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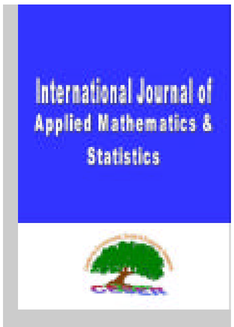

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Unification of Fusion Theories (UFT)

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ABSTRACT

Since no fusion theory neither rule fully satisfy all needed applications, the author proposes a Unification of Fusion Theories and a combination of fusion rules in solving problems/applications. For each particular application, one selects the most appropriate model, rule(s), and algorithm of implementation.
We are working in the unification of the fusion theories and rules, which looks like a cooking recipe, better we'd say like a logical chart for a computer programmer, but we don't see another method to comprise/unify all things.
The unification scenario presented herein, which is now in an incipient form, should periodically be updated incorporating new discoveries from the fusion and engineering research.

Keywords: Distributive lattice, Boolean algebra, Conjunctive rule, Disjunctive rule, Partial and Total conflicts, Weighted Operator (WO), Proportional Conflict Redistribution (PCR) rules, Murphy's average rule, Dempster-Shafer Theory (DST), Yager's rule, Transferable Belief Model (TBM), Dubois-Prade's rule (DP), Dezert-Smarandache Theory (DSmT), static and dynamic fusion

ACM Classification: Artificial Intelligence (I.2.3).

1. INTRODUCTION

Each theory works well for some applications, but not for all.

We extend the power and hyper-power sets from previous theories to a Boolean algebra obtained by the closure of the frame of discernment under union, intersection, and complement of sets (for non-exclusive elements one considers as complement the fuzzy or neutrosophic complement). All bbas and rules are redefined on this Boolean algebra.

A similar generalization has been previously used by Guan-Bell (1993) for the Dempster-Shafer rule using propositions in sequential logic, herein we reconsider the Boolean algebra for all fusion rules and theories but using sets instead of propositions, because generally it is harder to work in sequential logic with summations and inclusions than in the set theory.

2. FUSION SPACE

For n ≥ 2 let È = {$è_1$, $è_2$, …, $è_n$} be the frame of discernment of the fusion problem/application under consideration. Then (È, χ, 1, C), È closed under these three operations: union, intersection, and complementation of sets respectively, forms a Boolean algebra. With respect to the partial ordering relation, the inclusion φ, the minimum element is the empty set f , and the maximal element is the total ignorance I = $\bigcup_{i=1}^{n} q i$.

Similarly one can define: (È, χ, 1, \) for sets, È closed with respect to each of these operations: union, intersection, and difference of sets respectively.

(È, χ, 1, C) and (È, χ, 1, \) generate the same super-power set $S^{È}$ closed under χ, 1, C, and \ because for any A, B 0 $S^{È}$ one has CA = I \ A and reciprocally A\B = A1CB.

If one considers propositions, then (È, ω, ϖ, 5) forms a Lindenbaum algebra in sequential logic, which is isomorphic with the above (È, χ, 1, C) Boolean algebra.

By choosing the frame of discernment È closed under χ only one gets DST, Yager's, TBM, DP theories. Then making È closed under both χ, 1 one gets DSm theory. While, extending È for closure under χ, 1, and C one also includes the complement of set (or negation of proposition if working in sequential logic); in the case of non-exclusive elements in the frame of discernment one considers a fuzzy or neutrosophic complement. Therefore the super-power set (È, χ, 1, C) includes all the previous fusion theories.

The power set $2^{È}$, used in DST, Yager's, TBM, DP, which is the set of all subsets of È, is also a Boolean algebra, closed under χ, 1, and C, but does not contain intersections of elements from È.

The Dedekind distributive lattice $D^{È}$, used in DSmT, is closed under χ, 1, and if negations/complements arise they are directly introduced in the frame of discernment, say È', which is then closed under χ, 1. Unlike others, DSmT allows intersections, generalizing the previous theories.

The Unifying Theory contains intersections and complements as well.

Let's consider a frame of discernment È with exclusive or non-exclusive hypotheses, exhaustive or non-exhaustive, closed or open world (all possible cases).

We need to make the remark that in case when these n ≥ 2 elementary hypotheses $è_1$, $è_2$, …, $è_n$ are *exhaustive and exclusive* one gets the Dempster-Shafer Theory, Yager's, Dubois-Prade Theory, Dezert-Smarandache Theory, while for the case when the hypotheses are *non-exclusive* one gets Dezert-Smarandache Theory, but for *non-exhaustivity* one gets TBM.

An exhaustive frame of discernment is called *close world*, and a non-exhaustive frame of discernment is called *open world* (meaning that new hypotheses might exist in the frame of discernment that we are not aware of). È may be finite or infinite.

Let m_j: $S^{È}$ → [0. 1], 1 j s, be s 2 basic belief assignments,

(when bbas are working with crisp numbers),

or with subunitary subsets, m_j: $S^{È}$ → P([0. 1]), where P([0. 1]) is the set of all subsets of the interval [0,1] (when dealing with very imprecise information).

Normally the sum of crisp masses of a bba, m(.), is 1, i.e. $\sum_{X \in S^{\wedge}T} m(X) = 1$.

3. INCOMPLETE AND PARACONSISTENT INFORMATION

For incomplete information the sum of a bba components can be less than 1 (not enough information known), while in paraconsistent information the sum can exceed 1 (overlapping contradictory information).

The masses can be normalized (i.e. getting the sum of their components =1), or not (sum of components < 1 in incomplete information; or > 1 in paraconsistent information).

For a bba valued on subunitary subsets one can consider, as normalization of m(.),

either $\sum_{X \in S^{\wedge}T} \sup\{m(X)\} = 1$,

or that there exist crisp numbers x 0 X for each X 0 S^{È} such that $\sum_{\substack{X \in S^{\wedge}T \\ x \in X}} m(x) = 1$.

Similarly, for a bba m(.) valued on subunitary subsets dealing with paraconsistent and incomplete information respectively:

a) for incomplete information, one has $\sum_{X \in S^{\wedge}T} \sup\{m(X)\} < 1$,

b) while for paraconsistent information one has $\sum_{X \in S^{\wedge}T} \sup\{m(X)\} > 1$ and there do not exist crisp numbers x 0 X for each X 0 S^{È} such that $\sum_{\substack{X \in S^{\wedge}T \\ x \in X}} m(x) = 1$.

4. SPECIFICITY CHAINS

We use the min principle and the precocious/prudent way of computing and transferring the conflicting mass.

Normally by transferring the conflicting mass and by normalization we diminish the specificity. If A1B is empty, then the mass is moved to a less specific element A (also to B), but if we have a pessimistic view on A and B we move the mass m(A1B) to AχB (entropy increases, imprecision increases), and even more if we are very pessimistic about A and B: we move the conflicting mass to the total ignorance in a closed world, or to the empty set in an open world.

Specificity Chains:

a) From specific to less and less specific (in a closed world):

(A1B) δ A δ (AχB) δ I or (A1B) δ B δ (AχB) δ I.

Also from specific to unknown (in an open world):

A1B → ϕ.

b) And similarly for intersections of more elements: A1B1C, etc.

A1B1C δ A1B δ A δ (AχB) δ (AχBχC) δ I

or (A1B1C) δ (B1C) δ B δ (AχB) δ (AχBχC) δ I, etc. in a closed world.

Or A1B1C → ϕ in an open world.

c) Also in a closed world:

A1(BχC) δ BχC δ (BχC) δ (AχBχC) δ I or A1(BχC) δ A δ (AχB) δ (AχBχC) δ I.

Or A1(BχC) → ϕ in an open world.

5. STATIC AND DYNAMIC FUSION

According to Wu Li we have the following classification and definitions:

Static fusion means to combine all belief functions simultaneously.

Dynamic fusion means that the belief functions become available one after another sequentially, and the current belief function is updated by combining itself with a newly available belief function.

6. SCENARIO OF UNIFICATION OF FUSION THEORIES

Since everything depends on the application/problem to solve, this scenario looks like a logical chart designed by the programmer in order to write and implement a computer program, or like a cooking recipe.

Here it is the scenario attempting for a unification and reconciliation of the fusion theories and rules:

1) If all sources of information are reliable, then apply the conjunctive rule, which means consensus between them (or their common part):

2) If some sources are reliable and others are not, but we don't know which ones are unreliable, apply the disjunctive rule as a cautious method (and no transfer or normalization is needed).

3) If only one source of information is reliable, but we don't know which one, then use the exclusive disjunctive rule based on the fact that X_1 ξ X_2 ξ… ξX_n means either X_1 is reliable, or X_2, or and so on or X_n, but not two or more in the same time.

4) If a mixture of the previous three cases, in any possible way, use the mixed conjunctive-disjunctive rule.

As an example, suppose we have four sources of information and we know that: either the first two are telling the truth or the third, or the fourth is telling the truth.

The mixed formula becomes:

$m_{1\chi}(\phi) = 0$, and A0S^{È}\ϕ, one has $m_{1\chi}(A) = \sum_{\substack{X_1, X_2, X_3, X_4 \in S^\wedge\Theta \\ ((X_1 \cap X_2) \cup X_3) e \cup X_4 = A}} m_1(X_1) m_2(X_2) m_3(X_3) m_4(X_4)$.

5) If we know the sources which are unreliable, we discount them. But if all sources are fully unreliable (100%), then the fusion result becomes vacuum bba (i.e. m(È) = 1, and the problem is indeterminate. We need to get new sources which are reliable or at least they are not fully unreliable.

6) If all sources are reliable, or the unreliable sources have been discounted (in the <u>default case</u>), then use the DSm classic rule (which is commutative, associative, Markovian) on Boolean algebra (È, χ, 1, C), no

matter what contradictions (or model) the problem has. I emphasize that the super-power set $S^{\text{È}}$ generated by this Boolean algebra contains singletons, unions, intersections, and complements of sets.

7) If the sources are considered from a statistical point of view, use *Murphy's average rule* (and no transfer or normalization is needed).

8) In the case the model is not known (the default case), it is prudent/cautious to use the free model (i.e. all intersections between the elements of the frame of discernment are non-empty) and DSm classic rule on $S^{\text{È}}$, and later if the model is found out (i.e. the constraints of empty intersections become known), one can adjust the conflicting mass at any time/moment using the DSm hybrid rule.

9) Now suppose the model becomes known [i.e. we find out about the contradictions (= empty intersections) or consensus (= non-empty intersections) of the problem/application]. Then :

9.1) If an intersection A1B is not empty, we keep the mass m(A1B) on A1B, which means consensus (common part) between the two hypotheses A and B (i.e. both hypotheses A and B are right) [here one gets *DSmT*].

9.2) If the intersection A1B =f is empty, meaning contradiction, we do the following :

9.2.1) if one knows that between these two hypotheses A and B one is right and the other is false, but we don't know which one, then one transfers the mass m(A1B) to m(AχB), since AχB means at least one is right [here one gets *Yager's* if n=2, or *Dubois-Prade,* or *DSmT*];

9.2.2) if one knows that between these two hypotheses A and B one is right and the other is false, and we know which one is right, say hypothesis A is right and B is false, then one transfers the whole mass m(A1B) to hypothesis A (nothing is transferred to B);

9.2.3) if we don't know much about them, but one has an optimistic view on hypotheses A and B, then one transfers the conflicting mass m(A1B) to A and B (the nearest specific sets in the Specificity Chains) [using *Dempster's*, *PCR2-5*]

9.2.4) if we don't know much about them, but one has a pessimistic view on hypotheses A and B, then one transfers the conflicting mass m(A1B) to AχB (the more pessimistic the further one gets in the Specificity Chains: (A1B) δ A δ (AχB) δ I); this is also the default case [using *DP's*, *DSm hybrid rule*, *Yager's*];

if one has a very pessimistic view on hypotheses A and B then one transfers the conflicting mass m(A1B) to the total ignorance in a closed world [*Yager's*, *DSmT*], or to the empty set in an open world [*TBM*];

9.2.5.1) if one considers that no hypothesis between A and B is right, then one transfers the mass m(A1B) to other non-empty sets (in the case more hypotheses do exist in the frame of discernment) - different from A, B, AχB - for the reason that: if A and B are not right then there is a bigger chance that other hypotheses in the frame of discernment have a higher subjective probability to occur; we do this transfer in a **closed world** [DSm hybrid rule]; but, if it is an **open world**, we can transfer the mass m(A1B) to the empty set leaving room for new possible hypotheses [here one gets *TBM*];

9.2.5.2) if one considers that none of the hypotheses A, B is right and no other hypothesis exists in the frame of discernment (i.e. n = 2 is the size of the frame of discernment), then one considers the **open world** and one transfers the mass to the empty set [here *DSmT* and *TBM* converge to each other].

Of course, this procedure is extended for any intersections of two or more sets: A1B1C, etc. and even for mixed sets: A1 (BχC), etc.

If it is a dynamic fusion in a real time and associativity and/or Markovian process are needed, use an algorithm which transforms a rule (which is based on the conjunctive rule and the transfer of the conflicting mass) into an associative and Markovian rule by storing the previous result of the conjunctive rule and, depending of the rule, other data. Such rules are called quasi-associative and quasi-Markovian.

Some applications require the necessity of **decaying the old sources** because their information is considered to be worn out.

If some bba is not normalized (i.e. the sum of its components is < 1 as in incomplete information, or > 1 as in paraconsistent information) we can easily divide each component by the sum of the components and normalize it. But also it is possible to fusion incomplete and paraconsistent masses, and then normalize them after fusion. Or leave them unnormalized since they are incomplete or paraconsistent.

PCR5 does the most mathematically exact (in the fusion literature) redistribution of the conflicting mass to the elements involved in the conflict, redistribution which exactly follows the tracks of the conjunctive rule.

7. EXAMPLES

7.1. Bayesian Example:

Let È = {A, B, C, D, E} be the frame of discernment.

	A	B	C	D	E	A1B	A1C	A1D	A1E	B1C	B1D
						ϕ	= ϕ	= ϕ	= ϕ	Not known if = or ϕ	= ϕ
						Consensus between A and B	Contradiction between A and C, but optimistic in both of them	One right, one wrong, but don't know which one	A is right, B is wrong	Don't know the exact model	Unknown any relation between B and D.
m_1	0.2	0	0.3	0.4	0.1						
m_2	0.5	0.2	0.1	0	0.2						
m_{12}	0.10	0	0.03	0	0.02	0.04	0.17	0.20	0.09	0.06	0.08
						A1B	A, C	AχB	A	B1C We keep the mass 0.06 on B1C till we find out more information on the model.	BχD
m_r						0.04	0.107, 0.063	0.20	0.09	0.06	0.08
$\mathbf{m_{UFT}}$	0.32	0.04	0.12	0	0.03	0.04	0	0	0	0.06	0
m_{lower} (closed world)	0.10	0	0.03	0	0.02						
m_{lower} (open world)	0.10	0	0.03	0	0.02						
m_{middle} (default)	0.10	0	0.03	0	0.02						
m_{upper}	0.40	0.08	0.18	0.23	0.11						

Table 1. Bayesian Example using the Unified Fusion Theories rule regarding a mixed redistribution of partial conflicting masses (Part 1).

	B1E	C1D	C1E	D1E	AχB	AχC	AχD	AχE	BχC
	ϕ	= ϕ	= ϕ	= ϕ					
	The intersection is not empty, but neither B1E nor BχE interest us	Pessimistic in both C and D	Very pessimistic in both C and E	Both D and E are wrong					
m_1									
m_2									
m_{12}	0.02	0.04	0.07	0.08					
	B, E	CχD	AχBχCχD χE	A,B,C					
m_r	0.013, 0.007	0.04	0.07	0.027, 0.027, 0.027					
m_{UFT}	**0**	**0**	**0**	**0**	**0**	**0**	**0.20**	**0**	**0**
m_{lower} (closed world)									
m_{lower} (open world)									
m_{middle} (default)					0.04	0.17	0.20	0.09	0.06
m_{upper}									

Table 1. Bayesian Example using the Unified Fusion Theories rule regarding a mixed redistribution of partial conflicting masses (Part 2).

	BχD	BχE	CχD	CχE	DχE	AχBχCχDχE	ϕ
m_1							
m_2							
m_{12}							
m_r							
m_{UFT}	**0.08**	**0**	**0.04**	**0**	**0**	**0.07**	**0**
m_{lower} (closed world)						0.85	
m_{lower} (open world)							0.85
m_{middle} (default)	0.08	0.02	0.04	0.07	0.08		
m_{upper}							

Table 1. Bayesian Example using the Unified Fusion Theories rule regarding a mixed redistribution of partial conflicting masses (Part 3).

We keep the mass m_{12}(B1C) = 0.06 on B1C (eleventh column in Table 1, part 1) although we don't know if the intersection B1C is empty or not (this is considered the default model), since in the case when it is empty one considers an open world because m_{12}(ϕ)=0.06 meaning that there might be new possible hypotheses in the frame of discernment, but if B1C ϕ one considers a consensus between B and C.

Later, when finding out more information about the relation between B and C, one can transfer the mass 0.06 to BχC, or to the total ignorance I, or split it between the elements B, C, or even keep it on B1C.

m_{12}(A1C)=0.17 is redistributed to A and C using the PCR5:

a1/0.2 = c1/0.1 = 0.02(0.2+0.1),

whence a1 = 0.2(0.02/0.3) = 0.013,

c1 = 0.1(0.02/0.3) = 0.007.

a2/0.5 = c2/0.3 = 0.15(0.5+0.3),

whence a2 = 0.5(0.15/0.8) = 0.094,

c2 = 0.3(0.15/0.8) = 0.056.

Thus A gains a1+a2 = 0.013+0.0.094 = 0.107 and C gains c1+c2 = 0.007+0.056 = 0.063.

m_{12}(B1E)=0.02 is redistributed to B and E using the PCR5:

b/0.2 = e/0.1 = 0.02/(0.2+0.1),

whence b = 0.2(0.02/0.3) = 0.013,

e = 0.1(0.02/0.3) = 0.007.

Thus B gains 0.013 and E gains 0.007.

Then one sums the masses of the conjunctive rule $\mathbf{m_{12}}$ and the redistribution of conflicting masses $\mathbf{m_r}$ (according to the information we have on each intersection, model, and relationship between conflicting hypotheses) in order to get the mass of the Unification of Fusion Theories rule m_{UFT}.

$\mathbf{m_{UFT}}$, the Unification of Fusion Theories rule, is a combination of many rules and gives the optimal redistribution of the conflicting mass for each particular problem, following the given model and relationships between hypotheses; this extra-information allows the choice of the combination rule to be used for each intersection. The algorithm is presented above.

$\mathbf{m_{lower}}$, the lower bound believe assignment, the most pessimistic/prudent belief, is obtained by transferring the whole conflicting mass to the total ignorance (Yager's rule) in a closed world, or to the empty set (Smets' TBM) in an open world herein meaning that other hypotheses might belong to the frame of discernment.

$\mathbf{m_{middle}}$, the middle believe assignment, half optimistic and half pessimistic, is obtained by transferring the partial conflicting masses m_{12}(X1Y) to the partial ignorance XχY (as in Dubois-Prade theory or more general as in Dezert-Smarandache theory).

Another way to compute a middle believe assignment would be to average the $\mathbf{m_{lower}}$ and $\mathbf{m_{upper}}$.

$\mathbf{m_{upper}}$, the lower bound believe assignment, the most optimistic (less prudent) belief, is obtained by transferring the masses of intersections (empty or non-empty) to the elements in the frame of discernment using the PCR5 rule of combination, i.e. m_{12}(X1Y) is split to the elements X, Y (see Table 2). We use PCR5 because it is more exact mathematically (following the backwards the tracks of the conjunctive rule) than Dempster's rule, minC, and PCR1-4.

X	m_{12}(X)	A	B	C	D	E
A1B	0.040	0.020	0.020			
A1C	0.170	0.107		0.063		
A1D	0.200	0.111			0.089	
A1E	0.090	0.020 0.042				0.020 0.008
B1C	0.060		0.024	0.036		
B1D	0.080		0.027		0.053	
B1E	0.020		0.013			0.007
C1D	0.040			0.008	0.032	
C1E	0.070			0.036 0.005		0.024 0.005
D1E	0.080				0.053	0.027
Total	0.850	0.300	0.084	0.148	0.227	0.091

Table 2. Redistribution of the intersection masses to the singletons A, B, C, D, E using the PCR5 rule only, needed to compute the upper bound belief assignment m_{upper}.

7.2. Negation/Complement Example:

Let È = {A, B, C, D} be the frame of discernment. Since (È, χ, 1, C) is Boolean algebra, the super-power set $S^{È}$ includes complements/negations, intersections and unions. Let's note by C(B) the complement of B.

	A	B	D	C(B)	A1C {Later in the dynamic fusion process we find out that A1C is empty }	BχC=B	A1B	A1 C(B)=A
					= φ		= φ	φ
					Unknown relationship between A and C		Optimistic in both A and B.	Consensus between A and C(B), but Aδ C(B)
m_1	0.2	0.3	0	0.1	0.1	0.3		
m_2	0.4	0.1	0	0.2	0.2	0.1		
m_{12}	0.08	0.09	0	0.02	0.17	0.03	0.14	0.08
					AχC	B	A,B	A
m_r					0.17	0.03	0.082, 0.058	0.08
m_{UFT}	**0.277**	**0.32**	**0.035**	**0.02**	**0**	**0**	**0**	**0**
m_{lower} (closed world)	0.16	0.26	0	0.02	0	0		
m_{lower} (open world)	0.16	0.26	0	0.02	0	0		
m_{middle} (default)	0.16	0.26	0	0.02	0	0		
m_{upper}	0.296	0.23 0	0	0.12 6	0.219	0.129		

Table 3. Negation/Complement Example using the Unified Fusion Theories rule regarding a mixed redistribution of partial conflicting masses (Part 1).

	A1 (BχC)	B1 C(B)	B1(A1C)	C(B)1(A1C)	C(B)1(BχC) = C(B)1C	Bχ(A1C)=B
	= ϕ	= ϕ	= ϕ	= ϕ	= ϕ	
	At least one is right between A and BχC	B is right, C(B) is wrong	No relationship known between B and A1C (default case)	Very pessimistic on C(B) and A1C	Neither C(B) nor BχC are right	
m_1						
m_2						
m_{12}	0.14	0.07	0.07	0.04	0.07	
	Aχ(BχC) .	B	Bχ(A1C)=B	AχBχCχD	A, D	B
m_r	0.14	0.07	0.07	0.04	0.035, 0.035	
$\mathbf{m_{UFT}}$	**0**	**0**	**0**	**0**	**0**	**0**
m_{lower} (closed world)						
m_{lower} (open world)						
m_{middle} (default)						
m_{upper}						

Table 3. Negation/Complement Example using the Unified Fusion Theories rule regarding a mixed redistribution of partial conflicting masses (Part 2).

	AχB	AχC	AχD	BχC	BχD	CχD	AχBχC	AχBχCχD	ϕ
m_1									
m_2									
m_{12}									
m_r									
$\mathbf{m_{UFT}}$	**0**	**0.170**	**0**	**0**	**0**	**0**	**0.140**	**0.040**	**0**
m_{lower} (closed world)								0.56	
m_{lower} (open world)									0.56
m_{middle} (default)	0.14	0.17		0.03			0.14	0.11	
m_{upper}									

Table 3. Negation/Complement Example using the Unified Fusion Theories rule regarding a mixed redistribution of partial conflicting masses (Part 3).

Model of Negation/Complement Example:

A1B = ϕ, CδB, AδC(B).

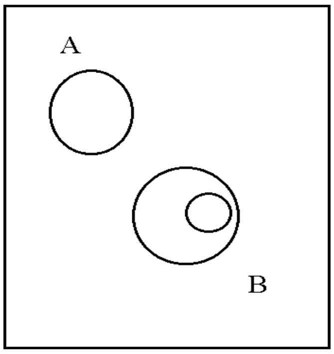

Fig. 1

m_{12}(A1B) = 0.14.

x1/0.2 = y1/0.1 = 0.02/0.3, whence x1 = 0.2(0.02/0.3) = 0.013, y1 = 0.1(0.02/0.3) = 0.007;

x2/0.4 = y2/0.3 = 0.12/0.7, whence x2 = 0.4(0.12/0.7) = 0.069, y2 = 0.3(0.12/0.7) = 0.051.

