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Abstract. Smarandache [48] introduced the concept of hypersoft set which is a generalization of soft set. This notion is more adaptable than soft set and more suited to challenges involving decision-making. Consequently, topology defined on the collection of hypersoft sets will be in great importance. In this paper, we introduce hypersoft topological spaces which are defined over an initial universe with a fixed set of parameters. The notions of hypersoft open sets, hypersoft closed sets, hypersoft neighborhood, hypersoft limit point, and hypersoft subspace are introduced and their basic properties are investigated. Finally, we introduce the concepts of hypersoft closure, hypersoft interior, hypersoft exterior, and hypersoft boundary and the relationship between them are discussed.

Keywords: hypersoft sets; hypersoft topology; hypersoft open sets; hypersoft closed sets; hypersoft neighborhood; hypersoft limit point; hypersoft closure; hypersoft interior; hypersoft exterior; hypersoft boundary.

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1. Introduction

In 1999, Molodtsov [30] developed the concept of a soft set to handle difficult problems in economics, engineering, and the environment, where no mathematical methods could effectively deal with the many types of uncertainty. Maji et al. in [25] developed various operators for soft set theory and conducted a more detailed theoretical analysis of soft set theory. Various operations analogous to union, intersection, complement, difference etc. in set theory have been discussed in the context of soft sets (see [5, 6, 10, 46]).

It is known that Topology is a branch of mathematics that has numerous applications in the physical and computer sciences. Topology is the study of qualitative properties of particular objects, known as topological spaces, that are invariant under specific transformations, known as continuous mappings. Open sets are commonly used to describe these characteristics. By replacing open sets with more general ones, the concept of topological space is frequently generalized. A classic example of this form of generalization is fuzzy topology, proposed by Chang [12] and later improved by Lowen [24]. Topological structures on soft sets, in a similar manner, are more generalized methods that can be used to measure the similarities and differences between the objects in a universe which are soft sets.

There are two versions of soft topology defined on soft sets, one by Shabir [47] and other by Čagman et al. [11]. The main difference between these approaches is that the first investigates a subcollection of all soft sets in an initial universe with a fixed set of parameters, whereas the second considers a subcollection of all soft subsets of a specific soft set in a universe. Based on these two definitions on soft topology, some concepts such as soft interior, soft closure, soft continuity, soft separation axioms etc. were introduced and studied by many authors (see for example [2–4, 7–9, 13–23, 26–29, 33–37, 40, 42–45, 49–54]).

In 2018, Smarandache [48] expanded the notion of a soft set to a hypersoft set by substituting the function with a multi-argument function described in the cartesian product with a different set of parameters. This concept is more adaptable than the soft set and more useful when it comes to making decisions. Recently, Musa and Asaad ([31, 32]) introduced a new idea of hypersoft sets called bipolar hypersoft sets and they investigated some of their bipolar hypersoft topological structures. Researchers have been drawn to hypersoft set structure because it is better suited to decision-making difficulties than soft set structure. Despite the fact that it is a new concept, numerous studies have been conducted, and the field of study continues to grow [1, 38, 39, 41].

Our paper is organized as follows: Section 2 contains some basic definitions related to hypersoft set which are required in our work. In section 3, we introduce hypersoft topological spaces which are defined over an initial universe with a fixed set of parameters and investigate the concepts of hypersoft neighborhood and hypersoft limit points. In section 4, the notions of hypersoft closure, hypersoft interior, hypersoft exterior, and hypersoft boundary are introduced associated with some of their properties. Furthermore, the relationships between all of the preceding concepts are studied, as well as several illustrated examples. The conclusion, on the other hand, is included in Section 5.

2. **Hypersoft Sets**

Here we recall some basic terminologies and results regarding hypersoft sets. For more details, the reader could refer to [39, 41].

Throughout the paper, let $\mathcal{U}$ be an initial universe, $\mathcal{P}(\mathcal{U})$ the power set of $\mathcal{U}$, and $E_1, E_2, ..., E_n$ the pairwise of disjoint sets of parameters. Let $A_i, B_i \subseteq E_i$ for $i = 1, 2, ..., n$.

Definition 2.1. [48] A pair $(\mathbb{F}, A_1 \times A_2 \times ... \times A_n)$ is called a hypersoft set over $\mathcal{U}$, where $\mathbb{F}$ is a mapping given by $\mathbb{F} : A_1 \times A_2 \times ... \times A_n \to \mathcal{P}(\mathcal{U})$.

Simply, we write the symbol $\mathcal{E}$ for $E_1 \times E_2 \times ... \times E_n$, and for the subsets of $\mathcal{E}:$ the symbols $\mathcal{A}$ for $A_1 \times A_2 \times ... \times A_n$, and $\mathcal{B}$ for $B_1 \times B_2 \times ... \times B_n$. Clearly, each element in $\mathcal{A}, \mathcal{B}$ and $\mathcal{E}$ is an n-tuple element.

We can represent a hypersoft set $(\mathbb{F}, \mathcal{A})$ as an ordered pair,

$$(\mathbb{F}, \mathcal{A}) = \{(\alpha, \mathbb{F}(\alpha)) : \alpha \in \mathcal{A}\}.$$

Definition 2.2. [39] For two hypersoft sets $(\mathbb{F}, \mathcal{A})$ and $(\mathbb{G}, \mathcal{B})$ over a common universe $\mathcal{U}$, we say that $(\mathbb{F}, \mathcal{A})$ is a hypersoft subset of $(\mathbb{G}, \mathcal{B})$ if

(1) $\mathcal{A} \subseteq \mathcal{B}$, and

(2) $\mathbb{F}(\alpha) \subseteq \mathbb{G}(\alpha)$ for all $\alpha \in \mathcal{A}$.

We write $(\mathbb{F}, \mathcal{A}) \mathrel{\widetilde{\subseteq}} (\mathbb{G}, \mathcal{B})$.

$(\mathbb{F}, \mathcal{A})$ is said to be a hypersoft superset of $(\mathbb{G}, \mathcal{B})$, if $(\mathbb{G}, \mathcal{B})$ is a hypersoft subset of $(\mathbb{F}, \mathcal{A})$. We denote it by $(\mathbb{F}, \mathcal{A}) \mathrel{\widetilde{\supseteq}} (\mathbb{G}, \mathcal{B})$.

Definition 2.3. [39] Two hypersoft sets $(\mathbb{F}, \mathcal{A})$ and $(\mathbb{G}, \mathcal{B})$ over a common universe $\mathcal{U}$ are said to be hypersoft equal if $(\mathbb{F}, \mathcal{A})$ is a hypersoft subset of $(\mathbb{G}, \mathcal{B})$ and $(\mathbb{G}, \mathcal{B})$ is a hypersoft subset of $(\mathbb{F}, \mathcal{A})$.

Definition 2.4. [39] Let $\mathcal{A} = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a set of parameters. The NOT set of $\mathcal{A}$ denoted by $\neg\mathcal{A}$ is defined by $\neg\mathcal{A} = \{\neg\alpha_1, \neg\alpha_2, ..., \neg\alpha_n\}$ where $\neg\alpha_i =$ not α_i for $i = 1, 2, ..., n$.

Proposition 2.5. *[31] For any subsets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{E}$.*

(1) $\neg(\neg\mathcal{A}) = \mathcal{A}$.

(2) $\neg(\mathcal{A} \cup \mathcal{B}) = \neg\mathcal{A} \cup \neg\mathcal{B}$.

(3) $\neg(\mathcal{A} \cap \mathcal{B}) = \neg\mathcal{A} \cap \neg\mathcal{B}$.

Definition 2.6. [39] The complement of a hypersoft set $(\mathbb{F}, \mathcal{A})$ is denoted by $(\mathbb{F}, \mathcal{A})^c$ and is defined by $(\mathbb{F}, \mathcal{A})^c = (\mathbb{F}^c, \mathcal{A})$ where $\mathbb{F}^c : \mathcal{A} \to \mathcal{P}(\mathcal{U})$ is a mapping given by $\mathbb{F}^c(\alpha) = \mathcal{U} \setminus \mathbb{F}(\alpha)$ for all $\alpha \in \mathcal{A}$.

Definition 2.7. [41] A hypersoft set $(\mathbb{F}, \mathcal{A})$ over $\mathcal{U}$ is said to be a relative null hypersoft set, denoted by $(\Phi, \mathcal{A})$, if for all $\alpha \in \mathcal{A}$, $\mathbb{F}(\alpha) = \phi$.

Definition 2.8. [41] A hypersoft set $(\mathbb{F}, \mathcal{A})$ over $\mathcal{U}$ is said to be a relative whole hypersoft set, denoted by $(\Psi, \mathcal{A})$, if for all $\alpha \in \mathcal{A}$, $\mathbb{F}(\alpha) = \mathcal{U}$.

Definition 2.9. [41] Difference of two hypersoft sets $(\mathbb{F}, \mathcal{A})$ and $(\mathbb{G}, \mathcal{B})$ over a common universe $\mathcal{U}$, is a hypersoft set $(\mathbb{H}, C)$, where $C = \mathcal{A} \cap \mathcal{B}$ and for all $\alpha \in C$, $\mathbb{H}(\alpha) = \mathbb{F}(\alpha) \setminus \mathbb{G}(\alpha)$. We write $(\mathbb{F}, \mathcal{A}) \setminus (\mathbb{G}, \mathcal{B}) = (\mathbb{H}, C)$.

Definition 2.10. [41] Union of two hypersoft sets $(\mathbb{F}, \mathcal{A})$ and $(\mathbb{G}, \mathcal{B})$ over a common universe $\mathcal{U}$, is a hypersoft set $(\mathbb{H}, C)$, where $C = \mathcal{A} \cap \mathcal{B}$ and for all $\alpha \in C$, $\mathbb{H}(\alpha) = \mathbb{F}(\alpha) \cup \mathbb{G}(\alpha)$. We write $(\mathbb{F}, \mathcal{A}) \tilde{\sqcup} (\mathbb{G}, \mathcal{B}) = (\mathbb{H}, C)$.

Definition 2.11. [39] Intersection of two hypersoft sets $(\mathbb{F}, \mathcal{A})$ and $(\mathbb{G}, \mathcal{B})$ over a common universe $\mathcal{U}$, is a hypersoft set $(\mathbb{H}, C)$, where $C = \mathcal{A} \cap \mathcal{B}$ and for all $\alpha \in C$, $\mathbb{H}(\alpha) = \mathbb{F}(\alpha) \cap \mathbb{G}(\alpha)$. We write $(\mathbb{F}, \mathcal{A}) \tilde{\sqcap} (\mathbb{G}, \mathcal{B}) = (\mathbb{H}, C)$.

3. Hypersoft Topological Spaces

Let $\mathcal{U}$ be an initial universe set and $\mathcal{E}$ be the non-empty set of parameters.

Definition 3.1. Let $(\mathbb{F}, \mathcal{E})$ be a hypersoft set over $\mathcal{U}$ and $u \in \mathcal{U}$. Then $u \in (\mathbb{F}, \mathcal{E})$ if $u \in \mathbb{F}(\alpha)$ for all $\alpha \in \mathcal{E}$. Note that for any $u \in \mathcal{U}$, $u \notin (\mathbb{F}, \mathcal{E})$, if $u \notin \mathbb{F}(\alpha)$ for some $\alpha \in \mathcal{E}$.

Definition 3.2. Let $\mathcal{Y}$ be a non-empty subset of $\mathcal{U}$. Then $(\Upsilon, \mathcal{E})$ denotes the hypersoft set over $\mathcal{U}$ defined by $\Upsilon(\alpha) = \mathcal{Y}$ for all $\alpha \in \mathcal{E}$.

