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Einstein Ordered Weighted Aggregation Operators for Pythagorean Fuzzy Hypersoft Set with its Application to Solve MCDM Problem

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ABSTRACT The Pythagorean fuzzy hypersoft set is the most generalized form of the Pythagorean fuzzy soft set used to resolve indeterminate and inexplicit information in the decision-making procedure, considering the parameters' multi-sub-attributes. Aggregation operators execute a dynamic role in considering the two prospect sequences and eliminating anxieties from this perception. The hybrid form of Pythagorean fuzzy sets with hypersoft sets has appeared as a supportive structure in fuzzy mathematics and ensued as a convenient perspective in decision-making. This paper prolongs the Einstein-weighted ordered aggregation operators for the Pythagorean fuzzy hypersoft set, which proficiently contracts with tentative and confusing data. Experts are using the Pythagorean fuzzy hypersoft set in their quest to report indefinite and specific decision-making processes. It is an effective technique for enlarging unsure facts in decision-making. Some operational laws for the Pythagorean fuzzy hypersoft set have been projected in light of Einstein's operations. Two innovative Einstein-ordered aggregation operators were established based on operational laws: Pythagorean fuzzy hypersoft Einstein-ordered weighted average and Pythagorean fuzzy hypersoft Einstein-ordered weighted geometric operators with their essential properties. Multi-criteria decision-making is imperative in overcoming barriers to real-world complications. However, conventional methods of multi-criteria decision-making regularly provide inconsistent results. The extended model appraisals recognized score values to regulate robotic agri-farming equated to prevalent methods, which is more useful for agribusiness. A numerical illustration of decision-making complications in real-life farming is deliberated to authenticate the established method's supremacy and applicability. Based on the anticipated aggregation operators, a robust multi-criteria decision-making method has been presented, which delivers the most appropriate outcomes compared to existing multi-criteria decision-making techniques. The consequences show that the intended approach is more effective and stable in handling rough information based on the Pythagorean fuzzy hypersoft set.

INDEX TERMS: hypersoft set; Pythagorean fuzzy hypersoft set; Einstein operators; Pythagorean fuzzy hypersoft Einstein-ordered weighted average operator; Pythagorean fuzzy hypersoft Einstein ordered weighted geometric operator; multi-criteria decision making.

I. INTRODUCTION

Multi-criteria decision making (MCDM) is deliberated as the most suitable method for verdict the adequate alternative from all conceivable choices, following measures or attributes. Most decisions are taken when the intentions and confines are generally unspecified or unclear in everyday surroundings. To overwhelm such vagueness and worries, Zadeh presented the idea of the fuzzy set (FS) [1], a fundamental tool to knob the insignificances and hesitations in decision-making (DM). Experts mainly consider a membership degree (MD) and a non-membership degree (NMD) in the DM procedure that FS cannot grip. Turksen [2] presented the interval-valued FS (IVFS) with fundamental operations. Atanassov [3] overcame the abovementioned limitations and developed the intuitionistic fuzzy set (IFS). Wang and Liu [4]

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introduced several operations such as Einstein product, Einstein sum, etc., and aggregation operators (AOs) for IFS. Garg and Kaur [5] protracted the idea of IFS and settled the cubic IFS (CIFS). Xu [6] protracted the notion of IFS and dropped a methodology for comparing two intuitionistic fuzzy values using score and accuracy functions. Moreover, he developed the average, geometric, and hybrid AOs with their basic properties. Garg [7] improved the cosine similarity measures (SM) for IFS and planned a DM technique. Lin et al. [8] presented a novel MCDM technique for IFS. De et al. [9] defined some elementary operations such as concentration, dilation, and normalization for IFS.

The systems stated overhead have been well-recognized by the professionals. Still, the obtainable IFS cannot grip the untimely and inexplicit facts because it is deliberated to envision the linear inequality among the MD and NMD. If experts choose MD = 0.6 and NMD = 0.7, then the IFS cannot deal with it because $0.6 + 0.7 \geq 1$. Yager [10] offered the Pythagorean fuzzy set (PFS) to resolve the inadequacy mentioned above by modifying the elementary state $\kappa + \delta \leq 1$ to $\kappa^2 + \delta^2 \leq 1$ and established some consequences related to score and accuracy functions. Xiao and Ding [11] proposed the innovative divergence measure for PFS considering the Jensen–Shannon divergence. Thao and Smarandache [12] presented the entropy measure for PFS and established an MCDM approach using their developed measure. Zhang et al. [13] familiarized some novel SM for PFS and verified that the settled SM are competent compared to prevailing similarity measures. Rahman et al. [14] planned geometric AO for PFS and showed the multi-attribute group decision-making (MAGDM) technique via their planned operator. Zhang and Xu [15] lengthy the method for order of preference by similarity to the ideal solution (TOPSIS) for PFS to resolve MCDM complications. Wei and Lu [16] offered the Pythagorean fuzzy power AOs and essential characteristics. Using their presented operators, they also established a DM technique to resolve multi-attribute decision-making (MADM). Wang and Li [17] demonstrated the operational interaction laws for Pythagorean fuzzy numbers (PFNs), and settled power Bonferroni mean operators. Zhang [18] offered an innovative DM approach established on SM to solve MCGDM obstacles for the PFS. Peng and Yuan [19] proposed the AOs for PFS and demonstrated a DM technique using their planned methodology.

All of the above techniques have broad applications, but these theories have some limitations on parametric chemistry due to their ineffectiveness. Molodtsov [20] introduced the soft sets (SS) theory and defined some elementary operations with their features to handle misperception and haziness. Maji et al. [21] extended the idea of SS and established many elementary and binary operations. Cagman and Enginoglu [22] introduced the fuzzy parametrized SS with necessary operations and possessions. They also prolonged the DM method based on their proposed theory and used it to resolve uncertain complications. Ali et al. [23] protracted from the idea of SS and defined several elementary operations for SS with their fundamental properties. Maji et al. [24] developed the fuzzy soft set with desirable properties by absorbing two general concepts, FS and SS. Roy and Maji [25] settled on a novel decision-making approach for FSS recognizing incomplete information. Cagman et al. [26] defined the AOs for FSS and presented a DM method using their proposed operators. Feng et al. [27] introduced the adjustable procedure to FSS and offered the weighted FSS with its application in DM. Maji et al. [28] developed the intuitionistic fuzzy soft set (IFSS) and some crucial operations with their essential possessions. Arora and Garg [29] settled the AOs for IFSS and offered an MCDM method using their settled operators. Çağman and Karataş [30] demarcated some operations for IFSS with their properties and constructed a DM technique based on their established operations. Muthukumar and Krishnan [31] introduced the SM and weighted SM for IFSS. They also discussed the basic operations for IFSS and presented a DM methodology using their projected methods.

Peng et al. [32] merged two well-known theories, PFS and SS, and developed the Pythagorean fuzzy soft sets (PFSS). Zulqarnain et al. [33] presented some operational laws for PFSS and prolonged the AOs for PFSS. Athira et al. [34] introduced the entropy measure, Hamming, and Euclidean distances for PFSS. Zulqarnain et al. [35] established the operational interaction laws for PFSS and protracted the interaction AOs based on established operational laws. They also established the DM methodology using their developed interaction AOs with their green supplier chain management application. Athira et al. [36] introduced the entropy measure for PFSS. Zulqarnain et al. [37-38] prolonged the Einstein-ordered operational laws for PFSS and introduced the Einstein-ordered weighted ordered average and geometric AOs for PFSS. They also established DM methods to solve complex real-life difficulties. Naeem e al. [39] prolonged the MCGDM technique for PFSS considering the TOPSIS, VIKOR, and AOs. Zulqarnain et al. [40] settled the TOPSIS method for PFSS using correlation coefficient (CC) and developed the MADM methodology to resolve DM obstacles. Ullah et al. [41] presented the dice SM for T-spherical fuzzy sets (T-SFS) and settled the TOPSIS technique and entropy measure for T-SFS. Ali et al. [42] offered innovative AOs for complex T-SFS and established a MADM method based on proven AOs.

Smarandache [43] formulated the notion of the hypersoft set (HSS), which incorporates several sub-attributes into the attribution function f, which is definite to the cartesian product with the n attribute. Smarandche HSS is the most appropriate philosophy that handles several sub-attributes of assumed parameters related to SS and other prevalent models. Several HSS leeway and DM techniques are advocated. Rahman et al. [44] planned the DM techniques founded on SM for the possibility intuitionistic fuzzy hypersoft set (IFHSS). Zulqarnain et al. [45-46] protracted the TOPSIS method based on the CC and AOs for IFHSS. Rahman et al. [47] developed the novel MCDM technique using a rough approximation of fuzzy HSS and employed their established approach for supplier selection in the construction industry. Zulqarnain et al. [48] protracted the impression of IFHSS to Pythagorean fuzzy hypersoft set (PFHSS) with essential operations. Saeed et al. [49] presented a decision support model for the diagnoses of COVID-19 patients using complex fuzzy hypersoft sets (CFHS). Rahman et al. [50] introduced the interval-valued CFHS with some fundamental operations. Rahman et al. [51] developed a speculative context of convexity cum concavity on fuzzy HSS which is a more comprehensive and operative notion to contract with optimization concerning complications. Zulqarnain et al. [52] developed the correlation-based TOPSIS approach for PFHSS and utilized their established technique to select the most appropriate face mask. Siddique et al. [53] offered the AOs for PFHSS and proposed an MCDM method using their proposed operators.

A. MOTIVATION AND DRAWBACK OF EXISTING APPROACHES

The PFHSS is an amalgam logical configuration of SS, IFSS, PFSS, and IFHSS are dominant scientific tools for allocating anonymous and restricted data. It has been identified that AOs are imperious in DM, so collectively assessed facts from unlike causes can be collected in a distinctive valuation. To the unsurpassed our consideration, Einstein ordered AOs with hybridization with a PFS, and HSS has no presence in the literature. Still, existing AOs for PFHSS cannot expertly deal with uncertain and imprecise information during the DM process. Moreover, the model states that the whole MD (NMD) is self-determining its NMD (MD). Hence, agreeing to these replicas, the consequences are not productive, so no proper inclination is indicated for substitutes. So, how to integrate these PFHSS over Einstein-ordered operations is a fascinating subject. We will introduce the Einstein-ordered AOs for PFHSS, such as PFHSEOWA and PFHSEOWG operators. The developed Einstein-weighted ordered AOs are proficient compared to prevailing amalgam organizations of FS. The above replicas have inferred that the general MD (NMD) is liberated of its compatible NMD (MD) values. As a result, the consequences of these AOs are inconsistent, and no substitute for alternatives is given. Therefore, incorporating these PFHSNs through Einstein-ordered AOs is an interesting subject. The methodologies chosen in [37, 38] cannot accommodate the multi-sub-attributes of the alternatives. Similarly, the AOs presented in [53, 54] is inadequate to examine the data with a reflective intelligence for higher notion and correct inferences. A boosted categorization method captivates investigators to crash inexplicable and insufficient facts. Contrary to the exploration effects, PFHSS plays a vigorous part in DM by assembling many cradles into a particular value.

B. CONTRIBUTION

PFHSS is an innovative hybrid configuration of PFSS. The enriched organization method has captivated detectives to crack confusing and deficient data. Regarding findings, PFHSS shows a dynamic part in DM by gathering various foundations into one value. So, to encourage existing investigation on PFHSS, we will state Einstein-ordered weighted AOs based on asymmetrical facts. The main intent of this study is as follows:

1. The AOs of PFHSS is a well-known competent estimators of AOs. In some cases, aspects of prevalent AOs did not respond to precise outcomes during DM. To overcome these particular difficulties, reviews of these AOs are essential. We state an advanced Einstein algorithm for the Pythagorean fuzzy hypersoft number (PFHSN).
2. The PFHSS capably contracts the multifaceted concerns seeing the multi sub-attributes of the deliberated parameters in the DM procedure. To preserve this benefit in concentration, we prolong the Einstein-ordered weighted AOs for PFHSS.
3. Pythagorean fuzzy hypersoft Einstein ordered weighted average (PFHSEOWA) and Pythagorean fuzzy hypersoft Einstein ordered weighted geometric (PFHSEOWG) operators were introduced with their

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necessary properties using developed operational laws.

4. A novel algorithm using the planned operators to resolve the DM problem is established to resolve MCDM issues under the PFHSS scenario.
5. A comparative analysis of advanced MCDM techniques and current approaches has been presented to consider utility and superiority.

The association of this paper is assumed as follows: The second section of this thesis involves some basic notions that support us in growing the organization of the subsequent study. In section 3, some novel Einstein operational laws for PFHSN have been proposed. Also, in the same section, PFHSEOWA and PFHSEOWG operators have been introduced based on our developed operators' basic properties. In section 4, an MCDM approach has been constructed using the anticipated AOs. A mathematical illustration has been discussed in the same section to confirm the pragmatism of the proven method. Furthermore, a brief comparative analysis has been delivered to endorse the competency of the developed methodology in section 5.

II. PRELIMINARIES

This section contains some elementary descriptions, making a structure available for the succeeding study.

A. DEFINITION 1 [20]

Let U and $\mathbb{N}$ be the universe of discourse and set of attributes, respectively. Let $\mathcal{P}(U)$ be the power set of U and $A \subseteq \mathbb{N}$. A pair (Ω, A) is called a SS over U, and its mapping is expressed as follows:

$$\Omega: A \rightarrow \mathcal{P}(U)$$

Also, it can be defined as follows:

$$(\Omega, A) = \{\Omega(t) \in \mathcal{P}(U): t \in \mathbb{N}, \Omega(t) = \emptyset \; \boldsymbol{if} \; t \notin A\}$$

B. DEFINITION 2 [43]

Let U be a universe of discourse and $\mathcal{P}(U)$ be a power set of U and $t = \{t_1, t_2, t_3,..., t_n\}$,(n ≥ 1) and T_i denoted the set of attributes and their consistent sub-attributes, such as $T_i \cap T_j = \varphi$, where $i \neq j$ for each $n \geq 1$ and $i, j \in \{1,2,3 \ldots n\}$. Assume $T_1 \times T_2 \times T_3 \times \ldots \times T_n = \ddot{A} = \{d_{1h} \times d_{2k} \times \cdots \times d_{nl}\}$ is a collection of sub-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and $\alpha, \beta, \gamma \in$ N. Then the pair $(\mathcal{F}, T_1 \times T_2 \times T_3 \times \ldots \times T_n = (\Omega, \ddot{A})$ is known as HSS and defined as follows:

$$\Omega: T_1 \times T_2 \times T_3 \times \ldots \times T_n = \ddot{A} \rightarrow \mathcal{P}(U)$$

It is also defined as

$$(\Omega, \ddot{A}) = \{\check{d}, \Omega_{\ddot{A}}(\check{d}): \check{d} \in \ddot{A}, \; \Omega_{\ddot{A}}(\check{d}) \in \mathcal{P}(U)\}$$

C. DEFINITION 3 [43]

Let U be a universe of discourse and $\mathcal{P}(U)$ be a power set of U and $t = \{t_1, t_2, t_3,..., t_n\}$,(n ≥ 1) and T_i denoted the set of attributes and their consistent sub-attributes, such as $T_i \cap T_j = \varphi$, where $i \neq j$ for each $n \geq 1$ and $i, j \in \{1,2,3 \ldots n\}$. Assume $T_1 \times T_2 \times T_3 \times \ldots \times T_n = \ddot{A} = \{d_{1h} \times d_{2k} \times \cdots \times d_{nl}\}$ is a collection of sub-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and $\alpha, \beta, \gamma \in$ N. Then the pair $(\mathcal{F}, T_1 \times T_2 \times T_3 \times \ldots \times T_n = (\Omega, \ddot{A})$ is known as IFHSS and defined as follows:

$$\Omega: T_1 \times T_2 \times T_3 \times \ldots \times T_n = \ddot{A} \rightarrow IFS^U$$

It is also defined as

$$(\Omega, \ddot{A}) = \left\{\left(\check{d}, \Omega_{\ddot{A}}(\check{d})\right): \check{d} \in \ddot{A}, \; \Omega_{\ddot{A}}(\check{d}) \in IFS^U \in [0,1]\right\}$$, where $\Omega_{\ddot{A}}(\check{d}) = \left\{\langle \zeta, \kappa_{\Omega(\check{d})}(\zeta), \delta_{\Omega(\check{d})}(\zeta)\rangle : \zeta \in U\right\}$, where $\kappa_{\Omega(\check{d})}(\zeta)$ and $\delta_{\Omega(\check{d})}(\zeta)$ represents MD and NMD, respectively, such as $\kappa_{\Omega(\check{d})}(\zeta), \delta_{\Omega(\check{d})}(\zeta) \in [0,1]$, and $0 \leq \kappa_{\Omega(\check{d})}(\zeta) + \delta_{\Omega(\check{d})}(\zeta) \leq 1$.

