Neutrosophic Soft Bitopological Spaces

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Abstract: In this paper, we built bitopological space on the concept of neutrosophic soft set, we defined the basic topological concepts of this spaces which are N_3-(bi)*-open set, N_3-(bi)*-closed set, (bi)*-neutrosophic soft interior, (bi)*-neutrosophic soft closure, (bi)*-neutrosophic soft boundary, (bi)*-neutrosophic soft exterior and we introduced their properties. In addition, we investigated the relations of these basic topological concepts with their counterparts in neutrosophic soft topological spaces and we introduced many examples.

Keywords : Neutrosophic soft bitopological Spaces, star bineutrosophic soft open set, star bi neutrosophic soft closed set.

1. **Introduction**

The concept of soft set is defined by Molodtsov [1] as follows: Let M be an initial universe set and E be a set of parameters. Let P(M) denotes the set of all the subsets of M. Consider $B\neq\emptyset$, $B\subseteq E$. The collection (β,B) is termed to be the soft set, where β is a mapping by $\beta:B\rightarrow P(M)$. Smarandache [2] introduced neutrosophic set as a generalization of fuzzy set [3] and intuitionistic fuzzy set [4]. P. K. Maji [5] defended the concept of neutrosophic soft set by combining the concept of neutrosophic set and soft set. This the concept is defined as follows: let M be an initial universe set and E be a set of parameters. Let P(M) denote the set of all the neutrosohpic sets of M. Consider $B\neq\emptyset$, $B\subseteq E$. The collection (β,B) is termed to be the soft neutrosophic set, where β is a mapping by $\beta:B\rightarrow P(M)$. This concept has been modified (see [6,7]). The concept of neutrosohpic soft topological space was introduced by T. Bera [8]. Al-Nafee [9] introduced the concept of neutrosohpic soft topological space depending on a new family of neutrosohpic soft sets ($N_3(M)$). Our work in this research is presented based on the neutrosohpic soft topological space which was built from the elements of family $N_3(M)$ ([9]). Other theoretical studies on these concepts were presented by a number of researchers, for example, Narmada, Georgiou, Taha, Cageman, Al-Nafee, Evanzalin and Salama, (see [10, 11, 22, 13, 14, 15, 16, 17, 18, 19, 20]).

Kelly, [21] introduced the concept of bitopological space. This concept is introduced as an extension of topological space. This concept has been introduced with interest in fuzzy set, soft set and neutrosophic set (see [22, 23, 24, 25]). Therefore, we find it important and necessary to build a bitopological spaces on the concept of neutrosophic soft set. In this paper, bitopological space on the concept of neutrosophic soft set is built, the basic topological concepts of this spaces which are N_3-(bi)*-open set, N_3-(bi)*-closed set, (bi)*-neutrosophic soft interior, (bi)*-neutrosophic soft closure, (bi)*-neutrosophic soft boundary, (bi)*-neutrosophic soft exterior are defined, the relations of these basic topological concepts with their counterparts in neutrosophic soft topological spaces are investigated and many examples on this concepts are given.

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2. Preliminary

In this section, we will refer to the basic definitions required in our work.

2.1. Definition [26]

The neutrosohpic set N over M is defined as follows:

$$N=\{< m, H_N(m), G_N(m), J_N(m) >) : m \in M\}.$$

where, the functions H,G,J : $M \to]-0,+1[$ and $-0 \leq H_N(m) + G_N(m) + J_N(m) \leq +3$

2.2. Definition [9]

Let β_B and $\mu_B \in N_3(M)$ such that;

$\beta_B = \{(r, \{< m^{(H_{\beta_B(r)}(m), G_{\beta_B(r)}(m), J_{\beta_B(r)}(m))} > : m \in M\}), r \in E_M\}$.

$\mu_B = \{(r, \{< m^{(H_{\mu_B(r)}(m), G_{\mu_B(r)}(m), J_{\mu_B(r)}(m))} > : m \in M\}), r \in E_M\}$. Then:

- $\widetilde{M}_B = \{(r, \{< m^{(1,1,0)} > : m \in M\}), r \in E_M\}$ [Absolute neutrosohpic soft set].
- $\widetilde{\emptyset}_B = \{(r, \{< m^{(0,0,1)} > : m \in M\}), r \in E_M\}$ [Null neutrosohpic soft set].
- $(\beta_B)^C = (r, \{< m^{(1-H_{\beta_B(r)}(m), 1-G_{\beta_B(r)}(m), 1-J_{\beta_B(r)}(m))} > : m \in M\}), r \in E_M\}$ [Complement of β_B].
- $\beta_B \sqsubseteq \mu_B \leftrightarrow \{(r, \{< m^{(H_{\beta_B(r)}(m) \leq H_{\mu_B(r)}(m), G_{\beta_B(r)}(m) \leq G_{\mu_B(r)}(m), J_{\beta_B(r)}(m) \geq J_{\mu_B(r)}(m))} > : m \in M\}), r \in E_M\}$.
- $\beta_B \sqcup \mu_B = \{(r, \{< m^{(H_{\beta_B(r)}(m) \vee H_{\mu_B(r)}(m), G_{\beta_B(r)}(m) \vee G_{\mu_B(r)}(m), J_{\beta_B(r)}(m) \wedge J_{\mu_B(r)}(m))} >) : m \in M\}, r \in E_M\}$.
- $\beta_B \sqcap \mu_B = \{(r, \{< m^{(H_{\beta_B(r)}(m) \wedge H_{\mu_B(r)}(m), G_{\beta_B(r)}(m) \wedge G_{\mu_B(r)}(m), J_{\beta_B(r)}(m) \vee J_{\mu_B(r)}(m))} > : m \in M\}), r \in E_M\}$.

Note that: "E_M is the set of all possible paramerers under consideration with respect to M ($B \subseteq E_M$)".

2.3. Definition [9]

Let $T \subseteq (N_3(M))$. The collection T is called a neutrosophic soft topology on M, if the following conditions are true:

1) $\widetilde{M}_B$, $\widetilde{\emptyset}_B$ belong to T.
2) If $\beta_{j_B} \in T$; $j \in J$, then $\sqcup_{j \in J} \beta_{j_B} \in T \ \forall j \in J$.
3) If $\beta_B, \mu_B \in T$, then $\beta_B \sqcap \mu_B \in T$.

Then the triplet (M,B,T) is a neutrosophic soft topological space or (N_3-Top for short). Members of T are called a neutrosohpic soft open sets (N_3-T-open for short) and their complements are a neutrosohpic soft open sets (N_3-T-closed for short).

The neutrosophic soft interior of $\beta_B \in N_3(M)$ ($(\beta_B)^0$ for short) is defended as:

$$(\beta_B)^0 = \sqcup\{(\omega_B): \omega_B \text{ is a } N_3\text{-T-open set}, \omega_B \sqsubseteq \beta_B\}.$$

The neutrosophic soft cloture of $\beta_B \in N_3(M)$ ($\overline{(\beta_B)}$ for short) is defended as:

$$\overline{(\beta_B)} = \sqcap\{(\omega_B): \omega_B \text{ is a } N_3\text{-T-closed set}, \beta_B \sqsubseteq \omega_B\}.$$

2.4. Example [9]

Let $M = \{m_1, m_2, m_3\}$, $B \subseteq E_M$, where $B = \{r\}$ and $\beta_B, \mu_B, \gamma_B \in N_3(M)$.

Such that

$$\beta_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(0,0,1)} >, < m_3^{(0,0,1)} >\})\}.$$

$$\mu_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(0,0,1)} >, < m_3^{(1,1,0)} >\})\}.$$

$$\gamma_B = \{(r, \{< m_1^{(0,0,1)} >, < m_2^{(0,0,1)} >, < m_3^{(1,1,0)} >\})\}.$$

Then, $T_2 = \{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B, \mu_B\}$ is a neutrosohpic soft topology on M.

Note that: 1) $(\beta_B \sqcup \gamma_B)^0 \not\sqsubseteq (\beta_B)^0 \sqcup (\gamma_B)^0$. 2) $\gamma_B \not\sqsubseteq (\gamma_B)^0$.

2.5. Example [9]

Let $M = \{m_1, m_2, m_3\}$, $B \subseteq E_M$, where $B = \{r\}$ and $\beta_B, \mu_B, \gamma_B, \delta_B \in N_3(M)$.

Such that

$$\beta_B = \{(r, \{< m_1^{(1, 1, 0)} >, < m_2^{(0, 0, 1)} >, < m_3^{(0, 0, 1)} >\})\}.$$

$$\mu_B = \{(r, \{< m_1^{(1, 1, 0)} >, < m_2^{(0, 0, 1)} >, < m_3^{(1, 1, 0)} >\})\}.$$

$$\gamma_B = \{(r, \{< m_1^{(1, 1, 0)} >, < m_2^{(1, 1, 0)} >, < m_3^{(0, 0, 1)} >\})\}.$$

$$\delta_B = \{(r, \{< m_1^{(0, 0, 1)} >, < m_2^{(1, 1, 0)} >, < m_3^{(1, 1, 0)} >\})\}.$$

Then, $T_2 = \{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B, \mu_B, \gamma_B\}$ is a neutrosohpic soft topology on M.

Note that:

1) $\overline{(\beta_B)} \sqcup \overline{(\delta_B)} \not\sqsubseteq \overline{(\beta_B \sqcap \delta_B)}$. 2) $\overline{(\beta_B)} \not\sqsubseteq \beta_B$.

