# A Study Of Some Neutrosophic Clean Rings

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**Abstract**

The objective of this paper is to introduce a necessary and sufficient condition for a neutrosophic ring to be clean. This work proves the equivalence between case of classical clean ring $R$ and the corresponding neutrosophic ring $R(I)$,refined neutrosophic ring $R(I_1, I_2)$, and n-refined neutrosophic ring $R_n(I)$.

**Keywords:** refined neutrosophic ring, n-refined neutrosophic ring ,classical clean ring, neutrosophic clean ring.

## 1.Introduction

Neutrosophy is a new kind of generalized logic founded by Smarandache to handle the indeterminacy and uncertainity in science and life.

The concept of neutrosophic rings was defined by Kandasamy and Samarandache in [1], as an extention of any algebraic ring $R$.

These rings played an important role in algebraic studies such as as number theory [2,3],functional analysis [4], and algebraic geometry [5].

Recently, neutrosophic rings have been expanded to refined neutrosophic rings [6], and n-refined neutrosophic rings [7].

In [11-13], we find many theorems about the structure of idempotents and special sets of these rings.

In [9], the concept of neutrosophic clean ring was studied with many elementary interesting properties.

This motivates us to extend the previous works about clean rings to refined and n-refined neutrosophic rings, where we show the equivalence between clean conditions in any ring $R$and its corresponding neutrosophic ring $R(I)$, refined neutrosophic ring $R(I_1, I_2)$, and n-refined neutrosophic ring $R_n(I)$. All rings are considered with unity 1.

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## 2. Preliminaries.

**Definition 1.**[1]

Let R be a ring, I be the indeterminacy with property $I^2 = I$ , then the neutrosophic ring is defined as follows:

$R(I) = \{a + bI; a, b \in R\}$.

**Definition 2**. [6]

(a) The element I can be split into two indeterminacies $I_1$ , $I_2$ with conditions:

$$I_1^{\ 2} = I_1 , I_2^{\ 2} = I_2 , I_1 I_2 = I_2 I_1 = I_1.$$

(b) If X is a set then $X(I_1, I_2) = \{(a, bI_1, cI_2): a , b , c \in X \}$ is called the refined neutrosophic set generated by X , $I_1, I_2$.

(c) Let (R,+,×) be a ring, ($R(I_1, I_2)$ , + ,×) is called a refined neutrosophic ring generated by R ,$I_1, I_2$.

**Definition 3**.

Let (R,+,×) be a ring and $I_k; 1 \leq k \leq n$ be n indeterminacies. We define $R_n$(I)=$\{a_0 + a_1 I + \cdots + a_n I_n ;\ a_i \in R\}$ to be n-refined neutrosophic ring.

Addition and multiplication on $R_n$(I) are defined as:

$\sum_{i=0}^{n} x_i I_i + \sum_{i=0}^{n} y_i I_i = \sum_{i=0}^{n} (x_i + y_i) I_i , \sum_{i=0}^{n} x_i I_i \times \sum_{i=0}^{n} y_i I_i = \sum_{i,j=0}^{n} (x_i \times y_j) I_i I_j$.

Where × is the multiplication defined on the ring R.

**Definition 4**. [10]

Let $R$ be any ring, $x$ be an arbitrary element, then.

a). $x$ is called a unit if there exists $y \in R$, such that $xy = yx = 1$.

b). $x$ is called an idempotent if and only if $x^2 = x$.

**Theorem 5.**[10]

Let $R(I)$ a neutrosophic ring, then.

a). $x = a + bI$ is idempotent if and only if $a, a + b$ are idempotents in $R$.

b). $x = a + bI$ is unit if and only if $a, a + b$ units in $R$.

**Theorem 6.**[10]

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Let $R(I_1, I_2)$ be a refined neutrosophic ring, then.

a). $x = (a, bI_1, cI_2)$ is an idempotent if and only if $a, a + c, a + b + c$ are idempotents in $R$.

b). $x = (a, bI_1, cI_2)$ is a unit if and only if $a, a + c, a + b + c$ are units in $R$.