D. DEFINITION 4 [48]

Let U be a universe of discourse and $\mathcal{P}(U)$ be a power set of U and $t = \{t_1, t_2, t_3,..., t_n\}$,(n ≥ 1) and T_i represented the set of attributes and their corresponding sub-attributes, such as $T_i \cap T_j = \varphi$, where $i \neq j$ for each $n \geq 1$ and $i, j \in \{1,2,3 \ldots n\}$. Assume $T_1 \times T_2 \times T_3 \times \ldots \times T_n = \ddot{A} = \{d_{1h} \times d_{2k} \times \cdots \times d_{nl}\}$ is a collection of sub-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and $\alpha, \beta, \gamma \in$ N. Then the pair $(\mathcal{F}, T_1 \times T_2 \times T_3 \times \ldots \times T_n = (\Omega, \ddot{A})$ is known as PFHSS and defined as follows:

$$\Omega: T_1 \times T_2 \times T_3 \times \ldots \times T_n = \ddot{A} \rightarrow PFS^U$$

It is also defined as

$$(\Omega, \ddot{A}) = \left\{\left(\check{d}, \Omega_{\ddot{A}}(\check{d})\right): \check{d} \in \ddot{A}, \; \Omega_{\ddot{A}}(\check{d}) \in PFS^U \in [0,1]\right\}$$, where $\Omega_{\ddot{A}}(\check{d}) = \left\{\langle \zeta, \kappa_{\Omega(\check{d})}(\zeta), \delta_{\Omega(\check{d})}(\zeta)\rangle : \zeta \in U\right\}$, where $\kappa_{\Omega(\check{d})}(\zeta)$ and $\delta_{\Omega(\check{d})}(\zeta)$ represents the MD and NMD, respectively, such as $\kappa_{\Omega(\check{d})}(\zeta), \delta_{\Omega(\check{d})}(\zeta) \in [0,1]$, and $0 \leq \left(\kappa_{\Omega(\check{d})}(\zeta)\right)^2 + \left(\delta_{\Omega(\check{d})}(\zeta)\right)^2 \leq 1$.

The PFHSN can be stated as $\mathcal{F} = \left\{\left(\kappa_{\Omega(\check{d})}(\zeta), \delta_{\Omega(\check{d})}(\zeta)\right)\right\}$.

E. EXAMPLE 1

Consider the universe of discourse $\mathcal{U} = \{\delta_1, \delta_2\}$ and $\mathfrak{L} = \{\ell_1 = Teaching\ methdology, \ell_2 = Subjects, \ell_3 = Classes\}$ represents the set of attributes, and their corresponding sub-attributes are given as teaching methodology = $L_1 = \{d_{11} = project\ base, d_{12} = \ class\ discussion\}$Subjects = $L_2 = \{d_{21} = Mathematics, d_{22} = Computer\ Science, d_{23} = \ Statistics\}$And Classes = $L_3 = \{d_{31} = Masters, d_{32} = Doctorol\}$. Let $\ddot{A} = L_1 \times L_2 \times L_3$ be a set of attributes

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$\dddot{A} = L_1 \times L_2 \times L_3 = \{d_{11}, d_{12}\} \times \{d_{21}, d_{22}, d_{23}\} \times \{d_{31}, d_{32}\}$

$$= \begin{Bmatrix} (d_{11}, d_{21}, d_{31}), (d_{11}, d_{21}, d_{32}), (d_{11}, d_{22}, d_{31}), (d_{11}, d_{22}, d_{32}), (d_{11}, d_{23}, d_{31}), (d_{11}, d_{23}, d_{32}), \\ (d_{12}, d_{21}, d_{31}), (d_{12}, d_{21}, d_{32}), (d_{12}, d_{22}, d_{31}), (d_{12}, d_{22}, d_{32}), (d_{12}, d_{23}, d_{31}), (d_{12}, d_{23}, d_{32}), \end{Bmatrix}$$

$\dddot{A} = \{\check{d}_1, \check{d}_2, \check{d}_3, \check{d}_4, \check{d}_5, \check{d}_6, \check{d}_7, \check{d}_8, \check{d}_9, \check{d}_{10}, \check{d}_{11}, \check{d}_{12}\}$

Then the PFHSS over $\mathcal{U}$ is given as follows

$$(\mathcal{F}, \dddot{A}) = \begin{Bmatrix} (\check{d}_1, (\delta_1, (.6, .3)), (\delta_2, (.5, .7))), (\check{d}_2, (\delta_1, (.6, .7)), (\delta_2, (.7, .5))), (\check{d}_3, (\delta_1, (.4, .8)), (\delta_2, (.3, .7))), \\ (\check{d}_4, (\delta_1, (.6, .5)), (\delta_2, (.5, .6))), (\check{d}_5, (\delta_1, (.7, .3)), (\delta_2, (.4, .8))), (\check{d}_6, (\delta_1, (.5, .4)), (\delta_2, (.6, .5))), \\ (\check{d}_7, (\delta_1, (.5, .6)), (\delta_2, (.4, .5))), (\check{d}_8, (\delta_1, (.2, .5)), (\delta_2, (.3, .9))), (\check{d}_9, (\delta_1, (.4, .6)), (\delta_2, (.8, .5))), \\ (\check{d}_{10}, (\delta_1, (.7, .4)), (\delta_2, (.7, .2))), (\check{d}_{11}, (\delta_1, (.4, .5)), (\delta_2, (.5, .3))), (\check{d}_{12}, (\delta_1, (.5, .7)), (\delta_2, (.4, .7))) \end{Bmatrix}$$

For readers' appropriateness, the PFHSN $\Omega_{\dddot{A}}(\check{d}) = \{\langle \zeta, \kappa_{\Omega(\check{d})}(\zeta), \delta_{\Omega(\check{d})}(\zeta) \rangle : \zeta \in U\}$, can be inscribed as $\mathfrak{J}_{\check{d}_{ij}} = \langle \kappa_{\Omega(\check{d})}(\zeta), \delta_{\Omega(\check{d})}(\zeta) \rangle$. The score function [53] for $\mathfrak{J}_{\check{d}_{ij}}$ is articulated as follows:

$S(\mathfrak{J}_{\check{d}_{ij}}) = {\kappa_{\mathcal{F}(\check{d}_{ij})}}^2 - {\delta_{\mathcal{F}(\check{d}_{ij})}}^2$, $S(\mathfrak{J}_{\check{d}_{ij}}) \in [-1, 1]$ (2.1)

However, sometimes the score function cannot provide suitable outcomes to compute the PFHSNs. It is difficult to conclude which alternative is more convenient. To intimidate such obstacles, the accuracy function had been settled.

$\mathrm{A}(\mathfrak{J}_{\check{d}_{ij}}) = {\kappa_{\mathcal{F}(\check{d}_{ij})}}^2 + {\delta_{\mathcal{F}(\check{d}_{ij})}}^2$, $\mathrm{A}(\mathfrak{J}_{\check{d}_{ij}}) \in [-1, 1]$.

Comparison rules are defined as follows for the comparison of two PFHSNs $\mathfrak{J}_{\check{d}_{ij}}$ and $\mathfrak{T}_{\check{d}_{ij}}$

1. If $S(\mathfrak{J}_{\check{d}_{ij}}) > S(\mathfrak{T}_{\check{d}_{ij}})$, then $\mathfrak{J}_{\check{d}_{ij}} > \mathfrak{T}_{\check{d}_{ij}}$.
2. If $S(\mathfrak{J}_{\check{d}_{ij}}) = S(\mathfrak{T}_{\check{d}_{ij}})$, then
   - If $\mathrm{A}(\mathfrak{J}_{\check{d}_{ij}}) > \mathrm{A}(\mathfrak{T}_{\check{d}_{ij}})$, then $\mathfrak{J}_{\check{d}_{ij}} > \mathfrak{T}_{\check{d}_{ij}}$
   - If $\mathrm{A}(\mathfrak{J}_{\check{d}_{ij}}) = \mathrm{A}(\mathfrak{T}_{\check{d}_{ij}})$, then $\mathfrak{J}_{\check{d}_{ij}} = \mathfrak{T}_{\check{d}_{ij}}$.

F. DEFINITION 5. [53]

Let $\mathfrak{J}_{\check{d}_k} = (\kappa_{\check{d}_k}, \delta_{\check{d}_k})$, $\mathfrak{J}_{\check{d}_{11}} = (\kappa_{\check{d}_{11}}, \delta_{\check{d}_{11}})$, and $\mathfrak{J}_{\check{d}_{12}} = (\kappa_{\check{d}_{12}}, \delta_{\check{d}_{12}})$ signifies the PFHSNs and $\partial > 0$. Then, algebraic operational laws for PFHSNs can be stated as follows:

1. $\mathfrak{J}_{\check{d}_{11}} \oplus \mathfrak{J}_{\check{d}_{12}} = \left\langle \sqrt{{\kappa_{\check{d}_{11}}}^2 + {\kappa_{\check{d}_{12}}}^2 - {\kappa_{\check{d}_{11}}}^2 {\kappa_{\check{d}_{12}}}^2}, \delta_{\check{d}_{11}} \delta_{\check{d}_{12}} \right\rangle$
2. $\mathfrak{J}_{\check{d}_{11}} \otimes \mathfrak{J}_{\check{d}_{12}} = \left\langle \kappa_{\check{d}_{11}} \kappa_{\check{d}_{12}}, \sqrt{{\mathcal{J}_{\check{d}_{11}}}^2 + {\mathcal{J}_{\check{d}_{12}}}^2 - {\delta_{\check{d}_{11}}}^2 {\delta_{\check{d}_{12}}}^2} \right\rangle$
3. $\partial \mathfrak{J}_{\check{d}_k} = \left\langle \sqrt{1 - \left(1 - {\kappa_{\check{d}_k}}^2\right)^{\partial}}, {\delta_{\check{d}_k}}^{\partial} \right\rangle$
4. $\mathfrak{J}_{\check{d}_k}^{\partial} = \left\langle {\kappa_{\check{d}_k}}^{\partial}, \sqrt{1 - \left(1 - {\delta_{\check{d}_k}}^2\right)^{\partial}} \right\rangle$

The AOs for PFHSNs are based on algebraic operational laws defined by Zulqarnain et al. [53], such as follows:

$$PFHSWA\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \left\langle \sqrt{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - {\kappa_{\check{d}_{ij}}}^2\right)^{\theta_i}\right)^{\lambda_j}}, \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\delta_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j} \right\rangle$$

$$PFHSWG\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \left\langle \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\kappa_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}, \sqrt{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - {\delta_{\check{d}_{ij}}}^2\right)^{\theta_i}\right)^{\lambda_j}} \right\rangle$$

G. DEFINITION 6 [54]

Let $\mathfrak{J}_{\check{d}_k} = (\kappa_{\check{d}_k}, \delta_{\check{d}_k})$, $\mathfrak{J}_{\check{d}_{11}} = (\kappa_{\check{d}_{11}}, \delta_{\check{d}_{11}})$, and $\mathfrak{J}_{\check{d}_{12}} = (\kappa_{\check{d}_{12}}, \delta_{\check{d}_{12}})$ signifies the PFHSNs and $\partial > 0$. Then, Einstein's operational laws for PFHSNs can be stated as follows:

1. $\mathfrak{J}_{\check{d}_{11}} \oplus_{\varepsilon} \mathfrak{J}_{\check{d}_{12}} = \left\langle \frac{\sqrt{(1+a^2_{\check{d}_{12}}) - (1-a^2_{\check{d}_{12}})}}{\sqrt{(1+a^2_{\check{d}_{12}}) + (1-a^2_{\check{d}_{12}})}}, \frac{\sqrt{2 b^2_{\check{d}_{12}}}}{\sqrt{(2-b^2_{\check{d}_{12}}) + b^2_{\check{d}_{12}}}} \right\rangle$
2. $\mathfrak{J}_{\check{d}_{11}} \otimes_{\varepsilon} \mathfrak{J}_{\check{d}_{12}} = \left\langle \frac{\sqrt{2\alpha^2_{\check{d}_{12}}}}{\sqrt{(2-\alpha^2_{\check{d}_{12}}) + \alpha^2_{\check{d}_{12}}}}, \frac{\sqrt{(1+b^2_{\check{d}_{12}}) - (1-b^2_{\check{d}_{12}})}}{\sqrt{(1+b^2_{\check{d}_{12}}) + (1-b^2_{\check{d}_{12}})}} \right\rangle$
3. $\partial \mathfrak{J}_{\check{d}_k} = \left\langle \frac{\sqrt{\left(1+{a_{\check{d}_k}}^2\right)^{\partial} - \left(1-{a_{\check{d}_k}}^2\right)^{\partial}}}{\sqrt{\left(1+{a_{\check{d}_k}}^2\right)^{\partial} + \left(1-{a_{\check{d}_k}}^2\right)^{\partial}}}, \frac{\sqrt{2({b_{\check{d}_k}}^2)^{\partial}}}{\sqrt{\left(2-{b_{\check{d}_k}}^2\right)^{\partial} + ({b_{\check{d}_k}}^2)^{\partial}}} \right\rangle$
4. ${\mathfrak{J}_{\check{d}_k}}^{\partial} = \left\langle \frac{\sqrt{2({a_{\check{d}_k}}^2)^{\partial}}}{\sqrt{\left(2-{a_{\check{d}_k}}^2\right)^{\partial} + ({a_{\check{d}_k}}^2)^{\partial}}}, \frac{\sqrt{\left(1+{b_{\check{d}_k}}^2\right)^{\partial} - \left(1-{b_{\check{d}_k}}^2\right)^{\partial}}}{\sqrt{\left(1+{b_{\check{d}_k}}^2\right)^{\partial} + \left(1-{b_{\check{d}_k}}^2\right)^{\partial}}} \right\rangle$.

The Einstein average AO for PFHSNs is based on Einstein's operational laws defined by Zulqarnain et al. [54], such as follows:

PFHSEWA $\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \oplus_{j=1}^{m} \lambda_j \left(\oplus_{i=1}^{n} \theta_i \mathfrak{J}_{\check{d}_{ij}}\right)$

$$= \left\langle \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-b^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b^2_{\check{d}_{ij}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle.$$

The prevalent AOs for PFHSS only weigh Pythagorean fuzzy hypersoft arguments. Meanwhile, existing Einstein-weighted ordered AOs only consider the ordered positions of the Pythagorean fuzzy soft arguments, not the Pythagorean fuzzy hypersoft arguments themselves. The Einstein-ordered operators

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of these PFSS cannot accommodate subattributes of alternatives. To overcome the shortcomings, we focus on developing new Einstein-ordered AOs for PFHSS in the upcoming section.

III. EINSTEIN ORDERED WEIGHTED AGGREGATION OPERATORS FOR PYTHAGOREAN FUZZY HYPERSOFT SET

The consequent section will propose the Einstein operational laws for PFHSS and introduce the PFHSEOWA and PFHSEOWG operators with their properties

1) DEFINITION

Let $\mathfrak{J}_{\check{d}_{ij}} = \left(a_{\check{d}_{ij}}, b_{\check{d}_{ij}}\right)$ be an assortment of PFHSNs. Then the PFHSEOWA operator is demarcated as:

$$\text{PFHSEOWA}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \oplus_{j=1}^{m}\lambda_j\left(\oplus_{i=1}^{n}\theta_i\mathfrak{J}_{\check{d}_{r(i)\,s(j)}}\right) \quad (3.1)$$

Where θ_i, λ_j denote the weighted vectors such that $\theta_i > 0, \sum_{i=1}^{n}\theta_i = 1$ and $\lambda_j > 0,\ \sum_{j=1}^{n}\lambda_j = 1$.