Note:

There are modifications in Example 4.12 and 4.13 in [9]. So we have corrected them as in Example 2.4 and 2.5.

3. Neutrosophic soft bitopological space

In this section, we defined the neutrosophic soft bitopological space or (N_3-Bi-Top for short) on the concept of neutrosophic soft set and the basic topological concepts of this spaces which are N_3-biopen and N_3-biclosed.

3.1. Definition

Let (M,B,T_1) and (M,B,T_2) be two N_3-Top spaces defined on M. Then (M,B,T_1,T_2) is called a neutrosophic soft bitopological space or (N_3-Bi-Top for short).

3.2. Example

Let $M = \{m_1,m_2\}$, $B \subseteq E_M$, where $B = \{r\}$ and $\beta_B, \mu_B \in N_3(M)$ such that

$$\beta_B = \{(r, \{< m_1^{(0.6, 0.2, 0.5)} >, < m_2^{(0.5, 0.4, 0.9)} >\})\}.$$

$$\mu_B = \{(r, \{< m_1^{(0.6, 0.2, 0.4)} >, < m_2^{(0.6, 0.4, 0.7)} >\})\}.$$

Then, $T_1 = \{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B\}$ is an N_3-Top on M and $T_2 = \{\widetilde{\emptyset}_B, \widetilde{M}_B, \mu_B\}$ is an N_3-Top on M.

Therefore, (M,B,T_1,T_2) is an N_3-Bi-Top space.

3.3. Definition

Let (M,B,T_1,T_2) be an N_3-Bi-Top space. The members of (M,B,T_1,T_2) are called bineutrosohpic soft open sets (N_3-biopen for short) and their complements are bineutrosohpic soft closed sets (N_3-biclosed for short).

3.4. Remark

a) Every neutrosophic soft open (closed) set in (M,B,T_1) or (M,B,T_2) is an N_3-biopen (N_3-biclosed) set.

b) Every N_3-Bi-Top space (M,B,T_1,T_2) induces two N_3-Top spaces as (M,B,T_1) and (M,B,T_2).

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c) If (M,B,T_1) is an N_3-Top space then (M,B,T_1,T_1) is an N_3-Bi-Top space.

3.5. Theorem

If (M,B,T_1,T_2) is an N_3-Bi-Top space, then $(M,B,T_1 \cap T_2)$ is an N_3-Top space.

Proof

Let (M,B,T_1,T_2) be an N_3-Bi-Top space.

(1) Clearly that $\widetilde{\emptyset}_B, \widetilde{M}_B \in (T_1 \cap T_2)$.

(2) Let $\beta_B, \mu_B \in (T_1 \cap T_2)$, then $\beta_B, \mu_B \in T_1$ and $\beta_B, \mu_B \in T_2$. This implies that, $\beta_B \sqcap \mu_B \in T_1$ and $\beta_B \sqcap \mu_B \in T_1$. Therefore, $\beta_B \sqcap \mu_B \in (T_1 \cap T_2)$.

(3) Let $\beta_{j_B} \in (T_1 \cap T_2)$; $j \in J$. Then $\beta_{j_B} \in T_1$ and $\beta_{j_B} \in T_2$; $j \in J$. Therefore $\sqcup_{j\in J} \beta_{j_B} \in T_1$ and $\sqcup_{j\in J} \beta_{j_B} \in T_2$ $\forall j \in J$. Thus, we have $\sqcup_{j\in J} \beta_{j_B} \in (T_1 \cap T_2)$. Hence, $(M,B,T_1 \cap T_2)$ is an N_3-Top space.

3.6. Remark

If we take the operation of union instead of the operation of intersection, then the above theorem is not generally correct.

3.7. Example

Let $M = \{m_1, m_2\}$, $B \subseteq E_M$, where $B = \{r\}$ and $\beta_B, \mu_B \in N_3(M)$ such that

$$\beta_B = \{(r, \{< m_1{}^{(0.3, 0.5, 0.7)} >, < m_2{}^{(0.2, 0.4, 0.6)} >\})\}.$$

$$\mu_B = \{(r, \{< m_1{}^{(0.5, 0.7, 0.8)} >, < m_2{}^{(0.3, 0.6, 0.8)} >\})\}.$$

Then, $T_1 = \{\widetilde{\emptyset}_B, \widetilde{M}_B, \mu_B\}$ is an N_3-Top on M and $T_2 = \{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B\}$ is an N_3-Top on M. Thus, (M,B,T_1,T_2) is an N_3-Bi-Top space. But, $(M,B,T_1 \cup T_2)$ is not an N_3-Top space. Because, $\beta_B \sqcup \mu_B$ does not belong to $(T_1 \cup T_2)$.

