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Fuzzy Neutrosophic Soft HX Subring

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Abstract: In this paper, we define the notion of fuzzy nutrosophic soft HX subring of a HX ring and some of their related properties investigated. We define the intersection, cartesian product of an fuzzy neutrosophic soft subset of an fuzzy nutrosophic soft HX ring and we show some results.

Keywords: fuzzy neutrosophic soft set, fuzzy HX ring, soft set, fuzzy neutrosophic soft ring.

1.Introduction

In 1965 Zadah [10] introduced the conceipt of Fuzzy subset of a set X as a function from X into the closed interval [0,1] and studied their properties. The idea of 'Intuitionistic Fuzzy Set' was first introduced by Atanassov [1]. In 1999 [7] Molodtsov initiated the novel concept of soft set theory which is a completely new approach for modeling vagueness and uncertainty. Further P. K. Maji, R. Biswas and A. R. Roy [4] introduced the concept of fuzzy soft sets in 2001. Also P. K. Maji, R. Biswas and A. R. Roy [5] defined and clarified the concept of intuitionistic fuzzy soft set . In 2014, R. Muthuraj and M. Muthuraman [8] defined the notion of intuitionistic fuzzy HX ring of a HX ring and investigated some of their related properties with the necessity and possibility operators of an intuitionistic fuzzy HX ring. The concept of neutrosophic set (NS) was first introduced by Smarandache [9] which is a generalization of classical sets, fuzzy set, intuitionistic fuzzy set etc. Later, Maji[6] has introduced a combined concept neutrosophic soft set (NSS). In 2016 the concept of neutrosophic soft ring is introduced by Bera and Mahapatra [2] and some basic properties related to it are established. In this paper, we introduce the notion of fuzzy neutrosophic soft HX subring, we discussed some related properties of fuzzy neutrosophic soft HX subring of a HX ring .

2. Preliminaries: We recall some basic definitions related to neutrosophic soft ring.

Definition (2.1) [10]:

A Fuzzy neutrosophic set A on the universe set X is defined as $A = \{< x, T_A(x), I_A(x), F_A(x) >: x \in X\}$ $where\ T, I, F: X \to [0,1]$ and $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$

Definition (2.2) [13]:

Let X be a universe sets and E be a set of parameters. A pair (F, E) is called soft set over X if and only if F is a function from E into the sets of all subsets of X , i.e. $: E \to P(X)$, where $P(X)$ is the power set of X .

Definition(2.3)[14]

Let U be the initial universe set and E be a set of parameters and $(FNSS(U))$ denote the set of all fuzzy neutrosophic sets of U , then the fuzzy neutrosophic soft set H over U is a set defined by a set valued function f_H representing a mapping $f_H: E \to \big(FNSS(U)\big)$, where f_H is called approximate function of the neutrosophic soft set H .On the other words the neutrosophic soft set is a parmetrized family of some elements of the set $(FNSS(U))$ and therefor H can be written as a set of ordered pairs ,

$$H = \{\big(e, \{< x, T_{fH(e)}(x), I_{fH(e)}(x), F_{fH(e)}(x) >: x \in U\}\big): e \in E\}$$

Definition(2.4)[9]

A neutrosophic set A=$\{< x, T_A(x), I_A(x), F_A(x) >: x \in R\}$ over a ring $(R, +, .)$ is called a neutrosophic subring of $(R, +, .)$ if the followings hold

i) $\begin{cases} T_A(x+y) \ge \min\{T_A(x), T_A(y)\} \\ I_A(x+y) \le \max\{I_A(x), I_A(y)\} \\ F_A(x+y) \le \max\{F_A(x), F_A(y)\} \end{cases}$, ii) $\begin{cases} T_A(-x) \ge T_A(x) \\ I_A(-x) \le I_A(x) \\ F_A(-x) \le F_A(x) \end{cases}$

$iii)$ $\begin{cases} T_A(x.y) \ge \min\{T_A(x), T_A(y)\} \\ I_A(x.y) \le \max\{I_A(x), I_A(y)\} \\ F_A(x.y) \le \max\{F_A(x), F_A(y)\} \end{cases}$

for all $x, y \in R$.

Definition(2.5)[9]

An NSS $H = \{(e, \{< x, T_{fH(e)}(x), I_{fH(e)}(x), F_{fH(e)}(x) >: x \in U\}) : e \in E\}$ over a ring $(R, +, .)$ is called neutrosophic soft ring if $f_{H(e)}$is a neutrosophic sub ring of $(R, +, .)$ for each e$\in E$.

