



n- Refined Neutrosophic Rings

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Abstract

The aim of this paper is to introduce the concept of n-refined neutrosophic ring as a generalization of refined neutrosophic ring. Also, we present concept of n-refined polynomial ring. We study some basic concepts related to these rings such as AH-subrings, AH-ideals, AH-factors, and AH-homomorphisms.

Keywords: n-Refined neutrosophic ring, AH-ideal, AHS-ideal, AH-homomorphism, n-Refined neutrosophic polynomial ring.

1.Introduction

Neutrosophy as a new branch of philosophy founded by F.Smarandache became a useful tool in algebraic studies. Many neutrosophic algebraic structures were defined and studied such as neutrosophic groups, neutrosophic rings, and neutrosophic vector spaces. (See [1,2,3,4,5,6]). Refined neutrosophic theory was introduced by Smarandache in 2013 when he extended the neutrosophic set / logic / probability to refined [n-valued] neutrosophic set / logic / probability respectively, i.e. the truth value T is refined/split into types of sub-truths such as $(T_1, T_2, \dots)$, similarly indeterminacy I is refined/split into types of sub-indeterminacies $(I_1, I_2, \dots)$ and the falsehood F is refined/split into sub-falsehood $(F_1, F_2, \dots)$ [10]. In [9], Smarandache proposed a way to split the Indeterminacy element I into n sub-indeterminacies $I_1, I_2, \dots, I_n$. This idea is very interesting and helps to define new generalizations of refined neutrosophic algebraic structures.

For our purpose we define multiplication operation between indeterminacies $I_1, I_2, \dots, I_n$ as follows:

$I_m I_s = I_{\min(m,s)}$. For examples if $n = 4$ we get

$I_4 I_2 = I_2, I_1 I_2 = I_1, I_2 I_3 = I_2$. If $n = 6$ we get $I_2 I_4 = I_2, I_1 I_4 = I_1, I_4 I_5 = I_4$. If $n = 2$ we get $I_1 I_2 = I_1$ (2-refined neutrosophic ring).

AH-substructures were firstly defined in [1]. AH-ideal in a neutrosophic ring $R(I)$ has the form $P+QI$, where P,Q are ideals in the ring R. We can understand these substructures as two sections, each one is ideal (in rings). These ideals are interesting since they have properties which are similar to classical ideals and they lead us to study the concept of AHS-homomorphisms which are ring homomorphisms but not neutrosophic homomorphisms. In this article we aim to define these ideals in n-refined neutrosophic rings too.

2. Preliminaries

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Definition 2.1: [7]

Let $(R, +, \times)$ be a ring, $R(I) = \{a + bI : a, b \in R\}$ is called the neutrosophic ring where I is a neutrosophic element with condition $I^2 = I$.

Remark 2.2: [4]

The element I can be split into two indeterminacies I_1, I_2 with conditions:

$$I_1^2 = I_1, I_2^2 = I_2, I_1 I_2 = I_2 I_1 = I_1.$$

Definition 2.3: [4]

If X is a set then $X(I_1, I_2) = \{(a, bI_1, cI_2) : a, b, c \in X\}$ is called the refined neutrosophic set generated by X, I_1, I_2 .

Definition 2.4: [4]

Let $(R, +, \times)$ be a ring, $(R(I_1, I_2), +, \times)$ is called a 2-refined neutrosophic ring generated by R, I_1, I_2 .

Theorem 2.5: [4]

Let $(R(I_1, I_2), +, \times)$ be a 2-refined neutrosophic ring then it is a ring.

In the following we remind the reader about some AH-substructures.

Definition 2.6: [2]

Let $(R(I_1, I_2), +, \cdot)$ be a refined neutrosophic ring and P_0, P_1, P_2 be ideals in the ring R then the set $P = (P_0, P_1 I_1, P_2 I_2) = \{(a, bI_1, cI_2) : a \in P_0, b \in P_1, c \in P_2\}$ is called a refined neutrosophic AH-ideal.

If $P_0 = P_1 = P_2$ then P is called a refined neutrosophic AHS-ideal.

Definition 2.7: [1]

Let R be a ring and $R(I)$ be the related neutrosophic ring and

$$P = P_0 + P_1 I = \{a_0 + a_1 I : a_0 \in P_0, a_1 \in P_1\}; P_0, P_1 \text{ are two subsets of } R.$$

(a) We say that P is an AH-ideal if P_0, P_1 are ideals in the ring R .

(b) We say that P is an AHS-ideal if $P_0 = P_1$.