Thus, A gains 0.013+0.069 = 0.082 and B gains 0.007+0.051 = 0.058.

For the upper belief assignment m_{upper} one considered all resulted intersections from results of the conjunctive rule as empty and one transferred the partial conflicting masses to the elements involved in the conflict using PCR5.

All elements in the frame of discernment were considered non-empty.

7.3. Example with Intersection:

Look at this:

Suppose A={x<0.4}, B={0.3<x<0.6}, C={x>0.8}. The frame of discernment T={A, B, C} represents the possible cross section of a target, and there are two sensors giving the following bbas:

m1(A)=0.5, m1(B)=0.2, m1(C)=0.3.

m2(A)=0.4, m2(B)=0.4, m2(C)=0.2.

	A	B	C	A1B= {.3<x<.4}	AχC	BχC
m_1	.5	.2	.3			
m_2	.4	.4	.2			
m_1&m_2 DSmT	.20	.08	.06	.28	.22	.16

We have a DSm hybrid model (one intersection A&B=nonempty).

This example proves the necessity of allowing intersections of elements in the frame of discernment. [Shafer's model doesn't apply here.]

Dezert-Smarandache Theory of Uncertain and Paradoxist Reasoning (DSmT) is the only theory which accepts intersections of elements.

7.4. Another Multi-Example of UFT:

Cases:

1. Both sources reliable: use conjunctive rule [default case]:

1.1. A1B ϕ:

1.1.1. Consensus between A and B; mass 6 A1B;

1.1.2. Neither A1B nor AχB interest us; mass 6 A, B;

1.2. A1B=ϕ:

1.2.1. Contradiction between A and B, but optimistic in both of them; mass 6 A, B;

1.2.2. One right, one wrong, but don't know which one; mass 6 AχB;

1.2.3. Unknown any relation between A and B [default case]; mass 6 AχB;

1.2.4. Pessimistic in both A and B; mass 6 AχB;

1.2.5. Very pessimistic in both A and B;

1.2.5.1. Total ignorance ε AχB; mass 6 AχBχCχD (total ignorance);

1.2.5.2. Total ignorance = AχB; mass 6 ϕ (open world);

1.2.6. A is right, B is wrong; mass 6 A;

1.2.7. Both A and B are wrong; mass 6 C, D;

1.3. Don't know if A1B = or ϕ (don't know the exact model); mass 6 A1B (keep the mass on intersection till we find out more info) [default case];

2. One source reliable, other not, but not known which one: use disjunctive rule; no normalization needed.

3. S1 reliable, S2 not reliable 20%: discount S2 for 20% and use conjunctive rule.

	A	B	AχB	A1B	ϕ (open world)	AχBχCχD	C	D
S1	.2	.5	.3					
S2	.4	.4	.2					
S1&S2	.24	.42	.06	.28				
S1 or S2	.08	.20	.72	0				
UFT 1.1.1	.24	.42	.06	.28				
UFT 1.1.2 (PCR5)	.356	.584	.060	0				
UFT 1.2.1	.356	.584	.060	0				
UFT 1.2.2	.24	.42	.34	0				
UFT 1.2.3	.24	.42	.34	0				
UFT 1.2.4	.24	.42	.34	0				
UFT 1.2.5.1	.24	.42	.06	0	0	.28		
UFT 1.2.5.2	.24	.42	.06	0	.28			
80% S2	.32	.32	.16			.20		
UFT 1.2.6	.52	.42	.06					
UFT 1.2.7	.24	.42	.06	0			.14	.14
UFT 1.3	.24	.42	.06	.28				
UFT 2	.08	.20	.72	0				
UFT 3	.232	.436	.108	.224		0		

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On The Inverse Problem of A Vibrating Membrane of Arbitrary Shape

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Abstarct

A study on the inverse problem of a transversely vibrating membrane of varying material properties has been carried out. Using a finite difference model the eigenvalue equation for the inverse problem of an anisotropic, non-homogenous membrane of rectangular, circular and elliptical shape is formulated. The basic relations for reconstructing the variation of stiffness and mass of such a membrane from eigenvalues and eigenvectors are established. Reconstruction with a varying number of eigenvectors for a membrane of rectangular and elliptical shape is discussed. Also a procedure which can be used to reconstruct the membranes is proposed and an example illustrating the procedure given.

1. INTRODUCTION

The inverse problem of a vibrating system involves determining the physical properties of a system from eigenvector and eigenvalue data. Earlier work on the topic involved formulating and solving the inverse problem of a simply connected spring mass system but work has also been carried out in solving the inverse problems associated with vibrating rods, beams, multiple connected spring mass systems etc. The inverse problem for vibrating rods and strings has been analysed by Barcilon [**1**] and

Gladwell [**3**] who have also shown how a discrete model of a beam subject to three different end conditions can be reconstructed. Recently Burak [**2**] studied the two dimensional model of homogenous membranes.

In the present paper, previous work on vibrating membranes has been extended to the inverse problem of a vibrating non-homogenous anisotropic membrane of different shapes, using a finite difference method so that parameters of the membrane can be reconstructed. A finite difference model of the membrane is first proposed where stiffness is considered to vary from element to element. This finite difference model is then used to formulate the eigenvalue equation for the inverse problem of a rectangular membrane, then the eigenvalue equation for the inverse problem of a circular and an elliptical membrane. A study is then carried out by varying the number of eigenvectors and determining the conditions of reconstruction with such variation. Finally a method of membrane reconstruction is proposed and an example given.

2. FORMULATION OF EIGENVALUE EQUATION

The differential equation for the small transverse vibration of a non-uniform membrane is given by

$$\frac{\partial}{\partial x}\left[Q(x,y)\frac{\partial w(x,y,t)}{\partial x}\right]+\frac{\partial}{\partial y}\left[P(x,y)\frac{\partial w(x,y,t)}{\partial y}\right]=\rho(x,y)\frac{\partial^2 w(x,y,t)}{\partial t^2} \tag{1}$$

where Q and P are the force components per unit length acting along the X-axis and Y-axis respectively. The space and time components in the above equation may be separated by

$$W(x,y,t)=u(x,y)\sin\omega t \tag{2}$$

yielding

$$\frac{\partial}{\partial x}\left[Q(x,y)\frac{\partial u(x,y)}{\partial x}\right]+\frac{\partial}{\partial y}\left[P(x,y)\frac{\partial u(x,y)}{\partial y}\right]+\lambda u(x,y)\rho(x,y)=0 \tag{3}$$

where $\lambda=\omega^2$

The membrane for which the eigenvalue equation is formulated here is considered to be anisotropic and non-homogenous. Since the membrane is anisotropic the finite difference model of the membrane is considered to have elements whose stiffness varies in both vertical and horizontal direction. The stiffnesses for an element ij along horizontal and vertical directions are denoted as K_{ij}^H and K_{ij}^V. Further the membrane is considered to be non-homogenous the mass of each element in the finite difference model of the membrane is considered to be different and for an element ij denoted by m_{ij}. The displacement of an element ij is denoted by U_{ij}.

For the present finite difference model, Eq.3 can be written as

(4)
$$K^{H}_{i+1j}\left(U_{i+2j}-U_{i+1j}\right)\left(U_{i+1j}-U_{ij}\right)+K^{H}_{i-1j}\left(U_{ij}-U_{i-1j}\right)\left(U_{i-1j}-U_{i-2j}\right)$$
$$+K^{v}_{ij+1}\left(U_{ij+2}-U_{ij+1}\right)\left(U_{ij+1}-U_{ij}\right)+K^{v}_{ij-1}\left(U_{ij}-U_{ij-1}\right)\left(U_{ij-1}-U_{ij-2}\right)+\lambda U_{ij}m_{ij}=0$$

where i=1,2,3…m and j=1,2,3…n for m rows and n columns of elements. Since the values of the stiffnesses and displacements are real and mass is real and positive the eigenvalue is expected to be real.

Equation 4 is the finite difference equation of an element of a membrane which can be used to determine the eigenvalue equation of the membrane.

2.1 EQUATION FOR A RECTANGULAR MEMBRANE

In the case of a rectangular membrane, Eq.4 can be written as

(5)
$$KU-l\,M\widetilde{U}=0$$

where we have denoted successively

$$K=diag\left\lfloor A_{ij}\;\; A_{ij+1}\;\; A_{ij+2}\cdots\right\rfloor,$$

$$A_{ij}=\begin{bmatrix}\begin{bmatrix}K^{A}_{ij}\\ K^{B}_{ij}\end{bmatrix} & & \\ & \begin{bmatrix}K^{A}_{i+1j}\\ K^{B}_{i+1j}\end{bmatrix} & \\ & & \ddots \\ & & \end{bmatrix},$$

$$K^{A}_{ij}=\begin{bmatrix}-K^{H}_{i-1j}\\ K^{H}_{i+1j}\end{bmatrix},$$

$$K^{B}_{ij}=\begin{bmatrix}-K^{V}_{ij-1}\\ K^{V}_{ij+1}\end{bmatrix}$$

while

$$U=\left[\hat{U}_{ij}\quad \hat{U}_{ij+1}\quad \hat{U}_{ij+2}\cdots\right]^{T},$$

$$\hat{U}_{ij}=\left[\dot{U}_{ij}\quad \dot{U}_{i+1j}\quad \dot{U}_{i+2j}\cdots\right]^{T},$$

$$\dot{U}_{ij}=\left[U'_{ij}\quad U''_{ij}\right]$$

$$U'_{ij}=\left[\widehat{U}_{i-1j}\quad \widehat{U}_{i+1j}\right],$$

$$U''_{ij} = \left[\breve{U}_{ij-1} \quad \breve{U}_{ij+1} \right] ,$$

$$\widehat{U}_{ij} = \left(U_{i+1j} - U_{ij} \right)\left(U_{ij} - U_{i-1j} \right),$$

$$\breve{U}_{ij} = \left(U_{ij+1} - U_{ij} \right)\left(U_{ij} - U_{ij-1} \right),$$

and

$$M = diag\left\lfloor M_{ij} \quad M_{ij+1} \quad M_{ij+2} \cdots \right\rfloor,$$

with

$$M_{ij} = diag\left\lfloor m_{ij} \quad m_{i+1j} \quad m_{i+2j} \cdots \right\rfloor,$$

and

$$\tilde{U} = \left[\overline{U}_{ij} \quad \overline{U}_{ij+1} \quad \overline{U}_{ij+2} \cdots \right]^T$$

with

$$\overline{U}_{ij} = \left[U_{ij} \quad U_{i+1j} \quad U_{i+2j} \cdots \right]^T$$

It is to be noted that in order to obtain the eigenvalue equation of a given membrane the above equation needs to be modified using the condition at its fixed boundary, where the displacement is zero.

2.2 EQUATION FOR A CIRCULAR AND ELLIPTICAL MEMBRANE

For a circular membrane using Eq.4 the eigenvalue equation is given by

$$\left[K_{LD} K_{TD} \right] \left[U_{LD} U_{TD} \right]^T - \lambda \left[M_{LD} M_{TD} \right] \left[\overleftrightarrow{U}_{LD} \overleftrightarrow{U}_{TD} \right]^T = 0 \qquad (6)$$

where

$$K_{LD} = diag\left\lfloor A_{ij} \quad A_{i+1j-1} \quad A_{i+2j-2} \cdots \right\rfloor$$

$$A_{ij} = \begin{bmatrix} \begin{bmatrix} K_{ij}^A \\ K_{ij}^B \end{bmatrix} & & \\ & \begin{bmatrix} K_{i+1j+1}^A \\ K_{i+1j+1}^B \end{bmatrix} & \\ & & \ddots \end{bmatrix},$$

and

$$U_{LD} = \left[\ddot{U}_{ij} \quad \ddot{U}_{i+1\,j-1} \quad \ddot{U}_{i+2\,j-2} \cdots \right]^T ,$$

$$\ddot{U}_{ij} = \left[\dot{U}_{ij} \quad \dot{U}_{i+1\,j+1} \quad \dot{U}_{i+2\,j+2} \cdots \right]^T$$

Also,

$$K_{TD} = diag\left\lfloor A_{ij} \quad A_{i+1\,j+1} \quad A_{i+2\,j+2} \cdots \right\rfloor$$

$$A_{ij} = \begin{bmatrix} \begin{bmatrix} K_{ij}^A \\ K_{ij}^B \end{bmatrix} & & \\ & \begin{bmatrix} K_{i+1\,j-1}^A \\ K_{i+1\,j-1}^B \end{bmatrix} & \\ & & \ddots \\ & & \end{bmatrix},$$

and,

$$U_{TD} = \left[\dddot{U}_{ij} \quad \dddot{U}_{i+1\,j+1} \quad \dddot{U}_{i+2\,j+2} \cdots \right]^T ,$$

$$\dddot{U}_{ij} = \left[\dot{U}_{ij} \quad \dot{U}_{i+1\,j-1} \quad \dot{U}_{i+2\,j-2} \cdots \right]^T$$

where the values of K_{ij}^A and K_{ij}^B and $\dot{U}_{ij}$ are the same as that for a rectangular membrane given in the earlier section.

Here,

$$M_{LD} = diag\left\lfloor M'_{ij} \quad M'_{i+1\,j-1} \quad M'_{i+2\,j-2} \cdots \right\rfloor,$$

$$M'_{ij} = diag\left\lfloor m_{ij} \quad m_{i+1\,j+1} \quad m_{i+2\,j+2} \cdots \right\rfloor,$$

$$\overleftrightarrow{U}_{LD} = \left[\vec{U}_{ij} \quad \vec{U}_{i+1\,j-1} \quad \vec{U}_{i+2\,j-2} \cdots \right]^T ,$$

$$\vec{U}_{ij} = \left[U_{ij} \quad U_{i+1\,j+1} \quad U_{i+2\,j+2} \cdots \right]^T ,$$

and

$$M_{TD} = diag\left\lfloor M''_{ij} \quad M''_{i+1\,j+1} \quad M''_{i+2\,j+2} \cdots \right\rfloor,$$

$$M''_{ij} = diag\left\lfloor m_{ij} \quad m_{i+1\,j-1} \quad m_{i+2\,j-2} \cdots \right\rfloor,$$

$$\overleftrightarrow{U}_{TD} = \left[\overleftarrow{U}_{ij} \quad \overleftarrow{U}_{i+1\,j+1} \quad \overleftarrow{U}_{i+2\,j+2} \cdots \right]^T ,$$

$$\overleftarrow{U}_{ij} = \left[U_{ij} \quad U_{i+1\,j-1} \quad U_{i+2\,j-2} \cdots \right]^T .$$

The above equation can also be considered as the eigenvalue equation of an elliptical membrane where each quadrant of the elliptical membrane is considered separately. In order to obtain the eigenvalue equation of a membrane with fixed boundary the above equation needs to be modified using the zero displacement boundary condition.

3. RELATIONS FOR RECONSTRUCTION

The eigenvalue equations developed in the earlier section can now be used to reconstruct the parameters of the membrane. This procedure of reconstruction involves determining the vertical and horizontal stiffness and the mass of each element. It is to be noted that forces acting between adjacent elements in a given direction are the same. Thus stiffness multiplied by displacement between adjacent elements are same. Such relation between elements combined with the equilibrium equations already determined allow us to reconstruct the parameters of the membrane. Equation 4 can be used to get the values of the stiffnesses.

In order to reconstruct a membrane the number of eigenvectors required need to be ascertained. A study is conducted here by varying the number of known eigenvectors for a rectangular and elliptical membrane and by comparing the equations available with the number of unknowns. This allows us to investigate whether for a given number of eivenvectors the membrane can be reconstructed or not. The minimum number of eigenvectors required for reconstruction without restriction and the number of eigenvectors required for reconstruction for practical purposes is thus determined.

4. RECTANGULAR MEMBRANE

In order to determine the number of eigenvectors required for a rectangular membrane let us divide the membrane into m by n elements (Fig.1). It is assumed that at least one eigenvalue and total mass of the membrane is known. It is required that the value of at least one eigenvalue is known since the eigenvalues can be determined only as a multiple of their actual values as can be seen from Eq.5. Also the value of the mass needs to be known since otherwise we can find the mass and and stiffness of elements only in relation with each other and not their actual values.

Since the number of unknown parameters in an element is three (horizontal and vertical stiffness and mass) the total number of unknowns for a membrane is therefore 3mn.

4.1.1 Case when one eigenvector is known

The number of equations available when one eigenvector is known is $mn + [(m-1)n] + [(n-1)m] + 1$.

In order to determine the parameters of the membrane, we have

$$mn + [(m-1)n] + [(n-1)m] + 1 \geq 3mn$$

$$\Rightarrow m + n \leq 1$$

Since the value of m or n cannot be less than 1 a rectangular membrane cannot be reconstructed using one eigenvector.

4.1.2 Case when two eigenvectors are known

The number of equations available when two eigenvectors are known is $2mn + [(m-1)n] + [(n-1)m] + 1$ while the number of unknowns include the second eigenvalue in addition to the total number of unknown parameters of the membrane.

In order to be able to determine the parameters of the membrane we have,

$$2mn + [(m-1)n] + [(n-1)m] + 1 \geq 3mn + 1$$

i.e., $m + n \leq mn$

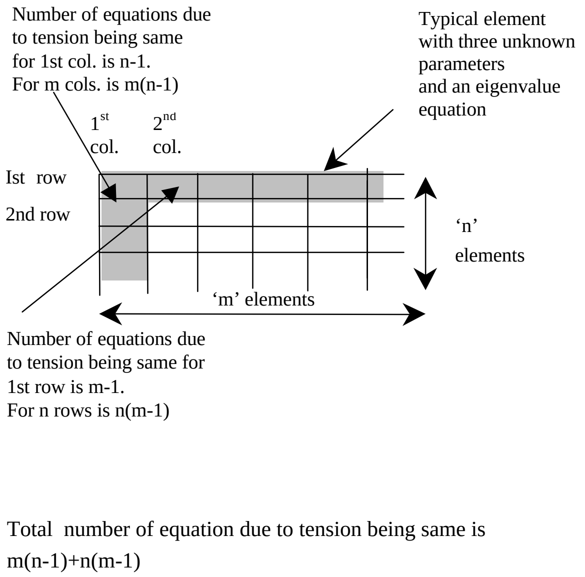

Figure 1. Determining number of unknowns and equations available for a rectangular membrane.

The above inequality is not valid when m=1 and n=1, m=2 and n=1 or m=1 and n=2.

4.1.3. Case when three eigenvectors are known

The number of equations available when three eigenvector are known is $3mn+[(m-1)n]+[(n-1)m]+1$ and the number of unknowns is 3mn+2.

In order to determine the parameters of the membrane, we therefore have

$$3mn+[(m-1)n]+[(n-1)m]+1\geq 3mn+2$$

i.e., $m+n+1\leq 2mn$

In this case also the above inequality is not valid when m=1 and n=1.

It is found that if we increase the number of eigenvectors greater than two the membrane can be reconstructed except when m=1 and n=1.

Hence for a rectangular membrane reconstruction is possible for a non-homogenous membrane with two different stiffness in axial directions when the number of known eigenvectors is three or greater while for practical purposes reconstruction can be done using only two known eigenvectors.

4.2 ELLIPTICAL MEMBRANE

In order to reconstruct an elliptical membrane let us divide the elements with all sides equal (Fig.2). Again we consider that we know at least one eigenvector and the mass of the membrane for the same reasons as earlier given when considering a rectangular membrane. Let us consider that the number of elements along the major axis be h while the number of elements along the minor axis is b.

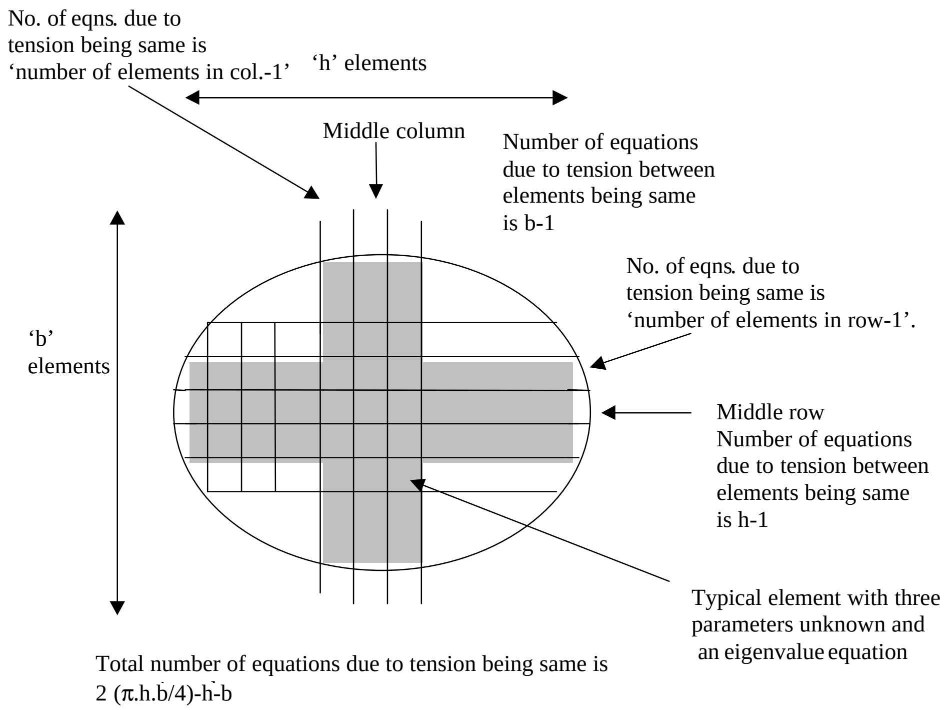

Figure 2. Determining the number of knowns and equations available for an elliptical membrane.

The total number of elements for an ellipse can be given approximately as $\pi hb/4$. Hence the total number of unknowns can be given approximately as $3\pi hb/4$.

4.2.1 Case when one eigenvector is known

The number of equations available for reconstruction when one eigenvector is known is approximately given by $(\pi hb/4)+(\pi hb/4-h)+(\pi bh/4-b)+1$

$$\Rightarrow (3\pi hb/4)-h-b+1$$

In order to determine the parameters of the membrane, we have

$$1+(3\pi\mathrm{hb}/4)-\mathrm{h}-\mathrm{b} > \text{or} \cong (3\pi\mathrm{hb}/4)$$

$$\Rightarrow \mathrm{h}+\mathrm{b} < \text{or} \cong 1$$

Since h and b cannot be less than 1 the membrane cannot be reconstructed with one eigenvector.

4.2.2 *Case when two eigenvectors are known*

When two eigenvectors are known the number of equations available for reconstruction is approximately $\pi hb - h - b + 1$ and the number of unknowns increase to include the second eigenvalue in addition to the unknown parameters of the membrane.

In order to reconstruct the membrane, we have

$$1 + \pi hb - h - b > \text{or} \cong (3\pi hb / 4) + 1$$

$$(\pi hb / 4) - h - b > \text{or} \cong 0$$

Clearly, the above inequality is not satisfied if b=1 and h has any value or if b=2 and h=2.

4.2.3 Case when three eigenvectors are known

When three eigenvectors are known the number of equations available for reconstruction is approximately $(5\pi hb/4) - h - b + 1$ and the number of unknowns is approximately $(3\pi hb/4) + 2$.

In order to reconstruct the membrane, we have approximate relation

$$1 + (5\pi hb / 4) - h - b > \text{or} \cong (3\pi hb / 4) + 2$$

$$(\pi hb / 2) - h - b + 1 > \text{or} \cong 0$$

Clearly, the values of b=1 and h=1 or 2 the above inequality is not satisfied.