Definition 3.3. Let $(\mathbb{F}, \mathcal{E})$ be a hypersoft set over $\mathcal{U}$ and $\mathcal{Y}$ be a non-empty subset of $\mathcal{U}$. Then the sub hypersoft set of $(\mathbb{F}, \mathcal{E})$ over $\mathcal{Y}$ denoted by $(\mathbb{F}_{\mathcal{Y}}, \mathcal{E})$, is defined as $\mathbb{F}_{\mathcal{Y}}(\alpha) = \mathcal{Y} \cap \mathbb{F}(\alpha)$ for each $\alpha \in \mathcal{E}$.

In other words $(\mathbb{F}_{\mathcal{Y}}, \mathcal{E}) = (\Upsilon, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}, \mathcal{E})$.

Definition 3.4. Let $\mathcal{T}_{\mathcal{H}}$ be the collection of hypersoft sets over $\mathcal{U}$, then $\mathcal{T}_{\mathcal{H}}$ is said to be a hypersoft topology on $\mathcal{U}$ if

(1) $(\Phi, \mathcal{E})$, $(\Psi, \mathcal{E})$ belong to $\mathcal{T}_{\mathcal{H}}$,
(2) the intersection of any two hypersoft sets in $\mathcal{T}_{\mathcal{H}}$ belongs to $\mathcal{T}_{\mathcal{H}}$,
(3) the union of any number of hypersoft sets in $\mathcal{T}_{\mathcal{H}}$ belongs to $\mathcal{T}_{\mathcal{H}}$.

Then $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ is called a hypersoft topological space over $\mathcal{U}$.

Definition 3.5. Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$, then the members of $\mathcal{T}_{\mathcal{H}}$ are said to be hypersoft open sets in $\mathcal{U}$.

Example 3.6. Let $\mathcal{U} = \{h_1, h_2\}$, $E_1 = \{e_1, e_2\}$, $E_2 = \{e_3\}$, and $E_3 = \{e_4\}$. Let $\mathcal{T}_{\mathcal{H}} = \{(\Phi, \mathcal{E}), (\Psi, \mathcal{E}), (\mathbb{F}_1, \mathcal{E}), (\mathbb{F}_2, \mathcal{E}), (\mathbb{F}_3, \mathcal{E})\}$ where $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E})$, and $(\mathbb{F}_3, \mathcal{E})$ are hypersoft sets over $\mathcal{U}$, defined as follows

$$(\mathbb{F}_1, \mathcal{E}) = \{((e_1, e_3, e_4), \{h_1\}), ((e_2, e_3, e_4), \{h_2\})\}.$$
$$(\mathbb{F}_2, \mathcal{E}) = \{((e_1, e_3, e_4), \{h_1\}), ((e_2, e_3, e_4), \mathcal{U}\}.$$
$$(\mathbb{F}_3, \mathcal{E}) = \{((e_1, e_3, e_4), \mathcal{U}), ((e_2, e_3, e_4), \{h_2\})\}.$$

Then the collection $\mathcal{T}_{\mathcal{H}}$ forms a hypersoft topology on $\mathcal{U}$.

Definition 3.7. Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$. A hypersoft set $(\mathbb{F}, \mathcal{E})$ over $\mathcal{U}$ is said to be a hypersoft closed set in $\mathcal{U}$, if its complement $(\mathbb{F}, \mathcal{E})^c$ belongs to $\mathcal{T}_{\mathcal{H}}$.

Proposition 3.8. *Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$. Then*

(1) $(\Phi, \mathcal{E})$, $(\Psi, \mathcal{E})$ are hypersoft closed set over $\mathcal{U}$,

(2) the union of any two hypersoft closed sets is a hypersoft closed set over $\mathcal{U}$,

(3) the intersection of any number of hypersoft closed sets is a hypersoft closed set over $\mathcal{U}$.

Proof. Follows from the definition of hypersoft topological spaces and De Morgan's laws.

Definition 3.9. Let $\mathcal{U}$ be an initial universe, $\mathcal{E}$ be the set of parameters, and $\mathcal{T}_{\mathcal{H}} = \{(\Phi, \mathcal{E}), (\Psi, \mathcal{E})\}$. Then $\mathcal{T}_{\mathcal{H}}$ is called the hypersoft indiscrete topology on $\mathcal{U}$ and $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ is said to be a hypersoft indiscrete space over $\mathcal{U}$.

Definition 3.10. Let $\mathcal{U}$ be an initial universe, $\mathcal{E}$ be the set of parameters, and $\mathcal{T}_{\mathcal{H}}$ be the collection of all hypersoft sets which can be defined over $\mathcal{U}$. Then $\mathcal{T}_{\mathcal{H}}$is called the hypersoft discrete topology on $\mathcal{U}$ and $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ is said to be a hypersoft discrete space over $\mathcal{U}$.

Definition 3.11. Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_1}, \mathcal{E})$ and $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_2}, \mathcal{E})$ be two hypersoft topological spaces over $\mathcal{U}$. If $\mathcal{T}_{\mathcal{H}_1} \,\widetilde{\sqsubseteq}\, \mathcal{T}_{\mathcal{H}_2}$, then $\mathcal{T}_{\mathcal{H}_2}$ is said to be finer than $\mathcal{T}_{\mathcal{H}_1}$. If $\mathcal{T}_{\mathcal{H}_1} \,\widetilde{\sqsubseteq}\, \mathcal{T}_{\mathcal{H}_2}$ or $\mathcal{T}_{\mathcal{H}_2} \,\widetilde{\sqsubseteq}\, \mathcal{T}_{\mathcal{H}_1}$, then $\mathcal{T}_{\mathcal{H}_1}$ and $\mathcal{T}_{\mathcal{H}_2}$ are said to be comparable hypersoft topologies over $\mathcal{U}$.

Proposition 3.12. *Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_1}, \mathcal{E})$ and $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_2}, \mathcal{E})$ be two hypersoft topological spaces on $\mathcal{U}$, then $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_1} \widetilde{\sqcap}\, \mathcal{T}_{\mathcal{H}_2}, \mathcal{E})$ is a hypersoft topological space over $\mathcal{U}$.*

Proof.

i. $(\Phi, \mathcal{E})$, $(\Psi, \mathcal{E})$ belong to $\mathcal{T}_{\mathcal{H}_1} \,\widetilde{\sqcap}\, \mathcal{T}_{\mathcal{H}_2}$.

ii. Let $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E}) \,\widetilde{\in}\, \mathcal{T}_{\mathcal{H}_1} \,\widetilde{\sqcap}\, \mathcal{T}_{\mathcal{H}_2}$. Then $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E}) \,\widetilde{\in}\, \mathcal{T}_{\mathcal{H}_1}$ and $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E}) \,\widetilde{\in}\, \mathcal{T}_{\mathcal{H}_2}$. Since $(\mathbb{F}_1, \mathcal{E}) \,\widetilde{\sqcap}\, (\mathbb{F}_2, \mathcal{E}) \,\widetilde{\in}\, \mathcal{T}_{\mathcal{H}_1}$ and $(\mathbb{F}_1, \mathcal{E}) \,\widetilde{\sqcap}\, (\mathbb{F}_2, \mathcal{E}) \,\widetilde{\in}\, \mathcal{T}_{\mathcal{H}_2}$, so $(\mathbb{F}_1, \mathcal{E}) \,\widetilde{\sqcap}\, (\mathbb{F}_2, \mathcal{E}) \,\widetilde{\in}\, \mathcal{T}_{\mathcal{H}_1} \,\widetilde{\sqcap}\, \mathcal{T}_{\mathcal{H}_2}$.

iii. Let $\{(\mathbb{F}_i, \mathcal{E}) \mid i \in I)\}$ be a family of hypersoft sets in $\mathcal{T}_{\mathcal{H}_1} \,\widetilde{\sqcap}\, \mathcal{T}_{\mathcal{H}_2}$. Then $(\mathbb{F}_i, \mathcal{E}) \,\widetilde{\in}\, \mathcal{T}_{\mathcal{H}_1}$ and $(\mathbb{F}_i, \mathcal{E}) \,\widetilde{\in}\, \mathcal{T}_{\mathcal{H}_2}$, for all $i \in I$, so $\widetilde{\sqcup}_{i\in I}\, (\mathbb{F}_i, \mathcal{E}) \,\widetilde{\in}\, \mathcal{T}_{\mathcal{H}_1}$ and $\widetilde{\sqcup}_{i\in I}\, (\mathbb{F}_i, \mathcal{E}) \,\widetilde{\in}\, \mathcal{T}_{\mathcal{H}_2}$. Therefore, $\widetilde{\sqcup}_{i\in I}(\mathbb{F}_i, \mathcal{E}) \,\widetilde{\in}\, \mathcal{T}_{\mathcal{H}_1} \,\widetilde{\sqcap}\, \mathcal{T}_{\mathcal{H}_2}$.

Thus $\mathcal{T}_{\mathcal{H}_1} \widetilde{\sqcap}\, \mathcal{T}_{\mathcal{H}_2}$ defines a hypersoft topology on $\mathcal{U}$ and $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_1} \widetilde{\sqcap}\, \mathcal{T}_{\mathcal{H}_2}, \mathcal{E})$ is a hypersoft topological space over $\mathcal{U}$.

Remark 3.13. The union of two hypersoft topologies on $\mathcal{U}$ may not be a hypersoft topology on $\mathcal{U}$.

Example 3.14. Let $\mathcal{U} = \{h_1, h_2, h_3, h_4\}$, $E_1 = \{e_1, e_2\}$, $E_2 = \{e_3\}$, and $E_3 = \{e_4\}$. Let $\mathcal{T}_{\mathcal{H}_1} = \{(\Phi, \mathcal{E}), (\Psi, \mathcal{E}), (\mathbb{F}_1, \mathcal{E}), (\mathbb{F}_2, \mathcal{E}), (\mathbb{F}_3, \mathcal{E})\}$ and $\mathcal{T}_{\mathcal{H}_2} = \{(\Phi, \mathcal{E}), (\Psi, \mathcal{E}), (\mathbb{G}_1, \mathcal{E}), (\mathbb{G}_2, \mathcal{E}),$

$(\mathbb{G}_3, \mathcal{E})\}$ be two hypersoft topologies defined on $\mathcal{U}$ where $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E})$, $(\mathbb{F}_3, \mathcal{E})$, $(\mathbb{G}_1, \mathcal{E})$, $(\mathbb{G}_2, \mathcal{E})$, and $(\mathbb{G}_3, \mathcal{E})$ are hypersoft sets over $\mathcal{U}$, defined as follows

$(\mathbb{F}_1, \mathcal{E}) = \{((e_1, e_3, e_4), \{h_3, h_4\}), ((e_2, e_3, e_4), \{h_2, h_3\})\}.$
$(\mathbb{F}_2, \mathcal{E}) = \{((e_1, e_3, e_4), \{h_1, h_2, h_3\}), ((e_2, e_3, e_4), \{h_1, h_4\})\}.$
$(\mathbb{F}_3, \mathcal{E}) = \{((e_1, e_3, e_4), \{h_3\}), ((e_2, e_3, e_4), \phi)\}.$

and

$(\mathbb{G}_1, \mathcal{E}) = \{((e_1, e_3, e_4), \{h_3, h_4\}), ((e_2, e_3, e_4), \{h_1, h_3, h_4\})\}.$
$(\mathbb{G}_2, \mathcal{E}) = \{((e_1, e_3, e_4), \{h_1, h_2\}), ((e_2, e_3, e_4), \{h_2, h_4\})\}.$
$(\mathbb{G}_3, \mathcal{E}) = \{((e_1, e_3, e_4), \phi), ((e_2, e_3, e_4), \{h_4\}\}.$

Then $\mathcal{T}_{\mathcal{H}_1} \widetilde{\sqcup} \mathcal{T}_{\mathcal{H}_2} = \{(\Phi, \mathcal{E}), (\Psi, \mathcal{E}), (\mathbb{F}_1, \mathcal{E}), (\mathbb{F}_2, \mathcal{E}), (\mathbb{F}_3, \mathcal{E}), (\mathbb{G}_1, \mathcal{E}), (\mathbb{G}_2, \mathcal{E}), (\mathbb{G}_3, \mathcal{E})\}$.