2) THEOREM

Let $\mathfrak{J}_{\check{d}_{ij}} = \left(a_{\check{d}_{ij}}, b_{\check{d}_{ij}}\right)$ be an assortment of PFHSNs, then the conquered aggregated value from Definition I is given as

$$\text{PFHSEOWA}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \oplus_{j=1}^{m}\lambda_j\left(\oplus_{i=1}^{n}\theta_i\mathfrak{J}_{\check{d}_{r(i)\,s(j)}}\right)$$

$$= \left\langle \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+a^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-a^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+a^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-a^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-b^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle \quad (3.2)$$

Where θ_i, λ_j be the weights such as $\theta_i > 0, \sum_{i=1}^{n}\theta_i = 1$ and $\lambda_j > 0, \sum_{j=1}^{n}\lambda_j = 1$ and $\mathfrak{r}, \mathfrak{s}$ are permutations such that $\mathfrak{J}_{\check{d}_{\mathfrak{r}(i-1)\mathfrak{s}(j)}} \geq \mathfrak{J}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}$ and $\mathfrak{J}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j-1)}} \geq \mathfrak{J}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}$ $\forall i, j$.

PROOF: The mentioned above theorem can be verified by employing mathematical induction.

For n = 1, we get $\theta_i = 1$.

$$\text{PFHSEOWA}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \oplus_{j=1}^{m}\lambda_j\mathfrak{J}_{\check{d}_{r(1)\,s(j)}}$$

$$= \left\langle \frac{\sqrt{\prod_{j=1}^{m}(1+a^2_{\check{d}_{r(1)\,s(j)}})^{\lambda_j} - \prod_{j=1}^{m}(1-a^2_{\check{d}_{r(1)\,s(j)}})^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}(1+a^2_{\check{d}_{r(1)\,s(j)}})^{\lambda_j} + \prod_{j=1}^{m}(1-a^2_{\check{d}_{r(1)\,s(j)}})^{\lambda_j}}}, \frac{\sqrt{2\prod_{j=1}^{m}(b^2_{\check{d}_{r(1)\,s(j)}})^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}(2-b^2_{\check{d}_{r(1)\,s(j)}})^{\lambda_j} + \prod_{j=1}^{m}(b^2_{\check{d}_{r(1)\,s(j)}})^{\lambda_j}}} \right\rangle$$

$$= \left\langle \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{1}(1+a^2_{\check{d}_{r(i)\,s(j)}})^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{1}(1-a^2_{\check{d}_{r(i)\,s(j)}})^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{1}(1+a^2_{\check{d}_{r(i)\,s(j)}})^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{1}(1-a^2_{\check{d}_{r(i)\,s(j)}})^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{1}(b^2_{\check{d}_{r(i)\,s(j)}})^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{1}(2-b^2_{\check{d}_{r(i)\,s(j)}})^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{1}(b^2_{\check{d}_{r(i)\,s(j)}})^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

For $m = 1$, we get $\lambda_j = 1$.

$$\text{PFHSEOWA}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \oplus_{i=1}^{n}\theta_i\mathfrak{J}_{\check{d}_{r(i)\,s(1)}}$$

$$= \left\langle \frac{\sqrt{\prod_{i=1}^{n}\left(1+a^2_{\check{d}_{r(i)\,s(1)}}\right)^{\theta_i} - \prod_{i=1}^{n}\left(1-a^2_{\check{d}_{r(i)\,s(1)}}\right)^{\theta_i}}}{\sqrt{\prod_{i=1}^{n}\left(1+a^2_{\check{d}_{r(i)\,s(1)}}\right)^{\theta_i} + \prod_{i=1}^{n}\left(1-a^2_{\check{d}_{r(i)\,s(1)}}\right)^{\theta_i}}}, \frac{\sqrt{2\prod_{i=1}^{n}\left(b^2_{\check{d}_{r(i)\,s(1)}}\right)^{\theta_i}}}{\sqrt{\prod_{i=1}^{n}\left(2-b^2_{\check{d}_{r(i)\,s(1)}}\right)^{\theta_i} + \prod_{i=1}^{n}\left(b^2_{\check{d}_{r(i)\,s(1)}}\right)^{\theta_i}}} \right\rangle$$

$$= \left\langle \frac{\sqrt{\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1+a^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1-a^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1+a^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1-a^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{2\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(b^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(2-b^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(b^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

So, it is true for n = 1 and m = 1.

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Assume that Theorem 2 holds for n = δ_2 , m = $\delta_1 + 1$ and for n = $\delta_2 + 1$, m = δ_1

$$\oplus_{j=1}^{\delta_1+1}\lambda_j(\oplus_{i=1}^{\delta_2}\theta_i\Im_{\check{d}_{r(i)\,s(j)}}) =$$

$$\left\langle \sqrt{\frac{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1+\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1-\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1+\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1-\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{2\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(\mathscr{b}^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(2-\mathscr{b}^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(\mathscr{b}^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

$$\oplus_{j=1}^{\delta_1}\lambda_j(\oplus_{i=1}^{\delta_2+1}\theta_i\Im_{\check{d}_{r(i)\,s(j)}}) =$$

$$\left\langle \sqrt{\frac{\prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(1+\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(1-\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}{\prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(1+\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(1-\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{2\prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(\mathscr{b}^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(2-\mathscr{b}^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(\mathscr{b}^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

Now, we verify equation 3.2 for m = $\delta_1 + 1$ and n = $\delta_2 + 1$

$$\oplus_{j=1}^{\delta_1+1}\lambda_j(\oplus_{i=1}^{\delta_2+1}\theta_i\Im_{\check{d}_{r(i)\,s(j)}}) = \oplus_{j=1}^{\delta_1+1}\lambda_j(\oplus_{i=1}^{\delta_2}\theta_i\Im_{\check{d}_{r(i)\,s(j)}}\oplus\theta_{i+1}\Im_{\check{d}_{r(\delta_2+1)\,s(j)}})$$

$$= (\oplus_{j=1}^{\delta_1+1}\oplus_{i=1}^{\delta_2}\theta_i\lambda_j\,\Im_{\check{d}_{r(i)\,s(j)}})(\oplus_{j=1}^{\delta_1+1}\lambda_j\theta_{i+1}\Im_{\check{d}_{r(\delta_2+1)\,s(j)}}) =$$

$$\left\langle \frac{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1+\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1-\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1+\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1-\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}} \oplus \frac{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\left(1+\alpha^2_{\check{d}_{r(\delta_2+1)\,s(j)}}\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j} - \prod_{j=1}^{\delta_1+1}\left(\left(1-\alpha^2_{\check{d}_{r(\delta_2+1)\,s(j)}}\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\left(1+\alpha^2_{\check{d}_{r(\delta_2+1)\,s(j)}}\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\left(1-\alpha^2_{\check{d}_{r(\delta_2+1)\,s(j)}}\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j}}}, \right.$$

$$\left. \frac{\sqrt{2\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(\mathscr{b}^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(2-\mathscr{b}^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(\mathscr{b}^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}} \oplus \frac{\sqrt{2\prod_{j=1}^{\delta_1+1}\left(\left(\mathscr{b}^2_{\check{d}_{r(\delta_2+1)\,s(j)}}\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\left(2-\mathscr{b}^2_{\check{d}_{r(\delta_2+1)\,s(j)}}\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\left(\mathscr{b}^2_{\check{d}_{r(\delta_2+1)\,s(j)}}\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j}}} \right\rangle$$

$$=$$

$$\left\langle \sqrt{\frac{r\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(1+\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(1-\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(1+\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(1-\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{2\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(\mathscr{b}^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(2-\mathscr{b}^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(\mathscr{b}^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

$$= \oplus_{j=1}^{\delta_1+1}\lambda_j(\oplus_{i=1}^{\delta_2+1}\theta_i\Im_{\check{d}_{r(i)\,s(j)}})$$

So, it is true for m = $\delta_1 + 1$ and n = $\delta_2 + 1$.

3) EXAMPLE

Let $\mathcal{R} = \{\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4\}$ be a group of experts with weights $\theta_i = (0.1, 0.3, 0.3, 0.3)^T$. The group of experts is going to designate the attraction of a company's under-considered set of attributes $A = \{d_1 = lawn,\ d_2 = security\ system\}$ with their corresponding sub-attributes Lawn = $d_1 = \{d_{11} = with\ grass, d_{12} = without\ grass\}$Security system = $d_2 = \{d_{21} = guards, d_{22} = cameras\}$. Let $\ddot{A} = d_1 \times d_2$ be a set of sub-attributes $\ddot{A} = d_1 \times d_2 = \{d_{11}, d_{12}\} \times \{d_{21}, d_{22}\} = \{(d_{11}, d_{21}), (d_{11}, d_{22}), (d_{12}, d_{21}), (d_{12}, d_{22})\}$

$\ddot{A} = \{\check{d}_1, \check{d}_2, \check{d}_3, \check{d}_4\}$ characterizes the set sub-attributes with weights $\lambda_j = (0.2, 0.2, 0.2, 0.4)^T$. The hypothetical score values in the form of PFHSNs for all attributes $(\mathcal{H}, \ddot{A}) = (\alpha_{ij}, \mathscr{b}_{ij})_{4\times 4}$ given as:

$$(\mathcal{H}, \ddot{A}) = \begin{bmatrix} (0.5, 0.8) & (0.7, 0.5) & (0.4, 0.6) & (0.7, 0.4) \\ (0.5, 0.6) & (0.9, 0.1) & (0.3, 0.7) & (0.4, 0.5) \\ (0.4, 0.8) & (0.7, 0.5) & (0.4, 0.6) & (0.3, 0.5) \\ (0.3, 0.7) & (0.6, 0.5) & (0.5, 0.4) & (0.5, 0.7) \end{bmatrix}$$

As we know that

PFHSEOWA $(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{nm}}) =$

$$\left\langle \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha^{2}_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha^{2}_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha^{2}_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha^{2}_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{b}^{2}_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\mathcal{b}^{2}_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{b}^{2}_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

PFHSEOWA $(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{44}})$

$$= \left\langle \frac{\sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1+\alpha^{2}_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{4}\left(\prod_{i=1}^{n}\left(1-\alpha^{4}_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1+\alpha^{2}_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-\alpha^{2}_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{2\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(\mathcal{b}^{2}_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(2-\mathcal{b}^{2}_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(\mathcal{b}^{2}_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

$$= \left\langle \frac{\sqrt{\begin{array}{c}\{(1.25)^{0.1}(1.25)^{0.3}(1.16)^{0.3}(1.09)^{0.3}\}^{0.2}\{(1.49)^{0.1}(1.81)^{0.3}(1.49)^{0.3}(1.36)^{0.3}\}^{0.2}\\ \{(1.16)^{0.1}(1.09)^{0.3}(1.16)^{0.3}(1.25)^{0.3}\}^{0.2}\{(1.49)^{0.1}(1.16)^{0.3}(1.09)^{0.3}(1.25)^{0.3}\}^{0.4}\\ - \\ [\{(0.75)^{0.1}(0.75)^{0.3}(0.84)^{0.3}(0.91)^{0.3}\}^{0.2}\{(0.51)^{0.1}(0.19)^{0.3}(0.51)^{0.3}(0.64)^{0.3}\}^{0.2}\\ \{(0.84)^{0.1}(0.91)^{0.3}(0.84)^{0.3}(0.75)^{0.3}\}^{0.2}\{(0.51)^{0.1}(0.84)^{0.3}(0.91)^{0.3}(0.75)^{0.3}\}^{0.4}\end{array}}}{\sqrt{\begin{array}{c}\{(1.25)^{0.1}(1.25)^{0.3}(1.16)^{0.3}(1.09)^{0.3}\}^{0.2}\{(1.49)^{0.1}(1.81)^{0.3}(1.49)^{0.3}(1.36)^{0.3}\}^{0.2}\\ \{(1.16)^{0.1}(1.09)^{0.3}(1.16)^{0.3}(1.25)^{0.3}\}^{0.2}\{(1.49)^{0.1}(1.16)^{0.3}(1.09)^{0.3}(1.25)^{0.3}\}^{0.4}\\ + \\ [\{(0.75)^{0.1}(0.75)^{0.3}(0.84)^{0.3}(0.91)^{0.3}\}^{0.2}\{(0.51)^{0.1}(0.19)^{0.3}(0.51)^{0.3}(0.64)^{0.3}\}^{0.2}\\ \{(0.84)^{0.1}(0.91)^{0.3}(0.84)^{0.3}(0.75)^{0.3}\}^{0.2}\{(0.51)^{0.1}(0.84)^{0.3}(0.91)^{0.3}(0.75)^{0.3}\}^{0.4}\end{array}}}, \frac{\sqrt{2\left[\begin{array}{c}\{(0.64)^{0.1}(0.36)^{0.3}(0.64)^{0.3}(0.49)^{0.3}\}^{0.2}\{(0.25)^{0.1}(0.01)^{0.3}(0.25)^{0.3}(0.25)^{0.3}\}^{0.2}\\ \{(0.36)^{0.1}(0.49)^{0.3}(0.36)^{0.3}(0.16)^{0.3}\}^{0.2}\{(0.16)^{0.1}(0.25)^{0.3}(0.25)^{0.3}(0.49)^{0.3}\}^{0.4}\end{array}\right]}}{\sqrt{\begin{array}{c}\{(1.36)^{0.1}(1.64)^{0.3}(1.36)^{0.3}(1.51)^{0.3}\}^{0.2}\{(1.75)^{0.1}(1.99)^{0.3}(1.75)^{0.3}(1.75)^{0.3}\}^{0.2}\\ \{(1.64)^{0.1}(1.51)^{0.3}(1.64)^{0.3}(1.84)^{0.3}\}^{0.2}\{(1.84)^{0.1}(1.75)^{0.3}(1.75)^{0.3}(1.51)^{0.3}\}^{0.4}\\ + \\ \{(0.64)^{0.1}(0.36)^{0.3}(0.64)^{0.3}(0.49)^{0.3}\}^{0.2}\{(0.25)^{0.1}(0.01)^{0.3}(0.25)^{0.3}(0.25)^{0.3}\}^{0.2}\\ \{(0.36)^{0.1}(0.49)^{0.3}(0.36)^{0.3}(0.16)^{0.3}\}^{0.2}\{(0.16)^{0.1}(0.25)^{0.3}(0.25)^{0.3}(0.49)^{0.3}\}^{0.4}\end{array}}} \right\rangle$$

$$= \left\langle \frac{\sqrt{\begin{array}{c}(1.0324)(1.0897)(1.0309)(1.0734)- \\ [(0.9616)(0.8350)(0.9638)(0.9105)]\end{array}}}{\sqrt{\begin{array}{c}(1.0324)(1.0897)(1.0309)(1.0734)+ \\ [(0.9616)(0.8350)(0.9638)(0.9105)]\end{array}}}, \frac{\sqrt{2[(0.8695)(0.6247)(0.7909)(0.6116)]}}{\sqrt{\begin{array}{c}(1.0822)(1.1270)(1.1061)(1.2313)+ \\ (0.8695)(0.6247)(0.7909)(0.6116)\end{array}}} \right\rangle$$

$= \langle 0.5263\,, 0.5225 \rangle.$

A. PROPERTIES OF PFHSEOWA OPERATOR

1) IDEMPOTENCY

If $\mathfrak{I}_{\check{d}_{r(i)\,s(j)}} = \mathfrak{I}_{\check{d}_k} = \left(\alpha_{\check{d}_{r(i)\,s(j)}}, \mathcal{b}_{\check{d}_{r(i)\,s(j)}}\right) \forall i, j,$ then PFHSEOWA $(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{nm}}) = \mathfrak{I}_{\check{d}_k}$

PROOF: As we know that

PFHSEOWA $(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{nm}})$

$$= \left\langle \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha^{2}_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha^{2}_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha^{2}_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha^{2}_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{b}^{2}_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\mathcal{b}^{2}_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{b}^{2}_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

$$= \left\langle \frac{\sqrt{\left(\left(1+\alpha^{2}_{\check{d}_{r(i)\,s(j)}}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}-\left(\left(1-\alpha^{2}_{\check{d}_{r(i)\,s(j)}}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}}{\sqrt{\left(\left(1+\alpha^{2}_{\check{d}_{r(i)\,s(j)}}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}+\left(\left(1-\alpha^{2}_{\check{d}_{r(i)\,s(j)}}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}}, \frac{\sqrt{2\left(\left(\mathcal{b}^{2}_{\check{d}_{r(i)\,s(j)}}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}}{\sqrt{\left(\left(2-\mathcal{b}^{2}_{\check{d}_{r(i)\,s(j)}}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}+\left(\left(\mathcal{b}^{2}_{\check{d}_{r(i)\,s(j)}}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}} \right\rangle$$

$$= \left\langle \frac{\sqrt{\left(1+\alpha^{2}_{\check{d}_{r(i)\,s(j)}}\right)-\left(1-\alpha^{2}_{\check{d}_{r(i)\,s(j)}}\right)}}{\sqrt{\left(1+\alpha^{2}_{\check{d}_{r(i)\,s(j)}}\right)+\left(1-\alpha^{2}_{\check{d}_{r(i)\,s(j)}}\right)}}, \frac{\sqrt{2\mathcal{b}^{2}_{\check{d}_{r(i)\,s(j)}}}}{\sqrt{\left(2-\mathcal{b}^{2}_{\check{d}_{r(i)\,s(j)}}\right)+\left(\mathcal{b}^{2}_{\check{d}_{r(i)\,s(j)}}\right)}} \right\rangle$$

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$= \left\langle a_{\check{d}_{r(i)\,s(j)}}, b_{\check{d}_{r(i)\,s(j)}} \right\rangle$

$= \Im_{\check{d}_k}$.