Definition 2.8: [2]

(a) Let $f: R(I_1, I_2) \rightarrow T(I_1, I_2)$ be an AHS-homomorphism we define AH-Kernel of f by : $AH - Ker f = \{(a, bI_1, cI_2) : a, b, c \in Ker f_R\} = (Ker f_R, Ker f_R I_1, Ker f_R I_2)$

(b) let $S = (S_0, S_1 I_1, S_2 I_2)$ be a subset of $R(I_1, I_2)$, then : $f(S) = (f_R(S_0), f_R(S_1) I_1, f_R(S_2) I_2) = \{(f_R(a_0), f_R(a_1) I_1, f_R(a_2) I_2) : a_i \in S_i\}$.

(c) let $S = (S_0, S_1 I_1, S_2 I_2)$ be a subset of $T(I_1, I_2)$. Then

$$f^{-1}(S) = (f_T^{-1}(S_0), f_T^{-1}(S_1) I_1, f_T^{-1}(S_2) I_2).$$

Definition 2.9: [2]

Let $f: R(I_1, I_2) \rightarrow T(I_1, I_2)$ be an AHS-homomorphism we say that f is an AHS-isomorphism if it is a bijective map and $R(I_1, I_2)$, $T(I_1, I_2)$ are called AHS-isomorphic refined neutrosophic rings.

It is easy to see that f_R will be an isomorphism between R , T .

Theorem 2.10 :

Let $f: R(I_1, I_2) \rightarrow T(I_1, I_2)$ be an AHS-homomorphism then we have :

- (a) $AH\text{-}Ker f$ is an AHS-ideal of $R(I_1, I_2)$.
- (b) If P is a refined neutrosophic AH-ideal of $R(I_1, I_2)$, $f(P)$ is a refined neutrosophic AH-ideal of $T(I_1, I_2)$.
- (c) If P is a refined neutrosophic AHS-ideal of $R(I_1, I_2)$, $f(P)$ is a refined neutrosophic AHS-ideal of $T(I_1, I_2)$.

3. n-Refined neutrosophic rings

Definition 1.3:

Let $(R, +, \times)$ be a ring and $I_k; 1 \leq k \leq n$ be n indeterminacies. We define $R_n(I) = \{a_0 + a_1I + \dots + a_nI_n; a_i \in R\}$ to be n -refined neutrosophic ring. If $n=2$ we get a ring which is isomorphic to 2-refined neutrosophic ring $R(I_1, I_2)$.

Addition and multiplication on $R_n(I)$ are defined as:

$$\sum_{i=0}^n x_i I_i + \sum_{i=0}^n y_i I_i = \sum_{i=0}^n (x_i + y_i) I_i, \sum_{i=0}^n x_i I_i \times \sum_{i=0}^n y_i I_i = \sum_{i,j=0}^n (x_i \times y_j) I_i I_j.$$

Where $\times$ is the multiplication defined on the ring R .

It is easy to see that $R_n(I)$ is a ring in the classical concept and contains a proper ring R .

Definition 2.3:

Let $R_n(I)$ be an n -refined neutrosophic ring, it is said to be commutative if $xy = yx$ for each $x, y \in R_n(I)$, if there is $1 \in R_n(I)$ such $1 \cdot x = x \cdot 1 = x$, then it is called an n -refined neutrosophic ring with unity.

Theorem 3.3:

Let $R_n(I)$ be an n -refined neutrosophic ring. Then

- (a) R is commutative if and only if $R_n(I)$ is commutative,
- (b) R has unity if and only if $R_n(I)$ has unity,
- (c) $R_n(I) = \sum_{i=0}^n R I_i = \{\sum_{i=0}^n x_i I_i; x_i \in R\}$.

Proof:

(a) Holds directly from the definition of multiplication on $R_n(I)$.

(b) If 1 is a unity of R then for each $a_0 + a_1I + \dots + a_nI^n \in R_n(I)$ we have

$$1 \cdot (a_0 + a_1I + \dots + a_nI^n) = (a_0 + a_1I + \dots + a_nI^n) \cdot 1 = a_0 + a_1I + \dots + a_nI^n \text{ so } 1 \text{ is the unity of } R_n(I).$$

(c) It is obvious that $\sum_{i=0}^n RI_i \leq R(I)$. Conversely assume that $a_0 + a_1I + \dots + a_nI_n \in R_n(I)$ then by the definition we have that $a_0 + a_1I + \dots + a_nI_n \in \sum_{i=0}^n RI_i$. Thus the proof is complete.