4.2.4 Case when four eigenvectors are known

When four eigenvectors are known the number of equations available for reconstruction is approximately $(6\pi hb/4) - h - b + 1$ and the number of unknowns approximately $(3\pi hb/4) + 3$.

In order to reconstruct the membrane, we have approximate relation

$$1 + (6\pi hb / 4) - h - b > \text{or} \cong (3\pi hb / 4) + 3$$

$$(3\pi hb / 4) - h - b + 2 > \text{or} \cong 0$$

The above inequality is satisfied by all values of h and b except for h=1 and b=1.

It is found that in all cases where the number of known eigenfactors is greater than four the membrane can be reconstructed except when m=1.

Hence for an elliptical membrane reconstruction cannot be carried out if one eigenvector is known. If two eigenvectors are known we can determine the properties of a non-homogenous anisotropic membrane for all cases except when b=1 for any h and when b=2 and h=2 while if three eigenvectors are known the properties of a non-homogenous member can be determined for all cases except when there are two elements along the major axis and one element along the minor axis. When four or more eigenvectors are known the component can be reconstructed for all cases of a non-homogenous membrane.

Studies similar to that of an elliptical membrane has been carried out considering a circular membrane. Such studies show that reconstruction of a circular membrane cannot be carried out if one eigenvector is known. If two eigenvectors are known we can determine the properties of a non-homogenous circular membrane for all cases except when there are two elements along the diameter while if three or more eigenvectors are known properties of a non-homogenous membrane can be determined for all cases.

4.3 SUMMARY OF RESULTS ON NUMBER OF EIGENVECTORS REQUIRED FOR RECONSTRUCTION

The study conducted by varying the number of eigenvectors and determining the conditions of reconstruction has been summarised in Table 1 given below. The reconstruction of a square and circular membrane has not been shown here but has been added to Table 1. Such results can be easily determined by making proper substitutions to the results of a rectangular or elliptical membrane.

Table 1. Summary of study on reconstruction of a non-homogenous, anisotropic membrane with varying eigenvectors.

Number of Known Eigenvectors	One	Two	Three	Four
Square	Reconstruction not possible	Reconstruction possible without restriction		
Rectangle	Reconstruction not possible	Reconstruction not possible when number of elements along length is two and along breadth is one.	Reconstruction possible without restriction	
Circle	Reconstruction not possible	Reconstruction is not possible when there are two elements along diameter.	Reconstruction possible without restriction.	
Ellipse	Reconstruction not possible	Reconstruction is not possible when minor axis has only one element or if both minor and major axis has two elements.	Reconstruction is not possible when minor axis has one element and major axis has two elements.	Reconstruction possible without restriction.

5. RECONSTRUCTION PROCEDURE

In the earlier section the number of eigenvectors required for reconstruction of a rectangular, circular and elliptical membrane was discussed. As can be seen from Table 1 for practical purposes reconstruction, for the shapes considered, can be carried out using two eigenvectors. Here a technique for reconstruction of a non-homogenous anisotropic membrane from a single eigenvalue and two eigenvectors is discussed.

A membrane is reconstructed here knowing only the eigenvalue corresponding to the first eigenvector. To start reconstruction a value for the eigenvalue corresponding to the second eigenvector is assumed. The first element is chosen as a boundary element with two sides fixed. Using the equilibrium equations of elements and equations obtained by equating tension of adjacent elements the stiffnesses and masses of all elements can be found out. The equilibrium equation of the last element to be considered is not used for finding out the stiffnesses and masses. Instead the fact that the summation of the masses of the elements gives the mass of the membrane is used to determine the stiffness and mass of the elements. The values of the stiffness and mass of the last element obtained is then substituted into the equilibrium equation of the last element considered. The values on the left hand and right hand side of the equation is then compared. If the values are not same, another value of the second eigenvalue is chosen and the whole process repeated. Different values are substituted till a value of the second eigenvalue is obtained for which the stiffnesses and masses of the last element when substituted into the equilibrium equation gives equal left hand and right values. This value is the required value of the second eigenvalue and the corresponding stiffnesses and masses of the elements and the correct values of stiffnesses and masses of the membrane.

6. EXAMPLE

The procedure of determining the parameters of a membrane from displacements and frequencies discussed can be illustrated by an example. Consider a square membrane 8 units (of length) on each side. The total mass of the membrane is 15 units (of mass). The displacements of elements the membrane is given in Fig. 3 for a frequency of 0.8. Fig. 4 gives the displacements of elements of the membrane for a different frequency. The variation of stiffness of the membrane and the mass of the elements needs to be determined.

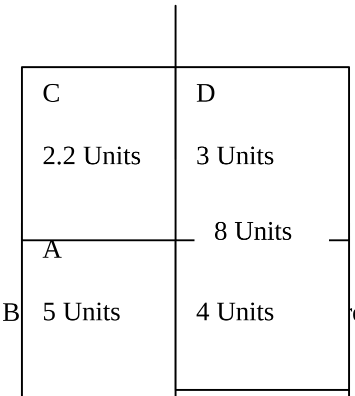

Figure 3. Displacement of elements A, B … equency of 0.8.

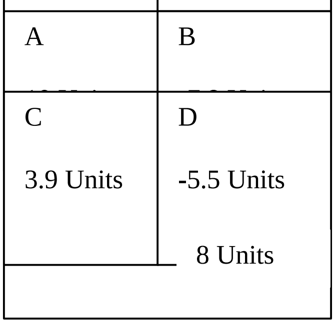

Figure 4. Displacement of elements A, B, C and D of membrane for unknown frequency.

Let the stiffnesses of the elements A, B, C, D are denoted by Kax, Kay,Kbx, Kby, Kcx, Kcy, Kdx, Kdy as shown in Fig.5. The masses of elements A, B, C and D are denoted as Ma, Mb, Mc and Md.

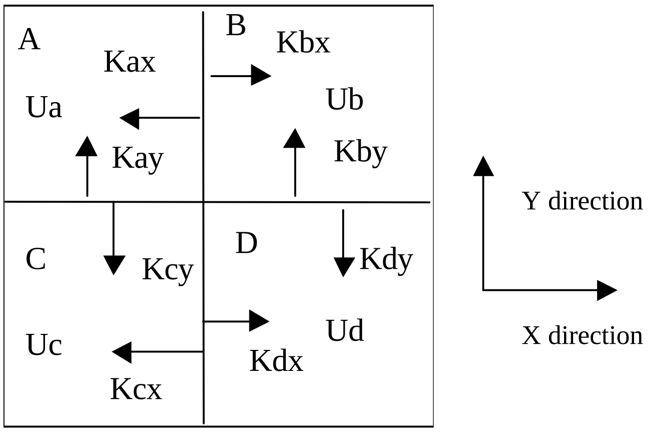

Figure 5 Variation of stiffness

Using the displacements given when frequency is 0.8 the increase in length of elements in both x and y directions can be determined. The values are obtained as:

Xa=Increase in length of element A in x direction= 3.447, Xb=Increase in length of element B in x direction= 2.534, Xb/Xa=0.735.

Xd=Increase in length of element D in x direction= 1.645, Xc=Increase in length of element C in x direction= 1.013, Xc/Xd=0.616.

Yb=Increase in length of element B in y direction= 2.534, Yd=Increase in length of element D in y direction=1.667, Yd/Yb= 0.658.

Yc=Increase in length of element C in y direction= 1.415, Ya=Increase in length of element A in y direction= 3.827, Yc/Ya= 0.37.

Using the increase in lengths during displacement in the x and y directions the ratio of stiffnesses of the elements can be determined since tension along x any y directions of the membrane remains same. Thus the ratio of stiffnesses Kax/Kbx=0.735, Kdx/Kcx=0.616, Kby/Kdy=0.658 and Kay/Kcy=0.37.

Using the displacements of elements of the membrane when λ=0.8 the equations of equilibrium of the elements can be given as:

Equilibrium equation of A

(14) $4\,Kbx + 6.16\,Kcy + 4\,Ma=0$

Equilibrium equation of B

(14) $5\,Kax + 3\,Kdy + 3.2\,Mb=0$

Equilibrium equation of C

(14) $3\,Kby + 1.76\,Kcx + 2.4\,Md=0$

Equilibrium equation of D

(14) $14\,Kay + 2.4\,Kdx+1.76\,Mc=0$

Substituting values of ratio of stiffnesses the above equations change to

(14) 5.44 Kax + 6.16 Kcy + 4 Ma=0

(14) 5Kax + 3 Kdy + 3.2 Mb=0

(14) 4Kby + 1.76 Kcx + 2.4 Md=0

(14) 5.18 Kcy + 2.4 Kdx + 1.76 Mc=0

Using Eqs. 11 to 14 the stiffnesses can be determined in terms of masses as

Kby=4.41 Ma + 3.61 Mb + 2.49 Md – 2.17 Mc

Kcy=2.69 Ma – 1.58 Mb + 2 Md – 1.75 Mc

Kax=-3.78 Ma + 1.79 Mb –2.27 Md + 1.98 Mc

Kdx=-5.8 Ma + 5.05 Mb – 4.32 Md + 3.04 Mc.

The displacements of the membrane for an unknown frequency can also be used to obtain a different set of equilibrium equations for the elements. The equations obtained are:

Equilibrium equation of A

(15) 138.84 Kbx + 23.79 Kcy + 10 λ Ma=0

Equilibrium equation of B

(16) 178 Kax + 12.65 Kdy + 7.8 λ Mb=0

Equilibrium equation of C

(17) 17.94 Kby + 36.66 Kcx + 5.5 λ Md=0

Equilibrium equation of D

(18) 61 Kay + 51.7 Kdx + 3.9 λ Mc=0

Substituting ratios of stiffness obtained the equations change to

(19) 188.87 Kax + 23.79 Kcy + 10 λ Ma=0

(20) 178 Kax + 19.23 Kdy + 7.8 λ Mb=0

(21) 17.94Kby + 59.55 Kcx + 5.5 λ Md=0

(22) 22.55 Kcy + 51.7 Kdx + 3.9 λ Mc=0

It is to be noted that another relation can be obtained by summing up the masses.

Thus Eq.23 is given as:

(23) Ma+Mb+Mc+Md=15.

Different values of λ can be substituted and the values Ma, Mb, Mc and Md determined. These values are then substituted in Eq.22 and the L.H.S. and R.H.S. are determined. When the value of λ=4.665 is substituted, using Eqs. 19, 20, 21 & 23 the values of masses are obtained as Ma=0.249, Mb=1.524, Mc=6.6, Md=6.628. These values when substituted in Eq.22 gives equal L.H.S. and R.H.S. values. The corresponding values of stiffnesses are obtained as Kby=2.324, Kcy=0.014, Kax=0.167, Kdx=2.312, Kbx=0.227, Kdy=3.532, Kcx=3.756, Kay=0.005. These values of masses and stiffnesses obtained are the required parameters of the membrane.

7. CONCLUSIONS

The paper presents a method of formulating and solving the inverse problem of an anisotropic, non-homogenous, fixed membrane of a rectangular, circular and elliptical shape. A finite difference model of an anisotropic, non-homogenous membrane is developed. The eigenvalue equations for the inverse problem of a rectangular, circular and elliptical membrane is formulated using the finite difference model developed. A study is carried out by varying the known number of eigenvectors and determining the conditions of reconstruction for a non-homogenous anisotropic rectangular and elliptical membrane. Also a technique of the reconstruction procedure and an example illustrating the process is given.

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Chebyshev Approximation for Numerical Solution of Differential-Algebraic Equation

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Abstract

In this paper, we apply the Chebyshev approximation to solve a differential-algebraic equation. First we calculate power series of the given differential-algebraic equations(DAEs) system then transform it into Chebyshev series form, which give an arbitrary order for solving differential-algebraic equation numerically.

Keywords: Differential- Algebraic Equation, Power Series, Chebyshev Approximation

1. INTRODUCTION

Differential-algebraic equations(DAEs) arises from many mathematical models as e.g., simulation of electric circuits, chemical reactions subject to invariants, modelling chemical engineering systems. The numerical and anlytical treatment of differential-algebraic equations(DAEs) has been the subject of intensive research activity for years [1], [5], [6]. The application of ordinary differential equation(ODE) methods to these systems presents numerical difficulties. Resently, several numerical methods were applied with some success to DAEs. Among them are the backward differentiation formulas (BDF) [4], [9], multistep methods [5], one-leg methods [10], implicit Runge-Kutta methods [12], [14], Rosenbrock methods[13], [15], extrapolation methods[4] and Padé approximation method[18],[19], [20]. The well known codes for DAEs are LSODI [8], DASSL [11], LÝMEX [3] , RADAU5 [7] and DAEIS[16].

Differential- Algebraic Equation(DAEs) are special implicit differential equations of the form

$$F(x, y(x), y'(x)) = 0 \qquad (1)$$

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with singular $F_{y'}$, where F and y are of the same dimension. Here an in the following we denote partial derivatives by subscripts, so that $F_{y'} = \partial F / \partial y'$. Equation (1) is also called a fully implicit DAE system. We are here especially interested in semi-explicit systems, differential-equations with algebraic constraints of the form

$$y'(x) = f(x, y(x), z(x))$$
$$0 = g(x, y(x), z(x))$$

where y represents the differential variables and z the algebraic variables [17]. The numerical methods devised for DAEs take into account the structure of the underlying DAE. we will calculate power series of the given differential-algebraic equations(DAEs) system then transform it into Chebyshev series form, which give an arbitrary order for solving differential-algebraic equation numerically.

2. THE METHOD

A differential-algebraic equation has the form

$$F(x, y, y') = 0 \tag{2}$$

with initial values

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \tag{3}$$

where F and y is a vector function for which we assumed sufficient differentiability, and the initial values to be consistent, i.e.

$$F(y_0, y_0', x_0) = 0.$$

The solutions of (2) can be assumed that

$$y = y_0 + y_1 x + e x^2, \tag{4}$$

where e is a vector function which is the same size as y_0 and y_0'. Substitute (4) into (2) and neglect higher order term, we have the linear equation of e in the form

$$Ae = B \tag{5}$$

where A and B are constant matrixes. Solving equation (5), the coefficients of x^2 in (4) can be determined. Repeating above procedure for higher order terms, we can get the arbitrary order power series of the solutions for (2). The power series given by above procedure can be transformed into Chebyshev series and we have numerical solution of differential-algebraic equation in (2)[18,19,20].

3. CHEBYSHEV APPROXIMATION

Suppose that we are given a Power series $\sum_{i=0}^{\infty} c_i x^i$, representing a function $f(x)$, so that

$$f(x) = \sum_{i=0}^{\infty} c_i x^i.$$

This expansion is the fundamental starting point of any analysis using Chebyshev approximants. Throughout this work we reserve the notation $c_i = 0,1,2,\cdots$ for the given set of coefficients, and $f(x)$ is the associated function.

The Chebyshev polynomial of degree n is denoted $T_n(x)$, and is given by the explicit formula

$$T_n(x) = \cos(n \ \arccos x) \tag{6}$$

The Chebyshev polynomials are orthogonal in the interval $[-1,1]$ over a weight $(1-x^2)^{-1/2}$. In particular,

$$\int_{-1}^{1} \frac{T_i(x)T_j(x)}{\sqrt{1-x^2}}\,dx = \begin{cases} 0 & i \neq j \\ \pi/2 & i = j \neq 0 \\ \pi & i = j = 0 \end{cases} \tag{7}$$

The polynomial $T_n(x)$ has n zeros in the interval $[-1,1]$, and they are located at the points

$$x = \cos\left(\frac{\pi(k-\frac{1}{2}}{n}\right) \qquad k = 1, 2, \dots, n \tag{8}$$

In this same interval there are $n+1$ extrema(maxima and minima), located at

$$x = \cos\left(\frac{\pi k}{n}\right) \qquad k = 0,1,\dots,n \tag{9}$$

At all of the maxima $T_n(x) = 1$, while at all of the minima $T_n(x) = -1$; it is precisely this property that makes the Chebyshev polynomials so useful in polynomial approximation of functions.

The Chebyshev polynomials satisfy discrete orthogonality relation as well as the continuous one (7): If $x_k \quad (k = 1,\dots,m)$ are the m zeros of $T_m(x)$ given by (8), and if $i, j < m$, then

$$\sum_{k=1}^{m} T_i(x_k)T_j(x_k) = \begin{cases} 0 & i \neq j \\ m/2 & i = j \neq 0 \\ m & i = j = 0 \end{cases} \tag{10}$$

It is not too difficult to combine equations (6), (8) and (10) to prove the following theorem:

Theorem 3.1 If $f(x)$ is an arbitrary function in the interval $[-1,1]$, and if N coefficients c_j, $j = 0,\dots,N-1$, are defined by

$$c_j = \frac{2}{N}\sum_{k=1}^{N} f(x_k)T_j(x_k)$$

$$= \frac{2}{N}\sum_{k=1}^{N} f\left[\cos\left(\frac{\pi(k-\frac{1}{2}}{N}\right)\right]\cos\left(\frac{\pi j(k-\frac{1}{2}}{N}\right) \quad (11)$$

then the approximation formula

$$f(x) \approx \left[\sum_{k=0}^{N-1} c_k T_k(x)\right] - \frac{1}{2}c_0 \quad (12)$$

is exact for x equal to all of the N zeros of $T_N(x)$.

For a fixed N , equation (12) is a polynomial in x which approximates the function $f(x)$ in the interval $[-1,1]$. Suppose N is so large that (12) is virtually a perfect approximation of $f(x)$.

4. TEST PROBLEM

In this section, we consider the following differential-algebraic equation as a test problem.

$$\begin{bmatrix} 1 & -x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} + \begin{bmatrix} 1 & -(1+x) \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \sin x \end{bmatrix} \quad (14)$$

and initial values

$$\begin{bmatrix} v_1(0) \\ v_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} v_1'(0) \\ v_2'(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The exact solution is

$$\begin{bmatrix} v_1(x) \\ v_2(x) \end{bmatrix} = \begin{bmatrix} \exp(-x) + x\sin x \\ \sin x \end{bmatrix}.$$

Using the method for differential-algebraic equations(DAEs) [18,19,20], the power series solutions of (14) can be obtained as

$$v_1(x) = 1 - x + 1.5x^2 - 0.1666667x^3 - 0.125x^4 - 0.0083333x^5 + 0.0013888x^6 - 0.0001984x^7$$
$$- 0.0000128x^8 - 0.2755731.10^{-2}x^9$$
$$v_2(x) = x - 0.1666667x^3 + 0.0083x^5 - 0.0001984x^7 + 0.2755731.10^{-2}x^9.$$

The Power series $v_1(x)$ can be transformed into following Chebyshev series,

$$\begin{aligned}\tilde{v}_1 &= 1.7061157T_0(x) - 1.1303182T_1(x) + 0.69198133T_2(x)\\ &- 0.44336842\times10^{-1}T_3(x) - 0.13840061\times10^{-1}T_4(x) - 0.54292224\times10^{-3}T_5(x)\\ &+ 0.29296875\times10^{-3}T_6(x) - 0.31970796\times10^{-5}T_7(x) - 0.13563368\times10^{-5}T_8(x)\\ &- 0.10764578\times10^{-7}T_9(x)\end{aligned}$$

The Power series $v_2(x)$ can be transformed into following Chebyshev series,

$$\begin{aligned}\tilde{v}_2(x) &= 0.88010118T_1(x) + 0.039126700T_3(x) + 0.00049951947T_5(x)\\ &- 0.30033172\times10^{-5}T_7(x) + 0.10764578\times10^{-7}T_9(x)\end{aligned}$$

Substitute $T_i(x)$ $i = 0,\ldots,9$ and step size h into $\tilde{v}_1(x)$ and $\tilde{v}_2(x)$. We obtain numerical solution of (14).

We show Table 1 and Table 2 for the solution of (14) by above numerical method. The numerical values on Table 1 and Table 2 are coinciding with the exact solutions of (14).

Table 1. Numerical solution of $v_1(x)$ in (14).

x	Exact $v_1(x)$	$\tilde{v}_1(x) =$ Approx. $v_1(x)$	$\lvert v_1(x) - \tilde{v}_1(x)\rvert$
0.0	0.0000000	0.0000000	0.0000000
0.1	0.91482076	0.91482070	0.6×10^{-7}
0.2	0.85846462	0.85846456	0.6×10^{-7}
0.3	0.82947428	0.82947423	0.5×10^{-7}
0.4	0.82608739	0.82608732	0.7×10^{-7}
0.5	0.84624343	0.84624335	0.8×10^{-7}
0.6	0.88759712	0.88759704	0.8×10^{-7}
0.7	0.94753768	0.94753750	0.18×10^{-6}
0.8	1.0232138	1.0232134	0.4×10^{-6}
0.9	1.1115639	1.1115628	0.11×10^{-5}
1.0	1.2093504	1.2093473	0.31×10^{-5}

Table 2. Numerical solution of $v_2(x)$ in (14).

x	Exact $v_2(x)$	$\tilde{v}_2(x) =$ Approx. $v_2(x)$	$\lvert v_2(x) - \tilde{v}_2(x) \rvert$
0.0	0.00000000	0.00000000	0.000000
0.1	0.099833417	0.099833416	0.1×10^{-8}
0.2	0.19866933	0.19866934	0.1×10^{-7}
0.3	0.29552021	0.29552021	0.000000
0.4	0.38941834	0.38941833	0.1×10^{-7}
0.5	0.47942554	0.47942555	0.1×10^{-7}
0.6	0.56464247	0.56464248	0.1×10^{-7}
0.7	0.64421769	0.64421768	0.1×10^{-7}
0.8	0.71735609	0.71735609	0.000000
0.9	0.78332691	0.78332692	0.1×10^{-7}
1.0	0.84147098	0.84147101	0.3×10^{-7}

The graph of $v_1(x)$, $v_2(x)$ and their Chebyshev approximant are simultaneously shown in Fig. 1 and Fig. 2. As can be seen from the graphics, the accuracy of the approximation by using Chebyshev series which agree with given exact solution of the equation systems.

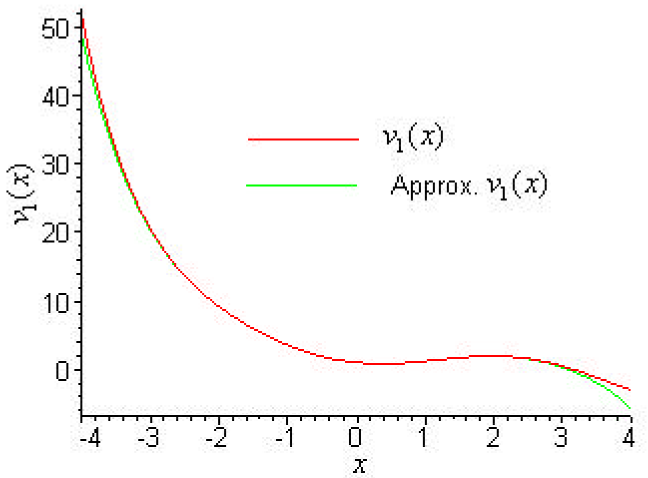

Fig. 1. Graph of $v_1(x)$ and its Chebyshev approximation.

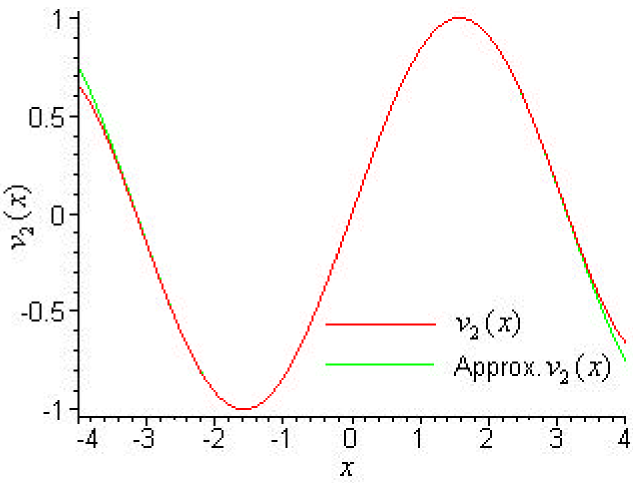

Fig. 2. Graph of $v_2(x)$ and its Chebyshev approximation.