2) BOUNDEDNESS

Let $\Im_{\check{d}_{r(i)\,s(j)}} = \left(a_{\check{d}_{r(i)\,s(j)}}, b_{\check{d}_{r(i)\,s(j)}}\right)$ be a collection PFHSNs and $\Im_{min} = min\left(\Im_{\check{d}_{r(i)\,s(j)}}\right)$, $\Im_{max} = max\left(\Im_{\check{d}_{r(i)\,s(j)}}\right)$. Then, $\Im_{min} \leq$ PFHSEOWA $\left(\Im_{\check{d}_{11}}, \Im_{\check{d}_{12}}, \ldots, \Im_{\check{d}_{nm}}\right) \leq \Im_{max}$.

PROOF: Let $g(y) = \sqrt{\frac{1-y^2}{1+y^2}}$, $y \in]0,1]$, then $\frac{d}{dy}\left(g(y)\right) = \frac{-2y}{(1+y^2)^2}\sqrt{\frac{1+y^2}{1-y^2}} < 0$. As $g(y) \leq \frac{d}{dy}\left(g(y)\right)$, then $g(y)$ is decreasing function on $]0,1]$. So, $a_{min} \leq a_{ij} \leq a_{max}$, $\forall\, i,j$. Hence, $g(a_{max}) \leq g\left(a_{\check{d}_{r(i)\,s(j)}}\right) \leq g(a_{min})$, $\forall\, i,j$.

$$\Rightarrow \sqrt{\frac{1-a_{max}^2}{1+a_{max}^2}} \leq \sqrt{\frac{1-a_{\check{d}_{r(i)\,s(j)}}^2}{1+a_{\check{d}_{r(i)\,s(j)}}^2}} \leq \sqrt{\frac{1-a_{min}^2}{1+a_{min}^2}}$$

Let θ_i and λ_j represents the weight vectors such as $\theta_i > 0$, $\sum_{i=1}^{n}\theta_i = 1$ and $\lambda_j > 0$, $\sum_{j=1}^{n}\lambda_j = 1$. We have

$$\Leftrightarrow \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-a_{max}^2}{1+a_{max}^2}\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-a_{\check{d}_{r(i)\,s(j)}}^2}{1+a_{\check{d}_{r(i)\,s(j)}}^2}\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-a_{min}^2}{1+a_{min}^2}\right)^{\theta_i}\right)^{\lambda_j}}$$

$$\Leftrightarrow \sqrt{\left(\left(\frac{1-a_{max}^2}{1+a_{max}^2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}} \leq \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-a_{\check{d}_{r(i)\,s(j)}}^2}{1+a_{\check{d}_{r(i)\,s(j)}}^2}\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{\left(\left(\frac{1-a_{min}^2}{1+a_{min}^2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}$$

$$\Leftrightarrow \sqrt{1+\left(\frac{1-a_{max}^2}{1+a_{max}^2}\right)} \leq \sqrt{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-a_{\check{d}_{r(i)\,s(j)}}^2}{1+a_{\check{d}_{r(i)\,s(j)}}^2}\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{1+\left(\frac{1-a_{min}^2}{1+a_{min}^2}\right)}$$

$$\Leftrightarrow \sqrt{\frac{2}{1+a_{max}^2}} \leq \sqrt{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-a_{\check{d}_{r(i)\,s(j)}}^2}{1+a_{\check{d}_{r(i)\,s(j)}}^2}\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{\frac{2}{1+a_{min}^2}}$$

$$\Leftrightarrow \sqrt{\frac{1+a_{min}^2}{2}} \leq \frac{1}{\sqrt{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-a_{\check{d}_{r(i)\,s(j)}}^2}{1+a_{\check{d}_{r(i)\,s(j)}}^2}\right)^{\theta_i}\right)^{\lambda_j}}} \leq \sqrt{\frac{1+a_{max}^2}{2}}$$

$$\Leftrightarrow \sqrt{1+a_{min}^2} \leq \sqrt{\frac{2}{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-a_{\check{d}_{r(i)\,s(j)}}^2}{1+a_{\check{d}_{r(i)\,s(j)}}^2}\right)^{\theta_i}\right)^{\lambda_j}}} \leq \sqrt{1+a_{max}^2}$$

$$\Leftrightarrow \sqrt{1+a_{min}^2-1} \leq \sqrt{\frac{2}{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-a_{\check{d}_{r(i)\,s(j)}}^2}{1+a_{\check{d}_{r(i)\,s(j)}}^2}\right)^{\theta_i}\right)^{\lambda_j}}-1} \leq \sqrt{1+a_{max}^2-1}$$

$$\Leftrightarrow \sqrt{a_{min}^2} \leq \sqrt{\frac{2}{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-a_{\check{d}_{r(i)\,s(j)}}^2}{1+a_{\check{d}_{r(i)\,s(j)}}^2}\right)^{\theta_i}\right)^{\lambda_j}}-1} \leq \sqrt{a_{max}^2}$$

$$\Leftrightarrow a_{min} \leq \sqrt{\frac{2}{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-a^{2}_{\breve{d}_{r(i)\,s(j)}}}{1+a^{2}_{\breve{d}_{r(i)\,s(j)}}}\right)^{\theta_i}\right)^{\lambda_j}}-1} \leq a_{max}$$

$$a_{min} \leq \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+a^{2}_{\breve{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-a^{2}_{\breve{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+a^{2}_{\breve{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-a^{2}_{\breve{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}} \leq a_{max} \tag{3.3}$$

Let $f(x) = \sqrt{\frac{2-x^2}{x^2}}$, $x \in \,]0,1]$, then $\frac{d}{dx}\left(f(x)\right) = \frac{-2}{x^3}\sqrt{\frac{x^2}{2-x^2}} < 0$. So, $f(x)$ is decreasing function on $]0,1]$. Since, $b_{min} \leq b_{ij} \leq b_{max}, \forall i,j$. Then, $f(b_{max}) \leq f\left(b_{ij}\right) \leq f(b_{min})$. So, $\sqrt{\frac{2-b^2_{max}}{b^2_{max}}} \leq \sqrt{\frac{2-b^2_{\breve{d}_{r(i)\,s(j)}}}{b^2_{\breve{d}_{r(i)\,s(j)}}}} \leq \sqrt{\frac{2-b^2_{min}}{b^2_{min}}}$. Where θ_i and λ_j represent the weights such as $\theta_i > 0$, $\sum_{i=1}^{n}\theta_i = 1$ and $\lambda_j > 0$, $\sum_{j=1}^{m}\lambda_j = 1$.

$$\Leftrightarrow \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-b^2_{max}}{b^2_{max}}\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-b^2_{\breve{d}_{r(i)\,s(j)}}}{b^2_{\breve{d}_{r(i)\,s(j)}}}\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-b^2_{min}}{b^2_{min}}\right)^{\theta_i}\right)^{\lambda_j}}$$

$$\Leftrightarrow \sqrt{\left(\left(\frac{2-b^2_{max}}{b^2_{max}}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}} \leq \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-b^2_{\breve{d}_{r(i)\,s(j)}}}{b^2_{\breve{d}_{r(i)\,s(j)}}}\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{\left(\left(\frac{2-b^2_{min}}{b^2_{min}}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}$$

$$\Leftrightarrow \sqrt{1+\left(\frac{2-b^2_{max}}{b^2_{max}}\right)} \leq \sqrt{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-b^2_{\breve{d}_{r(i)\,s(j)}}}{b^2_{\breve{d}_{r(i)\,s(j)}}}\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{1+\left(\frac{2-b^2_{min}}{b^2_{min}}\right)}$$

$$\Leftrightarrow \sqrt{\frac{2}{b^2_{max}}} \leq \sqrt{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-b^2_{\breve{d}_{r(i)\,s(j)}}}{b^2_{\breve{d}_{r(i)\,s(j)}}}\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{\frac{2}{b^2_{min}}}$$

$$\Leftrightarrow \sqrt{\frac{b^2_{min}}{2}} \leq \frac{1}{\sqrt{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-b^2_{\breve{d}_{r(i)\,s(j)}}}{b^2_{\breve{d}_{r(i)\,s(j)}}}\right)^{\theta_i}\right)^{\lambda_j}}} \leq \sqrt{\frac{b^2_{max}}{2}}$$

$$\Leftrightarrow b_{min} \leq \sqrt{\frac{2}{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-b^2_{\breve{d}_{r(i)\,s(j)}}}{b^2_{\breve{d}_{r(i)\,s(j)}}}\right)^{\theta_i}\right)^{\lambda_j}}} \leq b_{max}$$

$$b_{min} \leq \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b^2_{\breve{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-b^2_{\breve{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b^2_{\breve{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}} \leq b_{max} \tag{3.4}$$

Let PFHSEOWA $\left(\Im_{\breve{d}_{11}}, \Im_{\breve{d}_{12}}, \ldots, \Im_{\breve{d}_{nm}}\right) = \Im_{\breve{d}_k}$. Then, inequalities (3.3) and (3.4) can be written as $a_{min} \leq a \leq a_{max}$ *and* $b_{min} \leq b \leq b_{max}$. Thus, S $(\Im_{\breve{d}_k}) = a^2 - b^2 \leq a^2_{\max} - b^2_{\min}$ = S $(\Im_{\breve{d}_{k_{max}}})$ and S $(\Im_{\breve{d}_k})$ = $a^2 - b^2 \geq a^2_{\min} - b^2_{\max}$ = S $(\Im_{\breve{d}_{k_{min}}})$.

If S$(\Im_{\breve{d}_k}) <$ S $(\Im_{\breve{d}_{k_{max}}})$ and S$(\Im_{\breve{d}_k}) >$ S $(\Im_{\breve{d}_{k_{min}}})$. Then, we have

$$\Im_{\breve{d}_{k_{min}}} < \text{PFHSEOWA}\left(\Im_{\breve{d}_{11}}, \Im_{\breve{d}_{12}}, \ldots, \Im_{\breve{d}_{nm}}\right) < \Im_{\breve{d}_{k_{max}}} \tag{3.5}$$

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If S ($\Im_{\check{d}_k}$) = S ($\Im_{\check{d}_{k_{max}}}$), then we have $a^2 = a^2_{\max}$ and $b^2 = b^2_{\max}$. Thus, S ($\Im_{\check{d}_k}$) $= a^2 - b^2 = a^2_{\max} - b^2_{\max}$ = S ($\Im_{\check{d}_{k_{max}}}$). Therefore,

PFHSEOWA $\left(\Im_{\check{d}_{11}}, \Im_{\check{d}_{12}}, \ldots, \Im_{\check{d}_{nm}}\right) = \Im_{\check{d}_{k_{max}}}$. (3.6)

If S ($\Im_{\check{d}_k}$) = S ($\Im_{\check{d}_{k_{min}}}$). Then, we have $a^2 - b^2 = a^2_{\min} - b^2_{\min} \Rightarrow a^2 = a^2_{\min}$ and $b^2 = b^2_{\min}$.

Thus, A ($\Im_{\check{d}_k}$) $= a^2 + b^2 = a^2_{\min} + b^2_{\min}$ = A ($\Im_{\check{d}_{k_{min}}}$). So,

PFHSEOWA $\left(\Im_{\check{d}_{11}}, \Im_{\check{d}_{12}}, \ldots, \Im_{\check{d}_{nm}}\right) = \Im_{\check{d}_{k_{min}}}$ (3.7)

So, from inequalities (3.5), (3.6), and (3.7), we get $\Im_{\check{d}_{k_{min}}} \leq$ PFHSEOWA $\left(\Im_{\check{d}_{11}}, \Im_{\check{d}_{12}}, \ldots, \Im_{\check{d}_{nm}}\right) \leq \Im_{\check{d}_{k_{max}}}$.

3) HOMOGENEITY

Prove that PFHSEOWA $\left(\Im_{\check{d}_{11}}, \Im_{\check{d}_{12}}, \ldots, \Im_{\check{d}_{nm}}\right) = \partial$ PFHSEOWA $\left(\Im_{\check{d}_{11}}, \Im_{\check{d}_{12}}, \ldots, \Im_{\check{d}_{nm}}\right)$ for $\partial > 0$.

PROOF: Let $\Im_{\check{d}_{ij}}$ be a PFHSN and$\partial > 0$, then by

$$\partial\Im_{\check{d}_{r(i)\,s(j)}} = \left\langle \frac{\sqrt{\left(1+a_{\check{d}_{r(i)\,s(j)}}{}^2\right)^{\partial} - \left(1-a_{\check{d}_{r(i)\,s(j)}}{}^2\right)^{\partial}}}{\sqrt{\left(1+a_{\check{d}_{r(i)\,s(j)}}{}^2\right)^{\partial} + \left(1-a_{\check{d}_{r(i)\,s(j)}}{}^2\right)^{\partial}}}, \frac{\sqrt{2\left(b_{\check{d}_{r(i)\,s(j)}}{}^2\right)^{\partial}}}{\sqrt{\left(2-b_{\check{d}_{r(i)\,s(j)}}{}^2\right)^{\partial} + \left(b_{\check{d}_{r(i)\,s(j)}}{}^2\right)^{\partial}}} \right\rangle$$

So,

PFHSEOWA $(\partial\Im_{\check{d}_{11}}, \partial\Im_{\check{d}_{12}}, \ldots, \partial\Im_{\check{d}_{nm}})$

$$= \left\langle \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-b^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

$$= \left\langle \frac{\sqrt{\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial} - \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}}}{\sqrt{\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial} + \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}}}, \frac{\sqrt{\left(2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}}}{\sqrt{\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-b^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial} + \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}}} \right\rangle$$

= ∂ PFHSEOWA $(\Im_{\check{d}_{11}}, \Im_{\check{d}_{12}}, \ldots, \Im_{\check{d}_{nm}})$.