Definition 4.3:

(a) Let $R_n(I)$ be an n-refined neutrosophic ring and $P = \sum_{i=0}^n P_i I_i = \{a_0 + a_1I + \dots + a_nI_n : a_i \in P_i\}$ where P_i is a subset of R , we define P to be an AH-subring if P_i is a subring of R for all i , AHS-subring is defined by the condition $P_i = P_j$ for all i, j .

(b) P is an AH-ideal if P_i is a two sides ideal of R for all i , the AHS-ideal is defined by the condition $P_i = P_j$ for all i, j .

(c) The AH-ideal P is said to be null if $P_i = R$ or $P_i = \{0\}$ for all i .

Theorem 5.3:

Let $R_n(I)$ be an n-refined neutrosophic ring and P is an AH-ideal, $(P, +)$ is an abelian neutrosophic group with $k \leq n$ and $r.p \in P$ for all $p \in P$ and $r \in R$.

Proof :

Since P_i is abelian subgroup of $(R, +)$ and $r.x \in P_i$ for all $r \in R, x \in P_i$, the proof holds.

Remark 6.3:

We can define the right AH-ideal by the condition that P_i is a right ideal of R , the left AH-ideal can be defined as the same.

Definition 7.3:

Let $R_n(I)$ be an n-refined neutrosophic ring and $P = \sum_{i=0}^n P_i I_i, Q = \sum_{i=0}^n Q_i I_i$ be two AH-ideals then we define:

$$P+Q = \sum_{i=0}^n (P_i + Q_i) I_i, P \cap Q = \sum_{i=0}^n (P_i \cap Q_i) I_i.$$

Theorem 8.3:

Let $R_n(I)$ be an n-refined neutrosophic ring and $P = \sum_{i=0}^n P_i I_i, Q = \sum_{i=0}^n Q_i I_i$ be two AH-ideals then $P+Q, P \cap Q$ are AH-ideals. If P, Q are AHS-ideals then $P+Q, P \cap Q$ are AHS-ideals.

Proof :

Since $P_i + Q_i, P_i \cap Q_i$ are ideals of R then $P+Q, P \cap Q$ are AH-ideals of $R_n(I)$.

Definition 9.3:

Let $R_n(I)$ be an n-refined neutrosophic ring and $P = \sum_{i=0}^n P_i I_i$ be an AH-ideal then the AH-radical of P can be defined as $H-rad(P) = \sum_{i=0}^n (\sqrt{P_i}) I_i$.

Theorem 10.3:

The AH-radical of an AH-ideal is an AH-ideal.

Proof :

Since $\sqrt{P_i}$ is an ideal of R then $AH - Rad(P)$ is an AH-ideal of $R_n(I)$.

Definition 11.3:

Let $R_n(I)$ be an n -refined neutrosophic ring and $P = \sum_{i=0}^n P_i I_i$ be an AH-ideal, we define AH-factor $R(I)/P = \sum_{i=0}^n (R/P_i) I_i = \sum_{i=0}^n (x_i + P_i) I_i; x_i \in R$.

Theorem 12.3:

Let $R_n(I)$ be an n -refined neutrosophic ring and $P = \sum_{i=0}^n P_i I_i$ be an AH-ideal:

$R_n(I)/P$ is a ring with the following two binary operations

$$\sum_{i=0}^n (x_i + P_i) I_i + \sum_{i=0}^n (y_i + P_i) I_i = \sum_{i=0}^n (x_i + y_i + P_i) I_i,$$

$$\sum_{i=0}^n (x_i + P_i) I_i \times \sum_{i=0}^n (y_i + P_i) I_i = \sum_{i=0}^n (x_i \times y_i + P_i) I_i.$$

Proof :

Proof is similar to that of Theorem 3.9 in [1].

Definition 13.3:

(a) Let $R_n(I)$, $T_n(I)$ be two n -refined neutrosophic rings respectively, and $f_R: R \rightarrow T$ be a ring homomorphism. We define n -refined neutrosophic AHS-homomorphism as :

$$f: R_n(I) \rightarrow T_n(I); f(\sum_{i=0}^n x_i I_i) = \sum_{i=0}^n f_R(x_i) I_i.$$

(b) f is an n -refined neutrosophic AHS-isomorphism if it is a bijective n -refined neutrosophic AHS-homomorphism.

$$(c) \text{ AH-Ker } f = \sum_{i=0}^n \text{Ker}(f_R) I_i = \{ \sum_{i=0}^n x_i I_i; x_i \in \text{Ker } f_R \}.$$

Theorem 14.3:

Let $R_n(I)$, $T_n(I)$ be two n -refined neutrosophic rings respectively and f be an n -refined neutrosophic AHS-homomorphism $f: R_n(I) \rightarrow T_n(I)$. Then

(a) If $P = \sum_{i=0}^n P_i I_i$ is an AH- subring of $R_n(I)$ then $f(P)$ is an AH- subring of $T_n(I)$,

(b) If $P = \sum_{i=0}^n P_i I_i$ is an AHS- subring of $R_n(I)$ then $f(P)$ is an AHS- subring of $T_n(I)$,

(c) If $P = \sum_{i=0}^n P_i I_i$ is an AH-ideal of $R_n(I)$ then $f(P)$ is an AH-ideal of $f(R_n(I))$,

(d) $P = \sum_{i=0}^n P_i I_i$ is an AHS-ideal of $R_n(I)$ then $f(P)$ is an AHS-ideal of $f(R_n(I))$,

(e) $R_n(I)/AH - \text{Ker}(f)$ is AHS - isomorphic to $f(R(I))$,

(f) Inverse image of an AH-ideal P in $T_n(I)$ is an AH-ideal in $R(I)$.

Proof :

(a) Since $f(P_i)$ is a subring of T then $f(P)$ is an AH- subring of $T_n(I)$.

(b) Holds by a similar way to (a).

(c) Since $f(P_i)$ is an ideal of $f(R)$ then $f(P)$ is an AH-ideal of $f(R(I))$.

(d) It is similar to (c).

(e) We have $R/\text{Ker}(f_R) \cong f(R)$, by definition of AH-factor and $AH - \text{Ker}(f)$ we find that $R(I)/P \cong f(R(I))$.

(f) It is similar to the classical case.

Definition 15.3:

(a) Let $R(I)$ be a commutative n -refined neutrosophic ring, and $P = \sum_{i=0}^n P_i I_i$ be an AH-ideal, we define P to be a weak prime AH-ideal if P_i is a prime ideal of R for all i .

(b) P is called a weak maximal AH-ideal if P_i is a maximal ideal of R for all i .

(c) P is called a weak principal AH-ideal if P_i is a principal ideal of R for all i .

Theorem 16.3:

Let $R_n(I)$, $T_n(I)$ be two commutative n -refined neutrosophic rings with an n -refined neutrosophic AHS-homomorphism $f: R_n(I) \rightarrow T_n(I)$:

(a) If $P = \sum_{i=0}^n P_i I_i$ is an AHS-ideal of $R_n(I)$ and $\text{Ker}(f_R) \leq P_i \neq R_n(I)$:

(a) P is a weak prime AHS-ideal if and only if $f(P)$ is a weak prime AHS-ideal in $f(R_n(I))$.

(b) P is a weak maximal AHS-ideal if and only if $f(P)$ is a weak maximal AHS-ideal in $f(R_n(I))$.

(c) If $Q = \sum_{i=0}^n Q_i I_i$ is an AHS-ideal of $T_n(I)$ then it is a weak prime AHS-ideal if and only if $f^{-1}(Q)$ is a weak prime in $R_n(I)$.

(d) if $Q = \sum_{i=0}^n Q_i I_i$ is an AHS-ideal of $T_n(I)$ then it is a weak maximal AHS-ideal if and only if $f^{-1}(Q)$ is a weak maximal in $R_n(I)$.

Proof :

Proof is similar to that of Theorem 3.8 in [1].

Example 17.3:

Let $R = \mathbb{Z}$ be the ring of integers, $T = \mathbb{Z}_6$ be the ring of integers modulo 6 with multiplication and addition modulo 6, we have:

(a) $f_R: R \rightarrow T; f(x) = x \text{ mod } 6$ is a ring homomorphism, $\text{ker}(f_R) = 6\mathbb{Z}$, the corresponding AHS-homomorphism between $R_4(I)$, $T_4(I)$ is:

$$f: R_4(I) \rightarrow T_4(I); f(a + bI_1 + cI_2 + dI_3 + eI_4) = (a \text{ mod } 6) + (b \text{ mod } 6)I_1 + (c \text{ mod } 6)I_2 + (d \text{ mod } 6)I_3 + (e \text{ mod } 6)I_4; a, b, c, d, e \in \mathbb{Z}.$$

(b) $P = \langle 2 \rangle, Q = \langle 3 \rangle$ are two prime and maximal and principal ideals in R ,

$M = P + PI_1 + QI_2 + QI_3 + PI_4 = \{(2a + 2bI_1 + 3cI_2 + 3dI_3 + 2eI_4; a, b, c, d, e \in \mathbb{Z})\}$ is a weak prime/ maximal AH-ideal of $R_4(I)$.