If we take

$(\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcup} (\mathbb{G}_1, \mathcal{E}) = (\mathbb{H}, \mathcal{E}),$

then

$(\mathbb{H}, \mathcal{E}) = \{((e_1, e_3, e_4), \{h_3, h_4\}), ((e_2, e_3, e_4), \mathcal{U})\},$

but $(\mathbb{H}, \mathcal{E}) \widetilde{\widetilde{\notin}} \mathcal{T}_{\mathcal{H}_1} \widetilde{\sqcup} \mathcal{T}_{\mathcal{H}_2}$. Hence, $\mathcal{T}_{\mathcal{H}_1} \widetilde{\sqcup} \mathcal{T}_{\mathcal{H}_2}$ is not a hypersoft topology on $\mathcal{U}$.

Definition 3.15. Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$, $(\mathbb{F}, \mathcal{E})$ be a hypersoft set over $\mathcal{U}$ and $u \in \mathcal{U}$. Then $(\mathbb{F}, \mathcal{E})$ is said to be a hypersoft neighborhood of u if there exists a hypersoft open set $(\mathbb{G}, \mathcal{E})$ such that $u \in (\mathbb{G}, \mathcal{E}) \widetilde{\widetilde{\sqsubseteq}} (\mathbb{F}, \mathcal{E})$.

Proposition 3.16. *Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$, then*

(1) If $(\mathbb{F}, \mathcal{E})$ is a hypersoft neighborhood of $u \in \mathcal{U}$, then $u \in (\mathbb{F}, \mathcal{E})$.
(2) Each $u \in \mathcal{U}$ has a hypersoft neighborhood.
(3) If $(\mathbb{F}, \mathcal{E})$ and $(\mathbb{G}, \mathcal{E})$ are hypersoft neighborhoods of some $u \in \mathcal{U}$, then $(\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} (\mathbb{G}, \mathcal{E})$ is also a hypersoft neighborhood of u.
(4) If $(\mathbb{F}, \mathcal{E})$ is a hypersoft neighborhood of $u \in \mathcal{U}$ and $(\mathbb{F}, \mathcal{E}) \widetilde{\widetilde{\sqsubseteq}} (\mathbb{G}, \mathcal{E})$, then $(\mathbb{G}, \mathcal{E})$ is also a hypersoft neighborhood of $u \in \mathcal{U}$.

Proof.

(1) Follows from Definition 3.15.
(2) For any $u \in \mathcal{U}$, $u \in (\Psi, \mathcal{E})$ and since $(\Psi, \mathcal{E}) \widetilde{\widetilde{\in}} \mathcal{T}_{\mathcal{H}}$, so $u \in (\Psi, \mathcal{E}) \widetilde{\widetilde{\sqsubseteq}} (\Psi, \mathcal{E})$. Thus $(\Psi, \mathcal{E})$ is a hypersoft neighborhood of u.
(3) Let $(\mathbb{F}, \mathcal{E})$ and $(\mathbb{G}, \mathcal{E})$ be the hypersoft neighborhoods of $u \in \mathcal{U}$, then there exist $(\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}_2, \mathcal{E}) \widetilde{\widetilde{\in}} \mathcal{T}_{\mathcal{H}}$ such that $u \in (\mathbb{F}_1, \mathcal{E}) \widetilde{\widetilde{\sqsubseteq}} (\mathbb{F}, \mathcal{E})$ and $u \in (\mathbb{F}_2, \mathcal{E}) \widetilde{\widetilde{\sqsubseteq}} (\mathbb{G}, \mathcal{E})$. Now $u \in (\mathbb{F}_1, \mathcal{E})$ and $u \in (\mathbb{F}_2, \mathcal{E})$ implies that $u \in (\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E})$ and $(\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) \widetilde{\widetilde{\in}} \mathcal{T}_{\mathcal{H}}$. So we have $u \in (\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) \widetilde{\widetilde{\sqsubseteq}} (\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} (\mathbb{G}, \mathcal{E})$. Thus, $(\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} (\mathbb{G}, \mathcal{E})$ is a hypersoft neighborhood of u.

(4) Let $(\mathbb{F}, \mathcal{E})$ be a hypersoft neighborhood of $u \in \mathcal{U}$ and $(\mathbb{F}, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{G}, \mathcal{E})$. By definition, there exists a hypersoft open set $(\mathbb{F}_1, \mathcal{E})$ such that $u \in (\mathbb{F}_1, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{G}, \mathcal{E})$. Thus, $u \in (\mathbb{F}_1, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{G}, \mathcal{E})$. Hence, $(\mathbb{G}, \mathcal{E})$ is a hypersoft neighborhood of u.

Proposition 3.17. *Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$. For any hypersoft open set $(\mathbb{F}, \mathcal{E})$ over $\mathcal{U}$, $(\mathbb{F}, \mathcal{E})$ is a hypersoft neighborhood of each point of $\cap_{\alpha \in \mathcal{E}} \mathbb{F}(\alpha)$, that is, of each of its points.*

Proof. Let $(\mathbb{F}, \mathcal{E}) \tilde{\in} \mathcal{T}_{\mathcal{H}}$. For any $u \in \cap_{\alpha \in \mathcal{E}} \mathbb{F}(\alpha)$, we have $u \in \mathbb{F}(\alpha)$ for each $\alpha \in \mathcal{E}$. Thus $u \in (\mathbb{F}, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}, \mathcal{E})$ and so $(\mathbb{F}, \mathcal{E})$ is a hypersoft neighborhood of u.

Remark 3.18. The following example shows that the converse of Proposition 3.17 is not true in general.

Example 3.19. Consider $\mathcal{T}_{\mathcal{H}_1}$ given in Example 3.14 and let $(\mathbb{F}, \mathcal{E})$ be any hypersoft set defined as follows:

$(\mathbb{F}, \mathcal{E}) = \{((e_1, e_3, e_4), \{h_1, h_3, h_4\}), ((e_2, e_3, e_4), \{h_2, h_3\})\}$.

Then $(\mathbb{F}, \mathcal{E})$ is a hypersoft neighborhood of each point of $\cap_{\alpha \in \mathcal{E}} \mathbb{F}(\alpha)$, that is, of each of its points, but it is not a hypersoft open set.

Definition 3.20. Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$ and let $(\mathbb{F}, \mathcal{E})$ be a hypersoft set over $\mathcal{U}$. A point $u \in \mathcal{U}$ is called a hypersoft limit point of $(\mathbb{F}, \mathcal{E})$ if $(\mathbb{F}, \mathcal{E}) \tilde{\sqcap} ((\mathbb{G}, \mathcal{E}) \setminus \{u\})$ $\neq (\Phi, \mathcal{E})$ for every hypersoft open set $(\mathbb{G}, \mathcal{E})$ containing u. The set of all hypersoft limit points of $(\mathbb{F}, \mathcal{E})$ is denoted by $(\mathbb{F}, \mathcal{E})^d$.

Proposition 3.21. *Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$ and let $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E})$ be two hypersoft sets over $\mathcal{U}$. Then*

(1) $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})$ *implies* $(\mathbb{F}_1, \mathcal{E})^d \tilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})^d$.

(2) $((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^d \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^d \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E})^d$.

(3) $((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E}))^d = (\mathbb{F}_1, \mathcal{E})^d \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E})^d$.

Proof.

(1) Let $u \in (\mathbb{F}_1, \mathcal{E})^d$ so that u is a hypersoft limit point of $(\mathbb{F}_1, \mathcal{E})$. Then, by definition $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} ((\mathbb{G}, \mathcal{E}) \setminus \{u\}) \neq (\Phi, \mathcal{E})$ for every hypersoft open set $(\mathbb{G}, \mathcal{E})$ containing u. But $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})$, it follows that $(\mathbb{F}_2, \mathcal{E}) \tilde{\sqcap} ((\mathbb{G}, \mathcal{E}) \setminus \{u\}) \neq (\Phi, \mathcal{E})$. Thus, $u \in (\mathbb{F}_2, \mathcal{E})^d$. Therefore, $(\mathbb{F}_1, \mathcal{E})^d \tilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})^d$.

(2) Since $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})$. It follows from (1) that, $((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^d \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^d$ and $((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^d \tilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})^d$. Hence, $((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^d \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^d \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E})^d$.

(3) Since $(\mathbb{F}_1,\mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_1,\mathcal{E}) \widetilde{\sqcup} (\mathbb{F}_2,\mathcal{E})$ and $(\mathbb{F}_2,\mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_1,\mathcal{E}) \widetilde{\sqcup} (\mathbb{F}_2,\mathcal{E})$. By (1) we have $(\mathbb{F}_1,\mathcal{E})^d \widetilde{\sqsubseteq} ((\mathbb{F}_1,\mathcal{E}) \widetilde{\sqcup} (\mathbb{F}_2,\mathcal{E}))^d$ and $(\mathbb{F}_2,\mathcal{E})^d \widetilde{\sqsubseteq} ((\mathbb{F}_1,\mathcal{E}) \widetilde{\sqcup} (\mathbb{F}_2,\mathcal{E}))^d$. So, $(\mathbb{F}_1,\mathcal{E})^d \widetilde{\sqcup} (\mathbb{F}_2,\mathcal{E})^d \widetilde{\sqsubseteq} ((\mathbb{F}_1,\mathcal{E}) \widetilde{\sqcup} (\mathbb{F}_2,\mathcal{E}))^d$. Now, let $u \in ((\mathbb{F}_1,\mathcal{E}) \widetilde{\sqcup} (\mathbb{F}_2,\mathcal{E}))^d$. Then, $((\mathbb{F}_1,\mathcal{E}) \widetilde{\sqcup} (\mathbb{F}_2,\mathcal{E})) \widetilde{\sqcap} ((\mathbb{G},\mathcal{E}) \setminus \{u\}) \neq (\Phi,\mathcal{E})$ for every hypersoft open set $(\mathbb{G},\mathcal{E})$ containing u. Therefore, $(\mathbb{F}_1,\mathcal{E}) \widetilde{\sqcap} ((\mathbb{G},\mathcal{E}) \setminus \{u\}) \neq (\Phi,\mathcal{E})$ or $(\mathbb{F}_2,\mathcal{E}) \widetilde{\sqcap} ((\mathbb{G},\mathcal{E}) \setminus \{u\}) \neq (\Phi,\mathcal{E})$. Thus, $u \in (\mathbb{F}_1,\mathcal{E})^d$ or $u \in (\mathbb{F}_2,\mathcal{E})^d$ and then $u \in (\mathbb{F}_1,\mathcal{E})^d \widetilde{\sqcup} (\mathbb{F}_2,\mathcal{E})^d$. Therefore, $((\mathbb{F}_1,\mathcal{E}) \widetilde{\sqcup} (\mathbb{F}_2,\mathcal{E}))^d \widetilde{\sqsubseteq} (\mathbb{F}_1,\mathcal{E})^d \widetilde{\sqcup} (\mathbb{F}_2,\mathcal{E})^d$. Now, we have $((\mathbb{F}_1,\mathcal{E}) \widetilde{\sqcup} (\mathbb{F}_2,\mathcal{E}))^d = (\mathbb{F}_1,\mathcal{E})^d \widetilde{\sqcup} (\mathbb{F}_2,\mathcal{E})^d$.

Remark 3.22. The following example shows that the equality in Proposition 3.21 (2) does not hold in general.