4) DEFINITION

Let $\Im_{\check{d}_{ij}} = \left(a_{\check{d}_{ij}}, b_{\check{d}_{ij}}\right)$ be an assortment of PFHSNs. Then the PFHSEOWG operator is demarcated as:

$$\text{PFHSEOWG}\left(\Im_{\check{d}_{11}}, \Im_{\check{d}_{12}}, \ldots, \Im_{\check{d}_{nm}}\right) = \otimes_{j=1}^{m}\left(\otimes_{i=1}^{n}\left(\Im_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} \quad (3.8)$$

Where θ_i, λ_j denote the weighted vectors such that $\theta_i > 0$, $\sum_{i=1}^{n}\theta_i = 1$ and $\lambda_j > 0$, $\sum_{j=1}^{n}\lambda_j = 1$ and $\mathfrak{r}, \mathfrak{s}$ are permutations such that $\Im_{\check{d}_{\mathfrak{r}(i-1)\mathfrak{s}(j)}} \geq \Im_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}$ and $\Im_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j-1)}} \geq \Im_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}} \forall i, j$.

5) THEOREM

Let $\Im_{\check{d}_{ij}} = \left(a_{\check{d}_{ij}}, b_{\check{d}_{ij}}\right)$ be a collection of PFHSNs. Then the obtained aggregated value using Definition 4 is given as

PFHSEOWG $\left(\Im_{\check{d}_{11}}, \Im_{\check{d}_{12}}, \ldots, \Im_{\check{d}_{nm}}\right) = \otimes_{j=1}^{m}\left(\otimes_{i=1}^{n}\left(\Im_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(a^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-a^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(a^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+b^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-b^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+b^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-b^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle \quad (3.9)$$

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Where θ_i, λ_j denote the weighted vectors such that $\theta_i > 0, \sum_{i=1}^{n} \theta_i = 1$ and $\lambda_j > 0,\ \sum_{j=1}^{n} \lambda_j = 1$ and $\mathfrak{r}, \mathfrak{s}$ are permutations such that $\mathfrak{J}_{\check{d}_{\mathfrak{r}(i-1)\mathfrak{s}(j)}} \geq \mathfrak{J}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}$ and $\mathfrak{J}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j-1)}} \geq \mathfrak{J}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}} \forall i, j$.

PROOF: The mentioned above theorem can be verified using mathematical induction. For n = 1, we get $\theta_i = 1$.

$$\text{PFHSEOWG}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \otimes_{j=1}^{m} \left(\mathfrak{J}_{\check{d}_{\mathfrak{r}(1)\mathfrak{s}(j)}}\right)^{\lambda_j}$$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{m}\left(\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(1)\mathfrak{s}(j)}}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(2-\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(1)\mathfrak{s}(j)}}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(1)\mathfrak{s}(j)}}\right)^{\lambda_j}}}, \frac{\sqrt{\prod_{j=1}^{m}\left(1+\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(1)\mathfrak{s}(j)}}\right)^{\lambda_j}-\prod_{j=1}^{m}\left(1-\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(1)\mathfrak{s}(j)}}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(1+\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(1)\mathfrak{s}(j)}}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(1-\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(1)\mathfrak{s}(j)}}\right)^{\lambda_j}}} \right\rangle$$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(2-\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1+\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1-\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1+\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1-\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

For $m = 1$, we get $\lambda_j = 1$.

$$\text{PFHSEOWG}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \otimes_{i=1}^{n} \left(\mathfrak{J}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(1)}}\right)^{\theta_i}$$

$$= \left\langle \frac{\sqrt{2\prod_{i=1}^{n}\left(\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(1)}}\right)^{\theta_i}}}{\sqrt{\prod_{j=1}^{m}\left(2-\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(1)}}\right)^{\theta_i}+\prod_{j=1}^{m}\left(\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(1)}}\right)^{\theta_i}}}, \frac{\sqrt{\prod_{j=1}^{m}\left(1+\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(1)}}\right)^{\theta_i}-\prod_{j=1}^{m}\left(1-\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(1)}}\right)^{\theta_i}}}{\sqrt{\prod_{j=1}^{m}\left(1+\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(1)}}\right)^{\theta_i}+\prod_{j=1}^{m}\left(1-\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(1)}}\right)^{\theta_i}}} \right\rangle$$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(2-\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1+\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1-\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1+\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1-\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

So, Theorem 5 true for n = 1 and m = 1.

Assume that Theorem 5 holds for n = δ_2 , m = $\delta_1 + 1$ and for n = $\delta_2 + 1$, m = δ_1

$$\otimes_{j=1}^{\delta_1+1}\left(\otimes_{i=1}^{\delta_2}\left(\mathfrak{J}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}$$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(2-\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1+\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1-\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1+\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1-\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

$$\otimes_{j=1}^{\delta_1}\left(\otimes_{i=1}^{\delta_2+1}\left(\mathfrak{J}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}$$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(2-\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{\prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(1+\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(1-\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(1+\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(1-\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

Now, we prove Equation 3.9 for m = $\delta_1 + 1$ and n = $\delta_2 + 1$

$$\otimes_{j=1}^{\delta_1+1}\left(\otimes_{i=1}^{\delta_2+1}\left(\mathfrak{J}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j} = \otimes_{j=1}^{\delta_1+1}\left(\otimes_{i=1}^{\delta_2}\left(\mathfrak{J}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i} \otimes \left(\mathfrak{J}_{\check{d}_{\mathfrak{r}(\delta_2+1)\mathfrak{s}(j)}}\right)^{\theta_{i+1}}\right)^{\lambda_j}$$

$$= \left(\otimes_{j=1}^{\delta_1+1}\otimes_{i=1}^{\delta_2}\left(\mathfrak{J}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i\lambda_j}\right)\left(\otimes_{j=1}^{\delta_1+1}\left(\mathfrak{J}_{\check{d}_{\mathfrak{r}(\delta_2+1)\mathfrak{s}(j)}}\right)^{\lambda_j\theta_{i+1}}\right)$$

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$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(2-\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}} \otimes \frac{\sqrt{2\prod_{j=1}^{\delta_1+1}\left(\left(\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(\delta_2+1)\mathfrak{s}(j)}}\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\left(2-\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(\delta_2+1)\mathfrak{s}(j)}}\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j}+\prod_{j=1}^{\delta_1+1}\left(\left(\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(\delta_2+1)\mathfrak{s}(j)}}\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j}}}, \frac{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1+\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1-\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1+\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1-\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}} \otimes \frac{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\left(1+\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(\delta_2+1)\mathfrak{s}(j)}}\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j}-\prod_{j=1}^{\delta_1+1}\left(\left(1-\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(\delta_2+1)\mathfrak{s}(j)}}\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\left(1+\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(\delta_2+1)\mathfrak{s}(j)}}\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j}+\prod_{j=1}^{\delta_1+1}\left(\left(1-\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(\delta_2+1)\mathfrak{s}(j)}}\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j}}} \right\rangle$$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(2-\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(1+\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(1-\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(1+\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(1-\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

$$= \otimes_{j=1}^{\delta_1+1}\left(\otimes_{i=1}^{\delta_2+1}\left(\mathfrak{I}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}$$

So, it is valid for m = $\delta_1 + 1$ and n = $\delta_2 + 1$.

6) EXAMPLE:

Let $\mathcal{R} = \{\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4\}$ be a group of experts with weights $\theta_i = (0.1, 0.3, 0.3, 0.3)^T$. The group of specialists is going to designate the attraction of a company's under-considered set of attributes $A = \{d_1 = lawn,\ d_2 = security\ system\}$ with their corresponding sub-attributes Lawn = d_1 = $\{d_{11} = with\ grass, d_{12} = without\ grass\}$Security system = d_2 = $\{d_{21} = guards, d_{22} = cameras\}$. Let $\ddot{A}$ = $d_1 \times d_2$ be a set of sub-attributes $\ddot{A} = d_1 \times d_2 = \{d_{11}, d_{12}\} \times \{d_{21}, d_{22}\} = \{(d_{11}, d_{21}), (d_{11}, d_{22}), (d_{12}, d_{21}), (d_{12}, d_{22})\}$ $\ddot{A} = \{\check{d}_1, \check{d}_2, \check{d}_3, \check{d}_4\}$ denotes the set sub-attributes with weights $\lambda_j = (0.2, 0.2, 0.2, 0.4)^T$. The hypothetical score values for all attributes in PFHSNs form $(\mathcal{H}, \ddot{A}) = (a_{ij}, b_{ij})_{4\times4}$ given as:

$$(\mathcal{H}, \ddot{A}) = \begin{bmatrix} (0.5, 0.8) & (0.7, 0.5) & (0.4, 0.6) & (0.7, 0.4) \\ (0.5, 0.6) & (0.9, 0.1) & (0.3, 0.7) & (0.4, 0.5) \\ (0.4, 0.8) & (0.7, 0.5) & (0.4, 0.6) & (0.3, 0.5) \\ (0.3, 0.7) & (0.6, 0.5) & (0.5, 0.4) & (0.5, 0.7) \end{bmatrix}$$

Obtain the ordered position matrix:

$$(\mathcal{H}_{4\times4}, A) = \begin{bmatrix} (0.7, 0.4) & (0.7, 0.5) & (0.4, 0.6) & (0.5, 0.8) \\ (0.9, 0.1) & (0.4, 0.5) & (0.3, 0.7) & (0.5, 0.6) \\ (0.7, 0.5) & (0.4, 0.6) & (0.3, 0.5) & (0.4, 0.8) \\ (0.6, 0.5) & (0.5, 0.4) & (0.3, 0.7) & (0.5, 0.7) \end{bmatrix}$$

As we know that

PFHSEOWG $(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{nm}})$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

PFHSEOWG $(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{44}})$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(2-\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(\mathfrak{a}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1+\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1+\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-\mathfrak{b}^2_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

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$$= \left\langle \frac{\sqrt{2\begin{bmatrix}\{(0.49)^{0.1}(0.81)^{0.3}(0.49)^{0.3}(0.36)^{0.3}\}^{0.2}\{(0.49)^{0.1}(0.16)^{0.3}(0.16)^{0.3}(0.25)^{0.3}\}^{0.2}\\ \{(0.16)^{0.1}(0.09)^{0.3}(0.09)^{0.3}(0.09)^{0.3}\}^{0.2}\{(0.25)^{0.1}(0.25)^{0.3}(0.16)^{0.3}(0.25)^{0.3}\}^{0.4}\end{bmatrix}}}{\sqrt{\begin{matrix}\{(1.51)^{0.1}(1.19)^{0.3}(1.51)^{0.3}(1.64)^{0.3}\}^{0.2}\{(1.51)^{0.1}(1.84)^{0.3}(1.84)^{0.3}(1.75)^{0.3}\}^{0.2}\\ \{(1.84)^{0.1}(1.91)^{0.3}(1.91)^{0.3}(1.91)^{0.3}\}^{0.2}\{(1.75)^{0.1}(1.75)^{0.3}(1.84)^{0.3}(1.75)^{0.3}\}^{0.4}\\ +\\ \{(0.49)^{0.1}(0.81)^{0.3}(0.49)^{0.3}(0.36)^{0.3}\}^{0.2}\{(0.49)^{0.1}(0.16)^{0.3}(0.16)^{0.3}(0.25)^{0.3}\}^{0.2}\\ \{(0.16)^{0.1}(0.09)^{0.3}(0.09)^{0.3}(0.09)^{0.3}\}^{0.2}\{(0.25)^{0.1}(0.25)^{0.3}(0.16)^{0.3}(0.25)^{0.3}\}^{0.4}\end{matrix}}}, \frac{\sqrt{\begin{matrix}\{(1.16)^{0.1}(1.01)^{0.3}(1.25)^{0.3}(1.25)^{0.3}\}^{0.2}\{(1.25)^{0.1}(1.25)^{0.3}(1.36)^{0.3}(1.16)^{0.3}\}^{0.2}\\ \{(1.36)^{0.1}(1.49)^{0.3}(1.25)^{0.3}(1.49)^{0.3}\}^{0.2}\{(1.64)^{0.1}(1.36)^{0.3}(1.64)^{0.3}(1.49)^{0.3}\}^{0.4}\\ -\\ [\{(0.84)^{0.1}(0.99)^{0.3}(0.75)^{0.3}(0.75)^{0.3}\}^{0.2}\{(0.75)^{0.1}(0.75)^{0.3}(0.64)^{0.3}(0.84)^{0.3}\}^{0.2}\\ \{(0.64)^{0.1}(0.51)^{0.3}(0.75)^{0.3}(0.51)^{0.3}\}^{0.2}\{(0.36)^{0.1}(0.64)^{0.3}(0.36)^{0.3}(0.51)^{0.3}\}^{0.4}\end{matrix}}}{\sqrt{\begin{matrix}\{(1.16)^{0.1}(1.01)^{0.3}(1.25)^{0.3}(1.25)^{0.3}\}^{0.2}\{(1.25)^{0.1}(1.25)^{0.3}(1.36)^{0.3}(1.16)^{0.3}\}^{0.2}\\ \{(1.36)^{0.1}(1.49)^{0.3}(1.25)^{0.3}(1.49)^{0.3}\}^{0.2}\{(1.64)^{0.1}(1.36)^{0.3}(1.64)^{0.3}(1.49)^{0.3}\}^{0.4}\\ +\\ [\{(0.84)^{0.1}(0.99)^{0.3}(0.75)^{0.3}(0.75)^{0.3}\}^{0.2}\{(0.75)^{0.1}(0.75)^{0.3}(0.64)^{0.3}(0.84)^{0.3}\}^{0.2}\\ \{(0.64)^{0.1}(0.51)^{0.3}(0.75)^{0.3}(0.51)^{0.3}\}^{0.2}\{(0.36)^{0.1}(0.64)^{0.3}(0.36)^{0.3}(0.51)^{0.3}\}^{0.4}\end{matrix}}} \right\rangle$$

$= \langle 0.4923 , 0.5743\rangle.$

B. PROPERTIES OF PFHSEOWG OPERATOR

1) IDEMPOTENCY

Let $\mathfrak{I}_{\breve{d}_{ij}} = \left(\mathscr{a}_{\breve{d}_{ij}}, \mathscr{b}_{\breve{d}_{ij}}\right)$ be a collection of PFHSNs. If $\mathfrak{I}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}} = \mathfrak{I}_{\breve{d}_{\mathfrak{r}(1)\mathfrak{s}(1)}}$ are identical. Then

PFHSEOWG $\left(\mathfrak{I}_{\breve{d}_{11}}, \mathfrak{I}_{\breve{d}_{12}}, \ldots, \mathfrak{I}_{\breve{d}_{nm}}\right) = \mathfrak{I}$

PROOF: As we know that

PFHSEOWG $\left(\mathfrak{I}_{\breve{d}_{11}}, \mathfrak{I}_{\breve{d}_{12}}, \ldots, \mathfrak{I}_{\breve{d}_{nm}}\right) =$

$$\left\langle \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{a}^2_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\mathfrak{a}^2_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{a}^2_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathfrak{b}^2_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathfrak{b}^2_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathfrak{b}^2_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathfrak{b}^2_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}} \right\rangle$$

As $\mathfrak{I}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}} = \mathfrak{I}_{ij}$,

So

$$= \left\langle \frac{\sqrt{2\left(\left(\mathfrak{a}^2_{ij}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}}{\sqrt{\left(\left(2-\mathfrak{a}^2_{ij}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j} + \left(\left(\mathfrak{a}^2_{ij}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}}, \frac{\sqrt{\left(\left(1+\mathfrak{b}^2_{ij}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j} - \left(\left(1-\mathfrak{b}^2_{ij}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}}{\sqrt{\left(\left(1+\mathfrak{b}^2_{ij}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j} + \left(\left(1-\mathfrak{b}^2_{ij}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}} \right\rangle$$

$$= \left\langle \frac{\sqrt{2\mathfrak{a}^2_{ij}}}{\sqrt{\left(2-\mathfrak{a}^2_{ij}\right) + \left(\mathfrak{a}^2_{ij}\right)}}, \frac{\sqrt{\left(1+\mathfrak{a}^2_{ij}\right) - \left(1-\mathfrak{a}^2_{ij}\right)}}{\sqrt{\left(1+\mathfrak{a}^2_{ij}\right) + \left(1-\mathfrak{a}^2_{ij}\right)}} \right\rangle = \left\langle \mathfrak{a}_{ij}, \mathfrak{b}_{ij} \right\rangle = \mathfrak{I}.$$

2) BOUNDEDNESS

Let $\mathfrak{I}_{\breve{d}_{ij}} = \left(\mathscr{a}_{\breve{d}_{ij}}, \mathscr{b}_{\breve{d}_{ij}}\right)$ be a collection of PFHSNs. Where θ_i and λ_j indicate the weights such that $\theta_i > 0, \sum_{i=1}^{n}\theta_i = 1$ and $\lambda_j > 0,\ \sum_{j=1}^{n}\lambda_j = 1$. If $\mathfrak{I}_{min} = min\left(\mathfrak{I}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)$ and $\mathfrak{I}_{max} = max\left(\mathfrak{I}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)$. Then

$\mathfrak{I}_{min} \leq$ PFHSEOWG $\left(\mathfrak{I}_{\breve{d}_{11}}, \mathfrak{I}_{\breve{d}_{12}}, \ldots, \mathfrak{I}_{\breve{d}_{nm}}\right) \leq \mathfrak{I}_{max}$

PROOF: Let $f(x) = \sqrt{\frac{2-x^2}{x^2}}$, $x \in]0, 1]$, then $\frac{d}{dx}\left(f(x)\right) = \frac{-2}{x^3}\sqrt{\frac{x^2}{2-x^2}} < 0$. So, $f(x)$ is decreasing function on $]0, 1]$.