4. N_3-(bi)*-open set in neutrosophic soft bitopological space

In this section, N_3-(bi)*-open set, N_3-(bi)*-closed set, (bi)*-neutrosophic soft interior, (bi)*-neutrosophic soft closure, (bi)*-neutrosophic soft boundary, (bi)*-neutrosophic soft exterior are defined based on the idea of δ-open set which was defined in [27].

4.1. Definition

A subset $\beta_B \in N_3(M)$ of an N_3-Bi-Top space (M,B,T_1,T_2) is called star bineutrosophic soft open (N_3-(bi)*-open, for short) in (M,B,T_1,T_2) if and only if $\beta_B \sqsubseteq {\overline{(\beta_B)^{oT2}}^{(T1)}}^{oT2}$ and their complement is an N_3-(bi)*-closed set. The set of all N_3-(bi)*-open [N_3-(bi)*-closed] sets in (M,B,T_1,T_2) is denoted by $M^{(Bi)*-NSO}[M^{(Bi)*-NSC}]$ respectively.

4.2. Example

Let $M = \{m_1, m_2, m_3\}$, $B \subseteq E_M$, where $B = \{r\}$ and $\beta_B, \mu_B \in N_3(M)$ such that

$$\beta_B = \{(r, \{< m_1{}^{(1, 1, 0)} >, < m_2{}^{(0, 0, 1)} >, < m_3{}^{(0, 0, 1)} >\})\}.$$

$$\mu_B = \{(r, \{< m_1{}^{(1, 1, 0)} >, < m_2{}^{(1, 1, 0)} >, < m_3{}^{(0, 0, 1)} >\})\}.$$

$T_1 = \{\widetilde{\emptyset}_B, \widetilde{M}_B\}$ is an N_3-Top on M and $T_2 = \{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B, \mu_B\}$ is an N_3-Top on M. Thus, (M, B,T_1,T_2) is an N_3-Bi-Top space.

Note that:

$$\beta_B = \{(r, \{< m_1{}^{(1, 1, 0)} >, < m_2{}^{(0, 0, 1)} >, < m_3{}^{(0, 0, 1)} >\})\} \sqsubseteq {\overline{(\beta_B)^{oT2}}^{(T1)}}^{oT2} = \{(r, \{< m_1{}^{(1, 1, 0)} >, < m_2{}^{(1, 1, 0)} >, < m_3{}^{(1, 1, 0)} >\})\}.$$

$\therefore\ \beta_B \sqsubseteq \overline{(\beta_B)^{oT2}}^{(T1)^{oT2}}$.

μ_B = {(r, {< $m_1^{(1,1,0)}$ >, < $m_2^{(1,1,0)}$ > ,< $m_3^{(0,0,1)}$ >})} $\sqsubseteq \overline{(\mu_B)^{oT2}}^{(T1)^{oT2}}$ = {(r, {< $m_1^{(1,1,0)}$ >, < $m_2^{(1,1,0)}$ > ,< $m_3^{(1,1,0)}$ >})}.

$\therefore \mu_B \sqsubseteq \overline{(\mu_B)^{oT2}}^{(T1)^{oT2}}$.

γ_B = {(r, {< $m_1^{(1,1,0)}$ >, < $m_2^{(0,0,1)}$ >,< $m_3^{(1,1,0)}$ >})}.

{(r, {< $m_1^{(1,1,0)}$ >, < $m_2^{(0,0,1)}$ >,< $m_3^{(1,1,0)}$ >})} $\sqsubseteq \overline{(\gamma_B)^{oT2}}^{(T1)^{oT2}}$ = {(r, {< $m_1^{(1,1,0)}$ >, < $m_2^{(1,1,0)}$ > ,< $m_3^{(1,1,0)}$ >})}.

$\therefore \gamma_B \sqsubseteq \overline{(\gamma_B)^{oT2}}^{(T1)^{oT2}}$.

δ_B = {(r, {< $m_1^{(0,0,1)}$ >, < $m_2^{(1,1,0)}$ >,< $m_3^{(1,1,0)}$ >})}.

{(r, {< $m_1^{(0,0,1)}$ >, < $m_2^{(1,1,0)}$ >,< $m_3^{(1,1,0)}$ >})} $\not\sqsubseteq \overline{(\delta_B)^{oT2}}^{(T1)^{oT2}}$ = {(r, {< $m_1^{(0,0,1)}$ >, < $m_2^{(0,0,1)}$ > ,< $m_3^{(0,0,1)}$ >})}.

$\therefore \delta_B \not\sqsubseteq \overline{(\delta_B)^{oT2}}^{(T1)^{oT2}}$.

ε_B = {(r, {< $m_1^{(0,0,1)}$ >, < $m_2^{(1,1,0)}$ >,< $m_3^{(1,1,0)}$ >})}.