Example 3.23. Let us consider the hypersoft topological space $(\mathcal{U},\mathcal{T}_{\mathcal{H}_1},\mathcal{E})$ in Example 3.14 and let $(\mathbb{F},\mathcal{E})$ and $(\mathbb{G},\mathcal{E})$ are hypersoft sets defined as follows:

$(\mathbb{F},\mathcal{E}) = \{((e_1,e_3,e_4),\phi),((e_2,e_3,e_4),\{h_4\})\}$.
$(\mathbb{G},\mathcal{E}) = \{((e_1,e_3,e_4),\{h_2\}),((e_2,e_3,e_4),\{h_3\}$.

Then $(\mathbb{F},\mathcal{E})^d \widetilde{\sqcap} (\mathbb{G},\mathcal{E})^d = \{h_1\}$. But, $(\mathbb{F},\mathcal{E}) \widetilde{\sqcap} (\mathbb{G},\mathcal{E}) = (\Phi,\mathcal{E})$ and $((\mathbb{F},\mathcal{E}) \widetilde{\sqcap} (\mathbb{G},\mathcal{E}))^d = (\Phi,\mathcal{E})^d = \phi$. Hence, $((\mathbb{F}_1,\mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2,\mathcal{E}))^d \neq (\mathbb{F}_1,\mathcal{E})^d \widetilde{\sqcap} (\mathbb{F}_2,\mathcal{E})^d$.

Definition 3.24. Let $(\mathcal{U},\mathcal{T}_{\mathcal{H}},\mathcal{E})$ be a hypersoft space over $\mathcal{U}$ and $\mathcal{Y}$ be a non-empty subset of $\mathcal{U}$. Then

$$\mathcal{T}_{\mathcal{H}_{\mathcal{Y}}} = \{(\mathbb{F}_{\mathcal{Y}},\mathcal{E}) \mid (\mathbb{F},\mathcal{E}) \widetilde{\in} \mathcal{T}_{\mathcal{H}}\}$$

is said to be the relative hypersoft topology on $\mathcal{Y}$ and $(\mathcal{Y},\mathcal{T}_{\mathcal{H}_{\mathcal{Y}}},\mathcal{E})$ is called a hypersoft subspace of $(\mathcal{U},\mathcal{T}_{\mathcal{H}},\mathcal{E})$

It is easy to verify that $T_{\mathcal{H}_{\mathcal{Y}}}$ is a hypersoft topology on $\mathcal{Y}$.

Example 3.25. Any hypersoft subspace of a hypersoft indiscrete topological space is a hypersoft indiscrete topological space.

Example 3.26. Any hypersoft subspace of a hypersoft discrete topological space is a hypersoft discrete topological space.

Proposition 3.27. *Let $(\mathcal{Y},\mathcal{T}_{\mathcal{H}_{\mathcal{Y}}},\mathcal{E})$ be a hypersoft subspace of a hypersoft topological space $(\mathcal{U},\mathcal{T}_{\mathcal{H}},\mathcal{E})$ and $(\mathbb{F}_{\mathcal{Y}},\mathcal{E})$ be a hypersoft open set in $\mathcal{Y}$. If $(\Upsilon,\mathcal{E}) \widetilde{\in} \mathcal{T}_{\mathcal{H}}$ then $(\mathbb{F}_{\mathcal{Y}},\mathcal{E}) \widetilde{\in} \mathcal{T}_{\mathcal{H}}$.*

Proof. Let $(\mathbb{F}_{\mathcal{Y}},\mathcal{E})$ be a hypersoft open set in $\mathcal{Y}$, then there exists a hypersoft open set $(\mathbb{F},\mathcal{E})$ in $\mathcal{U}$ such that $(\mathbb{F}_{\mathcal{Y}},\mathcal{E}) = (\Upsilon,\mathcal{E}) \widetilde{\sqcap} (\mathbb{F},\mathcal{E})$. Now, if $(\Upsilon,\mathcal{E}) \widetilde{\in} \mathcal{T}_{\mathcal{H}}$ then $(\Upsilon,\mathcal{E}) \widetilde{\sqcap} (\mathbb{F},\mathcal{E}) \widetilde{\in} \mathcal{T}_{\mathcal{H}}$. Hence, $(\mathbb{F}_{\mathcal{Y}},\mathcal{E}) \widetilde{\in} \mathcal{T}_{\mathcal{H}}$.

Proposition 3.28. *Let $(\mathcal{Y}, \mathcal{T}_{\mathcal{H}_{\mathcal{Y}}}, \mathcal{E})$ and $(\mathcal{Z}, \mathcal{T}_{\mathcal{H}_{\mathcal{Z}}}, \mathcal{E})$ be two hypersoft subspaces of $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ and let $\mathcal{Y} \widetilde{\sqsubseteq} \mathcal{Z}$. Then $(\mathcal{Y}, \mathcal{T}_{\mathcal{H}_{\mathcal{Y}}}, \mathcal{E})$ is a hypersoft subspace of $(\mathcal{Z}, \mathcal{T}_{\mathcal{H}_{\mathcal{Z}}}, \mathcal{E})$.*

Proof. Let $(\mathbb{F}_{\mathcal{Y}}, \mathcal{E})$ be a hypersoft open set in $\mathcal{Y}$, then there exists a hypersoft open set $(\mathbb{F}, \mathcal{E})$ in $\mathcal{U}$ such that $(\mathbb{F}_{\mathcal{Y}}, \mathcal{E}) = (\Upsilon, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}, \mathcal{E})$, or equivalently, for each $\alpha \in \mathcal{E}$, $\mathbb{F}_{\mathcal{Y}}(\alpha) = \mathcal{Y} \cap \mathbb{F}(\alpha)$. Since $\mathcal{Y} \widetilde{\sqsubseteq} \mathcal{Z}$ then $\mathcal{Y} = \mathcal{Y} \widetilde{\sqcap} \mathcal{Z}$. Now, $\mathbb{F}_{\mathcal{Y}}(\alpha) = \mathcal{Y} \cap \mathbb{F}(\alpha) = (\mathcal{Y} \widetilde{\sqcap} \mathcal{Z}) \cap \mathbb{F}(\alpha) = \mathcal{Y} \widetilde{\sqcap} (\mathcal{Z} \cap \mathbb{F}(\alpha)) = \mathcal{Y} \cap \mathbb{F}_{\mathcal{Z}}(\alpha)$ so we have $\mathbb{F}_{\mathcal{Y}}(\alpha) = \mathcal{Y} \cap \mathbb{F}_{\mathcal{Z}}(\alpha)$, or equivalently, $(\mathbb{F}_{\mathcal{Y}}, \mathcal{E}) = (\Upsilon, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_{\mathcal{Z}}, \mathcal{E})$ where $(\mathbb{F}_{\mathcal{Z}}, \mathcal{E})$ is a hypersoft open set in $\mathcal{Z}$. Hence, $(\mathcal{Y}, \mathcal{T}_{\mathcal{H}_{\mathcal{Y}}}, \mathcal{E})$ is a hypersoft subspace of $(\mathcal{Z}, \mathcal{T}_{\mathcal{H}_{\mathcal{Z}}}, \mathcal{E})$.

4. Hypersoft Closure, Hypersoft Interior, Hypersoft Exterior, and Hypersoft Boundary

Definition 4.1. Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space and $(\mathbb{F}, \mathcal{E})$ be a hypersoft set over $\mathcal{U}$. The intersection of all hypersoft closed supersets of $(\mathbb{F}, \mathcal{E})$ is called the hypersoft closure of $(\mathbb{F}, \mathcal{E})$ and is denoted by $\overline{(\mathbb{F}, \mathcal{E})}$.

In other words, $\overline{(\mathbb{F}, \mathcal{E})} = \widetilde{\sqcap} \{(\mathbb{G}, \mathcal{E}) \mid (\mathbb{G}, \mathcal{E})^c \widetilde{\in} \mathcal{T}_{\mathcal{H}}, (\mathbb{G}, \mathcal{E}) \widetilde{\sqsupseteq} (\mathbb{F}, \mathcal{E})\}$.

Proposition 4.2. *Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space and $(\mathbb{F}, \mathcal{E})$ be a hypersoft set over $\mathcal{U}$. Then*

(1) $\overline{(\mathbb{F}, \mathcal{E})}$ is the smallest hypersoft closed set containing $(\mathbb{F}, \mathcal{E})$.
(2) $(\mathbb{F}, \mathcal{E})$ is a hypersoft closed set if and only if $(\mathbb{F}, \mathcal{E}) = \overline{(\mathbb{F}, \mathcal{E})}$.

Proof.

(1) Follows from Definition 4.1.
(2) Let $(\mathbb{F}, \mathcal{E})$ be a hypersoft closed set. So, $(\mathbb{F}, \mathcal{E})$ itself is the smallest hypersoft closed set over $\mathcal{U}$ containing $(\mathbb{F}, \mathcal{E})$ and hence $(\mathbb{F}, \mathcal{E}) = \overline{(\mathbb{F}, \mathcal{E})}$. Conversely, suppose that $(\mathbb{F}, \mathcal{E}) = \overline{(\mathbb{F}, \mathcal{E})}$. By (1.) $\overline{(\mathbb{F}, \mathcal{E})}$ is a hypersoft closed, so $(\mathbb{F}, \mathcal{E})$ is also a hypersoft closed set over $\mathcal{U}$

Proposition 4.3. *Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$ and let $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E})$ be two hypersoft sets over $\mathcal{U}$. Then*

(1) $\overline{(\Phi, \mathcal{E})} = (\Phi, \mathcal{E})$ and $\overline{(\Psi, \mathcal{E})} = (\Psi, \mathcal{E})$.
(2) $(\mathbb{F}_1, \mathcal{E}) \widetilde{\sqsubseteq} \overline{(\mathbb{F}_1, \mathcal{E})}$.
(3) $(\mathbb{F}_1, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})$ implies $\overline{(\mathbb{F}_1, \mathcal{E})} \widetilde{\sqsubseteq} \overline{(\mathbb{F}_2, \mathcal{E})}$.
(4) $\overline{((\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcup} (\mathbb{F}_2, \mathcal{E}))} = \overline{(\mathbb{F}_1, \mathcal{E})} \widetilde{\sqcup} \overline{(\mathbb{F}_2, \mathcal{E})}$.
(5) $\overline{((\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))} \widetilde{\sqsubseteq} \overline{(\mathbb{F}_1, \mathcal{E})} \widetilde{\sqcap} \overline{(\mathbb{F}_2, \mathcal{E})}$.
(6) $\overline{\overline{(\mathbb{F}_1, \mathcal{E})}} = \overline{(\mathbb{F}_1, \mathcal{E})}$.

Proof.

(1) Since $(\Phi, \mathcal{E})$ and $(\Psi, \mathcal{E})$ are hypersoft closed sets, then by Proposition 4.2 (2), we have $\overline{(\Phi, \mathcal{E})} = (\Phi, \mathcal{E})$ and $\overline{(\Psi, \mathcal{E})} = (\Psi, \mathcal{E})$.

(2) By Proposition 4.2 (1), $\overline{(\mathbb{F}_1, \mathcal{E})}$ is the smallest hypersoft closed set containing $(\mathbb{F}_1, \mathcal{E})$ and so $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqsubseteq} \overline{(\mathbb{F}_1, \mathcal{E})}$.

(3) By (2.), $(\mathbb{F}_2, \mathcal{E}) \tilde{\sqsubseteq} \overline{(\mathbb{F}_2, \mathcal{E})}$. Since $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})$, we have $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqsubseteq} \overline{(\mathbb{F}_2, \mathcal{E})}$. But $\overline{(\mathbb{F}_2, \mathcal{E})}$ is a hypersoft closed set. Thus, $\overline{(\mathbb{F}_2, \mathcal{E})}$ is a hypersoft closed set containing $(\mathbb{F}_1, \mathcal{E})$. Since $\overline{(\mathbb{F}_1, \mathcal{E})}$ is the smallest hypersoft closed set over $\mathcal{U}$ containing $(\mathbb{F}_1, \mathcal{E})$, so we have $\overline{(\mathbb{F}_1, \mathcal{E})} \tilde{\sqsubseteq} \overline{(\mathbb{F}_2, \mathcal{E})}$.