**Definition 7**. [9]

Let $R$ be a ring, then it is called clean if and only if for each $x \in R$, we have $x = e + l$, where $e$ is a unit and $l$ is an idempotent.

**Example 8.**

$Z_3 = \{0,1,2\}$ is clean, that is because.

$0 = 2 + 1$ ($2$ *is a unit and* $1$ *is an idempotent* ).

$1 = 1 + 0$ ($1$ *is a unit and* $0$ *is an idempotent* ).

$2 = 1 + 1$ ($1$ *is a unit and an idempotent too*).

**3. Main discussion.**

**Theorem 3.1**:

Let $R$ be any ring, $R(I)$be its corresponding neutrosophic ring.

$R(I)$is clean if and only if $R$ is clean.

Proof.

If $R(I)$ is clean, then $R$ is clean. That **is** because $R \subset R(I)$.

Conversely, suppose that $R$ is clean, we must prove that $R(I)$ is clean.

$\forall\ x = a + bI \in R(I)$, then $a, b, a + b \in R$.

By the assumption, we can write.

$a = e_1 + l_1, b = e_2 + l_2, a + b = e_3 + l_3$; where$e_i$ are units and $l_i$ are idempotents.

Now, we have $x = a + bI = a + (a + b - a)I = (e_1 + l_1) + I[(e_3 + l_3) - (e_1 + l_1)] = (e_1 + (\boldsymbol{e}_3 - e_1)I) + \big(l_1 + (l_3 - l_1)\big) = x_1 + x_2$.

$x_1$is a unit, that is because $e_1$, $(e_3 - e_1) + e_1 = e_3$ are units in $R$. Also, $x_2$ is an idempotent, that is because $l_1$, $(l_3 - l_1) + l_1 = l_3$are idempotents in $R$.

[See theorem 5].

Thus $x$ is a sum of an idempotent with a unit, hence $R(I)$ is a clean.

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**Example 3.2:**

We have shown that $Z_3$ is a clean ring.

According to the previous theorem $Z_3(I) = \{a + bI; a, b \in Z_3\} = \{0,1,2, I, 2I, 1 + I, 1 + 2I, 2 + I, 2 + 2I\}$ is clean.

Remark that $0 = 2 + 1$ , $1 = 1 + 0$ , $2 = 1 + 1$ , $I = (2 + 2I) + (1 + 2I)$ , $2I = (2 + 2I) + 1$ , $1 + I = (1) + (I)$ ,

$1 + 2I = (1 + I) + I$ , $2 + I = (3) + (I)$, $2 + 2I = (2 + 2I) + 0$. Where $\{2,1,2 + 2I, 1 + I\}$ are units and

$\{1,0,1 + 2I, I\}$are idempotents.

**Theorem 3.3:**

Let $R$ be any ring, $R(I_1, I_2)$ be its corresponding refined neutrosophic ring.

$R(I_1, I_2)$ is clean if and only if $R$ is clean.

Proof.

$R$ is a homomorophic image of $R(I_1, I_2)$, hence if $R(I_1, I_2)$ is clean, then $R$ is clean.

Conversely, assume that $R$ is clean, we must prove that $R(I_1, I_2)$ is clean.

$\forall\ x = (a, bI_1, cI_2) \in R(I_1, I_2)$, we have.

$a, a + c, a + b + c \in R$, hence $a = e_1 + l_1, a + c = e_2 + l_2, a + b + c = e_3 + l_3$; where $e_i$ are units and $l_i$ are idempotents.

Also, $x = (a, I_1[(a + b + c) - (a + c)], I_2[(a + c) - a]) = (e_1 + l_1, I_1[(e_3 + l_3) - (e_2 + l_2)], I_2[(e_2 + l_2) - (e_1 + l_1)]) = (e_1, [e_3 - e_2]I_1, [e_2 - e_1]I_2) + (l_1, [l_3 - l_2]I_1, [l_2 - l_1]I_2) = x_1 + x_2$.