As $\mathfrak{a}_{min} \leq \mathfrak{a}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}} \leq \mathfrak{a}_{max}$, $\forall i, j$, so $f(\mathfrak{a}_{max}) \leq f\left(\mathfrak{a}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right) \leq f(\mathfrak{a}_{min})$.

and, $\sqrt{\frac{2-\mathfrak{a}^2_{max}}{\mathfrak{a}^2_{max}}} \leq \sqrt{\frac{2-\mathfrak{a}^2_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}}{\mathfrak{a}^2_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}}} \leq \sqrt{\frac{2-\mathfrak{a}^2_{min}}{\mathfrak{a}^2_{min}}}$.

Let θ_i and λ_j exemplify the weight vectors such that$\theta_i > 0, \sum_{i=1}^{n}\theta_i = 1$ and $\lambda_j > 0,\ \sum_{j=1}^{n}\lambda_j = 1$. Then we have

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$$\Leftrightarrow \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-\mathfrak{a}_{max}^2}{\mathfrak{a}_{max}^2}\right)^{\theta_i}\right)^{\lambda_j}} \le \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-\mathfrak{a}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}{\mathfrak{a}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}\right)^{\theta_i}\right)^{\lambda_j}} \le \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-\mathfrak{a}_{min}^2}{\mathfrak{a}_{min}^2}\right)^{\theta_i}\right)^{\lambda_j}}$$

$$\Leftrightarrow \sqrt{\left(\left(\frac{2-\mathfrak{a}_{max}^2}{\mathfrak{a}_{max}^2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}} \le \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-\mathfrak{a}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}{\mathfrak{a}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}\right)^{\theta_i}\right)^{\lambda_j}} \le \sqrt{\left(\left(\frac{2-\mathfrak{a}_{min}^2}{\mathfrak{a}_{min}^2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}$$

$$\Leftrightarrow \sqrt{1+\left(\frac{2-\mathfrak{a}_{max}^2}{\mathfrak{a}_{max}^2}\right)} \le \sqrt{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-\mathfrak{a}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}{\mathfrak{a}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}\right)^{\theta_i}\right)^{\lambda_j}} \le \sqrt{1+\left(\frac{2-\mathfrak{a}_{min}^2}{\mathfrak{a}_{min}^2}\right)}$$

$$\Leftrightarrow \sqrt{\frac{2}{\mathfrak{a}_{max}^2}} \le \sqrt{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-\mathfrak{a}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}{\mathfrak{a}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}\right)^{\theta_i}\right)^{\lambda_j}} \le \sqrt{\frac{2}{\mathfrak{a}_{min}^2}}$$

$$\Leftrightarrow \sqrt{\frac{\mathfrak{a}_{min}^2}{2}} \le \frac{1}{\sqrt{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-\mathfrak{a}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}{\mathfrak{a}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}\right)^{\theta_i}\right)^{\lambda_j}}} \le \sqrt{\frac{\mathfrak{a}_{max}^2}{2}}$$

$$\Leftrightarrow \mathfrak{a}_{min} \le \sqrt{\frac{2}{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-\mathfrak{a}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}{\mathfrak{a}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}\right)^{\theta_i}\right)^{\lambda_j}}} \le \mathfrak{a}_{max}$$

$$\mathfrak{a}_{min} \le \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{a}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\mathfrak{a}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{a}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2\right)^{\theta_i}\right)^{\lambda_j}}} \le \mathfrak{a}_{max} \tag{3.10}$$

Let $g(y) = \sqrt{\frac{1-y^2}{1+y^2}}$, $y \in]0,1]$, then $\frac{d}{dy}\left(g(y)\right) = \frac{-2y}{(1+y^2)^2}\sqrt{\frac{1+y^2}{1-y^2}} < 0$. Which shows that $g(y)$ is decreasing function on $]0,1]$. So, $\mathfrak{b}_{min} \le \mathfrak{b}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}} \le \mathfrak{b}_{max}$.

Hence, $g(\mathfrak{b}_{max}) \le g\left(\mathfrak{b}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right) \le g(\mathfrak{b}_{min})$,

$$\Rightarrow \sqrt{\frac{1-\mathfrak{b}_{max}^2}{1+\mathfrak{b}_{max}^2}} \le \sqrt{\frac{1-\mathfrak{b}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}{1+\mathfrak{b}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}} \le \sqrt{\frac{1-\mathfrak{b}_{min}^2}{1+\mathfrak{b}_{min}^2}},$$

Let θ_i and λ_j signify the weight vectors such that $\theta_i > 0$, $\sum_{i=1}^{n}\theta_i = 1$ and $\lambda_j > 0$, $\sum_{j=1}^{n}\lambda_j = 1$. Then we have

$$\Leftrightarrow \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-\mathfrak{b}_{max}^2}{1+\mathfrak{b}_{max}^2}\right)^{\theta_i}\right)^{\lambda_j}} \le \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-\mathfrak{b}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}{1+\mathfrak{b}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}\right)^{\theta_i}\right)^{\lambda_j}} \le \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-\mathfrak{b}_{min}^2}{1+\mathfrak{b}_{min}^2}\right)^{\theta_i}\right)^{\lambda_j}}$$

$$\Leftrightarrow \sqrt{\left(\left(\frac{1-\mathfrak{b}_{max}^2}{1+\mathfrak{b}_{max}^2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}} \le \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-\mathfrak{b}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}{1+\mathfrak{b}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}\right)^{\theta_i}\right)^{\lambda_j}} \le \sqrt{\left(\left(\frac{1-\mathfrak{b}_{min}^2}{1+\mathfrak{b}_{min}^2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}$$

$$\Leftrightarrow \sqrt{1+\left(\frac{1-\mathfrak{b}_{max}^2}{1+\mathfrak{b}_{max}^2}\right)} \le \sqrt{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-\mathfrak{b}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}{1+\mathfrak{b}_{\breve{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}\right)^{\theta_i}\right)^{\lambda_j}} \le \sqrt{1+\left(\frac{1-\mathfrak{b}_{min}^2}{1+\mathfrak{b}_{min}^2}\right)}$$

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$$\Leftrightarrow \sqrt{\frac{2}{1+\mathfrak{b}_{max}^2}} \leq \sqrt{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-\mathfrak{b}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}{1+\mathfrak{b}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{\frac{2}{1+\mathfrak{b}_{min}^2}}$$

$$\Leftrightarrow \sqrt{\frac{1+\mathfrak{b}_{min}^2}{2}} \leq \frac{1}{\sqrt{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-\mathfrak{b}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}{1+\mathfrak{b}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}\right)^{\theta_i}\right)^{\lambda_j}}} \leq \sqrt{\frac{1+\mathfrak{b}_{max}^2}{2}}$$

$$\Leftrightarrow \sqrt{1+\mathfrak{b}_{min}^2} \leq \sqrt{\frac{2}{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-\mathfrak{b}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}{1+\mathfrak{b}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}\right)^{\theta_i}\right)^{\lambda_j}}} \leq \sqrt{1+\mathfrak{b}_{max}^2}$$

$$\Leftrightarrow \sqrt{1+\mathfrak{b}_{min}^2-1} \leq \sqrt{\frac{2}{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-\mathfrak{b}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}{1+\mathfrak{b}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}\right)^{\theta_i}\right)^{\lambda_j}}-1} \leq \sqrt{1+\mathfrak{b}_{max}^2-1}$$

$$\Leftrightarrow \sqrt{\mathfrak{b}_{min}^2} \leq \sqrt{\frac{2}{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-\mathfrak{b}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}{1+\mathfrak{b}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}\right)^{\theta_i}\right)^{\lambda_j}}-1} \leq \sqrt{\mathfrak{b}_{max}^2}$$

$$\Leftrightarrow \mathfrak{b}_{min} \leq \sqrt{\frac{2}{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-\mathfrak{b}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}{1+\mathfrak{b}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2}\right)^{\theta_i}\right)^{\lambda_j}}-1} \leq \mathfrak{b}_{max}$$

$$\mathfrak{b}_{min} \leq \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathfrak{b}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathfrak{b}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathfrak{b}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathfrak{b}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}^2\right)^{\theta_i}\right)^{\lambda_j}}} \leq \mathfrak{b}_{max} \quad (3.11)$$

Let PFHSEOWG $\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \mathfrak{J}$
Then, inequalities (3.10) and (3.11) can be written as

$\mathfrak{a}_{min} \leq \mathfrak{a} \leq \mathfrak{a}_{max}$ *and* $\mathfrak{b}_{min} \leq \mathfrak{b} \leq \mathfrak{b}_{max}$,

Thus, S $(\mathfrak{J})$ = $\mathfrak{a}^2 - \mathfrak{b}^2 \leq \mathfrak{a}_{\max}^2 - \mathfrak{b}_{\min}^2$ = S $(\mathfrak{J}_{max})$ and

S $(\mathfrak{J})$ = $\mathfrak{a}^2 - \mathfrak{b}^2 \geq \mathfrak{a}_{\min}^2 - \mathfrak{b}_{\max}^2$ = S $(\mathfrak{J}_{min})$.

If S$(\mathfrak{J})$ < S $(\mathfrak{J}_{max})$ and S$(\mathfrak{J})$ > S $(\mathfrak{J}_{min})$.

Then, we have

$\mathfrak{J}_{min}$ < PFHSEOWG $\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) < \mathfrak{J}_{max}$ (3.12)

If S $(\mathfrak{J})$ = S $(\mathfrak{J}_{max})$, then we have $\mathfrak{a}^2 = \mathfrak{a}_{\max}^2$ and $\mathfrak{b}^2 = \mathfrak{b}_{\max}^2$. Thus,

A$(\mathfrak{J})$ = $\mathfrak{a}^2 + \mathfrak{b}^2$ = $\mathfrak{a}_{\max}^2 + \mathfrak{b}_{\max}^2$ = A$(\mathfrak{J}_{max})$. Therefore,

PFHSEOWG $\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \mathfrak{J}_{max}$ (3.13)

If S $(\mathfrak{J})$ = S $(\mathfrak{J}_{min})$. Then we have $\mathfrak{a}^2 - \mathfrak{b}^2 = \mathfrak{a}_{\min}^2 - \mathfrak{b}_{\min}^2$. $\mathfrak{a}^2 = \mathfrak{a}_{\min}^2$ and $\mathfrak{b}^2 = \mathfrak{b}_{\min}^2$.

Thus, A $(\mathfrak{J})$ = $\mathfrak{a}^2 + \mathfrak{b}^2 = \mathfrak{a}_{\min}^2 + \mathfrak{b}_{\min}^2$ = A $(\mathfrak{J}_{min})$. Therefore,

PFHSEOWG $\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \mathfrak{J}_{min}$ (3.14)

So, from (3.12-3.14)
Hence, $\mathfrak{J}_{min} \leq$ PFHSEOWG $\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) \leq \mathfrak{J}_{max}$.

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3) (Commutativity). Prove that *PFHSEOWG* $\left(\partial\mathfrak{J}_{\check{d}_{11}}, \partial\mathfrak{J}_{\check{d}_{12}}, \ldots, \partial\mathfrak{J}_{\check{d}_{nm}}\right) = \partial$ *PFHSEOWG*$\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right)$, where $\partial > 0$.

PROOF: Let $\mathfrak{J}_{\check{d}_{ij}}$ be a PFHSN and $\partial > 0$, then we know that

$$\partial\mathfrak{J}_{\check{d}_{ij}} = \left\langle \frac{\sqrt{2(a_{\check{d}_{ij}}{}^{2})^{\partial}}}{\sqrt{\left(2-a_{\check{d}_{ij}}{}^{2}\right)^{\partial}+(a_{\check{d}_{ij}}{}^{2})^{\partial}}}, \frac{\sqrt{\left(1+b_{\check{d}_{ij}}{}^{2}\right)^{\partial}-\left(1-b_{\check{d}_{ij}}{}^{2}\right)^{\partial}}}{\sqrt{\left(1+b_{\check{d}_{ij}}{}^{2}\right)^{\partial}+\left(1-b_{\check{d}_{ij}}{}^{2}\right)^{\partial}}} \right\rangle$$

So,

$$\text{PFHSEOWG}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right)$$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(a^{2}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-a^{2}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(a^{2}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+b^{2}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-b^{2}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+b^{2}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-b^{2}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

$$= \left\langle \frac{\sqrt{\left(2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(a^{2}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}}}{\sqrt{\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-a^{2}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}+\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(a^{2}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}}}, \frac{\sqrt{\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+b^{2}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}-\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-b^{2}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}}}{\sqrt{\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+b^{2}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}+\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-b^{2}_{\check{d}_{\mathfrak{r}(i)\mathfrak{s}(j)}}\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}}} \right\rangle$$

$$= \partial\ \text{PFHSEOWG}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right).$$

IV. MULTI-CRITERIA DECISION-MAKING APPROACH BASED ON PROPOSED OPERATORS

To validate the implications of planned AOs, a DM approach has emerged to remove MCDM barriers. In addition, numerical information is delivered to endorse the convenience of the projected approach.

A. PROPOSED MCGDM APPROACH

Consider $\mathfrak{H} = \{\mathfrak{H}^1, \mathfrak{H}^2, \mathfrak{H}^3, \ldots, \mathfrak{H}^s\}$ be a set of substitutes $O = \{O_1, O_2, O_3, \ldots, O_r\}$ be a set of specialists. Let $\theta = (\theta_1, \theta_2, \theta_3, \ldots, \theta_n)^T$ be a weight vector for specialists such as $\theta_i > 0$, $\sum_{i=1}^{n}\theta_i = 1$. Let $\mathfrak{L} = \{d_1, d_2, \ldots, d_m\}$ be a set of parameters with multi sub-attributes such as $\mathfrak{L}' = \{(d\_1\rho\times d\_2\rho\times\ldots\times d\_m\rho)$ $\forall$ $\rho\in\{1, 2,\ldots, t\}\}$ with weight vector $\theta = (\theta_1, \theta_2, \theta_3, \ldots, \theta_n)^T$ such as $\theta_i > 0$, $\sum_{i=1}^{n}\theta_i = 1$, and can be identified as $\mathfrak{L}' = \left\{\check{d}_{\partial}: \partial \in \{1, 2, \ldots, m\}\right\}$. The group of specialists $\{\kappa^i: i = 1, 2,\ldots, n\}$ evaluate the alternatives $\{\mathfrak{H}^{(z)}: z = 1, 2, \ldots, s\}$ beneath the selected sub-attributes $\{\check{d}_{\partial}: \partial = 1, 2, \ldots, k\}$ in PFHSNs form such as $\left(\mathfrak{H}^{(z)}_{\check{d}_{ik}}\right)_{n\times m} = \left(\alpha_{\check{d}_{ij}}, b_{\check{d}_{ij}}\right)_{n*m}$, where $0 \leq \alpha_{\check{d}_{ij}}, b_{\check{d}_{ij}} \leq 1$ and $0 \leq \left(\alpha_{\check{d}_{ij}}\right)^2 + \left(b_{\check{d}_{ij}}\right)^2 \leq 1$ for all i, k. The team of specialists delivers their judgment for each alternate in PFHSNs form $\mathcal{L}_k$. The step-wise algorithm to obtain the most suitable alternative is given in the following.