(4) Since $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E})$ and $(\mathbb{F}_2, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E})$. By (3.), we have $\overline{(\mathbb{F}_1, \mathcal{E})} \tilde{\sqsubseteq} \overline{((\mathbb{F}_1, \mathcal{E})\tilde{\sqcup}(\mathbb{F}_2, \mathcal{E}))}$ and $\overline{(\mathbb{F}_2, \mathcal{E})} \tilde{\sqsubseteq} \overline{((\mathbb{F}_1, \mathcal{E})\tilde{\sqcup}(\mathbb{F}_2, \mathcal{E}))}$. Hence, $\overline{(\mathbb{F}_1, \mathcal{E})} \tilde{\sqcup} \overline{(\mathbb{F}_2, \mathcal{E})} \tilde{\sqsubseteq} \overline{((\mathbb{F}_1, \mathcal{E})\tilde{\sqcup}(\mathbb{F}_2, \mathcal{E}))}$. Now, since $\overline{(\mathbb{F}_1, \mathcal{E})}$ and $\overline{(\mathbb{F}_2, \mathcal{E})}$ are hypersoft closed sets, $\overline{(\mathbb{F}_1, \mathcal{E})} \tilde{\sqcup} \overline{(\mathbb{F}_2, \mathcal{E})}$ is also hypersoft closed. Also, $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqsubseteq} \overline{(\mathbb{F}_1, \mathcal{E})}$ and $(\mathbb{F}_2, \mathcal{E}) \tilde{\sqsubseteq} \overline{(\mathbb{F}_2, \mathcal{E})}$ implies that $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E}) \tilde{\sqsubseteq} \overline{(\mathbb{F}_1, \mathcal{E})} \tilde{\sqcup} \overline{(\mathbb{F}_2, \mathcal{E})}$. Thus, $\overline{(\mathbb{F}_1, \mathcal{E})} \tilde{\sqcup} \overline{(\mathbb{F}_2, \mathcal{E})}$ is a hypersoft closed containing $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E})$. Since $\overline{((\mathbb{F}_1, \mathcal{E})\tilde{\sqcup}(\mathbb{F}_2, \mathcal{E}))}$ is the smallest hypersoft closed set containing $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E})$, we have $\overline{((\mathbb{F}_1, \mathcal{E})\tilde{\sqcup}(\mathbb{F}_2, \mathcal{E}))} \tilde{\sqsubseteq} \overline{(\mathbb{F}_1, \mathcal{E})} \tilde{\sqcup} \overline{(\mathbb{F}_2, \mathcal{E})}$. Hence, $\overline{((\mathbb{F}_1, \mathcal{E})\tilde{\sqcup}(\mathbb{F}_2, \mathcal{E}))} = \overline{(\mathbb{F}_1, \mathcal{E})} \tilde{\sqcup} \overline{(\mathbb{F}_2, \mathcal{E})}$.

(5) Since $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})$, then $\overline{((\mathbb{F}_1, \mathcal{E})\tilde{\sqcap}(\mathbb{F}_2, \mathcal{E}))} \tilde{\sqsubseteq} \overline{(\mathbb{F}_1, \mathcal{E})}$ and $\overline{((\mathbb{F}_1, \mathcal{E})\tilde{\sqcap}(\mathbb{F}_2, \mathcal{E}))} \tilde{\sqsubseteq} \overline{(\mathbb{F}_2, \mathcal{E})}$. Therefore, $\overline{((\mathbb{F}_1, \mathcal{E})\tilde{\sqcap}(\mathbb{F}_2, \mathcal{E}))} \tilde{\sqsubseteq} \overline{(\mathbb{F}_1, \mathcal{E})} \tilde{\sqcap} \overline{(\mathbb{F}_2, \mathcal{E})}$.

(6) Since $\overline{(\mathbb{F}_1, \mathcal{E})}$ is a hypersoft closed set, therefore by Proposition 4.2 (2), we have $\overline{\overline{(\mathbb{F}_1, \mathcal{E})}} = \overline{(\mathbb{F}_1, \mathcal{E})}$.

Remark 4.4. The following example shows that the equality does not hold in Proposition 4.3 (5).

Example 4.5. Let us consider the hypersoft topological space $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_1}, \mathcal{E})$ in Example 3.14 and let $(\mathbb{F}, \mathcal{E})$, $(\mathbb{G}, \mathcal{E})$ in Example 3.23. Then

$\overline{(\mathbb{F}, \mathcal{E})} = (\mathbb{F}_1, \mathcal{E})^c$ and $\overline{(\mathbb{G}, \mathcal{E})} = (\mathbb{F}_3, \mathcal{E})^c$ and $\overline{(\mathbb{F}, \mathcal{E})} \tilde{\sqcap} \overline{(\mathbb{G}, \mathcal{E})} = (\mathbb{F}_1, \mathcal{E})^c$. Now, $(\mathbb{F}, \mathcal{E}) \tilde{\sqcap} (\mathbb{G}, \mathcal{E}) = (\Phi, \mathcal{E})$ and $\overline{((\mathbb{F}, \mathcal{E})\tilde{\sqcap}(\mathbb{G}, \mathcal{E}))} = \overline{(\Phi, \mathcal{E})} = (\Phi, \mathcal{E})$. Hence, $\overline{((\mathbb{F}, \mathcal{E})\tilde{\sqcap}(\mathbb{G}, \mathcal{E}))} \neq \overline{(\mathbb{F}, \mathcal{E})} \tilde{\sqcap} \overline{(\mathbb{G}, \mathcal{E})}$.

Definition 4.6. Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$, $(\mathbb{F}, \mathcal{E})$ be a hypersoft set over $\mathcal{U}$ and $u \in \mathcal{U}$. Then u is said to be a hypersoft interior point of $(\mathbb{F}, \mathcal{E})$ if there exists a hypersoft open set $(\mathbb{G}, \mathcal{E})$ such that $u \in (\mathbb{G}, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}, \mathcal{E})$.

Definition 4.7. Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$. Then hypersoft interior of hypersoft set $(\mathbb{F}, \mathcal{E})$ over $\mathcal{U}$ is denoted by $(\mathbb{F}, \mathcal{E})^o$ and is defined as the union of all hypersoft open set contained in $(\mathbb{F}, \mathcal{E})$.

In other words, $(\mathbb{F}, \mathcal{E})^o = \widetilde{\sqcup} \{(\mathbb{G}, \mathcal{E}) \mid (\mathbb{G}, \mathcal{E}) \widetilde{\in} \mathcal{T}_{\mathcal{H}}, (\mathbb{G}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}, \mathcal{E})\}$.

Proposition 4.8. *Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space and let $(\mathbb{F}, \mathcal{E})$ be a hypersoft set over $\mathcal{U}$. Then*

(1) $(\mathbb{F}, \mathcal{E})^o$ is the largest hypersoft open set contained in $(\mathbb{F}, \mathcal{E})$.
(2) $(\mathbb{F}, \mathcal{E})$ is a hypersoft open set if and only if $(\mathbb{F}, \mathcal{E}) = (\mathbb{F}, \mathcal{E})^o$.

Proof.

(1) Follows from Definition 4.7.
(2) Let $(\mathbb{F}, \mathcal{E})$ be a hypersoft open set. Then $(\mathbb{F}, \mathcal{E})$ is surely identical with the largest hypersoft open subset of $(\mathbb{F}, \mathcal{E})$. But by (1.), $(\mathbb{F}, \mathcal{E})^o$ is the largest hypersoft open subset of $(\mathbb{F}, \mathcal{E})$. Hence, $(\mathbb{F}, \mathcal{E}) = (\mathbb{F}, \mathcal{E})^o$. Conversely, let $(\mathbb{F}, \mathcal{E}) = (\mathbb{F}, \mathcal{E})^o$. By (1.), $(\mathbb{F}, \mathcal{E})^o$ is a hypersoft open set and therefore $(\mathbb{F}, \mathcal{E})$ is also hypersoft open set.

Proposition 4.9. *Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$ and let $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E})$ be two hypersoft sets over $\mathcal{U}$. Then*

(1) $(\Phi, \mathcal{E})^o = (\Phi, \mathcal{E})$ and $(\Psi, \mathcal{E})^o = (\Psi, \mathcal{E})$.
(2) $(\mathbb{F}_1, \mathcal{E})^o \widetilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})$.
(3) $(\mathbb{F}_1, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})$ implies $(\mathbb{F}_1, \mathcal{E})^o \widetilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})^o$.
(4) $(\mathbb{F}_1, \mathcal{E})^o \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E})^o = ((\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^o$.
(5) $(\mathbb{F}_1, \mathcal{E})^o \widetilde{\sqcup} (\mathbb{F}_2, \mathcal{E})^o \widetilde{\sqsubseteq} ((\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcup} (\mathbb{F}_2, \mathcal{E}))^o$.
(6) $((\mathbb{F}_1, \mathcal{E})^o)^o = (\mathbb{F}_1, \mathcal{E})^o$.

Proof.

(1) Since $(\Phi, \mathcal{E})$ and $(\Psi, \mathcal{E})$ are hypersoft open sets, then by Proposition 4.8 (2), we have $(\Phi, \mathcal{E})^o = (\Phi, \mathcal{E})$ and $(\Psi, \mathcal{E})^o = (\Psi, \mathcal{E})$.
(2) Let $u \in (\mathbb{F}_1, \mathcal{E})^o$ then u is a hypersoft interior point of $(\mathbb{F}_1, \mathcal{E})$. This implies that $(\mathbb{F}_1, \mathcal{E})$ is a hypersoft neighborhood of u. Then $u \in (\mathbb{F}_1, \mathcal{E})$. Hence, $(\mathbb{F}_1, \mathcal{E})^o \widetilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})$.
(3) Let $u \in (\mathbb{F}_1, \mathcal{E})^o$. Then u is a hypersoft interior point of $(\mathbb{F}_1, \mathcal{E})$ and so $(\mathbb{F}_1, \mathcal{E})$ is a hypersoft neighborhood of u. Since $(\mathbb{F}_1, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E})$ is also is a hypersoft neighborhood of u. This implies that $u \in (\mathbb{F}_2, \mathcal{E})^o$. Thus, $(\mathbb{F}_1, \mathcal{E})^o \widetilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})^o$.
(4) Since $(\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})$, we have by (3.), $((\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^o \widetilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^o$ and $((\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^o \widetilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})^o$. This implies that $((\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^o \widetilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^o \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E})^o$. Again let $u \in (\mathbb{F}_1, \mathcal{E})^o \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E})^o$. Then $u \in (\mathbb{F}_1, \mathcal{E})^o$ and $u \in (\mathbb{F}_2, \mathcal{E})^o$. Hence u is a hypersoft interior point of each of the

hypersoft sets $(\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}_2, \mathcal{E})$. It follows that $(\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}_2, \mathcal{E})$ are hypersoft neighborhoods of u so that their intersection $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E})$ is also is a hypersoft neighborhood of u. Hence, $u \in ((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^o$. Thus, $(\mathbb{F}_1, \mathcal{E})^o \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E})^o \tilde{\sqsubseteq} ((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^o$. Therefore, $(\mathbb{F}_1, \mathcal{E})^o \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E})^o = ((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^o$.

(5) By (3.), $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E})$ implies $(\mathbb{F}_1, \mathcal{E})^o \tilde{\sqsubseteq} ((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E}))^o$ and $(\mathbb{F}_2, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E})$ implies $(\mathbb{F}_2, \mathcal{E})^o \tilde{\sqsubseteq} ((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E}))^o$. Hence, $(\mathbb{F}_1, \mathcal{E})^o \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E})^o \tilde{\sqsubseteq} ((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E}))^o$.