5. CONCLUSION

A Chebyshev approximation method has proposed for solving differential-algebraic equations in this study. This method is very simple an effective for most of differential-algebraic equations.

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Stress and Flux Free Thermoelastic Vibrations in Plates of Anisotropic Media with Two Thermal Relaxation Times

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ABSTRACT

Analysis for the propagation of waves in homogenous heat conducting infinite monoclinic stress and flux free thermoelastic plate of finite thickness is developed. Frequency equations in closed form and isolate the mathematical conditions for symmetric (extensional) and antisymmetric (flexural) thermoelastic wave mode propagation in completely separate terms are derived. Higher symmetry material, such as orthotropic, transversely isotropic, cubic and isotropic is contained implicitly in the analysis. It is shown that the motion for SH modes decouple which is not effected by thermal variations from the rest of motion if propagation occurs along an in-plane axis of symmetry. Finally the Numerical solution of the frequency equations for zinc plate is carried out, and the dispersion curves for the first six modes are presented. The three motions namely, longitudinal, transverse and thermal of the medium are found dispersive and coupled with each other due to the thermal and anisotropic effects. The dispersion curves are get modified due to the thermal and anisotropic effects and is also influenced by the thermal relaxation time. Relevant results of previous investigations are deduced as special cases.

Keywords: Anisotropic, thermoelasticity, dispersion, frequency equation, thermal relaxation times

1. INTRODUCTION

Future advanced aerospace composite materials will be required to respond well to combined thermal and mechanical loading. These materials will be used for the next generation of aerospace structures, such as the proposed high-speed civil transport. Numerous technological situations at moderate and high temperatures such as distribution of temperature around propagating cracks, and temperature distribution in solid materials due to laser pulse train of a very short duration and many others physical experiments at low temperatures show the necessity of taking into account the wave structure of the heat transport.

Theory to include the effect of temperature change, known as the theory of thermoelasticity, has been well established Nowacki (1962,1975), Chadwick (1960). Classical theory of dynamic thermoelasticity that takes into account the coupling effects between temperature and strain fields involves the infinite thermal wave speed i.e. implies an immediate response to a temperature gradient and leads to a parabolic differential equation for the evolution of the temperature. In contrast, when relaxation effects are taken into account in the constitutive equation describing the heat flux, one has a hyperbolic equation, which implies a finite speed for heat transport. Hyperbolic heat transport has been receiving increasing attention both for theoretical motivations (analysis of thermal waves and second sound in dielectric solids, finite speed of heat transport, etc.) and for the analysis of some practical problems involving a fast supply of thermal energy (for instance, by a laser pulse or a chemical explosion, etc.). The literature dedicated to hyperbolic thermoelastic models is quite large and its detailed review can be found in Chandrasekharaiah (1986,1998).

Lord and Shulman (1967), and Green and Lindsay (1972) extended the coupled theory of thermoelasticity by introducing the thermal relaxation time in the constitutive equations. These new theories, which eliminate the paradox of infinite velocity of heat propagation, are called generalized theories of thermoelasticity. This generalized thermoelasticity theories admits finite speed for the propagation of thermoelastic disturbances has received much attention in recent years. The LS model introduces a single time constant to dictate the relaxation of thermal propagation, as well as the rate of change of strain rate and the rate of change of heat generation. In the GL theory, on the other hand, the thermal and thermo-mechanical relaxations are governed by two different time constants.

Chadwick and Seet (1970), and Chadwick (1979) investigated the thermoelastic wave propagation in transversely isotropic and homogeneous anisotropic heat conducting elastic materials, respectively. Waves types occurring in bounded anisotropic media are very complicated, and in thermoelasticity, the problem becomes even more complicated, because in thermoelasticity, solutions to both the heat conduction and thermoelasticity problems for anisotropic is required. These solutions are also to satisfy the thermal and mechanical boundary and interface conditions. Here, we mention that several authors Puri (1973; 1975), Agarwal (1978; 1979), Tao and Prevost (1984), Massalas and Kalpakidis (1987b) and Verma and Hasebe (1999) have considered the propagation of generalized thermoelastic waves in plates of isotropic media. Massalas and Kalpakidis (1987a) used the generalized theory of Lord and Shulman to study the characteristics of wave motion in a thin plate under plane stress state with mixed boundary conditions.

Banerjee and Pao (1974) extended this theory to anisotropic heat conducting elastic materials. Dhaliwal and Sherief (1980) treated the problem in more systematic manner. They derived governing field equations of generalized thermoelastic media and proved that these equations have a unique solution Hawwa and Nayfeh (1995) studied the propagation of harmonic waves in a laminated composite consisting of an arbitrary number of layered anisotropic plates. Verma (2001), Verma and Hasebe (2002) studied the propagation of generalized thermoelastic vibrations in infinite plates in the context of generalized thermoelasticity. Verma and Hasebe (2004) considered the propagation of plane harmonic thermoelastic waves in an infinite homogeneous orthotropic plate of finite thickness in the generalized theory of thermoelasticity with two thermal relaxation times is studied.

In this article, problem of plane harmonic thermoelastic waves in an infinite homogeneous monoclinic plate of finite thickness in the generalized theory of thermoelasticity with two thermal relaxation times is studied. Frequency equations in closed form and isolate the mathematical conditions for symmetric and antisymmetric thermoelastic wave mode propagation in completely separate terms is derived. Results of higher symmetry material, such as orthotropic, transversely isotropic, cubic and isotropic are contained implicitly in the analysis. Special cases of the frequency equations are also discussed. Numerical solution of the frequency equations for zinc plate is carried out, and the dispersion curves are presented. The three motions namely, longitudinal, transverse and thermal of the medium are found dispersive and coupled with each other due to the thermal and anisotropic effects. The dispersion curves are get modified due to the thermal and anisotropic effects and is also influenced by the thermal relaxation time. Relevant results of previous investigations are deduced as special cases.

2. FORMULATION

Consider an infinite monoclinic thermally conducting elastic plate at uniform temperature T_0 in the undisturbed state, having thickness $2d$, whose normal is aligned with the x_3-axis of a reference Cartesian coordinate system O (x_1, x_2, x_3). The mid plane of the plate is chosen to coincide with the x_1-x_2 plane. Let the faces of the plate be the planes $x_3 = \pm$ d.

The basic field equations of generalized thermoelasticity in the absence of body forces and heat sources of the plate (Tao and Prevost (1984)) are given by

$$\sigma_{ij,j} = \rho \ddot{u}_{,i}$$
(2.1)

$$K_{ij} T_{,ij} - \rho C_e (\dot{T} + \tau_0 \ddot{T}) = T_0 \beta_{ij} \dot{u}_{i,j}$$
(2.2)

where

$$\sigma_{ij} = c_{ijkl} e_{kl} - \beta_{ij} (T + \tau_1 \dot{T}) ,$$
(2.3)

$$\beta_{ij} = c_{ijkl} \alpha_{kl} , \quad i, j, k, l = 1,2, 3$$
(2.4)

ρ is the density, t is the time, u_i is the displacement in the x_i direction, K_{ij} are the thermal conductivities, C_e and τ_0 are respectively the specific heat at constant strain, and thermal relaxation time, σ_{ij} and e_{ij} are the stress and strain tensor respectively; β_{ij} are thermal moduli; α_{ij} is the thermal expansion tensor; T is temperature; and the fourth order tensor of the elasticity C_{ijkl} satisfies the (Green) symmetry conditions:

$$c_{ijkl} = c_{klij} = c_{ijlk} = c_{jikl} \text{ , and } \alpha_{ij} = \alpha_{ji} , \quad \beta_{ij} = \beta_{ji}$$
(2.5)

The parameter τ_1 and τ_0 are the thermal-mechanical relaxation time and the thermal relaxation time of the GL theory and satisfy the inequality $\tau_1 \geq \tau_0 \geq 0$. Comma notation is used for spatial derivatives and superposed dot represents differentiation with respect to time.

Strain-displacement relation

$$e_{ij} = (u_{i,j} + u_{j,i})/2 \,. \tag{2.6}$$

When specializing the equations (2.1)-(2.5) for monoclinic media in generalized thermoelasticity, the governing equations are

$$c_{11}u_{1,11} + c_{55}u_{1,33} + (c_{13} + c_{55})u_{3,13} + c_{16}u_{2,11} + c_{45}u_{2,33} - \beta_1(T + \tau_1\dot{T})_{,1} = \rho\ddot{u}_1 \,, \tag{2.7}$$

$$c_{16}u_{1,11} + c_{45}u_{1,33} + (c_{36} + c_{45})u_{3,13} + c_{66}u_{2,11} + c_{44}u_{2,33} - \beta_6(T + \tau_1\dot{T})_2 = \rho\ddot{u}_2 \,, \tag{2.8}$$

$$(c_{13} + c_{55})u_{1,13} + c_{55}u_{3,11} + c_{33}u_{33} + (c_{36} + c_{45})u_{2,13} - \beta_3(T + \tau_1\dot{T})_{,3} = \rho\ddot{u}_3 \tag{2.9}$$

$$K_1T_{,11} + K_2T_{,22} + K_3T_{,33} - \rho C_e(\hat{O} + \tau_0\ddot{T}) = T_0(\beta_1\dot{u}_{1,1} + \beta_6\dot{u}_{2,2} + \beta_3\dot{u}_{3,3}) \,. \tag{2.10}$$

3. ANALYSIS

Having identified the plane of incidence to be the $x_1 - x_3$ plane, then the solution for displacements and temperature for an angle of incidence θ, is proposed:

$$(u_j, T) = (U_j, U_4)\exp[i\xi(\sin\theta\, x_1 + \alpha x_3 - ct)],\; i = \sqrt{-1},\; j = 1, 2, 3, \tag{3.1}$$

where ξ is the wave number, c is the phase velocity (= ω / ξ), ω is the circular frequency, α is still an unknown parameters, U_j and U_4 are the constants related to the amplitudes of displacement u_1, u_2, u_3 and temperature T. Although solutions (3.1) are explicitly independent of x_2, an implicit dependence is contained in the transformation and the transverse displacement component u_2 is non-vanishing in equation (3.1). Substituting (3.1) in equations (2.7)-(2.10) leads to the coupled equations, the choice of solutions leads to four coupled equations

$$M_{mn}(\alpha)U_n = \mathbf{0} \qquad m,\; n = 1,2,3,4 \tag{3.2}$$

where

$$F_{11} = \sin^2\theta - \zeta^2,\; F_{12} = c_4\sin^2\theta,\; F_{13} = c_7\sin\theta,\; F_{14} = \sin\theta,\; F_{22} = c_3\sin^2\theta - \zeta^2$$

$$F_{23} = c_8\sin\theta\alpha,\; F_{24} = \bar{\beta}_6\sin\theta,\; F_{33} = c_2\sin^2\theta - \zeta^2,\; F_{34} = \bar{\beta}_3\alpha,$$

$$F_{41} = \varepsilon_1\tau_g\omega_1^*\zeta^2\sin\theta,\; F_{42} = \varepsilon_1\tau_g\bar{\beta}_6\omega_1^*\zeta^2\sin\theta,\; F_{43} = \varepsilon_1\tau_g\omega_1^*\zeta^2\bar{\beta}_3\alpha, \tag{3.3}$$

$$F_{44} = \sin^2\theta - \tau\omega_1^*\zeta^2,$$

and

$$M_{11} = F_{11} + c_2\alpha^2,\ M_{12} = F_{12} + c_5\alpha^2,\ M_{13} = F_{13}\alpha,\ M_{14} = F_{14}$$

$$M_{22} = F_{22} + c_6\alpha^2,\ M_{23} = F_{23}\alpha,\ M_{24} = F_{24}$$

$$M_{33} = F_{33} + c_1\alpha^2 \quad M_{34} = F_{34}\alpha$$

$$M_{41} = F_{41},\ M_{42} = F_{42},\ M_{43} = F_{43}\alpha,\ M_{44} = F_{44} + \bar{K}\alpha^2.$$

(3.4)

and

$$c_1 = \frac{c_{33}}{c_{11}},\ c_2 = \frac{c_{55}}{c_{11}},\ c_3 = \frac{c_{66}}{c_{11}},\ c_4 = \frac{c_{16}}{c_{11}},\ c_5 = \frac{c_{45}}{c_{11}},\ c_6 = \frac{c_{44}}{c_{11}},$$

$$c_7 = \frac{c_{13} + c_{55}}{c_{11}},\ c_8 = \frac{c_{36} + c_{45}}{c_{11}},\ \bar{\beta}_6 = \frac{\beta_6}{\beta_1}, \bar{\beta}_3 = \frac{\beta_3}{\beta_1},\ \bar{K} = \frac{K_3}{K_1},$$

$$\varepsilon_1 = \frac{T_0\beta_1^2}{\rho C_e c_{11}},\ \omega_1^* = \frac{C_e c_{11}}{K_1},\ \zeta^2 = \frac{c^2\rho}{c_{11}},\ \tau_g = \tau_1 + \frac{i}{\xi c},\ \tau = \tau_0 + \frac{i}{\xi c}. \qquad (3.5)$$

The system of relations (3.2) has a non-trivial solution if the determinant of the coefficients of U_1, U_2, U_3, and U_4 vanishes, which yields an algebraic equation relating α to *c*. We obtain an eighth-degree polynomial equation in α, which can be written as

$$C_0\alpha^8 + C_1\alpha^6 + C_2\alpha^4 + C_3\alpha^2 + C_4 = 0.$$

(3.6)

where

$$C_0 = (c_2c_6 - c_5^2)c_1(-\bar{K})$$

$$C_1 = A_{E1}(-\bar{K}) + (c_2c_6 - c_5^2)c_1F_{44} + (c_5^2 - c_2c_6)F_{34}F_{43}],$$

$$\begin{aligned} C_2 = {} & A_{E1}(-\bar{K}) + A_{E2}F_{44} + (c_1c_5F_{24} - c_1c_6F_{14} - c_5F_{23}F_{34} + c_6F_{13}F_{34})F_{41} \\ & + (C_1C_5F_{14} - c_1c_2F_{24} + c_2F_{23}F_{34} - c_5F_{13}F_{34})F_{42} + (c_2F_{23}F_{24} + 2c_5F_{12}F_{34} - c_2F_{22}F_{34} \\ & + c_6F_{13}F_{14} - c_6F_{11}F_{34} - c_5F_{14}F_{23} - c_5F_{13}F_{24})F_{43} \end{aligned}$$

$$\begin{aligned} C_3 = \{ & (A_{E3}(-\bar{K}) + A_{E2}F_{44} + [(F_{22}F_{13} - F_{12}F_{23})F_{34} + (c_1F_{12} - F_{13}F_{23} + c_5F_{33})F_{24} \\ & + (F_{23}^2 - c_1F_{22} - c_6F_{33})F_{14}]F_{41} + [(F_{11}F_{23} - F_{12}F_{13})F_{34} + (F_{13}^2 - c_1F_{11} - c_2F_{33})F_{24} \\ & + (c_1F_{12} - F_{13}F_{23} + c_5F_{33})F_{14}]F_{42} + [(F_{12}^2 - F_{11}F_{22})F_{34} + (F_{11}F_{23} - F_{12}F_{13})F_{24} \\ & + (F_{22}F_{13} - F_{12}F_{23})F_{14}]F_{43} \} \end{aligned}$$

$$C_4 = A_{E3}F_{44} - F_{33}(F_{14}F_{22} - F_{12}F_{24})F_{41} + F_{33}(F_{12}F_{14} - F_{11}F_{24})F_{42},$$

(3.7)

$$A_{E1} = (c_1c_6F_{11} - c_6F_{13}^2 + c_5F_{13}F_{23} - c_5^2F_{33} - c_2F_{23}^2 - 2c_1c_5F_{12} + c_5F_{21}F_{23} + c_2c_6F_{33}),$$

$$
\begin{aligned}
A_{E2} &= (F_{11}F_{23}^2 - F_{13}^2F_{22} + c_6F_{11}F_{33} + c_1F_{11}F_{22} - c_1F_{12}^2 + 2F_{12}F_{13}F_{23} + 2c_5F_{12}F_{33} + c_2F_{22}F_{33}) \\
A_{E3} &= (F_{11}F_{22} - F_{12}^2)F_{33},\ \Delta_E = (c_2c_6 - c_5^2)c_1,
\end{aligned}
$$

Eight different solutions are generated form the expression (3.6) for α as α_q, $q = 1, 2 \ldots 8$, having the properties)

$$\alpha_2 = -\alpha_1,\ \alpha_4 = -\alpha_3,\ \alpha_6 = -\alpha_5,\ \alpha_8 = -\ \ _7.$$

The displacements, temperature component ratios $_q = {}^{2q}/$, $_q = {}^{3q}/$ and $S\ \ {}^{U_q}/_{1}$ can be expressed as $V = \dfrac{L_1(\alpha_q)}{L(\ \ )}$, $W = \dfrac{L_2(\alpha_q)}{L(\ \ )}$ $S = \dfrac{L_3(\alpha_q)}{L(\ \ )}$

(3.9)

where

$$
\begin{aligned}
L_1(\alpha_q) &= M_{11}(\alpha_q)M_{23}(\alpha_q)M_{34}(\alpha_q) + M_{13}^2(\alpha_q)M_{24}(\alpha_q) + M_{12}(\alpha_q)M_{14}(\alpha_q)M_{33}(\alpha_q) \\
&\quad - M_{13}(\alpha_q)M_{14}(\alpha_q)M_{23}(\alpha_q) - M_{11}(\alpha_q)M_{24}(\alpha_q)M_{33}(\alpha_q) - M_{12}(\alpha_q)M_{13}(\alpha_q)M_{34}(\alpha_q) \\
L_2(\alpha_q) &= M_{13}^2(\alpha_q)M_{33}(\alpha_q) + M_{11}(\alpha_q)M_{23}(\alpha_q)M_{24}(\alpha_q) + M_{13}(\alpha_q)M_{14}(\alpha_q)M_{22}(\alpha_q) \\
&\quad - M_{12}(\alpha_q)M_{14}(\alpha_q)M_{23}(\alpha_q) - M_{12}(\alpha_q)M_{13}(\alpha_q)M_{24}(\alpha_q) - M_{11}(\alpha_q)M_{22}(\alpha_q)M_{22}(\alpha_q) \\
L_3(\alpha_q) &= M_{12}^2(\alpha_q)M_{33}(\alpha_q) + M_{13}^2(\alpha_q)M_{22}(\alpha_q) + M_{23}^2(\alpha_q)M_{11}(\alpha_q) \\
&\quad - 2M_{12}(\alpha_q)M_{13}(\alpha_q)M_{23}(\alpha_q) - M_{11}(\alpha_q)M_{22}(\alpha_q)M_{33}(\alpha_q)
\end{aligned}
$$

(3.10)

$$
\begin{aligned}
L(\alpha_q) &= M_{14}^2(\alpha_q)M_{23}(\alpha_q) + M_{12}(\alpha_q)M_{24}(\alpha_q)M_{33}(\alpha_q) + M_{13}(\alpha_q)M_{22}(\alpha_q)M_{34}(\alpha_q) \\
&\quad - M_{12}(\alpha_q)M_{23}(\alpha_q)M_{34}(\alpha_q) - M_{13}(\alpha_q)M_{23}(\alpha_q)M_{24}(\alpha_q) - M_{14}(\alpha_q)M_{22}(\alpha_q)M_{23}(\alpha_q)
\end{aligned}
$$

Combining expressions (3.10), with the stress-strain relations and using superposition, the formal solutions for the displacement and stress components can be written as

$$
\begin{bmatrix} u_1 & u_2 & u_3 & T \\ \sigma_{33} & \sigma_{13} & \sigma_{23} & T_{,x_3} \end{bmatrix} = \sum_{q=1}^{8} \begin{bmatrix} 1 & V_q & W_q & S_q \\ D_{1q} & D_{2q} & D_{3q} & D_{4q} \end{bmatrix} U_q e^{i\xi(x_1 + \alpha_q x - ct)}
$$

(3.11)

Suppressing the common factor $e^{i\xi(x_1 - ct)}$ and introducing a change of variable $\sigma_{ij}^* = {}^{\sigma_{ij}}/_{i\xi}$ for subsequent analyses in the expression (3.11) yields to

$$\begin{bmatrix} u_1 & u_2 & u_3 & T \end{bmatrix} = \sum_{q=1}^{8} \begin{bmatrix} 1 & V_q & W_q & S_q \end{bmatrix} U e^{i\xi\alpha_q x_3}$$

(3.12)

$$\begin{bmatrix} \sigma_{33}^{*} & \sigma_{13}^{*} & \sigma_{23}^{*} & T_{,x_3} \end{bmatrix} = \sum_{q=1}^{8} \begin{bmatrix} D_{1q} & D_{2q} & D_{3q} & D_{4q} \end{bmatrix} U e^{i\xi\alpha_q x_3}$$

(3.13)

$$D_{1q} = (c_7 - c_2)\sin\theta + (c_8 - c_5)V_q \sin\theta + c_1\alpha_q W_q + c_{11}^{-1} c\tau_g \beta_3 \Theta_q (i\xi)^{-1},$$

$$D_{2q} = c_2(\alpha_q + W_q \sin\theta) + c_5 \alpha_q V_q,$$

$$D_{3q} = c_5(\alpha_q + W_q \sin\theta) + c_6 \alpha_q V_q,$$

(3.14)

$$D_{4q} = \alpha_q \Theta_q \ , \quad q = 1, 2, 3, \ldots\ 8. \qquad (3.15)$$

Incorporating (3.8) - (3.15) and inspecting the resulting relations, we conclude that monoclinic symmetry implies the further restrictions.

$$W_{j+1} = -W_j,\ D_{2\,j+1} = -D_{2j},\ D_{3\,j+1} = -D_{3j},\ D_{4\,j+1} = -D_{4j}$$

(3.16)

$$V_{j+1} = V_j, \quad D_{1\,j+1} = D_{1j}, \quad S_{j+1} = S_j, \qquad j = 1,3,5,7 \qquad (3.17)$$

4. DISPERSION RELATION

Equations (3.12) and (3.13) can relate the displacement, temperature, stress and the temperature gradient at $x_3 = -d$ and $x_3 = d$ in the matrix equation form.