(c) $\text{Ker}(f_R) = 6Z \leq P, Q, f_R(P) = \{0, 2, 4\}, f_R(Q) = \{0, 3\},$

$f(M) = f(P) + f(P)I_1 + f(Q)I_2 + f(Q)I_3 + f(P)I_4$ which is a weak maximal/ prime/principal AH-ideal of $T_4(I)$.

(d) $AH - \text{Ker}(f) = 6Z + 6ZI_1 + 6ZI_2 + 6ZI_3 + 6ZI_4$ which is an AHS-ideal of $R_4(I)$.

(e) $R_4(I)/AH - \text{Ker}f = R/6Z + R/6Z I_1 + R/6Z I_2 + R/6Z I_3 + R/6Z I_4$ which is AHS-isomorphic to $f(R_4(I)) = T_4(I)$, since $R/6Z \cong T$.

Example 18.3:

Let $R = Z_8$ be a ring with addition and multiplication modulo 8.

(a) 3-refined neutrosophic ring related with R is $Z_{83}(I) = \{a + bI_1 + cI_2 + dI_3; a, b, c, d \in Z_8\}$.

(b) $P = \{0, 4\}$ is an ideal of R , $\sqrt{P} = \{0, 2, 4, 6\}$, $M = P + PI_1 + PI_2 + PI_3$ is an AHS-ideal of $Z_{83}(I)$,

$AH - \text{Rad}(M) = \sqrt{P} + \sqrt{P}I_1 + \sqrt{P}I_2 + \sqrt{P}I_3$ which is an AHS-ideal of $Z_{83}(I)$.

Example 19.3:

Let $R = Z_2$ the ring of integers modulo 2, let $n = 3$. The corresponding 3-refined neutrosophic ring is

$Z_{23}(I) = \{0, 1, I_1, I_2, I_3, 1 + I_1, 1 + I_2, 1 + I_3, I_1 + I_2, I_1 + I_3, I_1 + I_2 + I_3, I_2 + I_3, 1 + I_1 + I_2 + I_3, 1 + I_2 + I_3, 1 + I_1 + I_3, 1 + I_1 + I_2\}$.

4. n-Refined neutrosophic polynomial rings

Definition 1.4:

Let $R_n(I)$ be a commutative n-refined neutrosophic ring and $P: R_n(I) \rightarrow R_n(I)$ is a function defined as $P(x) = \sum_{i=0}^m a_i x^i$ such $a_i \in R_n(I)$, we call P a neutrosophic polynomial on $R_n(I)$.

We denote by $R_n(I)[x]$ to the ring of neutrosophic polynomials over $R_n(I)$.

Since $R_n(I)$ is a classical ring then $R_n(I)[x]$ is a classical ring.

Theorem 2.4:

Let $R(I)$ be a commutative n-refined neutrosophic ring. Then $R_n(I)[x] = \sum_{i=0}^n R[x]I_i$.

Proof :

Let $P(x) = \sum_{i=0}^n P_i(x)I^i \in \sum_{i=0}^n R[x]I^i$, by rearranging the previous sum we can write it as $P(x) = \sum_{i=0}^m a_i x^i \in R_n(I)[x]$.

Conversely, if $P(x) = \sum_{i=0}^n a_i x^i \in R_n(I)[x]$, then we can write it as

$P(x) = \sum_{i=0}^n P_i(x)I_i \in \sum_{i=0}^n R[x]I_i$, by the previous argument we find the proof.

Example 3.4:

Let $Z_3(I)$ be a 3-refined neutrosophic ring and $P(x) = I_1 + (2+I_1)x + (I_1+I_3)x^2$ a polynomial over $Z_{3n}(I)$, then we can write $P(x) = 2x + I_1(1+x+x^2) + I_2x^2$.

It is obvious that $R_n(I) \leq R_n(I)[x]$.

Definition 4.4:

Let $P(x) = \sum_{i=0}^n P_i(x)I^i$ a neutrosophic polynomial over $R_n(I)$ we define the degree of P by $\deg P = \max(\deg P_i)$.

5. Conclusion

In this paper we have defined the n -refined neutrosophic ring and n -refined neutrosophic polynomial ring, we have introduced and studied AH-structures such as:

AH-ideal, AHS-ideal, AH-weak principal ideal, AH-weak prime ideal. Authors hope that other n -refined neutrosophic algebraic structures will be defined in future research.

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