(6) By Proposition 4.8 (1), $(\mathbb{F}_1, \mathcal{E})^o$ is the hypersoft open set. Hence by (2.) of the same proposition $((\mathbb{F}_1, \mathcal{E})^o)^o = (\mathbb{F}_1, \mathcal{E})^o$.

Remark 4.10. The following example shows that the equality does not hold in Proposition 4.9 (5).

Example 4.11. Let us consider the hypersoft topological space $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_1}, \mathcal{E})$ in Example 3.14 and let $(\mathbb{F}, \mathcal{E})$ and $(\mathbb{G}, \mathcal{E})$ are hypersoft sets defined as follows:

$$(\mathbb{F}, \mathcal{E}) = \{((e_1, e_3, e_4), \{h_1, h_3, h_4\}), ((e_2, e_3, e_4), \{h_2, h_3\})\}.$$

$$(\mathbb{G}, \mathcal{E}) = \{((e_1, e_3, e_4), \mathcal{U}), ((e_1, e_3, e_4), \{h_1, h_4\})\}.$$

Then $(\mathbb{F}, \mathcal{E})^o = (\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{G}, \mathcal{E})^o = (\mathbb{F}_2, \mathcal{E})$ and $(\mathbb{F}, \mathcal{E})^o \tilde{\sqcup} (\mathbb{G}, \mathcal{E})^o = (\mathbb{F}_3, \mathcal{E})$. Now, $(\mathbb{F}, \mathcal{E}) \tilde{\sqcup} (\mathbb{G}, \mathcal{E}) = (\Psi, \mathcal{E})$ and $((\mathbb{F}, \mathcal{E}) \tilde{\sqcup} (\mathbb{G}, \mathcal{E}))^o = (\Psi, \mathcal{E})^o = (\Psi, \mathcal{E})$. Hence, $((\mathbb{F}, \mathcal{E}) \tilde{\sqcup} (\mathbb{G}, \mathcal{E}))^o \neq (\mathbb{F}, \mathcal{E})^o \tilde{\sqcup} (\mathbb{G}, \mathcal{E})^o$.

Proposition 4.12. *Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$ and let $(\mathbb{F}, \mathcal{E})$ be a hypersoft set over $\mathcal{U}$. Then $(\mathbb{F}, \mathcal{E})^o \tilde{\sqsubseteq} (\mathbb{F}, \mathcal{E}) \tilde{\sqsubseteq} \overline{(\mathbb{F}, \mathcal{E})}$.*

Proof. Follows from Proposition 4.3 (2) and Proposition 4.9 (2).

Proposition 4.13. *Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$ and let $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E})$ be two hypersoft sets over $\mathcal{U}$. Then*

(1) $(\overline{(\mathbb{F}_1, \mathcal{E})})^c = ((\mathbb{F}_1, \mathcal{E})^c)^o$.

(2) $((\mathbb{F}_1, \mathcal{E})^o)^c = \overline{((\mathbb{F}_1, \mathcal{E})^c)}$.

(3) $\overline{(\mathbb{F}_1, \mathcal{E})} = (((\mathbb{F}_1, \mathcal{E})^c)^o)^c$.

(4) $(\mathbb{F}_1, \mathcal{E})^o = (\overline{((\mathbb{F}_1, \mathcal{E})^c)})^c$.

(5) $((\mathbb{F}_1, \mathcal{E}) \setminus (\mathbb{F}_2, \mathcal{E}))^o \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^o \setminus (\mathbb{F}_2, \mathcal{E})^o$.

Proof. From the definitions of hypersoft closure and hypersoft interior, we have

(1) $\overline{(\mathbb{F}_1, \mathcal{E})} = \widetilde{\sqcap} \{(\mathbb{G}, \mathcal{E}) \mid (\mathbb{G}, \mathcal{E})^c \widetilde{\in} \mathcal{T}_{\mathcal{H}}, (\mathbb{G}, \mathcal{E}) \widetilde{\sqsupseteq} (\mathbb{F}_1, \mathcal{E})\}$.
$(\overline{(\mathbb{F}_1, \mathcal{E})})^c = [\widetilde{\sqcap} \{(\mathbb{G}, \mathcal{E}) \mid (\mathbb{G}, \mathcal{E})^c \widetilde{\in} \mathcal{T}_{\mathcal{H}}, (\mathbb{G}, \mathcal{E}) \widetilde{\sqsupseteq} (\mathbb{F}_1, \mathcal{E})\}]^c$.
$(\overline{(\mathbb{F}_1, \mathcal{E})})^c = \widetilde{\sqcup} \{(\mathbb{G}, \mathcal{E})^c \mid (\mathbb{G}, \mathcal{E})^c \widetilde{\in} \mathcal{T}_{\mathcal{H}}, (\mathbb{G}, \mathcal{E})^c \widetilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^c\} = ((\mathbb{F}_1, \mathcal{E})^c)^o$.

(2) $(\mathbb{F}_1, \mathcal{E})^o = \widetilde{\sqcup} \{(\mathbb{G}, \mathcal{E}) \mid (\mathbb{G}, \mathcal{E}) \widetilde{\in} \mathcal{T}_{\mathcal{H}}, (\mathbb{G}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})\}$.
$((\mathbb{F}_1, \mathcal{E})^o)^c = [\widetilde{\sqcup} \{(\mathbb{G}, \mathcal{E}) \mid (\mathbb{G}, \mathcal{E}) \widetilde{\in} \mathcal{T}_{\mathcal{H}}, (\mathbb{G}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})\}]^c$.
$((\mathbb{F}_1, \mathcal{E})^o)^c = \widetilde{\sqcap} \{(\mathbb{G}, \mathcal{E})^c \mid (\mathbb{G}, \mathcal{E}) \widetilde{\in} \mathcal{T}_{\mathcal{H}}, (\mathbb{F}_1, \mathcal{E})^c \widetilde{\sqsubseteq} (\mathbb{G}, \mathcal{E})^c\} = \overline{((\mathbb{F}_1, \mathcal{E})^c)}$.

(3) Obtained from (1.) by taking the hypersoft complement.

(4) Obtained from (2.) by taking the hypersoft complement.

(5) $((\mathbb{F}_1, \mathcal{E}) \setminus (\mathbb{F}_2, \mathcal{E}))^o = ((\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E})^c)^o = (\mathbb{F}_1, \mathcal{E})^o \widetilde{\sqcap} ((\mathbb{F}_2, \mathcal{E})^c)^o = (\mathbb{F}_1, \mathcal{E})^o \widetilde{\sqcap}$ $(\overline{(\mathbb{F}_2, \mathcal{E})})^c = (\mathbb{F}_1, \mathcal{E})^o \setminus \overline{(\mathbb{F}_2, \mathcal{E})} \widetilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^o \setminus (\mathbb{F}_2, \mathcal{E})^o$.

Definition 4.14. Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$ and let $(\mathbb{F}, \mathcal{E})$ be a hypersoft set over $\mathcal{U}$. A point $u \in \mathcal{U}$ is said to be a hypersoft exterior point of $(\mathbb{F}, \mathcal{E})$ if and only if it is a hypersoft interior point of $(\mathbb{F}, \mathcal{E})^c$, that is, if and only if there exists a hypersoft open set $(\mathbb{G}, \mathcal{E})$ such that $u \in (\mathbb{G}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}, \mathcal{E})^c$. The set of all hypersoft exterior points of $(\mathbb{F}, \mathcal{E})$ is called the hypersoft exterior of $(\mathbb{F}, \mathcal{E})$ and is denoted by $(\mathbb{F}, \mathcal{E})^e$.

Thus $(\mathbb{F}, \mathcal{E})^e = ((\mathbb{F}, \mathcal{E})^c)^o$. It follows that $((\mathbb{F}, \mathcal{E})^c)^e = (((\mathbb{F}, \mathcal{E})^c)^c)^o = (\mathbb{F}, \mathcal{E})^o$.

We also have $(\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}, \mathcal{E})^e = (\Phi, \mathcal{E})$, that is, no point of $(\mathbb{F}, \mathcal{E})$ can be a hypersoft exterior point of $(\mathbb{F}, \mathcal{E})$.

Example 4.15. Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be the same as in Example 3.6. Let $(\mathbb{F}, \mathcal{E})$ be a hypersoft set defined as follows:

$$(\mathbb{F}, \mathcal{E}) = \{((e_1, e_3, e_4), \{h_2\}), ((e_2, e_3, e_4), \{h_1\})\}.$$

Then $(\mathbb{F}, \mathcal{E})^e = \{((e_1, e_3, e_4), \{h_1\}), ((e_2, e_3, e_4), \{h_2\})\}$.

Remark 4.16. Since $(\mathbb{F}, \mathcal{E})^e$ is the hypersoft interior of $(\mathbb{F}, \mathcal{E})^c$, it follows that $(\mathbb{F}, \mathcal{E})^e$ is the hypersoft open and is the largest hypersoft open set contained in $(\mathbb{F}, \mathcal{E})^c$.

Proposition 4.17. *Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space and let $(\mathbb{F}, \mathcal{E})$ be a hypersoft set over $\mathcal{U}$. Then*

$$(\mathbb{F}, \mathcal{E})^e = \widetilde{\sqcup} \{(\mathbb{G}, \mathcal{E}) \mid (\mathbb{G}, \mathcal{E}) \widetilde{\in} \mathcal{T}_{\mathcal{H}}, (\mathbb{G}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}, \mathcal{E})^c\}.$$

Proof. From the definitions of hypersoft interior and hypersoft exterior, we have $((\mathbb{F}, \mathcal{E})^c)^o = \widetilde{\sqcup} \{(\mathbb{G}, \mathcal{E}) \mid (\mathbb{G}, \mathcal{E}) \widetilde{\in} \mathcal{T}_{\mathcal{H}}, (\mathbb{G}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}, \mathcal{E})^c\} = (\mathbb{F}, \mathcal{E})^e$.

Proposition 4.18. *Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$ and let $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E})$ be two hypersoft sets over $\mathcal{U}$. Then*

(1) $(\Psi, \mathcal{E})^e = (\Phi, \mathcal{E})$ *and* $(\Phi, \mathcal{E})^e = (\Psi, \mathcal{E})$.
(2) $(\mathbb{F}_1, \mathcal{E})^e \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^c$.
(3) $(\mathbb{F}_1, \mathcal{E})^e = (((\mathbb{F}_1, \mathcal{E})^e)^c)^e$.
(4) $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})$ *implies* $(\mathbb{F}_2, \mathcal{E})^e \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^e$.
(5) $(\mathbb{F}_1, \mathcal{E})^o \tilde{\sqsubseteq} ((\mathbb{F}_1, \mathcal{E})^e)^e$.
(6) $((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E}))^e = (\mathbb{F}_1, \mathcal{E})^e \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E})^e$.
(7) $((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^e \tilde{\sqsupseteq} (\mathbb{F}_1, \mathcal{E})^e \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E})^e$.

Proof.

(1) $(\Psi, \mathcal{E})^e = ((\Psi, \mathcal{E})^c)^o = (\Phi, \mathcal{E})^o = (\Phi, \mathcal{E})$.
$(\Phi, \mathcal{E})^e = ((\Phi, \mathcal{E})^c)^o = (\Psi, \mathcal{E})^o = (\Psi, \mathcal{E})$.

(2) By definition, $(\mathbb{F}_1, \mathcal{E})^e = ((\mathbb{F}_1, \mathcal{E})^c)^o$ and by Proposition 4.9 (2), we have $((\mathbb{F}_1, \mathcal{E})^c)^o \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^c$. Hence, $(\mathbb{F}_1, \mathcal{E})^e \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^c$.