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ T \\ \sigma_{33}^{*} \\ \sigma_{13}^{*} \\ \sigma_{23}^{*} \\ T_{,x_3} \end{bmatrix} = \begin{bmatrix} 1 & i & 1 & i & 1 & i & 1 & i \\ V_1 & iV & V_3 & iV_3 & V_5 & iV_5 & V_7 & iV_7 \\ iW_1T_1 & W_2T_1' & iW_3T_3 & W_4T_3' & iW_5T_5 & W_6T_5' & iW_7T_7 & W_7T_7' \\ S_1 & iS_1 & S_3 & iS_3 & S_5 & iS_5 & S_7 & iS_7 \\ D_{11} & iD_{11} & D_{13} & iD_{13} & D_{15} & iD_{15} & D_{17} & iD_{17} \\ iD_{21}T_1 & D_{21}T_1' & iD_{23}T_3 & D_{23}T_3' & iD_{25}T_5 & D_{25}T_5' & iD_{27}T_7 & D_{27}T_7' \\ iD_{31}T_1 & D_{31}T_1' & iD_{33}T_3 & D_{33}T_3' & iD_{35}T_5 & D_{35}T_5' & iD_{37}T_7 & D_{37}T_7' \\ iD_{41}T_1 & D_{41}T_1' & iD_{43}T_3 & D_{43}T_3' & iD_{45}T_5 & D_{45}T_5' & iD_{47}T_7 & D_{47}T_7' \end{bmatrix} \begin{bmatrix} C_1(U_1 + U_2) \\ S_1(U_1 - U_2) \\ C_3(U_3 + U_4) \\ S_3(U_3 - U_4) \\ C_5(U_5 + U_6) \\ S_5(U_5 - U_6) \\ C_7(U_7 + U_8) \\ S_7(U_7 - U_8) \end{bmatrix}$$

(4.1)

where $T_r' = T^{-1}{}_r = \cot(\xi\alpha_r x_3)$ and $S_r = \sin(\xi\alpha_r x_3),\ C_r = c\,\mathrm{os}(\xi\alpha_r x_3)$

(4.2)

It is obtained by specializing (3.12) and (3.13) on these two locations and employing algebraic reductions and manipulations consisting of following steps. Firstly, we use various trigonometric identities and the relations (3.16) and (3.17) to recast (3.12) and (3.13) into its equivalent matrix form. Secondly, we specialize the equivalent (4.1) to the top and bottom surface of the plate, namely, $x_3 = -d$ and $x_3 = d$ in (4.1). For the bottom surface similar relations will result with the single exception that the radical *i* in the matrix of equation is replaced by *–i* as it explicitly appears. Thirdly, we invoke the appropriate boundary conditions on the plate surfaces, namely

$$\sigma^{\bullet}_{13} = \sigma^{\bullet}_{23} = \sigma^{\bullet}_{33} = T_{,x_3} = 0 \quad \text{at} \quad x_3 = -d \text{ and } x_3 = d \tag{4.3}$$

finally yield the eight equations relating the propagation amplitude $U_1, U_2 ... U_8$, whose determinant of coefficients after lengthy algebraic manipulations and reductions leads to the two uncoupled characteristic equations.

$$\sum_{k=1,3,5,7} \left[(-1)^{(\frac{k+1}{2})-1} D_{1k} G_k \tan^{\mp}(\gamma\alpha_k) \right] = 0 \tag{4.4}$$

Corresponding to symmetric and antisymmetric modes respectively with

$$G_1 = \begin{vmatrix} D_{23} & D_{25} & D_{27} \\ D_{33} & D_{35} & D_{37} \\ D_{43} & D_{45} & D_{47} \end{vmatrix}, \quad G_3 = \begin{vmatrix} D_{21} & D_{25} & D_{27} \\ D_{31} & D_{35} & D_{37} \\ D_{41} & D_{45} & D_{47} \end{vmatrix}, \quad G_5 = \begin{vmatrix} D_{21} & D_{23} & D_{27} \\ D_{31} & D_{33} & D_{37} \\ D_{41} & D_{43} & D_{47} \end{vmatrix},$$

$$G_7 = \begin{vmatrix} D_{21} & D_{23} & D_{25} \\ D_{31} & D_{33} & D_{35} \\ D_{41} & D_{43} & D_{45} \end{vmatrix} \qquad \gamma = \frac{\xi d}{2} = \frac{\omega d}{2c} \tag{4.5}$$

5. HIGHER SYMMETRY MATERIALS

The results obtained in the previous section, for monoclinic material in generalized thermoelasticity with two thermal relaxation times are also valid for higher symmetry classes such as orthotropic, transversely isotropic, cubic and isotropic. Monoclinic materials have no axis of symmetry in the plane of the layers. Knowing that higher symmetry materials are different from the monoclinic materials in that case each possesses two orthogonal principal axes in the plane of the layers. For higher symmetry cases, the matrix equation corresponding to (4.1) can be obtained as special cases as far as the propagation direction of the waves is off the axes of symmetry. When waves propagates along an axis of symmetry, the field equations are no longer coupled, and the motion can be modeled by more than one matrix equation of the type to (4.1).

5.1 Propagation off axes of symmetry

If x_1 and x_2 are chosen to coincide with the in-plane principal axes for orthotropic symmetry, then, propagation is for off-principal axes and further appropriate restrictions on the number of non-zero thermoelastic constants of the monoclinic case are

$$c_{16} = c_{26} = c_{36} = c_{45} = 0 \text{ and } \alpha_{12} = 0. \tag{5.1}$$

For materials possessing transverse isotropy, whose x_1 axis is normal to the plane of isotropy, the additional conditions imposed by symmetry, namely

$$c_{33} = c_{22},\ c_{13} = c_{12},\ c_{55} = c_{66},\ c_{22} - c_{23} = 2c_{44},$$

$$\mathrm{K}_1 = K_2, K_3,\ \alpha_1 = \alpha_2, \alpha_3,\ \beta_1 = \beta_2 = (c_{11} + c_{12})\alpha_1 + c_{13}\alpha_3,\ \beta_3 = 2c_{13}\alpha_1 + c_{33}\alpha_3$$

(5.2) and for cubic symmetry

$$c_{11} = c_{22} = c_{33},\ c_{13} = c_{12} = c_{23}, c_{44} = c_{55} = c_{66}$$

$$\mathrm{K}_1 = K_2 = K_3,\ \alpha_1 = \alpha_2 = \alpha_3 = \alpha_t,\ \beta_1 = \beta_2 = \beta_3 = \beta = (c_{11} + c_{12})\alpha_t.$$

(5.3)

Finally, for the isotropic case

$$c_{11} = c_{22} = c_{33} = \lambda + 2\mu,\ c_{13} = c_{12} = c_{23} = \lambda,\ c_{44} = c_{55} = c_{66} = \mu$$

$$\mathrm{K}_1 = K_2 = K_3,\ \alpha_1 = \alpha_2 = \alpha_3 = \alpha_t,\ \beta_1 = \beta_2 = \beta_3 = (3\lambda + 2\mu)\alpha_t$$

(5.4)

On particularize the relations (3.2) - (3.6) to the orthotropic case and simplification of the thermoelastic constants has implications for the analysis commencing at (3.5). Of greatest importance is the fact that M_{12}, M_{23} and M_{42} in (3.5) vanish. This means that SH wave motion decouple from the rest of the motion. As a consequence, (3.6), to reduces to

$$\alpha^6 + A'_1\alpha^4 + A'_2\alpha^2 + A'_3 = 0, \tag{5.5}$$

and

$$c_3 + c_6\alpha^2 - \zeta^2 = 0, \tag{5.6}$$

where

$$A'_1 = [(P(-\overline{K}) + \Delta_1 F_{44} - c_2 F_{34} F_{43}]/\Delta\ ,$$

$$A'_2 = [Q(-\overline{K}) + PF_{44} + (F_{13}F_{34} - c_1 F_{14})F_{41} + (F_{13}F_{14} - F_{11}F_{34})F_{43}]/\Delta,$$

$$A'_3 = (RF_{44} - F_{14}F_{33}F_{41})/\Delta\ ,$$

$$P = (c_1 c_6 F_{11} - c_6 F_{13}^2 + c_5 F_{13} F_{23} - c_5^2 F_{33} - c_2 F_{23}^2 - 2c_1 c_5 F_{12} + c_5 F_{21} F_{23} + c_2 c_6 F_{33}),$$

$$Q = (F_{11}F_{23}^2 - F_{13}^2 F_{22} + c_6 F_{11} F_{33} + c_1 F_{11} F_{22} - c_1 F_{12}^2 + 2F_{12}F_{13}F_{23} + 2c_5 F_{12} F_{33} + c_2 F_{22} F_{33}),$$

$$R = (F_{11}F_{22} - F_{12}^2)F_{33},$$

$$\Delta = -\overline{K}(c_2 c_6 - c_5^2)c_1,\ \Delta_1 = (c_2 c_6 - c_5^2)c_1$$

(5.7)

Notice that roots of (5.6) correspond to the SH motion, gives a purely transverse wave, which is not affected by the temperature. This wave propagates without dispersion or damping. Relation (5.5) corresponds to the sagittal plane waves. As for the sagittal plane motion we notice that for each α_q, $(q = 1, 2....6)$ proceeding as in the previous section (4.4) and (4,5) reduce to

$$\sum_{k=1,3,5}\left[(-1)^{(\frac{k+1}{2})-1} D_{1k} G_k \tan^{\mp}(\gamma\alpha_k)\right] = 0 \qquad (5.8)$$

$$\sin(2\gamma\alpha_7) = 0 \qquad (5.9)$$

with γ as defined in (4.5) and

$$G_1' = \begin{bmatrix} D_{23}' & D_{25}' \\ D_{43}' & D_{45}' \end{bmatrix}, G_3' = \begin{bmatrix} D_{21}' & D_{25}' \\ D_{41}' & D_{45}' \end{bmatrix}, G_5' = \begin{bmatrix} D_{21}' & D_{23}' \\ D_{41}' & D_{43}' \end{bmatrix} \qquad (5.10)$$

$$D_{1q}' = i\xi(C_{13} + C_{33}\alpha_q W_q') - \beta_3 S_q'$$
$$D_{2q}' = i\xi[C_{55}(\alpha_q + W_q')]$$
$$D_{4q}' = i\xi\alpha_q S_q'$$
$$W_q' = \frac{M_{11}M_{34} - M_{13}M_{14}}{M_{14}M_{33} - M_{13}M_{34}}, S_q' = \frac{M_{11}M_{43} - M_{13}M_{41}}{M_{14}M_{43} - M_{13}M_{44}}, \; q = 1,2,\ldots 6 \qquad (5.11)$$

6. SPECIAL CASES

6.1 *Classical Case*:

If $\varepsilon_1 = 0$, then thermal and elastic fields decoupled from each other and equation (3.6) becomes characteristic equation in the uncoupled thermoelasticity. In this case from (3.7) and (3.8) for monoclinic material, we have

$$F_{41} = F_{42} = F_{43} = 0 \; , \qquad (6.1)$$

and (3.6) reduces to

$$M_{44}(\alpha^6 + A_{E1}\alpha^4 + A_{E1}\alpha^2 + A_{E1}) = 0 \,. \quad \text{(for monoclinic material)} \qquad (6.2)$$

and $M_{41} = M_{43} = 0$

$$(c_1c_2\alpha^4 + (c_2F_{33} - c_1F_{11} - F_{13}^2)\alpha^2 + F_{11}F_{33})M_{44}(\alpha) = 0 \quad \text{(for orthotropic material)} \qquad (6.3)$$

where

$$M_{44} = 1 - \tau\omega_1^{*}\zeta^2 + \bar{K}\alpha^2 = 0\,, \qquad (6.4)$$

and $\alpha^6 + P_1\alpha^4 + P_2\alpha^2 + P_3 = 0$, $c_1c_2\alpha^4 + (c_2F_{33} - c_1F_{11} - F_{13}^2)\alpha^2 + F_{11}F_{33} = 0$ are a secular equation corresponds to the purely elastic monoclinic and orthotropic materials, which are obtained and discussed by Abubakar (1962), Nayfeh and Chementi (1991).

Equation (6.2) and (6.4) provide us

$$1 - \tau\omega_1^{*}\zeta^2 + \bar{K}\alpha^2 = 0\,, \qquad (6.5)$$

which corresponds to the thermal wave. Clearly it is influenced by the thermal relaxation time τ_0 in the Green –Lindsay theory.

6.2 Coupled Thermoelasticity

This case corresponds to no thermal relaxation time, i.e. $\tau_0 = \tau_1 = 0$ ***and hence*** $\tau = \tau_g = i/\omega$. ***In case, proceeding on the same lines, we again arrived at frequency equations of the form that is again in agreement with the corresponding result obtained*** *by* Chadwick (1960), Lockett (1985) and Verma (2001).

If $\tau_1 = \tau_0 \neq 0$, equations (4.4) and (5.8) become the frequency equations in the LS theory of generalized thermoelasticity of previous work Verma (1997).

7. NUMERICAL DISCUSSION

Numerical illustrations of the analytical characteristic equations are presented in the form of dispersion curves. Dispersion and Damping curves are plotted by taking γ (wave number) real and letting Ω be complex, then the phase velocity is defined as $c = \mathrm{Re}(\Omega)/\gamma$ and the imaginary part of Ω is measure the damping of the waves. One can also let c be real and let γ be complex. In this case the wave c corresponding to Re(γ) , and Im(γ) is a measure of the attenuation of the wave. Characteristic equation (5.8) are solved by considering crystal of zinc (Chadwick 1960) with the following properties

$$c_{11} = 1.628\times10^{11}\,\mathrm{N\,m^{-2}},\ c_{12} = 0.362\times10^{11}\,\mathrm{N\,m^{-2}},\ c_{13} = 0.508\times10^{11}\,\mathrm{N\,m^{-2}}$$
$$c_{33} = 0.627\times10^{11}\,\mathrm{N\,m^{-2}},\ c_{44} = 0.385\times10^{11}\,\mathrm{N\,m^{-2}},\ \rho = 7.14\times10^{3}\ \mathrm{kg\,m^{-3}}$$
$$\beta_1 = 5.75\times10^{6}\,\mathrm{N\,m^{-2}\ deg^{-1}},\ \beta_3 = 5.07\times10^{6}\,\mathrm{N\,m^{-2}\ deg^{-1}},\ C_e = 3.9\times10^{2}\ \mathrm{J\,kg\,m^{-1}\ deg},$$
$$K_1 = 1.24\times10^{2}\,\mathrm{W\,m^{-1}\ deg^{-1}},\ K_3 = 1.24\times10^{2}\,\mathrm{W\,m^{-1}\ deg^{-1}},\ T_0 = 296K,\ \varepsilon_1 = 0.0221.$$

$\left(\varepsilon_1 = \dfrac{T_0\beta_1^2}{\rho C_e c_{11}}\right)$ $\tau_0 = 2.10^{-7}s$ and taking different values of τ_1 keeping in mind $\tau_1 \geq \tau_0 \geq 0$. Dispersion and damping curves in the forms of variations γ wave number (dimensionless) versus Ω

frequency (dimensionless) are constructed at different values of times relaxation time ratios $\left(\frac{\tau_1}{\tau_0}\right)$ =2, 5, 10 and $\theta = \pi/2$ for the first six modes of representative orthotropic plate in Figs. 1-8 for generalized theory of thermoelasticity with two thermal relaxation times and Figs. 9-10 for classical theory of thermoelasticity (no relaxation time). The following non-dimension for the frequency and wave number has been used throughout in this section.

$$\Omega = \frac{\omega d}{\sqrt{c_{11}/\rho}}, \ \gamma = \frac{1}{2}\xi d \ .$$

The dispersive character of quasi-longitudinal, quasi-transverse and quasi-thermal modes can be seen in Figs. 9-10, and dissipation (damping) in Figs. 9-10 when τ_0 and τ_1 are set equal to zero are. Three waves namely, quasi-longitudinal (QL), quasi-transverse (QT) and quasi-thermal (T-mode) of the medium are found coupled with each other due to the thermal and anisotropic effects. The wave-like behavior of the quasi-thermal modes is characterized in the thermoelasticity theory with two thermal relaxation times. The dissipation is high for small wave number higher modes and much higher for lower modes than lower and dips down to local minima at certain values of wave number. Higher modes appear in both cases with γ increases. Lower modes (symmetric and antisymmetric) are found more influenced by the thermal relaxation times at low values of wave number. The effect of thermal relaxation times is observed to be small as the inclusion of thermal relaxation times increases the amount of dissipation.

The antisymmetric (flexural) and symmetric (extensional) modes have certain features in common with their equivalents. The two families of curves are uncoupled as the lower order antisymmetric and symmetric modes approach the Rayleigh velocity for large wave number, and. higher order plate modes asymptote to shear velocity for large wave number. The most striking features between the anisotropic and isotropic is the oscillatory manner in which they approach their asymptotic limits.

8. CONCLUSIONS

In the present work an attempt has been made to study the problem of thermoelastic wave propagation in plates of monoclinic materials in the context of generalized thermoelasticity with two thermal relaxation times. For the most general case, we have provided an exact formal solution for the displacements, temperature and thermal stresses in an infinite plate of finite thickness. We have solved exactly for the dispersion Eqs. (4.8) of thermoelastic waves on a stress and flux free boundary. We have obtained dispersion Eqs. (4.4) of thermoelastic stress and flux free waves, employing algebraic reductions to express the results in the compact form. Exploiting the monoclinic solution as a starting point we have shown how to construct dispersion Es. (5.8) for the plates having orthotropic, transversely isotropic, and cubic symmetry, whereby free waves may propagate in any arbitrary in-plane direction. The horizontally polarized SH wave (5.9) gets decoupled from the rest of motion and propagates without dispersion or damping, and is not affected by thermal variations on the same plate if propagation occurs along an in-plane axis of symmetry. The other three waves namely, quasi-longitudinal (QL), quasi-transverse (QT) and quasi-thermal (T-mode) of the medium are found coupled with each other due to the thermal and anisotropic effects. The phase velocity of the waves get modified due to the thermal and anisotropic effects and is also influenced by the thermal relaxation times.

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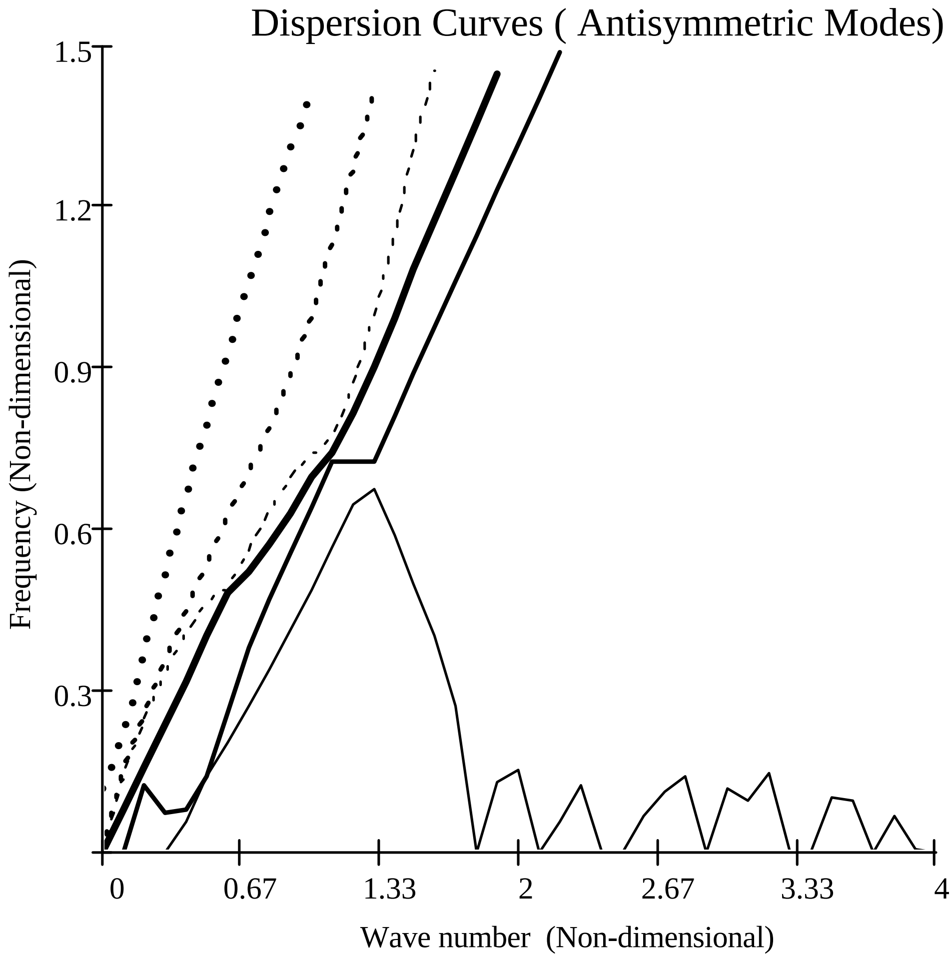

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Fig.1. Dispersion of antisymmetric modes of zinc plate in Generalized thermoelasticity when $t_0 = 2.10^{-7}s$ and $t_1 = 4.10^{-7}s$

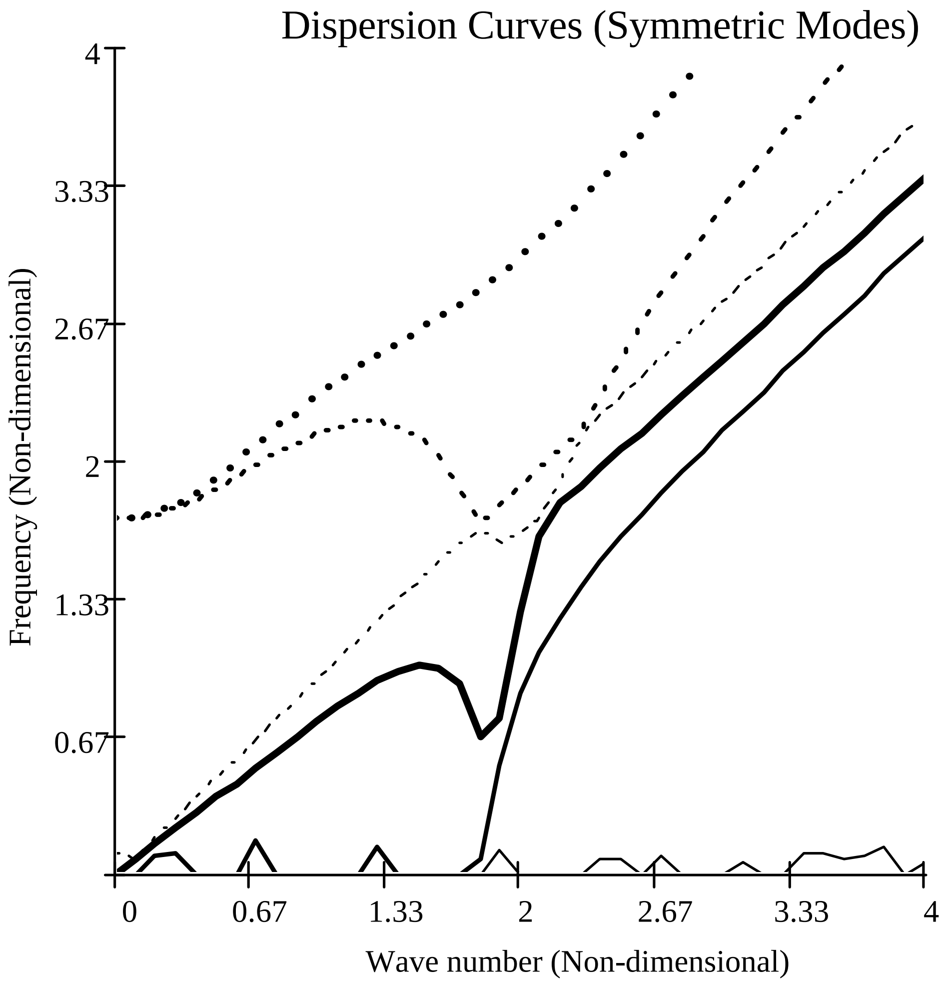

Fig.2. Dispersion of symmetric modes of zinc plate in Generalized thermoelasticity when $\tau_0 = 2.10^{-7} s$ and $\tau_1 = 4.10^{-7} s$