{(r, {< $m_1^{(0,0,1)}$ >, < $m_2^{(1,1,0)}$ >,< $m_3^{(1,0,0)}$ >})} $\not\sqsubseteq \overline{(\varepsilon_B)^{oT2}}^{(T1)^{oT2}}$ = {(r, {< $m_1^{(0,0,1)}$ >, < $m_2^{(0,0,1)}$ > ,< $m_3^{(0,0,1)}$ >})}.

$\therefore \varepsilon_B \not\sqsubseteq \overline{(\varepsilon_B)^{oT2}}^{(T1)^{oT2}}$.

ϑ_B = {(r, {< $m_1^{(0,0,1)}$ >, < $m_2^{(0,0,1)}$ >,< $m_3^{(1,1,0)}$ >})}.

{(r, {< $m_1^{(0,0,1)}$ >, < $m_2^{(0,0,1)}$ >,< $m_3^{(1,1,0)}$ >})} $\not\sqsubseteq \overline{(\vartheta_B)^{oT2}}^{(T1)^{oT2}}$ = {(r, {< $m_1^{(0,0,1)}$ >, < $m_2^{(0,0,1)}$ > ,< $m_3^{(0,0,1)}$ >})}.

$\therefore \vartheta_B \not\sqsubseteq \overline{(\vartheta_B)^{oT2}}^{(T1)^{oT2}}$.

In general in any N3-Bi-Top space, $\widetilde{\emptyset}_B$, $\widetilde{M}_B$ are clearly N3-(bi)*-open sets.

Hence:

$M^{(Bi)*-NSO}$ = {$\widetilde{\emptyset}_B$, $\widetilde{M}_B$, β_B, μ_B, γ_B}.

$M^{(Bi)*-NSC}$ = { {(r, {< $m_1^{(0,0,1)}$ >, < $m_2^{(1,1,0)}$ >,< $m_3^{(1,1,0)}$ >})},
{(r, {< $m_1^{(0,0,1)}$ >, < $m_2^{(0,0,1)}$ >,< $m_3^{(1,1,0)}$ >})},
{(r, {< $m_1^{(0,0,1)}$ >, < $m_2^{(1,1,0)}$ >,< $m_3^{(0,0,1)}$ >})},
$\widetilde{\emptyset}_B$,
$\widetilde{M}_B$ }.

4.3. Remark

Let β_B and μ_B be an N3-(bi)*-open sets, then $\beta_B \sqcap \mu_B$ is not necessary an N3-(bi)*-open set.

4.4. Example

Let M = {m_1, m_2, m_3, m_4, m_5}, B ⊆ E_M, where B = {r} and β_B, μ_B, γ_B, ε_B, ϑ_B, α_B ∈ N3(M).

Such that

β_B = {(r, {< $m_1^{(1,1,0)}$ >, < $m_2^{(0,0,1)}$ >, < $m_3^{(0,0,1)}$ >, < $m_4^{(0,0,1)}$ >})}.

μ_B = {(r, {< $m_1^{(0,0,1)}$ >, < $m_2^{(0,0,1)}$ >, < $m_3^{(0,0,1)}$ >, < $m_4^{(1,1,0)}$ >})}.

γ_B = {(r, {< $m_1^{(1,1,0)}$ >, < $m_2^{(0,0,1)}$ >, < $m_3^{(0,0,1)}$ >, < $m_4^{(1,1,0)}$ >})}.

ε_B = {(r, {< $m_1^{(0,0,1)}$ >, < $m_2^{(1,1,0)}$ >, < $m_3^{(1,1,0)}$ >, < $m_4^{(0,0,1)}$ >})}.

ϑ_B = {(r, {< $m_1^{(1,1,0)}$ >, < $m_2^{(1,1,0)}$ >, < $m_3^{(1,1,0)}$ >, < $m_4^{(0,0,1)}$ >})}.

α_B = {(r, {< $m_1^{(0,0,1)}$ >, < $m_2^{(1,1,0)}$ >, < $m_3^{(1,1,0)}$ >, < $m_4^{(1,1,0)}$ >})}.

T_1 = {$\widetilde{\emptyset}_B$, $\widetilde{M}_B$, β_B, μ_B, γ_B} is an N_3-Top on M and T_2 = {$\widetilde{\emptyset}_B$, $\widetilde{M}_B$, β_B, μ_B, γ_B, ε_B, ϑ_B, α_B} is an N_3-Top on M. Thus, (M,B,T_1,T_2) is an N_3-Bi-Top space. Then

ε_B and {(r, {< $m_1^{(1,1,0)}$ >, < $m_2^{(0,0,1)}$ >, < $m_3^{(1,1,0)}$ >, < $m_4^{(0,0,1)}$ >})} are an N_3-(bi)*-open sets, but the intersection of them {(r, {< $m_1^{(0,0,1)}$ >, < $m_2^{(0,0,1)}$ >, < $m_3^{(1,1,0)}$ >, < $m_4^{(0,0,1)}$ >})} is not an N_3-(bi)*-open set.