(3) $(((\mathbb{F}_1, \mathcal{E})^e)^c)^e = ((((\mathbb{F}_1, \mathcal{E})^c)^o)^c)^e = (((((\mathbb{F}_1, \mathcal{E})^c)^o)^c)^c)^o = (((\mathbb{F}_1, \mathcal{E})^c)^o)^o = ((\mathbb{F}_1, \mathcal{E})^c)^o = (\mathbb{F}_1, \mathcal{E})^e$.

(4) $(\mathbb{F}_1, \mathcal{E}) \tilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})$ then $(\mathbb{F}_2, \mathcal{E})^c \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^c$. Implies that $((\mathbb{F}_2, \mathcal{E})^c)^o \tilde{\sqsubseteq} ((\mathbb{F}_1, \mathcal{E})^c)^o$. So, $(\mathbb{F}_2, \mathcal{E})^e \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^e$.

(5) By (2.), we have $(\mathbb{F}_1, \mathcal{E})^e \tilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^c$. Then (4.) gives $((\mathbb{F}_1, \mathcal{E})^c)^e \tilde{\sqsubseteq} ((\mathbb{F}_1, \mathcal{E})^e)^e$. But $(\mathbb{F}_1, \mathcal{E})^o = ((\mathbb{F}_1, \mathcal{E})^c)^e$. Hence $(\mathbb{F}_1, \mathcal{E})^o \tilde{\sqsubseteq} ((\mathbb{F}_1, \mathcal{E})^e)^e$.

(6) $((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E}))^e = (((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E}))^c)^o = ((\mathbb{F}_1, \mathcal{E})^c \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E})^c)^o = ((\mathbb{F}_1, \mathcal{E})^c)^o \tilde{\sqcap} ((\mathbb{F}_2, \mathcal{E})^c)^o = (\mathbb{F}_1, \mathcal{E})^e \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E})^e$.

(7) $((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^e = (((\mathbb{F}_1, \mathcal{E}) \tilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))^c)^o = ((\mathbb{F}_1, \mathcal{E})^c \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E})^c)^o \tilde{\sqsupseteq} ((\mathbb{F}_1, \mathcal{E})^c)^o \tilde{\sqcup} ((\mathbb{F}_2, \mathcal{E})^c)^o = (\mathbb{F}_1, \mathcal{E})^e \tilde{\sqcup} (\mathbb{F}_2, \mathcal{E})^e$.

Definition 4.19. Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$, then hypersoft boundary of hypersoft set $(\mathbb{F}, \mathcal{E})$ over $\mathcal{U}$ is denoted by $(\mathbb{F}, \mathcal{E})^{b}$ and is defined as $(\mathbb{F}, \mathcal{E})^{b} = \overline{(\mathbb{F}, \mathcal{E})} \tilde{\sqcap} \overline{(\mathbb{F}, \mathcal{E})^c}$.

Example 4.20. Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ and $(\mathbb{F}, \mathcal{E})$ be the same as in Example 4.15, then $(\mathbb{F}, \mathcal{E})^{b} = (\mathbb{F}, \mathcal{E})$.

Remark 4.21. From Definition 4.19 it follows that the hypersoft sets $(\mathbb{F}, \mathcal{E})$ and $(\mathbb{F}, \mathcal{E})^c$ have the same hypersoft boundary.

Proposition 4.22. *Let* $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ *be a hypersoft space and let* $(\mathbb{F}, \mathcal{E})$ *be a hypersoft set over* $\mathcal{U}$. *Then*

(1) $(\mathbb{F}, \mathcal{E})^{b} \tilde{\sqsubseteq} \overline{(\mathbb{F}, \mathcal{E})}$.
(2) $(\mathbb{F}, \mathcal{E})^{b} = \overline{(\mathbb{F}, \mathcal{E})} \setminus (\mathbb{F}, \mathcal{E})^o$.
(3) $((\mathbb{F}, \mathcal{E})^{b})^c = (\mathbb{F}, \mathcal{E})^o \tilde{\sqcup} (\mathbb{F}, \mathcal{E})^e$.

(4) $(\mathbb{F},\mathcal{E})^{o} = (\mathbb{F},\mathcal{E}) \setminus (\mathbb{F},\mathcal{E})^{b}$.

(5) $((\mathbb{F},\mathcal{E})^{o})^{b} \,\widetilde{\sqsubseteq}\, (\mathbb{F},\mathcal{E})^{b}$.

(6) $(\overline{(\mathbb{F},\mathcal{E})})^{b} \,\widetilde{\sqsubseteq}\, (\mathbb{F},\mathcal{E})^{b}$.

Proof.

(1) By definition, $(\mathbb{F},\mathcal{E})^{b} = \overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, \overline{((\mathbb{F},\mathcal{E})^{c})}$. Hence, $(\mathbb{F},\mathcal{E})^{b} \,\widetilde{\sqsubseteq}\, \overline{(\mathbb{F},\mathcal{E})}$.

(2) $(\mathbb{F},\mathcal{E})^{b} = \overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, \overline{((\mathbb{F},\mathcal{E})^{c})} = \overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, ((\mathbb{F},\mathcal{E})^{o})^{c} = \overline{(\mathbb{F},\mathcal{E})} \setminus (\mathbb{F},\mathcal{E})^{o}$.

(3) $((\mathbb{F},\mathcal{E})^{b})^{c} = [\overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, \overline{((\mathbb{F},\mathcal{E})^{c})}]^{c} = (\overline{(\mathbb{F},\mathcal{E})})^{c} \,\widetilde{\sqcup}\, (\overline{(\mathbb{F},\mathcal{E})^{c}})^{c} = ((\mathbb{F},\mathcal{E})^{c})^{o} \,\widetilde{\sqcup}\, (\mathbb{F},\mathcal{E})^{o} = (\mathbb{F},\mathcal{E})^{e} \,\widetilde{\sqcup}\, (\mathbb{F},\mathcal{E})^{o}$.

(4) $(\mathbb{F},\mathcal{E}) \setminus (\mathbb{F},\mathcal{E})^{b} = (\mathbb{F},\mathcal{E}) \,\widetilde{\sqcap}\, ((\mathbb{F},\mathcal{E})^{b})^{c} = (\mathbb{F},\mathcal{E}) \,\widetilde{\sqcap}\, ((\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcup}\, (\mathbb{F},\mathcal{E})^{e}) = ((\mathbb{F},\mathcal{E}) \,\widetilde{\sqcap}\, (\mathbb{F},\mathcal{E})^{o}) \,\widetilde{\sqcup}\, ((\mathbb{F},\mathcal{E}) \,\widetilde{\sqcap}\, (\mathbb{F},\mathcal{E})^{e}) = (\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcup}\, (\Phi,\mathcal{E}) = (\mathbb{F},\mathcal{E})^{o}$.

(5) $((\mathbb{F},\mathcal{E})^{o})^{b} = \overline{(\mathbb{F},\mathcal{E})^{o}} \,\widetilde{\sqcap}\, \overline{((\mathbb{F},\mathcal{E})^{o})^{c}} = \overline{(\mathbb{F},\mathcal{E})^{o}} \,\widetilde{\sqcap}\, \overline{\overline{(\mathbb{F},\mathcal{E})^{c}}} \,\widetilde{\sqsubseteq}\, \overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})^{c}} = (\mathbb{F},\mathcal{E})^{b}$.

(6) $(\overline{(\mathbb{F},\mathcal{E})})^{b} = \overline{\overline{(\mathbb{F},\mathcal{E})}} \,\widetilde{\sqcap}\, \overline{\overline{(\mathbb{F},\mathcal{E})})^{c}} \,\widetilde{\sqsubseteq}\, \overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})^{c}} = (\mathbb{F},\mathcal{E})^{b}$.

Proposition 4.23. *Let $(\mathcal{U},\mathcal{T}_{\mathcal{H}},\mathcal{E})$ be a hypersoft space over $\mathcal{U}$ and let $(\mathbb{F}_1,\mathcal{E})$, $(\mathbb{F}_2,\mathcal{E})$ be two hypersoft sets over $\mathcal{U}$. Then*

(1) $((\mathbb{F}_1,\mathcal{E}) \,\widetilde{\sqcup}\, (\mathbb{F}_2,\mathcal{E}))^{b} \,\widetilde{\sqsubseteq}\, (\mathbb{F}_1,\mathcal{E})^{b} \,\widetilde{\sqcup}\, (\mathbb{F}_2,\mathcal{E})^{b}$.

(2) $((\mathbb{F}_1,\mathcal{E}) \,\widetilde{\sqcap}\, (\mathbb{F}_2,\mathcal{E}))^{b} \,\widetilde{\sqsubseteq}\, (\mathbb{F}_1,\mathcal{E})^{b} \,\widetilde{\sqcup}\, (\mathbb{F}_2,\mathcal{E})^{b}$.

Proof.

(1) $((\mathbb{F}_1,\mathcal{E})\widetilde{\sqcup}(\mathbb{F}_2,\mathcal{E}))^{b} = [\overline{(\mathbb{F}_1,\mathcal{E})\widetilde{\sqcup}(\mathbb{F}_2,\mathcal{E})}]\widetilde{\sqcap}[\overline{((\mathbb{F}_1,\mathcal{E})\widetilde{\sqcup}(\mathbb{F}_2,\mathcal{E}))^{c}}] = [\overline{(\mathbb{F}_1,\mathcal{E})} \,\widetilde{\sqcup}\, \overline{(\mathbb{F}_2,\mathcal{E})}] \,\widetilde{\sqcap}\, [\overline{(\mathbb{F}_1,\mathcal{E})^{c}\widetilde{\sqcap}(\mathbb{F}_2,\mathcal{E})^{c}}] \,\widetilde{\sqsubseteq}\, [\overline{(\mathbb{F}_1,\mathcal{E})} \,\widetilde{\sqcup}\, \overline{(\mathbb{F}_2,\mathcal{E})}] \,\widetilde{\sqcap}\, [\overline{(\mathbb{F}_1,\mathcal{E})^{c}} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})^{c}}] = [\overline{(\mathbb{F}_1,\mathcal{E})} \,\widetilde{\sqcap}\, (\overline{(\mathbb{F}_1,\mathcal{E})^{c}} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})^{c}})] \,\widetilde{\sqcup}\, [\overline{(\mathbb{F}_2,\mathcal{E})} \,\widetilde{\sqcap}\, (\overline{(\mathbb{F}_1,\mathcal{E})^{c}} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})^{c}})] = [(\overline{(\mathbb{F}_1,\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_1,\mathcal{E})^{c}}) \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})^{c}}] \,\widetilde{\sqcup}\, [(\overline{(\mathbb{F}_2,\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})^{c}}) \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_1,\mathcal{E})^{c}}] \,\widetilde{\sqsubseteq}\, [(\mathbb{F}_1,\mathcal{E})^{b} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})^{c}}] \,\widetilde{\sqcup}\, [(\mathbb{F}_2,\mathcal{E})^{b} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_1,\mathcal{E})^{c}}] \,\widetilde{\sqsubseteq}\, (\mathbb{F}_1,\mathcal{E})^{b} \,\widetilde{\sqcup}\, (\mathbb{F}_2,\mathcal{E})^{b}$.