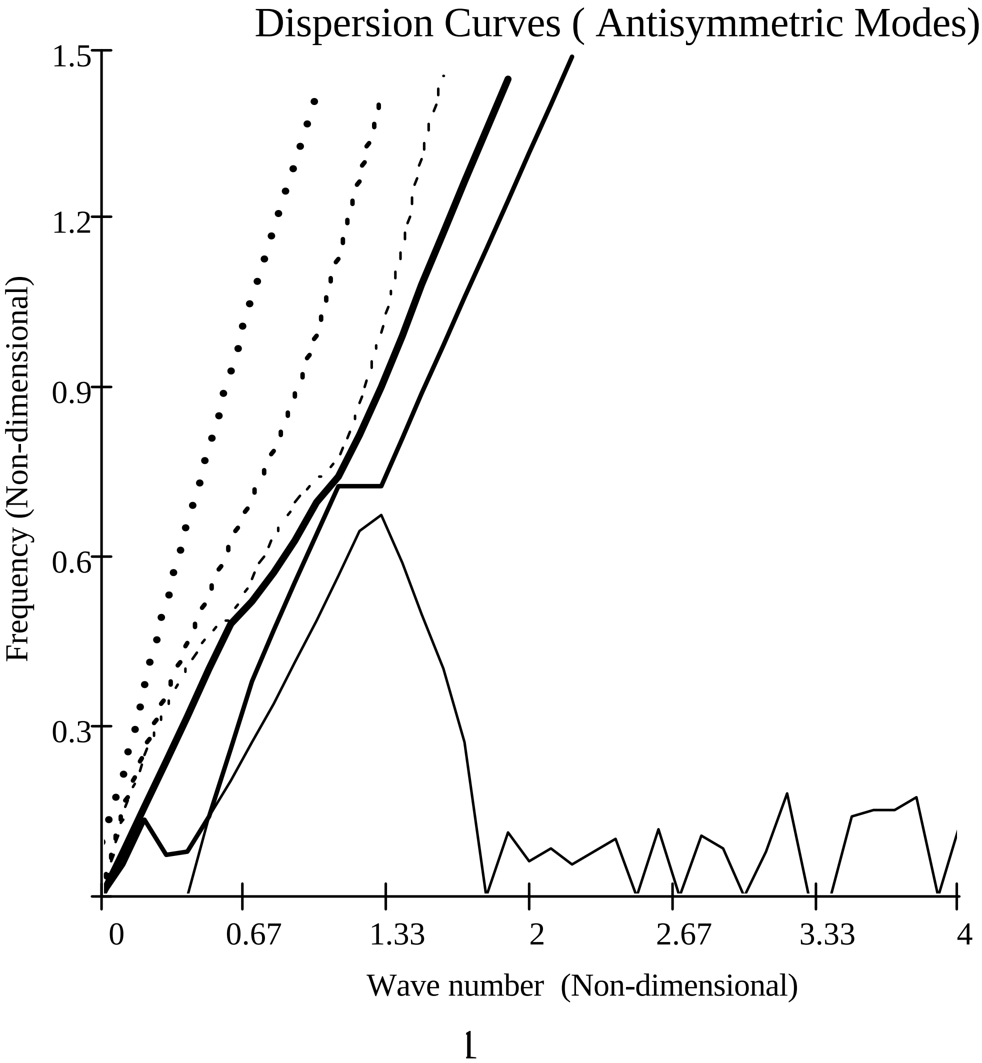

Fig.3. Dispersion of antisymmetric modes of zinc plate in Generalized thermoelasticity when $t_0 = 2.10^{-7}s$ and $t_1 = 10.10^{-7}s$

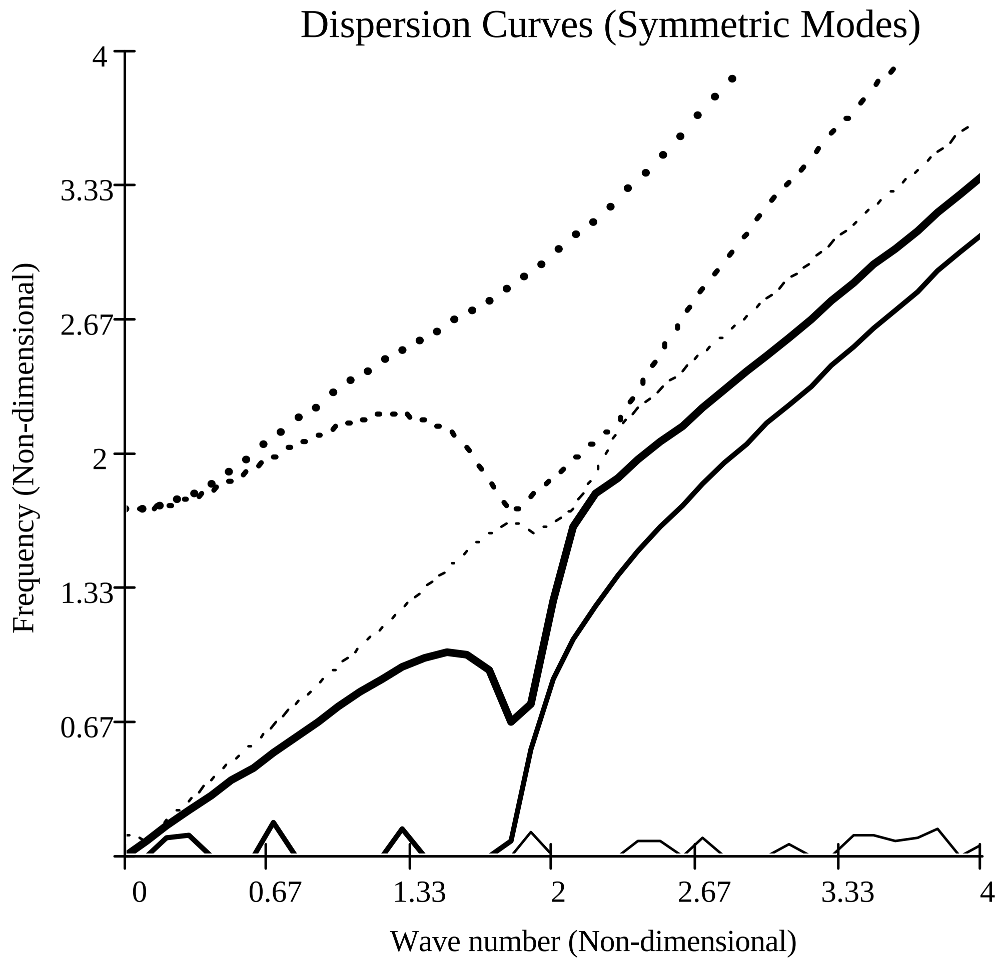

Fig.4. Dispersion of symmetric modes of zinc plate in Generalized thermoelasticity when $\tau_0 = 2.10^{-7} s$ and $\tau_1 = 10.10^{-7} s$

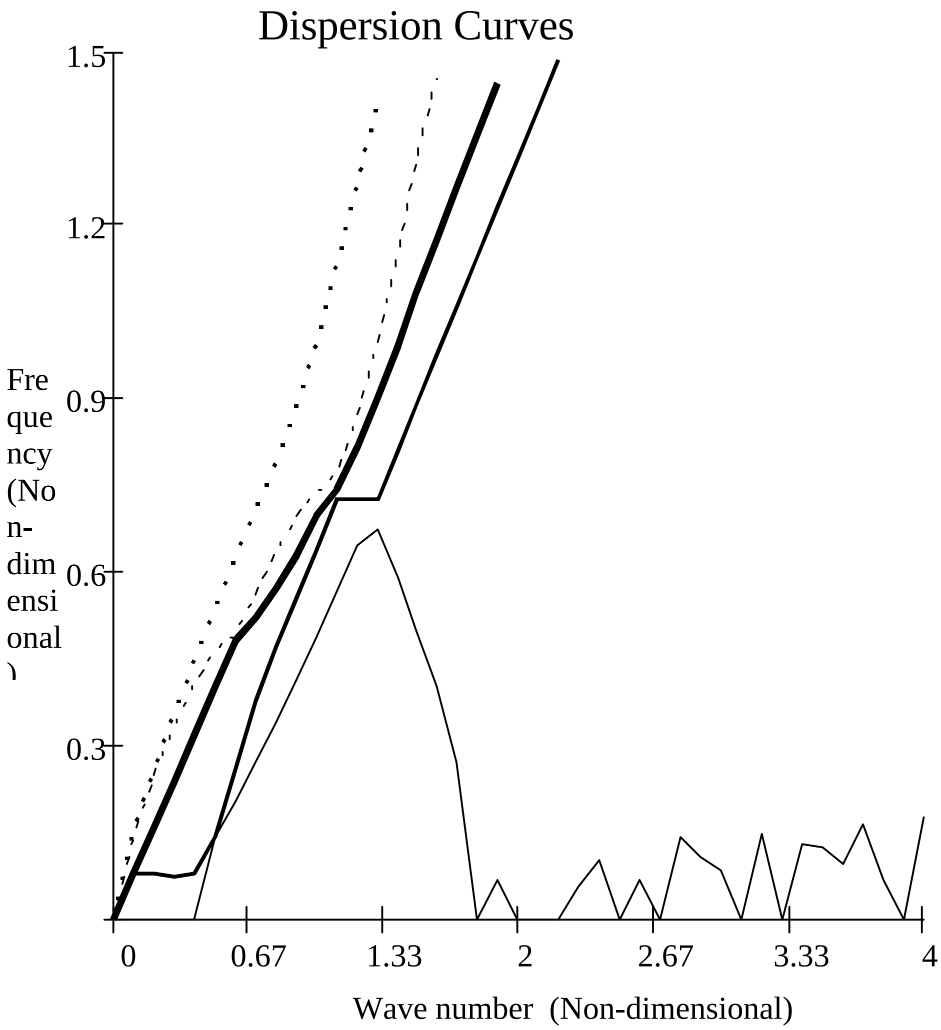

Fig.5. Dispersion of antisymmetric modes of zinc plate in Generalized thermoelasticity when $t_0 = 2.10^{-7}s$ and $t_1 = 2.10^{-6}s$

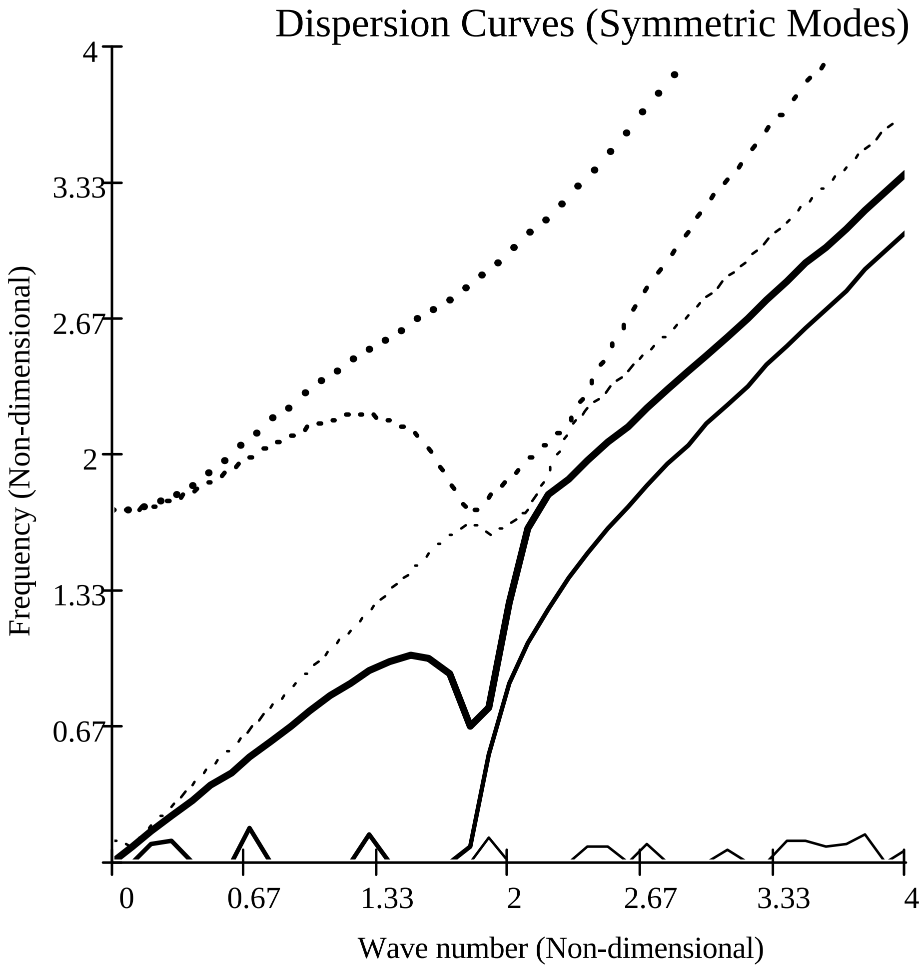

Fig.6. Dispersion of symmetric modes of zinc plate in Generalized thermoelasticity when $t_0 = 2.10^{-7} s$ and $t_1 = 2.10^{-6} s$

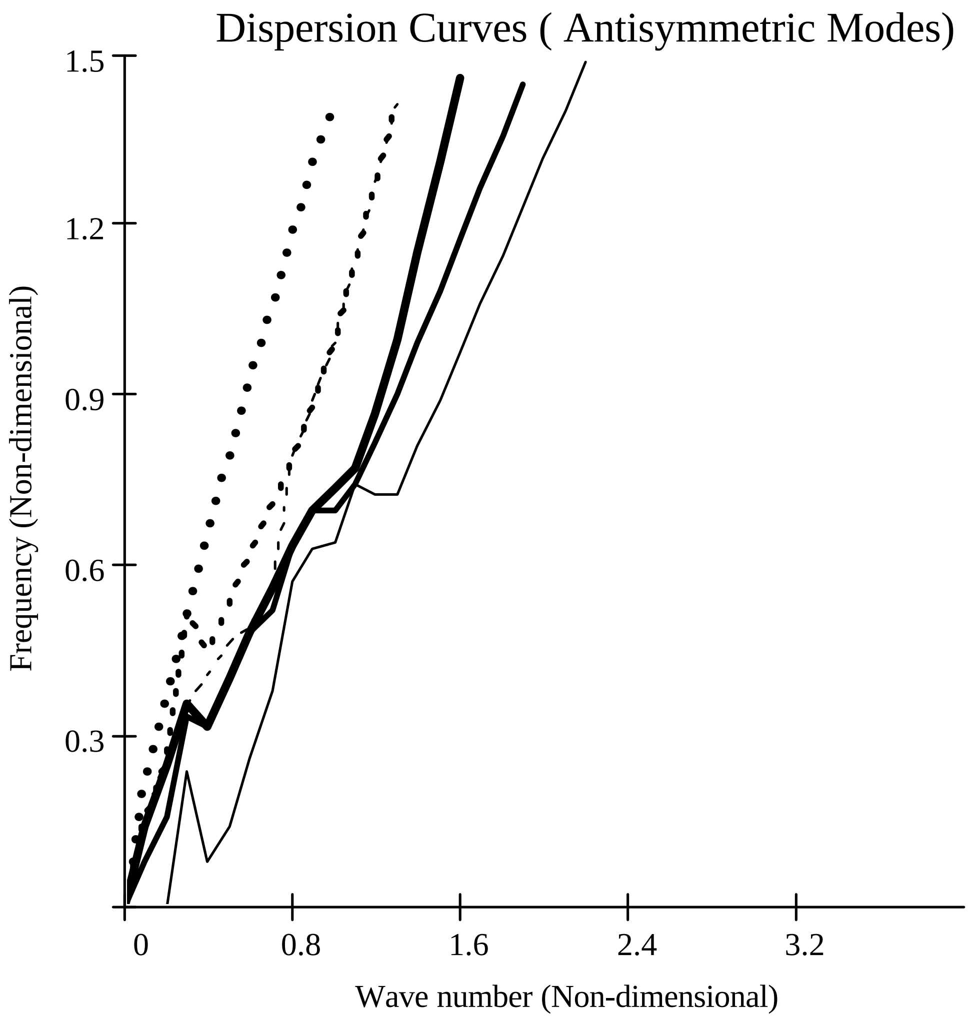

Fig.7. Dispersion of antisymmetric modes of zinc plate in Classical thermoelasticity when $t_0 = 0$ and $t_1 = 0$

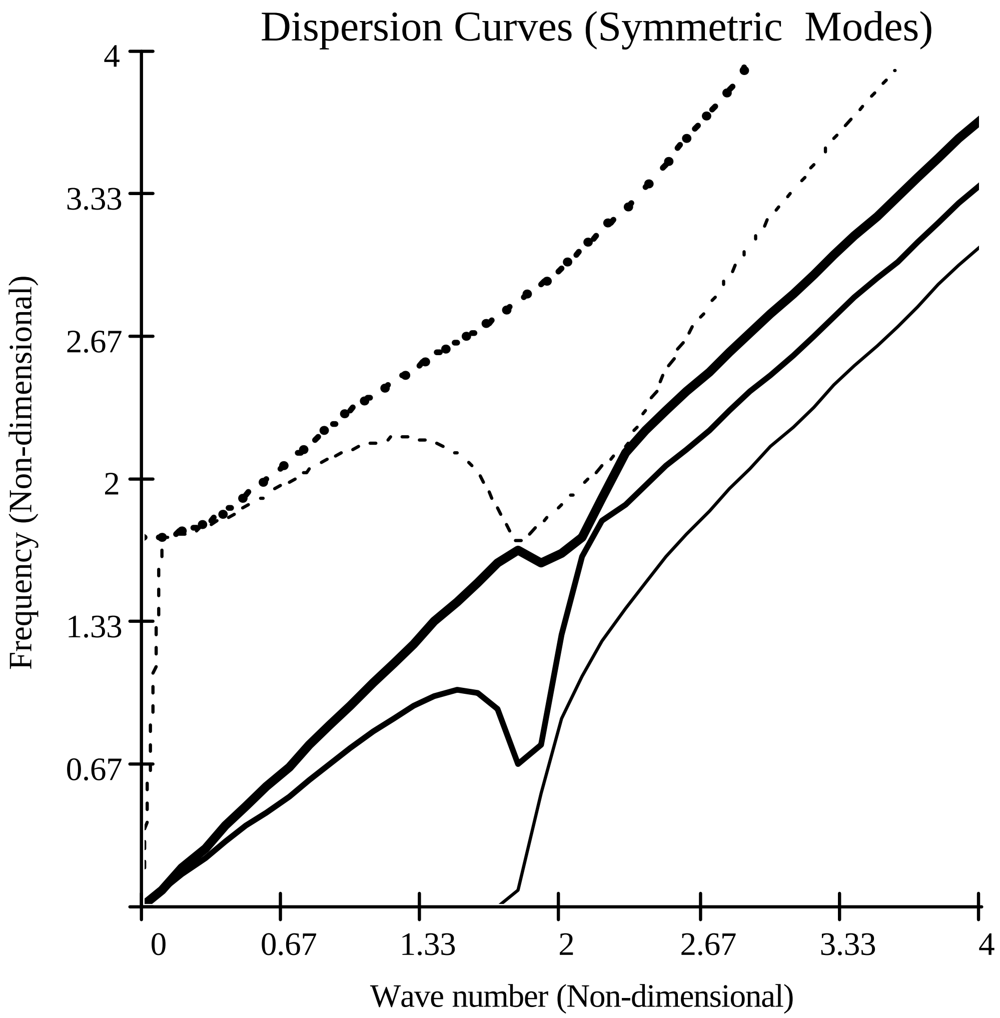

Fig.8. Dispersion of symmetric modes of zinc plate in Classical thermoelasticity when $t_0 = 0$ and $t_1 = 0$

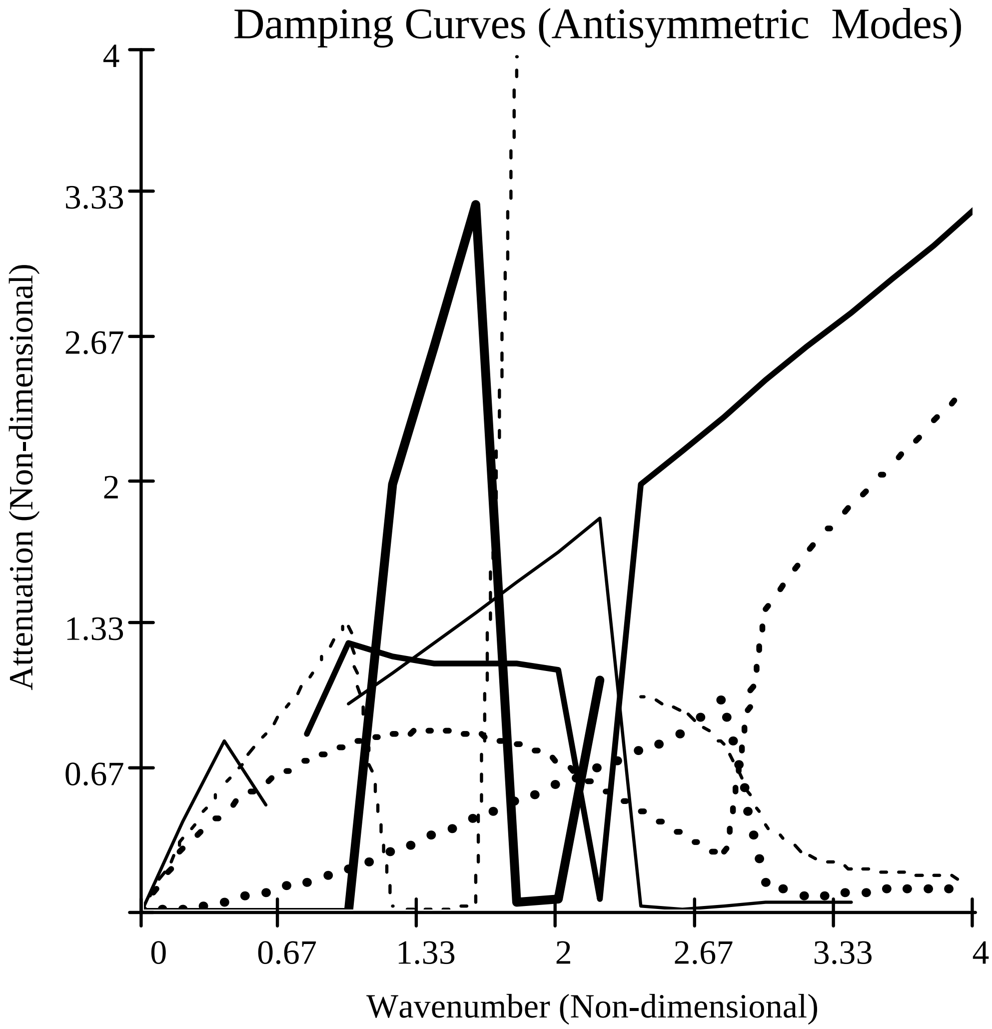

Fig.9. Damping of antisymmetric modes of zinc plate in Generalized thermoelasticity when $\tau_0 = 2.10^{-7} s$ and $\tau_1 = 4.10^{-7} s$

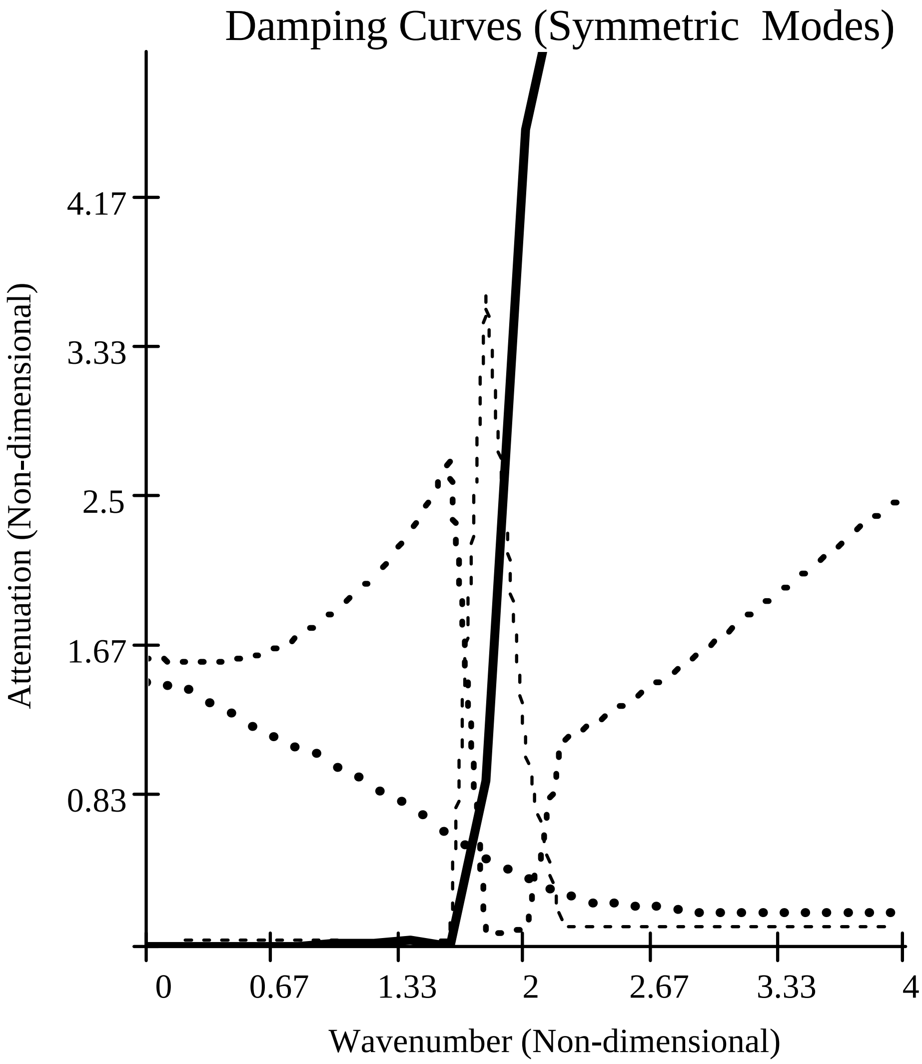

Fig.10. Damping of symmetric modes of zinc plate in Generalized thermoelasticity when $\tau_0 = 2.10^{-7} s$ and $\tau_1 = 4.10^{-7} s$

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Pitman Estimator to Weibull Scale Parameters

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ABSTRACT

The problem of estimation of ordered scale parameters in two Weibull populations is considered while shape parameters are assumed to be known and unequal. The Pitman estimator for scale parameters with different priors and loss functions are obtained and comparison is given between restricted and unrestricted estimators with respect to risks and biases by simulation.