4.5. Theorem

Let (M,B,T_1,T_2) be an N_3-Bi-Top space, then every neutrosophic soft open set in (M,B,T_2) is an N_3-(bi)*-open set in (M,B,T_1,T_2).

Proof

Let β_B be a neutrosophic soft open set in (M,B,T_2). Then $(\beta_B)^{oT2} = \beta_B$. Since $\beta_B \sqsubseteq \overline{(\beta_B)}^{T1}$, $\beta_B \sqsubseteq \overline{(\beta_B)^{oT2}}^{T1}$, $(\beta_B)^{oT2} \sqsubseteq \overline{(\beta_B)^{oT2}}^{T1\,oT2}$. Thererfor $\beta_B \sqsubseteq \overline{(\beta_B)^{oT2}}^{T1\,oT2}$ and thus β_B is an N_3-(bi)*-open set in (M,B,T_1,T_2).

4.6. Remark

The converse of above remark is not true in general. In Example 3.4 note that, {(r, {< $m_1^{(1,1,0)}$ >, < $m_2^{(0,0,1)}$ >, < $m_3^{(1,1,0)}$ >, < $m_4^{(0,0,1)}$ >})} is an N_3-(bi)*-open set in (M,B,T_1,T_2), but not a neutrosophic soft open set in (M,B,T_2).

4.7. Definition

If (M,B,T_1,T_2) is an N_3-(Bi)*-Top space and $\beta_B \in N_3(M)$, then the largest N_3-(bi)*-open set contained in β_B is called (bi)*-neutrosophic soft interior of β_B, ($(\beta_B)^{0(bi)*}$ for short). i.e.,

$$(\beta_B)^{0(bi)*} = \sqcup\{(\omega_B): \omega_B \text{ is a } N_3\text{-(bi)*-open set}, \omega_B \sqsubseteq \beta_B\}.$$

4.8. Theorem

Let (M,B,T_1,T_2) be an N_3-(Bi)*-Top space and $\beta_B \in N_3(M)$. Then β_B is an N_3-(bi)*-open set if and only if $\beta_B = (\beta_B)^{0(bi)*}$.

Proof

Let β_B be an N_3-(bi)*-open set. Then β_B is itself an N_3-(bi)*-open set which contains β_B. Therefore, β_B is the largest N_3-(bi)*-open set contained in β_B and $\beta_B = (\beta_B)^{0(bi)*}$. Conversely, suppose that $\beta_B = (\beta_B)^{0(bi)*}$,then β_B is the largest N_3-(bi)*-open set contained in β_B. Thus, β_B is an N_3-(bi)*-open set.

4.9. Theorem

Let $\beta_B, \mu_B \in N_3(M)$.

a) $(\beta_B)^{0(bi)*} \sqsubseteq \beta_B$.
b) $((\beta_B)^{0(bi)*})^{0(bi)*} = (\beta_B)^{0(bi)*}$.
c) $(\beta_B)^{0(bi)*} \sqsubseteq (\mu_B)^{0(bi)*}$; whenever $\beta_B \sqsubseteq \mu_B$.
d) $(\beta_B \sqcap \mu_B)^{0(bi)*} = (\beta_B)^{0(bi)*} \sqcap (\mu_B)^{0(bi)*}$.
e) $(\beta_B \sqcup \mu_B)^{0(bi)*} \sqsupseteq (\beta_B)^{0(bi)*} \sqcup (\mu_B)^{0(bi)*}$.
f) $(\widetilde{M}_B)^{0(bi)*} = \widetilde{M}_B$.
g) $(\widetilde{\emptyset}_B)^{0(bi)*} = \widetilde{\emptyset}_B$.

Proof

(a), (f), (g), (c) (Straightforward).

(b) Let $\mu_B = (\beta_B)^{0(bi)*}$. Then $\mu_B = (\mu_B)^{0(bi)*}$(from Theorem 4.8). Thus$((\beta_B)^{0(bi)*})^{0(bi)*} = (\beta_B)^{0(bi)*}$.

(d) Since, $(\beta_B \sqcap \mu_B)^{0(bi)*} \sqsubseteq (\beta_B)^{0(bi)*}$ and $(\beta_B \sqcap \mu_B)^{0(bi)*} \sqsubseteq (\mu_B)^{0(bi)*}$. Then, $(\beta_B \sqcap \mu_B)^{0(bi)*} \sqsubseteq (\beta_B)^{0(bi)*} \sqcap (\mu_B)^{0(bi)*}$...(1.