Bu using theorem 6, we get.

$x_1$is a unit, that is because $e_1$, $(e_2 - e_1) + e_1 = e_2, e_1 + (e_3 - e_2) + (e_2 - e_1) = e_3$ are units in $R$. Also, $x_2$ is an idempotent by a similar discussion.

This implies that $R(I_1, I_2)$ is clean.

**Example 3.4:**

$Z_2 = \{0,1\}$is clean ring, where $1 = 1 + 0, 0 = 1 + 1$.

$Z_2(I_1, I_2) = \{(0,0,0), (1,0,0), (0, I_1, 0), (0,0, I_2), (0, I_1, I_2), (1, I_1, 0), (1,0, I_2), (1, I_1, I_2)\}$ is clean too.

Remark that: $(0,0,0) = (1,0,0) + (1,0,0)$.

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$\mathbf{(1,0,0)=(1,0,0)+(0,0,0)}$

$\mathbf{(0,I_1,0)=(1,0,0)+(1,I_1,0)}$

$\mathbf{(0,0,I_2)=(1,0,0)+(1,0,I_2)}$

The rest can be computed by the same way.

**Theorem 3.5**:

Let $R$ be any ring, $R_n(I)$ be its corresponding n-refined neutrosophic ring. Then $R_n(I)$ is clean if and only if $R$ is clean.

Proof.

If $R_n(I)$ isclean, then $R$ is clean as a direct result of the homomorphism between $R_n(I)$ and $R$.

For the converse, suppose that $R$ is clean. We must prove that $R_n(I)$ is clean.

$$\forall\, x \in R_n(I); x = a_0 + a_1 I_1 + a_2 I_2 + \cdots + a_n I_n$$
$$= a_0 + I_1\left[\left(\sum_{i=0}^{n} a_i - \sum_{i\neq 1}^{n} a_i\right)\right] + I_1\left[\left(\sum_{i\neq 1}^{n} a_i - \sum_{i\neq 1,2}^{n} a_i\right)\right] + \cdots$$
$$+ [(a_{n-1} + a_n + a_0) - (a_{n-1} + a_0)]I_{n-1} + [(a_0 + a_n) - a_0]I_n$$

$$x = e_0 + l_0 + I_1[(e_1 + l_1) - (e_2 + l_2)] + I_2[(e_2 + l_2) - (e_3 + l_3)] + \cdots$$
$$+ I_{n-1}[(e_{n-1} + l_{n-1}) - (e_n + l_n)]$$
$$+ I_n[(e_n + l_n) - (e_0 + l_0)];\ e_i \ are\ units, l_i\ are\ idempotents.$$

$$x = \big[e_0 + I_1[e_1 - e_2] + I_2[e_2 - e_3] + \cdots + I_{n-1}[e_{n-1} - e_n] + I_n[e_n - e_0]\big] + \big[l_0 + I_1[l_1 - l_2] + I_2[l_2 - l_3] + \cdots + I_{n-1}[l_{n-1} - l_n] + I_n[l_n - l_0]\big] = x_1 + x_2$$

According to theorem ,$x_1$ is a unit, that is because $e_0$, $e_0(e_1 - e_2) + \cdots + (e_n - e_0) = e_1, e_0 + (e_2 - e_3) + (e_3 - e_4) + \cdots + (e_{n-1} - e_n) + (e_n - e_0) = e_2$ are units in $R$.

By the same method, we get that $x_2$ is an idempotent, hence $R_n(I)$ is clean.

## 4. Conclusion

In this article, we have determined necessary and sufficient condition for a neutrosophic ring to be clean. Also, we have found the equivalence between classical clean ring $\boldsymbol{R}$ and the corresponding neutrosophic ring $\boldsymbol{R(I)}$, refined neutrosophic ring $\boldsymbol{R(I_1, I_2)}$, and n-refined neutrosophic ring $\boldsymbol{R_n(I)}$.

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