Key words : Scale Parameter ; Estimation; Order Restriction; Pitman Estimator.

1. INTRODUCTION

The distribution has been named after the Swedish scientist, Weibull, who proposed it for the first time in 1939 in connection with his studies on strength of material. Weibull (1951) showed that the distribution is also useful in describing the 'wear-out' or fatigue failures. Kao (1959) used it as a model for ball bearing failures. Mann (1968) gives a variety of situations in which the distribution is used for other type of failure data.

Let $X_{i1}, X_{i2}, ..., X_{in}$, i=1,2 be independent random sample from the Weibull population Π_i. Π_i has the density function as defined in Johnson and Kotz (1970)

$$f(x, \alpha_i, \beta_i) = \frac{\alpha_i}{\beta_i}\left(\frac{x}{\beta_i}\right)^{\alpha_i - 1} \exp\left(-\frac{x}{\beta_i}\right)^{\alpha_i}, \qquad for\ \ x \geq 0,\ \ \beta_i, \alpha_i > 0, \qquad (1.1)$$

where α_i's are shape parameter and β_i 's are scale parameter.

We are interested in estimation of scale parameter β_1 and β_2 of two populations when it is assumed a priori that $\beta_1 \le \beta_2$. This type of problem arises in reliability engineering application for example, computer software usually contains errors called 'bugs'. A common technique of debugging is to turn the program until a bug appears and then correcting the fault and continuing the execution. If errors are corrected in stages without introducing new errors, the reliability of the program increases at each stage. If bugs are removed in two stages and β_i denotes the reliability at the ith stage, i= 1, 2 then we have $\beta_1 \le \beta_2$.

2. PITMAN ESTIMATOR

The Pitman estimators for simultaneous estimation of ordered location parameters for normal populations has been considered by Kumar and Sharma (1988, 1993). In this section we will obtain the analogue of Pitman estimators for scale parameters, which are generalized Bayes estimators.

Consider the estimation of β_1 and β_2, with no ordering on β_1 and β_2 and the parameters α_1 and α_2 are known, then

$$E\left(\sum_{j=1}^{n_i} X_{ij}\right)^{2/\alpha_i} = \beta_i^{\,2}\,\frac{\Gamma(n_i + 2/\alpha_i)}{\Gamma(n_i)} \text{ and } E\left(\sum_{j=1}^{n_i} X_{ij}\right)^{1/\alpha_i} = \beta_i\,\frac{\Gamma(n_i + 1/\alpha_i)}{\Gamma(n_i)}, \text{ for } i = 1,2$$

Let $\Theta_1 = \{(\beta_1,\beta_2) : \beta_1,\beta_2 > 0\}$ and $\Theta_2 = \{(\beta_1,\beta_2) : \beta_1 \le \beta_2\}$, in this section we will obtain the generalized Bayes estimators for scale parameters (β_1,β_2), with respect to squared error loss function: $L_1(d,\beta) = (d-\beta)^2$ and scale invariant squared error loss function: $L_2(d,\beta) = (d/\beta - 1)^2$, and prior distributions

$$g_1(\beta_1,\beta_2) = 1;\quad \forall\,(\beta_1,\beta_2)\in\Theta_1,\quad g_2(\beta_1,\beta_2) = \frac{1}{\beta_1\beta_2};\quad \forall\,(\beta_1,\beta_2)\in\Theta_1,\quad g_3(\beta_1,\beta_2) = 1;$$

$$\forall\,(\beta_1,\beta_2)\in\Theta_2 \text{ and } g_4(\beta_1,\beta_2) = \frac{1}{\beta_1\beta_2};\quad \forall\,(\beta_1,\beta_2)\in\Theta_2$$

The joint probability density function of **x** given β_1,β_2 from (1.1) is

$$f(\mathbf{x}/\beta_1,\beta_2) = \prod_{i=1}^{2}\prod_{j=1}^{n_i}\frac{\alpha_i}{\beta_i}\left(\frac{x_{ij}}{\beta_i}\right)^{\alpha_i-1}\exp\left(-\frac{x_{ij}}{\beta_i}\right)^{\alpha_i}$$

The biases and risks used for comparison are:

Absolute bias: $B_1(d,\beta) = E(d-\beta)$, Standardized bias: $B_2(d,\beta) = E(d/\beta - 1)$, Risk functions: $R_1(d,\beta) = E(d-\beta)^2$ and $R_2(d,\beta) = E(d/\beta - 1)^2$.

(A) Pitman Estimator with No Ordering

Here, the parameter space of interest is Θ_1 i.e., no ordering on β_1 and β_2, with respect to the loss function L_1, the Pitman estimators for β_1 and β_2 with prior g_1 are

$$d_{11}(\mathbf{x}) = \frac{\Gamma(n_1 - 2/\alpha_1)\left(\sum_{j=1}^{n_1} X_{1j}^{\alpha_1}\right)^{1/\alpha_1}}{\Gamma(n_1 - 1/\alpha_1)} \text{ and } d_{12}(\mathbf{x}) = \frac{\Gamma(n_2 - 2/\alpha_2)\left(\sum_{j=1}^{n_2} X_{2j}^{\alpha_2}\right)^{1/\alpha_2}}{\Gamma(n_2 - 1/\alpha_2)}$$

with prior g_2

$$d_{21}(\mathbf{x}) = \frac{\Gamma(n_1 - 1/\alpha_1)\left(\sum_{j=1}^{n_1} X_{1j}^{\alpha_1}\right)^{1/\alpha_1}}{\Gamma(n_1)} \text{ and } d_{22}(\mathbf{x}) = \frac{\Gamma(n_2 - 1/\alpha_2)\left(\sum_{j=1}^{n_2} X_{2j}^{\alpha_2}\right)^{1/\alpha_2}}{\Gamma(n_2)}.$$

Now consider the loss function L_2, then Generalized Bayes estimators for β_1 and β_2 with prior g_1 are

$$d_{31}(\mathbf{x}) = \frac{\Gamma(n_1)\left(\sum_{j=1}^{n_1} X_{1j}^{\alpha_1}\right)^{1/\alpha_1}}{\Gamma(n_1 + 1/\alpha_1)} \text{ and } d_{32}(\mathbf{x}) = \frac{\Gamma(n_2)\left(\sum_{j=1}^{n_2} X_{2j}^{\alpha_2}\right)^{1/\alpha_2}}{\Gamma(n_2 + 1/\alpha_2)}$$

with prior g_2

$$d_{41}(\mathbf{x}) = \frac{\Gamma(n_1 + 1/\alpha_1)\left(\sum_{j=1}^{n_1} X_{1j}^{\alpha_1}\right)^{1/\alpha_1}}{\Gamma(n_1 + 2/\alpha_1)} \text{ and } d_{42}(\mathbf{x}) = \frac{\Gamma(n_2 + 1/\alpha_2)\left(\sum_{j=1}^{n_2} X_{2j}^{\alpha_2}\right)^{1/\alpha_2}}{\Gamma(n_2 + 2/\alpha_2)}$$

Now we will calculate the risks and biases of Pitman estimators obtained so far.

Theorem 2.1.

(i) For estimators d_{11} and d_{12}

$$R_1(d_{1i},\beta_i)=\beta_i^2\left[1+\frac{\{\Gamma(n_i-2/\alpha_i)\}^2\Gamma(n_i+2/\alpha_i)}{\Gamma(n_i)\{\Gamma(n_i-1/\alpha_i)\}^2}-2\frac{\Gamma(n_i-2/\alpha_i)\Gamma(n_i+1/\alpha_i)}{\Gamma(n_i-1/\alpha_i)\Gamma(n_i)}\right],i=1,2$$

(ii) For estimators d_{21} and d_{22}

$$R_1(d_{2i},\beta_i)=\beta_i^2\left[\frac{\{\Gamma(n_i-1/\alpha_i)\}^2\Gamma(n_i+2/\alpha_i)}{\{\Gamma(n_i)\}^3}+1-2\frac{\Gamma(n_i-1/\alpha_i)\Gamma(n_i+1/\alpha_i)}{\{\Gamma(n_i)\}^2}\right],i=1,2$$

(iii) For estimators d_{31} and d_{32}

$$R_2(d_{3i},\beta_i)=\frac{\Gamma(n_i)\Gamma(n_i+2/\alpha_i)}{\{\Gamma(n_i+1/\alpha_i)\}^2}-1,\ i=1,2$$

(iv) For estimators d_{41} and d_{42}

$$R_2(d_{4i},\beta_i)=1-\frac{\{\Gamma(n_i+1/\alpha_i)\}^2}{\Gamma(n_i+2/\alpha_i)\Gamma(n_i)},\ i=1,2$$

Proof:

(i) $\mathrm{R}_1(d_{1\mathrm{i}},\beta_i)=\mathrm{E}\left(d_{1\mathrm{i}}(\mathbf{x})-\beta_{\mathrm{i}}\right)^2=E(d_{1\mathrm{i}}(\mathbf{x}))^2+\beta_{\mathrm{i}}^2-2\beta_{\mathrm{i}}E(d_{1\mathrm{i}}(\mathbf{x}))$

$$=\frac{\{\Gamma(n_i-2/\alpha_i)\}^2}{\{\Gamma(n_i-1/\alpha_i)\}^2}E\left(\sum X_{ij}^{\alpha_i}\right)^{2/\alpha_i}+\beta_{\mathrm{i}}^2-2\beta_{\mathrm{i}}\frac{\Gamma(n_i-2/\alpha_i)}{\Gamma(n_i-1/\alpha_i)}E\left(\sum X_{ij}^{\alpha_i}\right)^{1/\alpha_i}$$

$$=\beta_{\mathrm{i}}^2\left[\frac{\{\Gamma(n_i-2/\alpha_i)\}^2}{\{\Gamma(n_i-1/\alpha_i)\}^2}\frac{\Gamma(n_i+2/\alpha_i)}{\Gamma(n_i)}+1-2\frac{\Gamma(n_i-2/\alpha_i)}{\Gamma(n_i-1/\alpha_i)}\frac{\Gamma(n_i+1/\alpha_i)}{\Gamma(n_i)}\right]$$

(ii) $\mathrm{R}_1(d_{2\mathrm{i}},\beta_i)=\mathrm{E}\left(d_{2\mathrm{i}}(\mathbf{x})-\beta_{\mathrm{i}}\right)^2=E(d_{2\mathrm{i}}(\mathbf{x}))^2+\beta_{\mathrm{i}}^2-2\beta_{\mathrm{i}}E(d_{2\mathrm{i}}(\mathbf{x}))$

$$=\frac{\{\Gamma(n_i-1/\alpha_i)\}^2}{\{\Gamma(n_i)\}^2}E\left(\sum X_{ij}^{\alpha_i}\right)^{2/\alpha_i}+\beta_{\mathrm{i}}^2-2\beta_{\mathrm{i}}\frac{\Gamma(n_i-1/\alpha_i)}{\Gamma(n_i)}E\left(\sum X_{ij}^{\alpha_i}\right)^{1/\alpha_i}$$

$$=\beta_i^{\,2}\left[\frac{\{\Gamma(n_i-1/\alpha_i)\}^2}{\{\Gamma(n_i)\}^2}\frac{\Gamma(n_i+2/\alpha_i)}{\Gamma(n_i)}+1-2\frac{\Gamma(n_i-1/\alpha_i)\Gamma(n_i+1/\alpha_i)}{[\Gamma(n_i)]^2}\right]$$

(iii) $$R_2(d_{3i},\beta_i)=E\left(\frac{d_{3i}(\mathbf{x})}{\beta_i}-1\right)^2=\frac{1}{\beta_i^{\,2}}E(d_{3i}(\mathbf{x}))^2+1-\frac{2}{\beta_i}E(d_{3i}(\mathbf{x}))$$

$$=\frac{1}{\beta_i^{\,2}}\frac{\{\Gamma(n_i)\}^2}{\{\Gamma(n_i+1/\alpha_i)\}^2}E\left(\sum X_{ij}^{\alpha_i}\right)^{2/\alpha_i}+1-\frac{2}{\beta_i}\frac{\Gamma(n_i)}{\Gamma(n_i+1/\alpha_i)}E\left(\sum X_{ij}^{\alpha_i}\right)^{1/\alpha_i}$$

$$=\frac{\Gamma(n_i)\Gamma(n_i+2/\alpha_i)}{\{\Gamma(n_i+1/\alpha_i)\}^2}-1$$

(iv) $$R_2(d_{4i},\beta_i)=E\left(\frac{d_{4i}(\mathbf{x})}{\beta_i}-1\right)^2=\frac{1}{\beta_i^{\,2}}E(d_{4i}(\mathbf{x}))^2+1-\frac{2}{\beta_i}E(d_{4i}(\mathbf{x}))$$

$$=\frac{1}{\beta_i^{\,2}}\frac{\{\Gamma(n_i+1/\alpha_i)\}^2}{\{\Gamma(n_i+2/\alpha_i)\}^2}E\left(\sum X_{ij}^{\alpha_i}\right)^{2/\alpha_i}+1-\frac{2}{\beta_i}\frac{\Gamma(n_i+1/\alpha_i)}{\Gamma(n_i+2/\alpha_i)}E\left(\sum X_{ij}^{\alpha_i}\right)^{1/\alpha_i}$$

$$=1-\frac{\{\Gamma(n_i+1/\alpha_i)\}^2}{\Gamma(n_i+2/\alpha_i)\Gamma(n_i)}$$

Theorem 2.2.

(i) For estimators d_{11} and d_{12}, $B_1(d_{1i},\beta_i)=\beta_i\left[\dfrac{\Gamma(n_i-2/\alpha_i)\Gamma(n_i+1/\alpha_i)}{\Gamma(n_i-1/\alpha_i)\Gamma(n_i)}-1\right]$, $i=1,2$

(ii) For estimators d_{21} and d_{22}, $B_1(d_{2i},\beta_i)=\beta_i\left[\dfrac{\Gamma(n_i-1/\alpha_i)\Gamma(n_i+1/\alpha_i)}{\{\Gamma(n_i)\}^2}-1\right]$, $i=1,2$

(iii) For estimators d_{31} and d_{32}, $B_2(d_{3i},\beta_i)=0$, $i=1,2$

(iv) For estimators d_{41} and d_{42}, $B_2(d_{4i},\beta_i)=\dfrac{\{\Gamma(n_i+1/\alpha_i)\}^2}{\Gamma(n_i)\Gamma(n_i+2/\alpha_i)}-1$, $i=1,2$

Proof:

(i) $$B_1(d_{1i},\beta_i)=\left|E(d_{1i}-\beta_i)\right|=\left|E\left(\frac{\left(\sum X_{ij}^{\alpha_i}\right)^{1/\alpha_i}\Gamma(n_i-2/\alpha_i)}{\Gamma(n_i-1/\alpha_i)}-\beta_i\right)\right|$$

$$=\beta_i\left[\frac{\Gamma(n_i-2/\alpha_i)}{\Gamma(n_i-1/\alpha_i)}\frac{\Gamma(n_i+1/\alpha_i)}{\Gamma(n_i)}-1\right]$$

(ii) $$B_1(d_{2i},\beta_i)=\left|E(d_{2i}-\beta_i)\right|=\left|E\left(\frac{\left(\sum X_{ij}^{\alpha_i}\right)^{1/\alpha_i}\Gamma(n_i-1/\alpha_i)}{\Gamma(n_i)}-\beta_i\right)\right|$$

$$=\beta_i\left[\frac{\Gamma(n_i-1/\alpha_i)\Gamma(n_i+1/\alpha_i)}{\{\Gamma(n_i)\}^2}-1\right]$$

(iii) $$B_2(d_{3i},\beta_i)=\left|E\left(\frac{d_{3i}}{\beta_i}-1\right)\right|=\left|E\left(\frac{1}{\beta_i}\frac{\left(\sum X_{ij}^{\alpha_i}\right)^{1/\alpha_i}\Gamma(n_i)}{\Gamma(n_i+1/\alpha_i)}-1\right)\right|$$

$$=\frac{\Gamma(n_i)}{\beta_i\Gamma(n_i+1/\alpha_i)}\frac{\beta_i\Gamma(n_i+1/\alpha_i)}{\Gamma(n_i)}\beta_i-1=0$$

(iv) $$B_2(d_{4i},\beta_i)=\left|E\left(\frac{d_{4i}}{\beta_i}-1\right)\right|=\left|E\left(\frac{\left(\sum X_{ij}^{\alpha_i}\right)^{1/\alpha_i}\Gamma(n_i+1/\alpha_i)}{\beta_i\Gamma(n_i+2/\alpha_i)}-1\right)\right|=\frac{\{\Gamma(n_i+1/\alpha_i)\}^2}{\Gamma(n_i)\Gamma(n_i+2/\alpha_i)}-1$$

(B) Pitman Estimator with Ordering

Now, consider the parameter space of interest to be Θ_2, i.e., ordering $(\beta_1 \le \beta_2)$ is assumed on parameters.

Let the loss function be L_1. The generalized Bayes estimators for β_i with prior g_3 are

$$\delta_{1i}(\mathbf{x}) = \frac{\iint_{\Theta_2} \beta_i f(\mathbf{x}/\beta_1\beta_2)g(\beta_1,\beta_2)d\beta_1 d\beta_2}{\iint_{\Theta_2} f(\mathbf{x}/\beta_1\beta_2)g(\beta_1,\beta_2)d\beta_1 d\beta_2} \quad i=1,2$$

hence

$$\delta_{11}(\mathbf{x}) = \frac{Z_1^{1/\alpha_1}\int_0^\infty\int_0^{Z_2(t/Z_1)^{\alpha_2/\alpha_1}} e^{-t}t^{n_1-2/\alpha_1-1}e^{-u}u^{n_2-1/\alpha_2-1}du\,dt}{\int_0^\infty\int_0^{Z_2(t/Z_1)^{\alpha_2/\alpha_1}} e^{-t}t^{n_1-1/\alpha_1-1}e^{-u}u^{n_2-1/\alpha_2-1}du\,dt}$$

and

$$\delta_{12}(\mathbf{x}) = \frac{Z_2^{1/\alpha_2}\int_0^\infty\int_0^{Z_2(t/Z_1)^{\alpha_2/\alpha_1}} e^{-t}t^{n_1-1/\alpha_1-1}e^{-u}u^{n_2-2/\alpha_2-1}du\,dt}{\int_0^\infty\int_0^{Z_2(t/Z_1)^{\alpha_2/\alpha_1}} e^{-t}t^{n_1-1/\alpha_1-1}e^{-u}u^{n_2-1/\alpha_2-1}du\,dt}$$

with prior g_4

$$\delta_{2i}(\mathbf{x}) = \frac{\iint_{\Theta_2} \beta_i f(\mathbf{x}/\beta_1\beta_2)g(\beta_1,\beta_2)d\beta_1 d\beta_2}{\iint_{\Theta_2} f(\mathbf{x}/\beta_1\beta_2)g(\beta_1,\beta_2)d\beta_1 d\beta_2} \quad i=1,2$$

hence $\delta_{21}(\mathbf{x}) = \frac{Z_1^{1/\alpha_1}\int_0^\infty\int_0^{Z_2(t/Z_1)^{\alpha_2/\alpha_1}} e^{-t}t^{n_1-1/\alpha_1-1}e^{-u}u^{n_2-1}du\,dt}{\int_0^\infty\int_0^{Z_2(t/Z_1)^{\alpha_2/\alpha_1}} e^{-t}t^{n_1-1}e^{-u}u^{n_2-1}du\,dt}$

and $\delta_{22}(\mathbf{x}) = \frac{Z_2^{1/\alpha_2}\int_0^\infty\int_0^{Z_2(t/Z_1)^{\alpha_2/\alpha_1}} e^{-t}t^{n_1-1}e^{-u}u^{n_2-1/\alpha_2-1}du\,dt}{\int_0^\infty\int_0^{Z_2(t/Z_1)^{\alpha_2/\alpha_1}} e^{-t}t^{n_1-1}e^{-u}u^{n_2-1}du\,dt}$

Now consider the loss function L_2,

The Generalized Bayes estimators for β_i with prior g_3

$$\delta_{3i}(\mathbf{x}) = \frac{\iint_{\Theta_2} \frac{1}{\beta_i} f(\mathbf{x}/\beta_1\beta_2)g(\beta_1,\beta_2)d\beta_1 d\beta_2}{\iint_{\Theta_2} \frac{1}{\beta_i^2} f(\mathbf{x}/\beta_1\beta_2)g(\beta_1,\beta_2)d\beta_1 d\beta_2},\ i=1,2$$

hence

$$\delta_{31}(\mathbf{x}) = \frac{Z_1^{1/\alpha_1}\int_0^{\infty}\int_0^{Z_2(t/Z_1)^{\alpha_2/\alpha_1}} e^{-t}t^{n_1-1}e^{-u}u^{n_2-1/\alpha_2-1}du\,dt}{\int_0^{\infty}\int_0^{Z_2(t/Z_1)^{\alpha_2/\alpha_1}} e^{-t}t^{n_1+1/\alpha_1-1}e^{-u}u^{n_2-1/\alpha_2-1}du\,dt}$$

and $$\delta_{32}(\mathbf{x}) = \frac{Z_2^{1/\alpha_2}\int_0^{\infty}\int_0^{Z_2(t/Z_1)^{\alpha_2/\alpha_1}} e^{-t}t^{n_1-1/\alpha_1-1}e^{-u}u^{n_2-1}du\,dt}{\int_0^{\infty}\int_0^{Z_2(t/Z_1)^{\alpha_2/\alpha_1}} e^{-t}t^{n_1-1/\alpha_1-1}e^{-u}u^{n_2+1/\alpha_2-1}du\,dt}$$

with prior g_4

$$\delta_{4i}(\mathbf{x}) = \frac{\iint_{\Theta_2}\frac{1}{\beta_i} f(\mathbf{x}/\beta_1\beta_2)g(\beta_1,\beta_2)d\beta_1 d\beta_2}{\iint_{\Theta_2}\frac{1}{\beta_i^2} f(\mathbf{x}/\beta_1\beta_2)g(\beta_1,\beta_2)d\beta_1 d\beta_2},\ i=1,2$$

$$\delta_{41}(\mathbf{x}) = \frac{Z_1^{1/\alpha_1}\int_0^{\infty}\int_0^{Z_2(t/Z_1)^{\alpha_2/\alpha_1}} e^{-t}t^{n_1+1/\alpha_1-1}e^{-u}u^{n_2-1}du\,dt}{\int_0^{\infty}\int_0^{Z_2(t/Z_1)^{\alpha_2/\alpha_1}} e^{-t}t^{n_1+2/\alpha_1-1}e^{-u}u^{n_2-1}du\,dt}$$

and $$\delta_{42}(\mathbf{x}) = \frac{Z_2^{1/\alpha_2}\int_0^{\infty}\int_0^{Z_2(t/Z_1)^{\alpha_2/\alpha_1}} e^{-t}t^{n_1-1}e^{-u}u^{n_2+1/\alpha_2-1}du\,dt}{\int_0^{\infty}\int_0^{Z_2(t/Z_1)^{\alpha_2/\alpha_1}} e^{-t}t^{n_1-1}e^{-u}u^{n_2+2/\alpha_2-1}du\,dt}$$

All these estimators are involving incomplete gamma integrals hence can not be given in closed form, but these can be obtained by numerical integrations for given samples.