Since, $(\beta_B)^{0(bi)*} \sqsubseteq \beta_B$ and $(\mu_B)^{0(bi)*} \sqsubseteq \mu_B$, then $(\beta_B)^{0(bi)*} \sqcap (\mu_B)^{0(bi)*} \sqsubseteq \beta_B \sqcap \mu_B$. But $(\beta_B)^{0(bi)*} \sqcap (\mu_B)^{0(bi)*}$ is a N3-(bi)*-open subset of $\beta_B \sqcap \mu_B$. Therefore, from the detention, we have that $(\beta_B \sqcap \mu_B)^{0(bi)*} \sqsupseteq (\beta_B)^{0(bi)*} \sqcap (\mu_B)^{0(bi)*}$…(2.

Hence, $(\beta_B \sqcap \mu_B)^{0(bi)*} = (\beta_B)^{0(bi)*} \sqcap (\mu_B)^{0(bi)*}$.

(e) Since, $\beta_B \sqsubseteq (\beta_B \sqcup \mu_B)$ and $\mu_B \sqsubseteq (\beta_B \sqcup \mu_B)$, therefore $(\beta_B \sqcup \mu_B)^{0(bi)*} \sqsupseteq (\beta_B)^{0(bi)*}$and $(\beta_B \sqcup \mu_B)^{0(bi)*} \sqsupseteq (\mu_B)^{0(bi)*}$. So, $(\beta_B \sqcup \mu_B)^{0(bi)*} \sqsupseteq (\beta_B)^{0(bi)*} \sqcup (\mu_B)^{0(bi)*}$.

4.10. Example

Let us consider $\beta_B, \mu_B, \gamma_B \in$ N3(M) in Example 2.4. Such that, T2 = $\{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B, \mu_B\}$ is an N3-Top on M and T1 = $\{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B\}$ is an N3-Top on M. Thus, (M,B,T1,T2) is an N3-Bi-Top space.

Note that: 1) $(\beta_B \sqcup \gamma_B)^{0(bi)*} \not\sqsubseteq (\beta_B)^{0(bi)*} \sqcup (\gamma_B)^{0(bi)*}$. 2) $\gamma_B \not\sqsubseteq (\gamma_B)^{0(bi)*}$.

4.11. Definition

If (M,B,T1,T2) is an N3-(Bi)*-Top space and $\beta_B \in$ N3(M), then the intersection of all N3-(bi)*-closed sets containing β_B is called a (bi)*-neutrosophic soft cloture of β_B, ($\overline{(\beta_B)}^{(bi)*}$ for short). i.e.,

$$\overline{(\beta_B)}^{(bi)*} = \sqcap\{(\omega_B): \omega_B \text{ is an N3-(bi)*-closed set}, \beta_B \sqsubseteq \omega_B\}.$$

4.12. Theorem

Let $\beta_B, \mu_B \in$ N3(M).

a) $\beta_B \sqsubseteq \overline{(\beta_B)}^{(bi)*}$.
b) $\overline{\left(\overline{(\beta_B)}^{(bi)*}\right)}^{(bi)*} = \overline{(\beta_B)}^{(bi)*}$.
c) $\overline{(\beta_B)}^{(bi)*} \sqsubseteq \overline{(\mu_B)}^{(bi)*}$; whenever $\beta_B \sqsubseteq \mu_B$.
d) $\overline{(\beta_B \sqcap \mu_B)}^{(bi)*} \sqsubseteq \overline{(\beta_B)}^{(bi)*} \sqcap \overline{(\mu_B)}^{(bi)*}$.
e) $\overline{(\beta_B \sqcup \mu_B)}^{(bi)*} = \overline{(\beta_B)}^{(bi)*} \sqcup \overline{(\mu_B)}^{(bi)*}$.
f) $\overline{(\widetilde{M}_B)}^{(bi)*} = \widetilde{M}_B$.
g) $\overline{(\widetilde{\emptyset}_B)}^{(bi)*} = \widetilde{\emptyset}_B$.

Proof Straightforward.

4.13. Remark

In above theorem, it is not necessary the converse of (a) and (d) be true.

4.14. Example

Let us take, $\beta_B, \mu_B, \gamma_B, \delta_B \in$ N3(M) in Example 2.5.

T2 = $\{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B, \mu_B, \gamma_B\}$ is an N3-Top on M and T1 = $\{\widetilde{\emptyset}_B, \widetilde{M}_B\}$ is an N3-Top on M. Thus, (M,B,T1,T2) is an N3-Bi-Top space.

Note that:

1) $(\beta_B \sqcup \gamma_B)^{0(bi)*} \not\sqsubseteq (\beta_B)^{0(bi)*} \sqcup (\gamma_B)^{0(bi)*}$. 2) $\gamma_B \not\sqsubseteq (\gamma_B)^{0(bi)*}$.