Step 1. In light of the expert's opinion, develop the decision matrices in the form of PFHSNs $F = \left(\mathfrak{J}_{\check{d}_{ij}}\right)_{n*m}$ for each alternative.

$$\left(\mathfrak{H}^{(z)}_{\check{d}_{ik}}, \mathfrak{L}'\right)_{n\times\partial} = \begin{matrix} O_1 \\ O_2 \\ \vdots \\ O_n \end{matrix}\begin{pmatrix} \left(\alpha^{(z)}_{\check{d}_{11}}, b^{(z)}_{\check{d}_{11}}\right) & \left(\alpha^{(z)}_{\check{d}_{12}}, b^{(z)}_{\check{d}_{12}}\right) & \cdots & \left(\alpha^{(z)}_{\check{d}_{1\partial}}, b^{(z)}_{\check{d}_{1\partial}}\right) \\ \left(\alpha^{(z)}_{\check{d}_{21}}, b^{(z)}_{\check{d}_{21}}\right) & \left(\alpha^{(z)}_{\check{d}_{22}}, b^{(z)}_{\check{d}_{22}}\right) & \cdots & \left(\alpha^{(z)}_{\check{d}_{2\partial}}, b^{(z)}_{\check{d}_{2\partial}}\right) \\ \vdots & \vdots & \vdots & \vdots \\ \left(\alpha^{(z)}_{\check{d}_{n1}}, b^{(z)}_{\check{d}_{n1}}\right) & \left(\alpha^{(z)}_{\check{d}_{n2}}, b^{(z)}_{\check{d}_{n2}}\right) & \cdots & \left(\alpha^{(z)}_{\check{d}_{n\partial}}, b^{(z)}_{\check{d}_{n\partial}}\right) \end{pmatrix}$$

Step 2. to normalize the decision matrices, use the normalization rule.

$$M_{\check{d}_{ij}} = \begin{cases} \mathfrak{J}^{c}_{\check{d}_{ij}} = \left(b_{\check{d}_{ij}}, \alpha_{\check{d}_{ij}}\right) & \textit{cost type parameter} \\ \mathfrak{J}_{\check{d}_{ij}} = \left(\alpha_{\check{d}_{ij}}, b_{\check{d}_{ij}}\right) & \textit{benefit type parameter} \end{cases}$$

Step 3. Use the developed PFHSEOWA operator to collective the PFHSNs $\mathfrak{J}_{\check{d}_{ij}}$ for each alternative $\mathfrak{H} = \{\mathfrak{H}^1, \mathfrak{H}^2, \mathfrak{H}^3, \ldots, \mathfrak{H}^s\}$ into the decision matrix $\mathcal{L}_k$.

Step 4. Compute the score values employing Equation 2.1.

Step 5. Choose the most suitable alternative.

Step 6. Analyze the alternatives.

B. NUMERICAL EXAMPLE

Green agribusiness sponsors sustainable progression perceptions in agricultural science, such as encouraging food manufacture and fiber, while remembering economic and public constrictions to confirm the long-term capability of manufacture. For example, sustainable agricultural science moderates the use of fly spray dangerous to agronomists' and consumers' health. Strictly cultivation and intelligent farming are the core appliances of sustainable agricultural science. Growing manufacturing and raising livestock is a farming profession or business. Agricultural science comprises interrupting animals and developing crops to deliver nutrients and raw ingredients. Agricultural science was patented about 5,000 years ago, but the precise technique and origin are unreliable. Agricultural science is life sequence equipment, not just a business. This alternative statement must be a perpetual rehearsal, irrespective of age. Because of this land demolition, nutrition budgets will skyrocket, and we will devote more of our budget to meeting our daily nutritional needs. Farmers must focus on growing produces through farming machines to get out of this condition. The use of machines in agricultural science is a graphic of motivation beyond origins. Farming, as a business, will grow into a high-tech industry in the new era. Agricultural robots or agricultural robots are other terminologies for agricultural machines [56]. Five core alternates are interconnected to natural agri-business, such as Good crop production ($\mathfrak{H}^1$); Environmental protection ($\mathfrak{H}^2$); Natural resources availability ($\mathfrak{H}^3$); Food security and productivity ($\mathfrak{H}^4$); Availability of machines ($\mathfrak{H}^5$). In addition, above mentioned five alternatives are evaluated using four parameters. The attribute of robotic agriculture is given as follows: $\mathfrak{L} = \{d_1 = \text{Quality production}, d_2 = \text{Completion of a time} - \text{consuming project}, d_3 = \text{Consistent role in completing a project}, d_4 = \text{Limiting the need for manual labor}\}$. The conforming subattributes of the deliberated parameters $\text{Quality production} = d_1 = \{d_{11} = \text{High} - \text{quality production}, d_{12} = Low - \text{quality production}\}$, $\text{Completion of a time} - \text{consuming project} = d_2 = \{d_{21} = \text{Short} - \text{term}, d_{22} = \text{Long} - \text{term}, \}$, $\text{Consistent role in completing a project} = d_3 = \{d_{31} = \text{Project budgeting and forecasting}\}$, $\text{Limiting the need for manual labor} = d_4 = \{d_{41} = \text{Limiting the need for manual labor}\}$. Let $\mathfrak{L}' = d_1 \times d_2 \times d_3 \times d_4$ be a set of sub-attributes

$\mathfrak{L}' = d_1 \times d_2 \times d_3 \times d_4 = \{d_{11}, d_{12}\} \times \{d_{21}, d_{22}\} \times \{d_{31}\} \times \{d_{41}\}$
$= \begin{Bmatrix} (d_{11}, d_{21}, d_{31}, d_{41}), (d_{11}, d_{22}, d_{31}, d_{41}), \\ (d_{12}, d_{21}, d_{31}, d_{41}), \ (d_{12}, d_{22}, d_{31}, d_{41}) \end{Bmatrix}$, $\mathfrak{L}' = \{\check{d}_1, \check{d}_2, \check{d}_3, \check{d}_4\}$ be a set of all sub-attributes with weights $(0.2, 0.2, 0.2, 0.4)^T$. Let $\{O^1, O^2, O^3, O^4\}$ be a set of three experts with weights $(0.1, 0.3, 0.3, 0.3)^T$. To judge the optimum alternative, specialists provide their preferences in PFHSNs.

C. PFHSEOWA OPERATOR

Step 1. The expert's opinion in PFHSN form for each alternative is given in Table I- Table V.

TABLE I. DECISION MATRIX FOR $\mathfrak{H}^1$ IN THE FORM OF PFHSN

	$\check{d}_1$	$\check{d}_2$	$\check{d}_3$	$\check{d}_4$
O^1	(.8, .5)	(.7, .5)	(.6, .4)	(.7, .4)
O^2	(.6, .5)	(.9, .1)	(.7, .3)	(.4, .5)
O^3	(.8, .4)	(.7, .5)	(.6, .4)	(.3, .5)
O^4	(.7, .3)	(.6, .5)	(.4, .5)	(.5, .7)

TABLE II. DECISION MATRIX FOR $\mathfrak{H}^2$ IN THE FORM OF PFHSN

	$\check{d}_1$	$\check{d}_2$	$\check{d}_3$	$\check{d}_4$
O^1	(.7, .5)	(.8, .5)	(.6, .4)	(.8, .4)
O^2	(.6, .3)	(.9, .2)	(.8, .3)	(.7, .5)
O^3	(.5, .4)	(.6, .5)	(.6, .3)	(.3, .6)
O^4	(.7, .4)	(.6, .4)	(.7, .5)	(.5, .7)

TABLE III. DECISION MATRIX FOR $\mathfrak{H}^3$ IN THE FORM OF PFHSN

	$\check{d}_1$	$\check{d}_2$	$\check{d}_3$	$\check{d}_4$
O^1	(.7, .5)	(.7, .4)	(.6, .4)	(.8, .4)
O^2	(.6, .6)	(.9, .1)	(.6, .3)	(.4, .5)
O^3	(.8, .3)	(.7, .2)	(.6, .5)	(.4, .5)
O^4	(.7, .6)	(.3, .5)	(.4, .5)	(.5, .6)

TABLE IV. DECISION MATRIX FOR $\mathfrak{H}^4$ IN THE FORM OF PFHSN

	$\check{d}_1$	$\check{d}_2$	$\check{d}_3$	$\check{d}_4$
O^1	(.8, .5)	(.7, .5)	(.7, .4)	(.6, .4)
O^2	(.6, .4)	(.8, .1)	(.7, .3)	(.4, .7)
O^3	(.7, .4)	(.7, .5)	(.6, .4)	(.3, .5)
O^4	(.6, .3)	(.6, .3)	(.8, .5)	(.5, .6)

TABLE V. DECISION MATRIX FOR $\mathfrak{H}^5$ IN THE FORM OF PFHSN

	$\check{d}_1$	$\check{d}_2$	$\check{d}_3$	$\check{d}_4$
O^1	(.6, .5)	(.6, .5)	(.6, .4)	(.5, .4)
O^2	(.6, .4)	(.8, .1)	(.8, .3)	(.7, .5)
O^3	(.6, .4)	(.7, .3)	(.6, .4)	(.6, .5)
O^4	(.7, .4)	(.7, .5)	(.4, .5)	(.5, .8)

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Step 2. Calculate the Ordered PFHS decision matrices in the expert's opinion given in the following Table VI to Table X.

TABLE VI. ORDERED PFHS DECISION MATRIX FOR $\mathfrak{H}^1$

	$\breve{d}_1$	$\breve{d}_2$	$\breve{d}_3$	$\breve{d}_4$
O^1	$(.8,.5)$	$(.7,.5)$	$(.6,.4)$	$(.7,.4)$
O^2	$(.6,.5)$	$(.9,.1)$	$(.7,.3)$	$(.4,.5)$
O^3	$(.8,.4)$	$(.7,.5)$	$(.6,.4)$	$(.3,.5)$
O^4	$(.7,.3)$	$(.6,.5)$	$(.4,.5)$	$(.5,.7)$

TABLE VII. ORDERED PFHS DECISION MATRIX FOR $\mathfrak{H}^2$

	$\breve{d}_1$	$\breve{d}_2$	$\breve{d}_3$	$\breve{d}_4$
O^1	$(.7,.5)$	$(.8,.5)$	$(.6,.4)$	$(.8,.4)$
O^2	$(.6,.3)$	$(.9,.2)$	$(.8,.3)$	$(.7,.5)$
O^3	$(.5,.4)$	$(.6,.5)$	$(.6,.3)$	$(.3,.6)$
O^4	$(.7,.4)$	$(.6,.4)$	$(.7,.5)$	$(.5,.7)$

TABLE VIII. ORDERED PFHS DECISION MATRIX FOR $\mathfrak{H}^3$

	$\breve{d}_1$	$\breve{d}_2$	$\breve{d}_3$	$\breve{d}_4$
O^1	$(.7,.5)$	$(.7,.4)$	$(.6,.4)$	$(.8,.4)$
O^2	$(.6,.6)$	$(.9,.1)$	$(.6,.3)$	$(.4,.5)$
O^3	$(.8,.3)$	$(.7,.2)$	$(.6,.5)$	$(.4,.5)$
O^4	$(.7,.6)$	$(.3,.5)$	$(.4,.5)$	$(.5,.6)$

TABLE IX. ORDERED PFHS DECISION MATRIX FOR $\mathfrak{H}^4$

	$\breve{d}_1$	$\breve{d}_2$	$\breve{d}_3$	$\breve{d}_4$
O^1	$(.8,.5)$	$(.7,.5)$	$(.7,.4)$	$(.6,.4)$
O^2	$(.6,.4)$	$(.8,.1)$	$(.7,.3)$	$(.4,.7)$
O^3	$(.7,.4)$	$(.7,.5)$	$(.6,.4)$	$(.3,.5)$
O^4	$(.6,.3)$	$(.6,.3)$	$(.8,.5)$	$(.5,.6)$

TABLE X. ORDERED PFHS DECISION MATRIX FOR $\mathfrak{H}^5$

	$\breve{d}_1$	$\breve{d}_2$	$\breve{d}_3$	$\breve{d}_4$
O^1	$(.6,.5)$	$(.6,.5)$	$(.6,.4)$	$(.5,.4)$
O^2	$(.6,.4)$	$(.8,.1)$	$(.8,.3)$	$(.7,.5)$
O^3	$(.6,.4)$	$(.7,.3)$	$(.6,.4)$	$(.6,.5)$
O^4	$(.7,.4)$	$(.7,.5)$	$(.4,.5)$	$(.5,.8)$

Step 3. Obtain normalized ordered PFHS decision matrices as given in Table XI-Table-XV.

TABLE XI. NORMALIZED ORDERED PFHS DECISION MATRIX FOR $\mathfrak{H}^1$

	$\breve{d}_1$	$\breve{d}_2$	$\breve{d}_3$	$\breve{d}_4$
O^1	$(.5,.8)$	$(.7,.5)$	$(.4,.6)$	$(.7,.4)$
O^2	$(.5,.6)$	$(.9,.1)$	$(.3,.7)$	$(.4,.5)$
O^3	$(.4,.8)$	$(.7,.5)$	$(.4,.6)$	$(.3,.5)$
O^4	$(.3,.7)$	$(.6,.5)$	$(.5,.4)$	$(.5,.7)$

TABLE XII. NORMALIZED ORDERED PFHS DECISION MATRIX FOR $\mathfrak{H}^2$

	$\breve{d}_1$	$\breve{d}_2$	$\breve{d}_3$	$\breve{d}_4$
O^1	$(.5,.7)$	$(.8,.5)$	$(.4,.6)$	$(.8,.4)$
O^2	$(.3,.6)$	$(.9,.2)$	$(.3,.8)$	$(.7,.5)$
O^3	$(.4,.5)$	$(.6,.5)$	$(.3,.6)$	$(.3,.6)$
O^4	$(.4,.7)$	$(.6,.4)$	$(.5,.7)$	$(.5,.7)$

TABLE XIII. NORMALIZED ORDERED PFHS DECISION MATRIX FOR $\mathfrak{H}^3$

	$\breve{d}_1$	$\breve{d}_2$	$\breve{d}_3$	$\breve{d}_4$
O^1	$(.5,.7)$	$(.7,.4)$	$(.4,.6)$	$(.8,.4)$
O^2	$(.6,.6)$	$(.9,.1)$	$(.3,.6)$	$(.4,.5)$
O^3	$(.3,.8)$	$(.7,.2)$	$(.5,.6)$	$(.4,.5)$
O^4	$(.6,.7)$	$(.3,.5)$	$(.5,.4)$	$(.5,.6)$

TABLE XIV. NORMALIZED ORDERED PFHS DECISION MATRIX FOR $\mathfrak{H}^4$

	$\breve{d}_1$	$\breve{d}_2$	$\breve{d}_3$	$\breve{d}_4$
O^1	$(.5,.8)$	$(.7,.5)$	$(.4,.7)$	$(.6,.4)$
O^2	$(.4,.6)$	$(.8,.1)$	$(.3,.7)$	$(.4,.7)$
O^3	$(.4,.7)$	$(.7,.5)$	$(.4,.6)$	$(.3,.5)$
O^4	$(.3,.6)$	$(.6,.3)$	$(.5,.8)$	$(.5,.6)$

TABLE XV. NORMALIZED PFHS ORDERED DECISION MATRIX FOR $\mathfrak{H}^5$

	$\breve{d}_1$	$\breve{d}_2$	$\breve{d}_3$	$\breve{d}_4$
O^1	$(.5,.6)$	$(.6,.5)$	$(.4,.6)$	$(.5,.4)$
O^2	$(.4,.6)$	$(.8,.1)$	$(.3,.8)$	$(.7,.5)$
O^3	$(.4,.6)$	$(.7,.3)$	$(.4,.6)$	$(.6,.5)$
O^4	$(.4,.7)$	$(.7,.5)$	$(.5,.4)$	$(.5,.8)$

Step 4. Compute the aggregated values using the PFHSEOWA operator.

$$\text{PFHSEOWA}\left(\mathfrak{J}_{\breve{d}_{11}},\mathfrak{J}_{\breve{d}_{12}},\ldots,\mathfrak{J}_{\breve{d}_{nm}}\right)=\ \oplus_{j=1}^{m}\lambda_j\left(\oplus_{i=1}^{n}\theta_i\mathfrak{J}_{\breve{d}_{r(i)\,s(j)}}\right)$$

$$=\left\langle\frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha^2_{\breve{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha^2_{\breve{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha^2_{\breve{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha^2_{\breve{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}},\frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\ell^2_{\breve{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\ell^2_{\breve{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\ell^2_{\breve{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}\right\rangle$$

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$$\text{PFHSEOWA}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{44}}\right) = \left\langle \frac{\sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1+\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1+\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-\alpha^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{2\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(b^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(2-b^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(b^2_{\check{d}_{r(i)\,s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

The accumulated assessment for each alternative is given as follows: $\mathcal{L}_1 = \langle 0.5283, 0.5242\rangle$, $\mathcal{L}_2 = \langle 0.7105, 0.4250\rangle$, $\mathcal{L}_3 = \langle 0.5834, 0.4680\rangle$, $\mathcal{L}_4 = \langle 0.6521, 0.4253\rangle$, and $\mathcal{L}_5 = \langle 0.6260, 0.4583\rangle$.