(2) $((\mathbb{F}_1,\mathcal{E})\widetilde{\sqcap}(\mathbb{F}_2,\mathcal{E}))^{b} = [\overline{(\mathbb{F}_1,\mathcal{E})\widetilde{\sqcap}(\mathbb{F}_2,\mathcal{E})}]\widetilde{\sqcap}[\overline{((\mathbb{F}_1,\mathcal{E})\widetilde{\sqcap}(\mathbb{F}_2,\mathcal{E}))^{c}}] \,\widetilde{\sqsubseteq}\, [\overline{(\mathbb{F}_1,\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})}] \,\widetilde{\sqcap}\, [\overline{(\mathbb{F}_1,\mathcal{E})^{c}\widetilde{\sqcup}(\mathbb{F}_2,\mathcal{E})^{c}}] = [\overline{(\mathbb{F}_1,\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})}] \,\widetilde{\sqcap}\, [\overline{(\mathbb{F}_1,\mathcal{E})^{c}} \,\widetilde{\sqcup}\, \overline{(\mathbb{F}_2,\mathcal{E})^{c}}] = [\overline{(\mathbb{F}_1,\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_1,\mathcal{E})^{c}}] \,\widetilde{\sqcup}\, [\overline{(\mathbb{F}_1,\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})^{c}}] = [(\overline{(\mathbb{F}_1,\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_1,\mathcal{E})^{c}}) \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})}] \,\widetilde{\sqcup}\, [\overline{(\mathbb{F}_1,\mathcal{E})} \,\widetilde{\sqcap}\, (\overline{(\mathbb{F}_2,\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})^{c}})] = [(\mathbb{F}_1,\mathcal{E})^{b} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2,\mathcal{E})}] \,\widetilde{\sqcup}\, [\overline{(\mathbb{F}_1,\mathcal{E})} \,\widetilde{\sqcap}\, (\mathbb{F}_2,\mathcal{E})^{b}] \,\widetilde{\sqsubseteq}\, (\mathbb{F}_1,\mathcal{E})^{b} \,\widetilde{\sqcup}\, (\mathbb{F}_2,\mathcal{E})^{b}$.

Proposition 4.24. *Let $(\mathbb{F},\mathcal{E})$ be a hypersoft set of hypersoft space over $\mathcal{U}$. Then the following hold.*

(1) $(\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcup}\, (\mathbb{F},\mathcal{E})^{b} = \overline{(\mathbb{F},\mathcal{E})}$.

(2) $(\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcup}\, (\mathbb{F},\mathcal{E})^{e} \,\widetilde{\sqcup}\, (\mathbb{F},\mathcal{E})^{b} = (\Psi,\mathcal{E})$.

Proof.

(1) $(\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcup}\, (\mathbb{F},\mathcal{E})^{b} = (\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcup}\, [\overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})^{c}}] = [(\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcup}\, \overline{(\mathbb{F},\mathcal{E})}] \,\widetilde{\sqcap}\, [(\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcup}\, \overline{(\mathbb{F},\mathcal{E})^{c}}] = \overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, [(\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcup}\, ((\mathbb{F},\mathcal{E})^{o})^{c}] = \overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, (\Psi,\mathcal{E}) = \overline{(\mathbb{F},\mathcal{E})}$.

(2) By Proposition 4.22 (3), $(\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcup}\, (\mathbb{F},\mathcal{E})^{e} = ((\mathbb{F},\mathcal{E})^{b})^{c}$, then $(\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcup}\, (\mathbb{F},\mathcal{E})^{e} \,\widetilde{\sqcup}\, (\mathbb{F},\mathcal{E})^{b} = ((\mathbb{F},\mathcal{E})^{b})^{c} \,\widetilde{\sqcup}\, ((\mathbb{F},\mathcal{E})^{b}) = (\Psi,\mathcal{E})$.

Proposition 4.25. *Let $(\mathcal{U},\mathcal{T}_{\mathcal{H}},\mathcal{E})$ be a hypersoft space and let $(\mathbb{F},\mathcal{E})$ be a hypersoft set over $\mathcal{U}$. Then*

(1) $(\mathbb{F},\mathcal{E})$ is a hypersoft open set if and only if $(\mathbb{F},\mathcal{E}) \,\widetilde{\sqcap}\, (\mathbb{F},\mathcal{E})^{b} = (\Phi,\mathcal{E})$.

(2) $(\mathbb{F},\mathcal{E})$ is a hypersoft closed set if and only if $(\mathbb{F},\mathcal{E})^{b} \,\widetilde{\sqsubseteq}\, (\mathbb{F},\mathcal{E})$.

Proof.

(1) Let $(\mathbb{F},\mathcal{E})$ be a hypersoft open set. By Proposition 4.22 (3), $(\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqsubseteq}\, ((\mathbb{F},\mathcal{E})^{b})^{c}$. But $(\mathbb{F},\mathcal{E})^{o} = (\mathbb{F},\mathcal{E})$ since $(\mathbb{F},\mathcal{E})$ is a hypersoft open set. Hence, $(\mathbb{F},\mathcal{E}) \,\widetilde{\sqsubseteq}\, ((\mathbb{F},\mathcal{E})^{b})^{c}$. This implies that $(\mathbb{F},\mathcal{E}) \,\widetilde{\sqcap}\, (\mathbb{F},\mathcal{E})^{b} = (\Phi,\mathcal{E})$.
Conversely, let $(\mathbb{F},\mathcal{E}) \,\widetilde{\sqcap}\, (\mathbb{F},\mathcal{E})^{b} = (\Phi,\mathcal{E})$. Then $(\mathbb{F},\mathcal{E}) \,\widetilde{\sqcap}\, [\overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})^{c}}] = (\Phi,\mathcal{E})$ or $(\mathbb{F},\mathcal{E}) \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})^{c}} = (\Phi,\mathcal{E})$ or $\overline{(\mathbb{F},\mathcal{E})^{c}} \,\widetilde{\sqsubseteq}\, (\mathbb{F},\mathcal{E})^{c}$, which implies $(\mathbb{F},\mathcal{E})^{c}$ is a hypersoft closed set and hence $(\mathbb{F},\mathcal{E})$ is a hypersoft open set.

(2) Let $(\mathbb{F},\mathcal{E})$ be a hypersoft closed set. By Proposition 4.22 (1), $(\mathbb{F},\mathcal{E})^{b} \,\widetilde{\sqsubseteq}\, \overline{(\mathbb{F},\mathcal{E})}$. Since $(\mathbb{F},\mathcal{E})$ is a hypersoft closed set, then $\overline{(\mathbb{F},\mathcal{E})} = (\mathbb{F},\mathcal{E})$. This implies that $(\mathbb{F},\mathcal{E})^{b} \,\widetilde{\sqsubseteq}\, (\mathbb{F},\mathcal{E})$. Conversely, let $(\mathbb{F},\mathcal{E})^{b} \,\widetilde{\sqsubseteq}\, (\mathbb{F},\mathcal{E})$. Then $(\mathbb{F},\mathcal{E})^{b} \,\widetilde{\sqcap}\, (\mathbb{F},\mathcal{E})^{c} = (\Phi,\mathcal{E})$. Since $(\mathbb{F},\mathcal{E})^{b} = ((\mathbb{F},\mathcal{E})^{b})^{c}$, then we have $((\mathbb{F},\mathcal{E})^{b})^{c} \,\widetilde{\sqcap}\, (\mathbb{F},\mathcal{E})^{c} = (\Phi,\mathcal{E})$. By (1), $(\mathbb{F},\mathcal{E})^{c}$ is a hypersoft open set and hence $(\mathbb{F},\mathcal{E})$ is a hypersoft closed set.

Proposition 4.26. *let $(\mathbb{F},\mathcal{E})$ be a hypersoft set of a hypersoft space over $\mathcal{U}$. Then $(\mathbb{F},\mathcal{E})^{b}$ $= (\Phi,\mathcal{E})$ if and only if $(\mathbb{F},\mathcal{E})$ is a hypersoft open set and a hypersoft closed set.*

Proof. Suppose that $(\mathbb{F},\mathcal{E})^{b} = (\Phi,\mathcal{E})$ then $\overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})^{c}} = (\Phi,\mathcal{E})$ implies $\overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqsubseteq}\, ((\overline{(\mathbb{F},\mathcal{E})^{c}}))^{c} = (\mathbb{F},\mathcal{E})^{o}$. Since $(\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqsubseteq}\, (\mathbb{F},\mathcal{E})$ then $\overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqsubseteq}\, (\mathbb{F},\mathcal{E})$ and hence $\overline{(\mathbb{F},\mathcal{E})} = (\mathbb{F},\mathcal{E})$. This implies that $(\mathbb{F},\mathcal{E})$ is a hypersoft closed set. Again, $(\mathbb{F},\mathcal{E})^{b} = (\Phi,\mathcal{E})$ then $\overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})^{c}} = (\Phi,\mathcal{E})$ or $\overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, ((\mathbb{F},\mathcal{E})^{o})^{c} = (\Phi,\mathcal{E})$ then $\overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqsubseteq}\, (\mathbb{F},\mathcal{E})^{o}$. This implies that $(\mathbb{F},\mathcal{E})^{o} = (\mathbb{F},\mathcal{E})$. Hence $(\mathbb{F},\mathcal{E})$ is a hypersoft open set.
Conversely, suppose that $(\mathbb{F},\mathcal{E})$ is a hypersoft open set and a hypersoft closed set. Then $(\mathbb{F},\mathcal{E})^{b} = \overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})^{c}} = \overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, ((\mathbb{F},\mathcal{E})^{o})^{c} = (\mathbb{F},\mathcal{E}) \,\widetilde{\sqcap}\, (\mathbb{F},\mathcal{E})^{c} = (\Phi,\mathbb{G},\mathcal{E})$.

Proposition 4.27. *Let $(\mathcal{U},\mathcal{T}_{\mathcal{H}},\mathcal{E})$ be a hypersoft space and let $(\mathbb{F},\mathcal{E})$ be a hypersoft set over $\mathcal{U}$. Then*

(1) $(\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcap}\, (\mathbb{F},\mathcal{E})^{b} = (\Phi,\mathcal{E})$.

(2) $(\mathbb{F},\mathcal{E})^{e} \,\widetilde{\sqcap}\, (\mathbb{F},\mathcal{E})^{b} = (\Phi,\mathcal{E})$.

Proof.

(1) $(\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcap}\, (\mathbb{F},\mathcal{E})^{b} = (\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcap}\, [\overline{(\mathbb{F},\mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})^{c}}] = [(\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})}] \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})^{c}} = (\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcap}\, \overline{(\mathbb{F},\mathcal{E})^{c}} = (\mathbb{F},\mathcal{E})^{o} \,\widetilde{\sqcap}\, ((\mathbb{F},\mathcal{E})^{o})^{c} = (\Phi,\mathcal{E})$.

(2) $(\mathbb{F},\mathcal{E})^{e} \tilde{\sqcap} (\mathbb{F},\mathcal{E})^{b} = ((\mathbb{F},\mathcal{E})^{c})^{o} \tilde{\sqcap} [\overline{(\mathbb{F},\mathcal{E})} \tilde{\sqcap} \overline{(\mathbb{F},\mathcal{E})^{c}}] = (\overline{(\mathbb{F},\mathcal{E})})^{c} \tilde{\sqcap} [\overline{(\mathbb{F},\mathcal{E})} \tilde{\sqcap} \overline{(\mathbb{F},\mathcal{E})^{c}}]$
$= [(\overline{(\mathbb{F},\mathcal{E})})^{c} \tilde{\sqcap} \overline{(\mathbb{F},\mathcal{E})}] \tilde{\sqcap} \overline{(\mathbb{F},\mathcal{E})^{c}}] = (\Phi,\mathcal{E}) \tilde{\sqcap} \overline{(\mathbb{F},\mathcal{E})^{c}} = (\Phi,\mathcal{E})$.

5. Conclusions

In this paper, we have introduced the concept of hypersoft topological spaces which are defined over an initial universe with a fixed set of parameters. Some concepts such as hypersoft closure, hypersoft interior, hypersoft boundary, etc. which are based on our definition were introduced and studied and some relationships between them were discussed. For future trends, we can define the most important fundamental topological properties such as connectedness and compactness. Also, we can define hypersoft separation axioms by using ordinary point as well as hypersoft point.

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