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