4.15. Theorem

Let (M,B,T1,T2) be an N3-(Bi)*-Top space and $\beta_B \in$ N3(M).

a) $((\beta_B)^C)^{0(bi)*} = \left(\overline{(\beta_B)}^{(bi)*}\right)^C$.

b) $\overline{(\beta_B)^C}^{(bi)*} = \left((\beta_B)^{0(bi)*}\right)^C$.

Proof

(a) We know that, $\overline{(\beta_B)}^{(bi)*} = \sqcap\{\omega_B : (\omega_B)^C$ is a N3-(bi)*-open set, $\beta_B \sqsubseteq \omega_B\}$. So, we have that, $\left(\overline{(\beta_B)}^{(bi)*}\right)^C = \sqcup\{(\omega_B)^C$: $(\omega_B)^C$ is an N3-(bi)*- open set, $(\omega_B)^C \sqsubseteq (\beta_B)^C\} = ((\beta_B)^C)^{0(bi)*}$. Thus, $\left(\left(\beta_B\right)^C\right)^{0(bi)*} = \left(\overline{(\beta_B)}^{(bi)*}\right)^C$.

(b) If we take, $(\beta_B)^C$ instead of β_B in (a), we get that,

$\left(\overline{((\beta_B)^c)}^{(bi)*}\right)^C = (((\beta_B)^C)^C)^{0(bi)*} = ((\beta_B))^{0(bi)*}$. So, $\overline{(\beta_B)^C}^{(bi)*} = \left((\beta_B)^{0(bi)*}\right)^C$.

4.16. Theorem

If (M,T1,T2) is an N3-(Bi)*-Top space and $\beta_B \in$ N3(M), then β_B is an N3-(bi)*-closed set if and only if $\beta_B = \overline{(\beta_B)}^{(bi)*}$.

Proof

Let β_B be an N3-(bi)*-closed set, then β_B is itself an N3-(bi)*-closed set which contains β_B. Therefore, β_B is the intersection of all N3-(bi)*-closed sets containing β_B and $\beta_B = \overline{(\beta_B)}^{(bi)*}$.

Conversely, suppose that $\beta_B = \overline{(\beta_B)}^{(bi)*}$, then β_B is the intersection of all N3-(bi)*-closed sets containing β_B. Thus, β_B is an N3-(bi)*-closed set.

4.17. Definition

If (M,T1,T2) is an N3-(Bi)*-Top space and $\beta_B \in$ N3(M), then the (bi)*-neutrosophic soft exterior of β_B, (bi)*-ext(β_B) for short) is defined as, (bi)*-ext(β_B) = $((\beta_B)^C)^{0(bi)*}$.

4.18. Definition

If (M,B,T1,T2) is an N3-(Bi)*-Top space and $\beta_B \in$ N3(M), then the (bi)*-neutrosophic soft boundary of β_B, ((bi)*-br(β_B) for short) is defined as, (bi)*-br(β_B) = $\overline{(\beta_B)^C}^{(bi)*} \sqcap \overline{(\beta_B)}^{(bi)*}$.

4.19. Theorem

Assume that (M,B,T1,T2) is an N3-(Bi)*-Top space and $\beta_B \in$ N3(M).

- (bi)*-br $((\beta_B)^C)$ = (bi)*-ext(β_B) $\sqcup (\beta_B)^{0(bi)*}$.
- $\overline{(\beta_B)}^{(bi)*}$ = (bi)*-br $(\beta_B) \sqcup (\beta_B)^{0(bi)*}$.
- (bi)*-br $(\beta_B) \sqcap (\beta_B)^{0(bi)*} = \widetilde{\emptyset}_B$.
- (bi)*-br $(\beta_B)^{0(bi)*} \sqsubseteq$ (bi)*-br (β_B).

Proof Straightforward.

4.20. Theorem

Assume that (M,B,T1,T2) is an N3-(Bi)*-Top space and $\beta_B \in$ N3(M).

- $\beta_B \in M^{(Bi)*-NSO}$ if and only if (bi)*-br $(\beta_B) \sqcap \beta_B = \widetilde{\emptyset}_B$.
- $\beta_B \in M^{(Bi)*-NSC}$ if and only if (bi)*-br $(\beta_B) \sqsubseteq \beta_B$.

Proof Straightforward.

Conclusion

In this research, bitopological-space on the concept of neutrosophic soft set is built", the basic topological concepts of this spaces which are N3-(bi)*-open set, N3-(bi)*-closed set, (bi)*-neutrosophic soft interior, (bi)*-neutrosophic soft closure, (bi)*-neutrosophic soft boundary, (bi)*-neutrosophic soft exterior are defined and many examples on this concepts are given.

"This paper is just a beginning of a new structure and we have studied a few ideas only", "it will be necessary to carry out more theoretical research to establish a general framework for the practical application".

"we hope that the findings in this paper will help researchers enhance and promote the further study on neutrosophic soft bitopological-space"."

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