The risks and biases for these estimators δ_{ij}, $j=1,2, i=1,2,3,4$ can be defined as follows:

$$\mathrm{R}_k(\delta_{jk},\beta_k) = E(\delta_{jk}-\beta_k)^2 \text{ and } \mathrm{R}_k(\delta_{jk},\beta_k) = E\left(\frac{\delta_{jk}-\beta_k}{\beta_k^{\,i-1}}\right)^2 \quad \text{for } i=1,2;\ j=1,2,3,4;\ k=1,2,$$

$$\mathrm{B}_i(\delta_{jk},\beta_k) = \left|\mathrm{E}(\delta_{jk}-\beta_k)\right| \text{ and } \mathrm{B}_i(\delta_{jk},\beta_k) = \left|E\left(\frac{\delta_{jk}-\beta_k}{\beta_k}\right)\right| \text{ for } i=1,2;\ j=1,2,3,4;\ k=1,2,$$

Since estimators can be obtained only by numerical integration. So the risks and biases can be simulated only. Thus the comparison of these estimators is done only through simulation.

3. COMPARISON OF PITMAN ESTIMATORS

In this section a comparison is done between Pitman estimators, obtained in section 3 and in section 4 with respect to biases and risks. In case of Pitman estimators with ordering, the estimators were not obtainable in close form as they have incomplete gamma functions. So first these estimators were obtained by numerical integration. Then these numerically obtained estimators are compared with Pitman estimators with no ordering.

For the parameter sets a_1 = 3.0, a_2 = 3.5, b_1 = 1.0, b_2 = 1.5; the samples of different sizes are generated, these are replicated 50 times each and then again complete set is repeated 3 times. The sample sizes taken are as n_1= 5, n_2= 10, n_1= 10, n_2= 15, n_1= n_2=15. The risk R_1 and bias B_1 are simulated for d_{ji} and d_{ji} for $i, j = 1,2$. These results are given in table 3.1 and 3.2, which clearly show that Pitman estimators with ordering are better than Pitman estimators with no ordering. Further the risk R_2 and bias B_2 are simulated for d_{ji} and d_{ji} $i = 1,2,\ j = 3,4,$ and tabulated in Tables 3.3 and 3.4, which also clearly shows the same results that Pitman estimators with ordering improves over pitman estimators with no ordering.

The same exercise is further repeated new sets of parameters a_1 = 5.0, a_2 = 5.5, b_1 = 3.0, b_2 = 3.2 and a_1 = 3.0, a_2 = 3.5, b_1 = 5.0, b_2 = 5.2, the biases and risks for all estimators are tabulated in Tables 3.5 to 3.12.

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TABLE 3.1.

COMPARISON OF B_1 AND R_1FOR $d_{1i}(x)$ WITH $\delta_{1i}(x)$

β_1	β_2	α_1	α_2	n_1	n_2	$B_1(d_{11})$	$B_1(\delta_{11})$	$B_1(d_{12})$	$B_1(\delta_{12})$
1.0	1.2	3.0	3.5	5	10	0.686233	0.078609	1.072306	0.080351
						0.703567	0.086024	1.089167	0.103908
						0.684418	0.08286	1.06855	0.076724
1.0	1.2	3.0	3.5	10	15	1.160178	0.074397	0.664209	0.115959
						1.124014	0.079258	0.663111	0.148108
						1.084296	0.077646	0.594237	0.118536
1.0	1.2	3.0	3.5	15	15	1.414431	0.060328	1.377604	0.054353
						1.420226	0.051073	1.350952	0.072925
						1.42971	0.056697	1.377365	0.067411

β_1	β_2	α_1	α_2	n_1	n_2	$R_1(d_{11})$	$R_1(\delta_{11})$	$R_1(d_{12})$	$R_1(\delta_{12})$
1.0	1.2	3.0	3.5	5	10	0.533232	0.010312	1.178706	0.009536
						0.559314	0.010706	1.244093	0.016024
						0.550936	0.010455	1.173263	0.008757
1.0	1.2	3.0	3.5	5	10	1.288904	0.011088	0.466951	0.023249
						1.251385	0.007441	0.456391	0.017149
						1.197772	0.012348	0.444404	0.020456
1.0	1.2	3.0	3.5	5	10	2.233822	0.006027	1.954132	0.006013
						2.050657	0.004212	1.872917	0.008805
						2.080766	0.007318	1.912709	0.005667

TABLE 3.2.

COMPARISON OF B_1 AND R_1 FOR $d_{2i}(x)$ WITH $\delta_{2i}(x)$

β_1	β_2	α_1	α_2	n_1	n_2	$B_1(d_{21})$	$B_1(\delta_{21})$	$B_1(d_{22})$	$B_1(\delta_{22})$
1.0	1.2	3.0	3.5	5	10	0.578442	0.083158	0.947522	0.075319
						0.583073	0.091098	0.94929	0.097583
						0.585961	0.081412	0.946143	0.097566
1.0	1.2	3.0	3.5	10	15	0.763542	0.077334	0.73584	0.129383
						0.763998	0.078617	0.73557	0.135995
						0.768412	0.078613	0.751771	0.108461
1.0	1.2	3.0	3.5	15	15	0.823062	0.065356	1.01518	0.058933
						0.829006	0.075667	1.013602	0.070985
						0.827165	0.064701	1.016456	0.049949

β_1	β_2	α_1	α_2	n_1	n_2	$R_1(d_{21})$	$R_1(\delta_{21})$	$R_1(d_{22})$	$R_1(\delta_{22})$
1.0	1.2	3.0	3.5	5	10	0.338489	0.011206	0.898154	0.008465
						0.333619	0.010931	0.894964	0.014559
						0.340277	0.011731	0.898977	0.007737
1.0	1.2	3.0	3.5	5	10	0.594473	0.012939	0.556707	0.017196
						0.583652	0.009228	0.545965	0.025863
						0.584485	0.011712	0.546303	0.029669
1.0	1.2	3.0	3.5	5	10	0.677696	0.005962	1.030738	0.005577
						0.681388	0.004635	1.036708	0.003871
						0.683097	0.005712	1.031992	0.007036

TABLE 3.3

COMPARISON OF B_2 AND R_2FOR $d_{3i}(x)$ WITH $\delta_{3i}(x)$

β_1	β_2	α_1	α_2	n_1	n_2	$B_2(d_{31})$	$B_2(\delta_{31})$	$B_2(d_{32})$	$B_2(\delta_{32})$
1.0	1.2	3.0	3.5	5	10	0.686233	0.08967	0.893588	0.061391
						0.703567	0.086056	0.907639	0.082643
						0.684418	0.091607	0.890458	0.055323
1.0	1.2	3.0	3.5	10	15	1.147594	0.085392	0.525997	0.081072
						1.115354	0.078299	0.579485	0.109586
						1.093719	0.08794	0.524648	0.083038
1.0	1.2	3.0	3.5	15	15	1.47713	0.065946	1.156239	0.047776
						1.445315	0.058163	1.121878	0.041259
						1.414431	0.063346	1.148003	0.042685

β_1	β_2	α_1	α_2	n_1	n_2	$R_2(d_{31})$	$R_2(\delta_{31})$	$R_2(d_{32})$	$R_2(\delta_{32})$
1.0	1.2	3.0	3.5	5	10	0.457144	0.023253	0.832293	0.006715
						0.492683	0.012312	0.951865	0.008291
						0.501973	0.015391	0.957209	0.008645
1.0	1.2	3.0	3.5	5	10	1.392901	0.007629	0.337858	0.009691
						1.384804	0.011152	0.306408	0.009378
						1.293032	0.009441	0.392868	0.018211
1.0	1.2	3.0	3.5	5	10	2.058945	0.006192	1.31844	0.003116
						2.086975	0.005936	1.345531	0.004697
						2.080766	0.007796	1.32827	0.003595

TABLE 3.4

COMPARISON OF B_2 AND R_2FOR $d_{4i}(x)$ WITH $\delta_{4i}(x)$

β_1	β_2	α_1	α_2	n_1	n_2	$B_2(d_{41})$	$B_2(\delta_{41})$	$B_2(d_{42})$	$B_2(\delta_{42})$
1.0	1.2	3.0	3.5	5	10	0.686233	0.099578	0.893588	0.059042
						0.703567	0.092394	0.907639	0.081467
						0.684418	0.099749	0.890458	0.053164
1.0	1.2	3.0	3.5	5	10	1.160178	0.074827	0.553507	0.072176
						1.147594	0.086379	0.525997	0.074638
						1.084296	0.084619	0.495197	0.083229
1.0	1.2	3.0	3.5	5	10	1.419685	0.067786	1.141343	0.03899
						1.43077	0.060551	1.149229	0.053782
						1.423693	0.072705	1.144334	0.049606

β_1	β_2	α_1	α_2	n_1	n_2	$R_2(d_{31})$	$R_2(\delta_{31})$	$R_2(d_{32})$	$R_2(\delta_{32})$
1.0	1.2	3.0	3.5	5	10	0.457144	0.02699	0.832293	0.006466
						0.492683	0.014814	0.951865	0.007182
						0.501973	0.018139	0.957209	0.007637
1.0	1.2	3.0	3.5	5	10	1.392901	0.007737	0.337858	0.008153
						1.327465	0.012537	0.358638	0.016369
						1.235547	0.012637	0.291717	0.011523
1.0	1.2	3.0	3.5	5	10	2.233822	0.006228	1.357036	0.003472
						2.130339	0.005201	1.276243	0.002627
						2.058945	0.006521	1.31844	0.003078

TABLE 3.5

COMPARISON OF B_1 AND R_1 For $d_{1i}(x)$ WITH $\delta_{1i}(x)$

β_1	β_2	α_1	α_2	n_1	n_2	$B_1(d_{21})$	$B_1(\delta_{21})$	$B_1(d_{22})$	$B1(\delta_{22})$
3.0	3.2	5.0	5.5	5	10	1.166734	0.135811	1.600153	0.152464
						1.099196	0.141889	1.542128	0.110877
						1.124713	0.150918	1.561941	0.136577
3.0	3.2	5.0	5.5	5	10	1.790592	0.120026	1.037294	0.228295
						1.766139	0.116587	0.948943	0.203699
						1.778721	0.129413	1.085825	0.241275
3.0	3.2	5.0	5.5	5	10	2.206989	0.077369	1.948732	0.100929
						2.118959	0.086711	2.028905	0.110251
						2.158177	0.088082	1.961759	0.095009

β_1	β_2	α_1	α_2	n_1	n_2	$R_2(d_{31})$	$R_2(\delta_{31})$	$R_2(d_{32})$	$R_2(\delta_{32})$
3.0	3.2	5.0	5.5	5	10	1.506784	0.037656	7.530213	0.042547
						1.305913	0.033827	8.021944	0.041258
						1.486678	0.043114	7.708415	0.056197
3.0	3.2	5.0	5.5	5	10	3.275788	0.023231	1.204266	0.080667
						3.052725	0.032905	1.144245	0.061988
						3.247642	0.02655	1.29623	0.086552
3.0	3.2	5.0	5.5	5	10	4.91638	0.010831	11.854381	0.015163
						4.574408	0.014936	11.699264	0.010934
						4.706124	0.013004	12.07526	0.015641

TABLE 3.6

COMPARISON OF B_1 AND R_1For $d_{2i}(x)$ WITH $\delta_{2i}(x)$

β_1	β_2	α_1	α_2	n_1	n_2	$B_1(d_{21})$	$B_1(\delta_{21})$	$B_1(d_{22})$	$B1(\delta_{22})$
3.0	3.2	5.0	5.5	5	10	1.958317	0.145384	2.66665	0.145992
						1.98383	0.156819	2.656129	0.163514
						1.960215	0.157974	2.657402	0.165973
3.0	3.2	5.0	5.5	5	10	2.476136	0.146988	2.144109	0.185217
						2.477003	0.129688	2.137088	0.202355
						2.469031	0.129021	2.128544	0.220564
3.0	3.2	5.0	5.5	5	10	2.628072	0.078195	2.832233	0.100522
						2.633816	0.101706	2.833113	0.091905
						2.631319	0.116816	2.829565	0.124713

β_1	β_2	α_1	α_2	n_1	n_2	$R_2(d_{31})$	$R_2(\delta_{31})$	$R_2(d_{32})$	$R_2(\delta_{32})$
3.0	3.2	5.0	5.5	5	10	3.844099	0.037761	7.112075	0.032808
						3.908676	0.034522	7.146093	0.016668
						3.885021	0.040057	7.134618	0.025512
3.0	3.2	5.0	5.5	5	10	6.090461	0.023066	4.590513	0.070884
						6.103944	0.022545	4.686374	0.057278
						6.146476	0.035702	4.716953	0.045029
3.0	3.2	5.0	5.5	5	10	6.969557	0.027495	7.982009	0.026397
						6.940046	0.011694	7.989375	0.016731
						6.925348	0.012402	8.016562	0.015027

TABLE 3.7

COMPARISON OF B_2 AND R_2 For $d_{3i}(x)$ WITH $\delta_{3i}(x)$

β_1	β_2	α_1	α_2	n_1	n_2	$B_1(d_{21})$	$B_1(\delta_{21})$	$B_1(d_{22})$	$B1(\delta_{22})$
3.0	3.2	5.0	5.5	5	10	0.388911	0.051291	0.500048	0.044774
						0.366399	0.054368	0.481915	0.032926
						0.374904	0.056067	0.488107	0.040955
3.0	3.2	5.0	5.5	5	10	0.596864	0.040406	0.324155	0.063178
						0.585852	0.041541	0.306628	0.053811
						0.571591	0.049888	0.319864	0.054442
3.0	3.2	5.0	5.5	5	10	0.735663	0.026344	0.608979	0.031047
						0.708859	0.034578	0.605132	0.028417
						0.720512	0.039627	0.620653	0.038622

β_1	β_2	α_1	α_2	n_1	n_2	$R_2(d_{31})$	$R_2(\delta_{31})$	$R_2(d_{32})$	$R_2(\delta_{32})$
3.0	3.2	5.0	5.5	5	10	0.14922	0.004958	0.256077	0.003297
						0.136408	0.007266	0.280275	0.003994
						0.139238	0.004497	0.288825	0.003986
3.0	3.2	5.0	5.5	5	10	0.358897	0.003363	0.109242	0.004686
						0.345626	0.004699	0.097324	0.005017
						0.351685	0.002818	0.106264	0.004989
3.0	3.2	5.0	5.5	5	10	0.52808	0.002249	0.393404	0.002519
						0.503087	0.001355	0.406487	0.001565
						0.522903	0.001425	0.381296	0.001425

TABLE 3.8

COMPARISON OF B_2 AND R_2FOR $d_{4i}(x)$ WITH $\delta_{4i}(x)$

β_1	β_2	α_1	α_2	n_1	n_2	$B_1(d_{21})$	$B_1(\delta_{21})$	$B_1(d_{22})$	$B1(\delta_{22})$
3.0	3.2	5.0	5.5	5	10	0.388911	0.055278	0.500048	0.043337
						0.366399	0.058868	0.481915	0.031833
						0.374904	0.059519	0.488107	0.039811
3.0	3.2	5.0	5.5	5	10	0.596864	0.040964	0.324155	0.059215
						0.588713	0.039411	0.296545	0.053584
						0.592907	0.043709	0.33932	0.060298
3.0	3.2	5.0	5.5	5	10	0.726477	0.025123	0.636493	0.033112
						0.703308	0.037862	0.636404	0.029914
						0.735121	0.030852	0.629026	0.031443

β_1	β_2	α_1	α_2	n_1	n_2	$R_2(d_{31})$	$R_2(\delta_{31})$	$R_2(d_{32})$	$R_2(\delta_{32})$
3.0	3.2	5.0	5.5	5	10	0.139093	0.006451	0.259326	0.002627
						0.145101	0.004909	0.276858	0.002327
						0.165186	0.005709	0.266651	0.004003
3.0	3.2	5.0	5.5	5	10	0.327505	0.004329	0.095774	0.003656
						0.355094	0.003651	0.100237	0.004009
						0.358897	0.00343	0.109242	0.004071
3.0	3.2	5.0	5.5	5	10	0.546264	0.001209	0.37745	0.001568
						0.508268	0.001797	0.371351	0.001192
						0.503087	0.001434	0.406487	0.001492

TABLE 3.9
COMPARISON OF B_1 AND R_1 FOR $d_{1i}(x)$ WITH $\delta_{1i}(x)$

β_1	β_2	α_1	α_2	n_1	n_2	$B_1(d_{21})$	$B_1(\delta_{21})$	$B_1(d_{22})$	$B1(\delta_{22})$
5.0	5.2	3.0	3.5	5	10	3.493439	0.46442	4.745423	0.454504
						3.418675	0.486516	4.722131	0.449132
						3.555076	0.395493	4.854006	0.439837
5.0	5.2	3.0	3.5	5	10	5.262426	0.411677	2.857423	0.664787
						5.702573	0.330078	2.763005	0.661044
						5.833188	0.306555	2.750705	0.682506
5.0	5.2	3.0	3.5	5	10	7.351353	0.303015	5.924184	0.339269
						7.160379	0.304172	6.099613	0.376927
						7.233281	0.285471	5.852633	0.313212

β_1	β_2	α_1	α_2	n_1	n_2	$R_2(d_{31})$	$R_2(\delta_{31})$	$R_2(d_{32})$	$R_2(\delta_{32})$
5.0	5.2	3.0	3.5	5	10	9.692334	0.477942	22.726185	0.217402
						12.602381	0.283611	22.850366	0.322938
						13.427834	0.218591	22.228505	0.247083
5.0	5.2	3.0	3.5	5	10	30.912553	0.181657	8.549652	0.597858
						35.256227	0.19058	9.105396	0.690992
						34.832473	0.14414	8.909375	0.694121
5.0	5.2	3.0	3.5	5	10	49.129449	0.201393	35.043077	0.150874
						52.493443	0.110723	37.931927	0.196431
						53.487376	0.118946	35.098891	0.155638

TABLE 3.10

COMPARISON OF B_1 AND R_1 FOR $d_{2i}(x)$ WITH $\delta_{2i}(x)$

β_1	β_2	α_1	α_2	n_1	n_2	$B_1(d_{21})$	$B_1(\delta_{21})$	$B_1(d_{22})$	$B1(\delta_{22})$
5.0	5.2	3.0	3.5	5	10	2.895331	0.533485	4.097541	0.418634
						2.863628	0.437333	4.082888	0.388101
						2.872306	0.430204	4.108771	0.379425
5.0	5.2	3.0	3.5	5	10	3.85973	0.437529	3.185644	0.594407
						3.810825	0.337566	3.210283	0.567736
						3.808803	0.405243	3.232141	0.547225
5.0	5.2	3.0	3.5	5	10	4.148695	0.365314	4.410108	0.306694
						4.121115	0.311322	4.412278	0.272859
						4.117058	0.285787	4.411012	0.302254

β_1	β_2	α_1	α_2	n_1	n_2	$R_1(d_{21})$	$R_1(\delta_{21})$	$R_1(d_{22})$	$R_1(\delta_{22})$
5.0	5.2	3.0	3.5	5	10	9.276051	0.578024	16.825807	0.187443
						8.742676	0.411237	16.904664	0.171908
						8.319034	0.279012	16.892792	0.211646
5.0	5.2	3.0	3.5	5	10	14.910891	0.278819	10.223327	0.524307
						14.430525	0.196937	10.328086	0.544112
						14.421944	0.15374	10.402959	0.554303
5.0	5.2	3.0	3.5	5	10	17.21813	0.218683	19.452729	0.139388
						16.990181	0.157999	19.471148	0.123098
						16.954862	0.124648	19.461079	0.158379

TABLE 3.11

COMPARISON OF B_2 AND R_2 FOR $d_{3i}(x)$ WITH $\delta_{3i}(x)$

β_1	β_2	α_1	α_2	n_1	n_2	$B_2(d_{31})$	$B_2(\delta_{31})$	$B_2(d_{32})$	$B_2(\delta_{32})$
5.0	5.2	3.0	3.5	5	10	0.581954	0.136044	0.902357	0.064055
						0.646694	0.110042	0.885651	0.066959
						0.670553	0.114733	0.875932	0.065534
5.0	5.2	3.0	3.5	5	10	1.052485	0.090489	0.549504	0.106299
						1.140515	0.068921	0.531347	0.100708
						1.144154	0.082894	0.516017	0.097644
5.0	5.2	3.0	3.5	5	10	1.383655	0.075585	1.126633	0.057479
						1.460878	0.063746	1.120791	0.051708
						1.472238	0.058102	1.124199	0.056989

β_1	β_2	α_1	α_2	n_1	n_2	$R_2(d_{31})$	$R_2(\delta_{31})$	$R_2(d_{32})$	$R_2(\delta_{32})$
5.0	5.2	3.0	3.5	5	10	0.387693	0.026129	0.840465	0.006497
						0.56016	0.018845	0.859371	0.009102
						0.504095	0.015384	0.845058	0.009912
5.0	5.2	3.0	3.5	5	10	1.15105	0.011879	0.346333	0.016774
						1.340227	0.007789	0.324672	0.015308
						1.363536	0.009606	0.302757	0.014694
5.0	5.2	3.0	3.5	5	10	1.965178	0.009245	1.295972	0.004927
						2.185844	0.006647	1.277588	0.004348
						2.204314	0.005206	1.293212	0.005603

TABLE 3.12

COMPARISON OF B_2 AND R_2 FOR $d_{4i}(x)$ WITH $\delta_{4i}(x)$

β_1	β_2	α_1	α_2	n_1	n_2	$B_2(d_{41})$	$B_2(\delta_{41})$	$B_2(d_{42})$	$B_2(\delta_{42})$
5.0	5.2	3.0	3.5	5	10	0.670553	0.125641	0.875932	0.063026
						0.687924	0.118256	0.874324	0.073597
						0.664505	0.130564	0.892039	0.076311
5.0	5.2	3.0	3.5	5	10	1.052485	0.097052	0.549504	0.096344
						1.140515	0.073093	0.531347	0.090007
						1.144154	0.085918	0.516017	0.085349
5.0	5.2	3.0	3.5	5	10	1.383655	0.079571	1.126633	0.055203
						1.470271	0.065317	1.139266	0.056245
						1.432076	0.068927	1.173003	0.063666

β_1	β_2	α_1	α_2	n_1	n_2	$R_2(d_{41})$	$R_2(\delta_{41})$	$R_2(d_{42})$	$R_2(\delta_{42})$
5.0	5.2	3.0	3.5	5	10	0.387693	0.030649	0.840465	0.005842
						0.525821	0.020157	0.804772	0.008927
						0.507804	0.022441	0.832469	0.008489
5.0	5.2	3.0	3.5	5	10	1.236502	0.010235	0.316185	0.012554
						1.410249	0.008823	0.336738	0.013994
						1.393299	0.007119	0.329489	0.014584
5.0	5.2	3.0	3.5	5	10	1.965178	0.010028	1.295972	0.004644
						2.185844	0.007167	1.277588	0.003987
						2.204314	0.00567	1.293212	0.005223

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