Step 5. Compute score values using Equation 2.1, $S = a_{\mathcal{F}(\check{d}_{ij})}{}^2 - b_{\mathcal{F}(\check{d}_{ij})}{}^2$.

$S(\boldsymbol{\mathcal{L}}_1) = 0.0088$, $S(\boldsymbol{\mathcal{L}}_2) = 0.2855$, $S(\boldsymbol{\mathcal{L}}_3) = 0.1154$, $S(\boldsymbol{\mathcal{L}}_4) = 0.2268$, $S(\boldsymbol{\mathcal{L}}_5) = 0.1677$.

Step 6. Ranking of alternatives

$S(\boldsymbol{\mathcal{L}}_2) > S(\boldsymbol{\mathcal{L}}_4) > S(\boldsymbol{\mathcal{L}}_5) > S(\boldsymbol{\mathcal{L}}_3) > S(\boldsymbol{\mathcal{L}}_1)$. So, $\mathfrak{H}^2 > \mathfrak{H}^4 > \mathfrak{H}^5 > \mathfrak{H}^3 > \mathfrak{H}^1$.

Hence, the best alternative is $\mathfrak{H}^2$.

D. PFHSEOWG OPERATOR

Step 1, step 2, and step 3 are similar to the PFHSEOWA operator.

Step 4. Calculate the accumulated values using the PFHSEOWG operator.

$$\text{PFHSEOWG}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \left\langle \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(a^2_{\check{d}_{r(i)s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-a^2_{\check{d}_{r(i)s(j)}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(a^2_{\check{d}_{r(i)s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+b^2_{\check{d}_{r(i)s(j)}}\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-b^2_{\check{d}_{r(i)s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+b^2_{\check{d}_{r(i)s(j)}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-b^2_{\check{d}_{r(i)s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

$\text{PFHSEOWG}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{44}}\right)$

$$\mathcal{L}_1 = \left\langle \frac{\sqrt{2\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(a^2_{\check{d}_{r(i)s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(2-a^2_{\check{d}_{r(i)s(j)}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(a^2_{\check{d}_{r(i)s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}, \frac{\sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1+b^2_{\check{d}_{r(i)s(j)}}\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-b^2_{\check{d}_{r(i)s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1+b^2_{\check{d}_{r(i)s(j)}}\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-b^2_{\check{d}_{r(i)s(j)}}\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

The aggregated values for each alternate are given as follows: $\mathcal{L}_1 = \langle 0.4923, 0.5743\rangle$, $\mathcal{L}_2 = \langle 0.4821, 0.5994\rangle$, $\mathcal{L}_3 = \langle 0.4675, 0.5801\rangle$, $\mathcal{L}_4 = \langle 0.5421, 0.6512\rangle$, and $\mathcal{L}_5 = \langle 0.3203, 0.8382\rangle$.

Step 5. Use equation 2.1 $S = a_{\mathcal{F}(\check{d}_{ij})}{}^2 - b_{\mathcal{F}(\check{d}_{ij})}{}^2$ to determine the score values for all alternatives.

$S(\mathcal{L}_1) = -0.5999$, $S(\mathcal{L}_2) = -0.1068$, $S(\mathcal{L}_3) = -0.1179$, $S(\mathcal{L}_4) = -0.1301$, $S(\mathcal{L}_5) = -0.1322$.

Step 6. Ranking of alternatives

$S(\mathcal{L}_2) > S(\mathcal{L}_3) > S(\mathcal{L}_4) > S(\mathcal{L}_5) > S(\mathcal{L}_1)$. So, $\mathfrak{H}^2 > \mathfrak{H}^3 > \mathfrak{H}^4 > \mathfrak{H}^5 > \mathfrak{H}^1$.

Hence, $\mathfrak{H}^2$ the most suitable alternative.

The graphical demonstration of alternatives ranking is prearranged in the following figure I.

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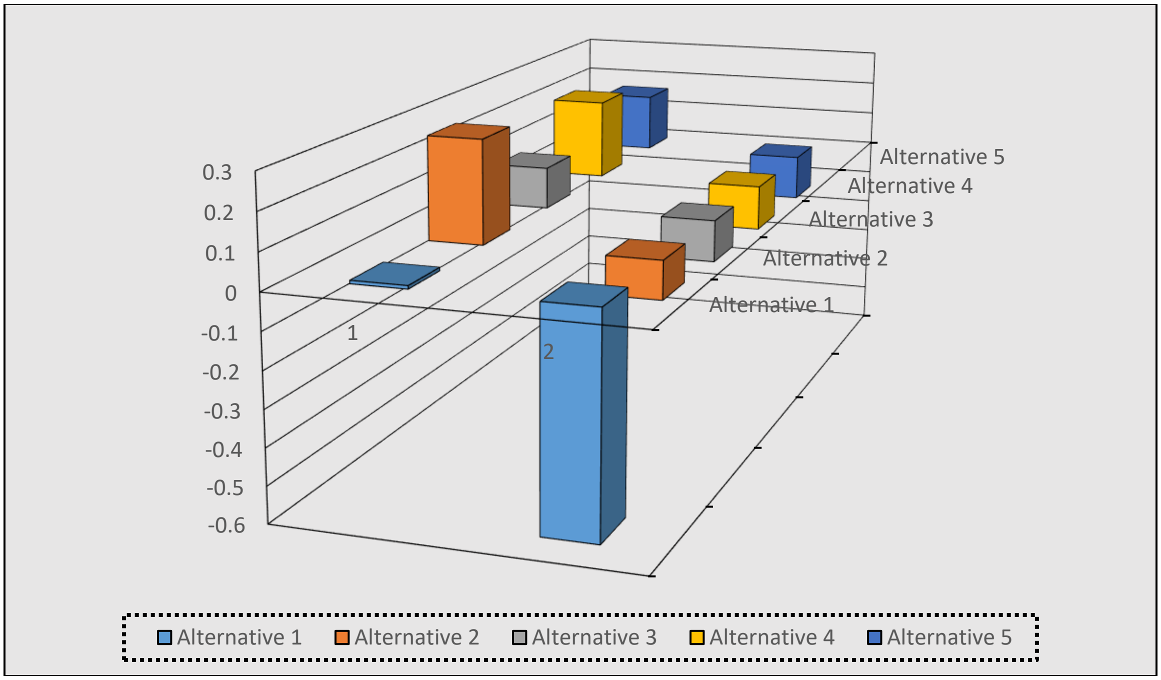

FIGURE I. GRAPHICAL REPRESENTATION OF THE OBTAINED OUTCOMES

V. COMPARATIVE STUDIES AND DISCUSSION

To validate the usefulness of the anticipated method, an assessment between the proposed model and the conventional techniques is planned in the subsequent section.

A. SUPREMACY OF THE PLANNED TECHNIQUE

The intended method is proficient and realistic; in the PFHSS setting, we construct an inventive MCDM model on the PFHSEOWA and PFHSEOWG operators. Our plan model is more endowed than prevalent methods and can produce the most refined implications in MCDM complications. The cooperative model is versatile and familiar, amending to evolving unpredictability, assurance, and output. Different models have particular ranking procedures, so there is a straight modification between the rankings of the anticipated methods to their expectations. This systematic study and assessment determined that the outcomes attained from prevailing procedures are irregularly equated to hybrid structures. Also, due to some favorable situations, many mixed FS, IFS, PFS, IFSS, PFSS, and IFHSS grow into special in PFHSS. It is easy to syndicate insufficient and ambiguous data in DM procedures. Data about substances can be articulated in it more perfectly and rationally. Imprecise and anxious facts are mixed in the DM procedure. Hence, our scheduled method will be more proficient, crucial, boosted, and enhanced than numerous mixed configurations of FS. Table XVI below presents the projected technique and the characteristic analysis of some prevailing models.

TABLE XVI. CHARACTERISTIC ANALYSIS OF DIFFERENT MODELS WITH A PLANNED MODEL

	Fuzzy information	Aggregated attributes information	Aggregated sub-attributes information of any attribute	Einstein Aggregated information
FS [1]	✓	×	×	×
IFWA [6]	✓	×	×	×
IFWG [6]	✓	×	×	×
PFWA [19]	✓	×	×	×
PFWG [10]	✓	×	×	×
IFSWA [29]	✓	✓	×	×
IFSWG [29]	✓	✓	×	×

PFSWA [33]	✓	✓	×	×
PFSWG [33]	✓	✓	×	×
PFSEWA [55]	✓	✓	×	✓
PFSEWG [55]	✓	✓	×	✓
IFHSWA [46]	✓	✓	✓	×
IFHSWG [46]	✓	✓	✓	×
PFHSWA [53]	✓	✓	✓	×
PFHSWG [53]	✓	✓	✓	×
Proposed PFHSEOWA	✓	✓	✓	✓
Proposed PFHSEOWG	✓	✓	✓	✓

B. DISCUSSION AND COMPARATIVE STUDIES

To demonstrate the practicality of the planned skill, we equate the attained consequences with some prevailing approaches under PFS, IFSS, PFSS, and IFHSS. A residue of all consequences is specified in Table XVII. Garg and Arora [29] settled that AOs for IFSS cannot accommodate the state when the sum of MD and NMD surpasses 1. Zulqarnain et al. [33] established AOs for PFSS that cannot accommodate the decision-makers selection when any attribute contains a further sub-attribute. Zulqarnain et al. [55] demonstrated that PFSEWA and PFSEWG operators deal with alternative parameterized values. But, these AOs fail to handle the scenario if any parameter contains a different sub-parameter. Meanwhile, our proposed Einstein AOs capably deal with such complicated environments. Furthermore, if any parameter has a sub-parameter, the PFHSS reduces to the PFSS. Similarly, if the sum of MD and NMD is less or equal to 1. Then, PFHSS was reduced to IFHSS.

TABLE XVII. COMPARATIVE ANALYSIS OF PROPOSED OPERATORS WITH EXISTING OPERATORS

Authors	AO	Alternatives ranking	Optimal choice
Arora & Garg [29]	IFSWA	$\mathfrak{H}^2 > \mathfrak{H}^4 > \mathfrak{H}^5 > \mathfrak{H}^1 > \mathfrak{H}^3$	$\mathfrak{I}^2$
Zulqarnain et al. [33]	PFSWA	$\mathfrak{H}^2 > \mathfrak{H}^4 > \mathfrak{H}^5 > \mathfrak{H}^3 > \mathfrak{H}^1$	$\mathfrak{I}^2$
Arora & Garg [29]	IFSWG	$\mathfrak{H}^2 > \mathfrak{H}^4 > \mathfrak{H}^5 > \mathfrak{H}^1 > \mathfrak{H}^3$	$\mathfrak{I}^2$
Zulqarnain et al. [33]	PFSWG	$\mathfrak{H}^2 > \mathfrak{H}^4 > \mathfrak{H}^1 > \mathfrak{H}^3 > \mathfrak{H}^5$	$\mathfrak{I}^2$
Zulqarnain et al. [55]	PFSEWA	$\mathfrak{H}^2 > \mathfrak{H}^3 > \mathfrak{H}^1 > \mathfrak{H}^4 > \mathfrak{H}^5$	$\mathfrak{I}^2$
Zulqarnain et al. [55]	PFSEWG	$\mathfrak{H}^2 > \mathfrak{H}^5 > \mathfrak{H}^4 > \mathfrak{H}^3 > \mathfrak{H}^1$	$\mathfrak{I}^2$
Siddique et al. [53]	PFHSWA	$\mathfrak{H}^2 > \mathfrak{H}^4 > \mathfrak{H}^5 > \mathfrak{H}^3 > \mathfrak{H}^1$	$\mathfrak{I}^2$
Siddique et al. [53]	PFHSWG	$\mathfrak{H}^2 > \mathfrak{H}^3 > \mathfrak{H}^5 > \mathfrak{H}^4 > \mathfrak{H}^1$	$\mathfrak{I}^2$
Proposed	PFHSEOWA	$\mathfrak{H}^2 > \mathfrak{H}^4 > \mathfrak{H}^5 > \mathfrak{H}^3 > \mathfrak{H}^1$	$\mathfrak{I}^2$
Proposed	PFHSEOWG	$\mathfrak{H}^2 > \mathfrak{H}^3 > \mathfrak{H}^4 > \mathfrak{H}^5 > \mathfrak{H}^1$	$\mathfrak{I}^2$

Thus, PFHSS is the most generalized form of PFSS. Hence, based on the abovementioned details, the anticipated operators in this work are more influential, consistent, and prosperous. It is observed that the most acceptable alternative is the same against each aggregation operator. But, existing IFSWA, PFSWA, IFSWG, PFSWG, PFSEWA, and PFSEWG operators are unable to cope with the multi-sub attributes of alternatives. On the other hand, our developed AOs competently deal with multi sub-attributes of the alternatives. Moreover, our planned model delivers the same best and worst attributes in both cases. Meanwhile, several other models delivered different worst alternatives.

VI. CONCLUSION

Decision-making is a pre-planned process for arranging and choosing logical preferences from multiple alternatives. DM is a multifaceted procedure because it can switch from one scene to another. So, it is imperious to judge the characteristics and limits of alternatives. Additionally, DM is a healthier method that improves the chances of indicating the most appropriate alternative. It is serious about differentiating how much real perspective data decision-makers need. The most operational approach in DM is paying close attention and focusing on your goals. In a real DM, assessing alternative facts as told by a professional is permanently incorrect, irregular, and impressive. Therefore, PFHSNs can be used to match this uncertain data. The core persistence of this work is to prolong the Einstein AOs for PFHSS. First, we extant the Einstein operational rules for the PFHSS. Considering the advanced operational rules, we have provided PFHSEOWA and PFHSEOWG operators with their required features for PFHSS. Furthermore, some fundamental properties such as idempotency, homogeneity, and boundedness of the established PFHSEOWA and PFHSEOWG operators are proposed. Also, a DM method has been intended to report MCDM complications using the permitted operators. We deliver a comprehensive mathematical interpretation to demonstrate the fixation method's strength. A comparative analysis has been presented to ensure the practicability of the planning model. Furthermore, based on the obtained results, it is resolute that the method proposed in this investigation is the most concrete and efficient to resolve the MCDM problem. Future exploration emphasizes emerging DM techniques, such as Einstein's hybrid AOs in the PFHSS setting. We are assured that these substantial developments and forecasts will help deliberate managerial exploration extents involved in the world's climate. Furthermore, anyone can extend the Einstein AOs and Einstein-ordered AOs to a q-rung orthopair fuzzy hypersoft environment with their decision-making techniques. These replicas can be employed as a presentation in operation study, artificial networking, sustainable supplier selection, Agri farming, etc.

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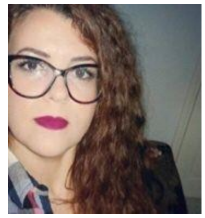

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