Proposition(2.6) [9]

An NSS $H = \{(e, \{< x, T_{fH(e)}(x), I_{fH(e)}(x), F_{fH(e)}(x) >: x \in U\}) : e \in E\}$ over the ring $(R, +, .)$ is called a neutrosophic soft ring iff following's hold

i) $\begin{cases} T_{fH(e)}(x - y) \geq \min\{T_A(x), T_A(y)\} \\ I_{fH(e)}(x - y) \leq \max\{I_A(x), I_A(y)\} \\ F_{fH(e)}(x - y) \leq \max\{F_A(x), F_A(y)\} \end{cases}$

ii) $\begin{cases} T_{fH(e)}(xy) \geq \min\{T_A(x), T_A(y)\} \\ I_{fH(e)}(xy) \leq \max\{I_A(x), I_A(y)\} \\ F_{fH(e)}(xy) \leq \max\{F_A(x), F_A(y)\} \end{cases}$

for each $x, y \in R$, $e \in E$

Definition(2.7)[15] Let R be a ring and a non empty set $N \subset 2^R - \{\Phi\}$ on two binary operations (+) and (.) , then N is said to be HX ring of R if N is a ring with respect to the algebraic operations defined by

i) $A + B = \{a + b: a \in A, \ and \ \ b \in B\}$, which its null element is denoted by Q , and the negative element of A is denoted by $-A$.

ii) $A.B = \{a.b: \ a \in A \ , b \in B\}$

iii) $A.(B + C) = A.B + A.C \quad and \quad (B + C).A = B.A + C.A$

3 Fuzzy Neutrosophic Soft HX subring

In this section, we define the concepts of the fuzzy neutrosophic soft HX set, fuzzy neutrosophic soft HX subring ,anti fuzzy neutrosophic soft HX subring with some of their properties and results.

Definition(3.1) Let R be a ring, (u, E) be fuzzy soft set defined on R and $N \subset 2^R - \{\Phi\}$ be a HX ring on R . Define fuzzy soft HX set $(\lambda^u_{f(e)})$ such that

$$\lambda^u_{f(e)}(A) = \max\{u_e(x) : for\ all\ x \in A \subset R \ \ , e \in E\}$$

Example(3.2)

Let $R = (Z_3, +, .)$ be a ring and $E = \{e_1, e_2\}$ and let $u: R \to I$ .Let (f, E) a fuzzy soft set where $f: E \to I^u$ defined as

$$u_{f(e_1)}(0) = 0.3 \quad , u_{f(e_1)}(1) = 0.7 \quad , u_{f(e_1)}(2) = 0.2$$

$$u_{f(e_2)}(0) = 0.6 \quad , u_{f(e_2)}(1) = 0.5 \quad , u_{f(e_2)}(2) = 0.3$$

Now , let $N = \{A, B, C\}$ $where\ A = \{0\}\ , B = \{1\}\ \ , C = \{2\}$ and (λ^u, E) $where$

$$\lambda^u_{f(e_1)}(A) = \max\{u_{f(e_1)}(x) : \forall x \in A\}, \quad \lambda^u_{f(e_2)} = \max\{u_{f(e_2)}(x) : \forall x \in A\}$$

$$\lambda^u_{f(e_1)}(A) = 0.3 \quad , \lambda^u_{f(e_1)}(B) = 0.7 \quad , \lambda^u_{f(e_1)}(C) = 0.2$$

$$\lambda^u_{f(e_2)}(A) = 0.6 \quad , \lambda^u_{f(e_2)}(B) = 0.5 \quad , \lambda^u_{f(e_2)}(C) = 0.3$$

then
$\lambda^u_{f(e_1)} = \{(A, 0.3), (B, 0.7), (C, 0.2)\}, \lambda^u_{f(e_2)} = \{(A, 0.6), (B, 0.5), (C, 0.3)\}$
is a fuzzy soft HX set.

Definition(3.3) Let R be a ring and
$H = \{(e, \{< x, T_{fH(e)}(x), I_{fH(e)}(x), F_{fH(e)}(x) >: x \in U\}) : e \in E\}$ be a

fuzzy neutrosophic soft set defined on R and $N \subset 2^R\_\{\emptyset\}$ be HX ring on R. Define fuzzy neutrosophic soft HX set $\lambda^H = \{< A, \lambda^T_{fH(e)}(A), \lambda^I_{fH(e)}(A), \lambda^F_{fH(e)}(A) >: A \in N, and\ \lambda^T_{fH(e)}(A) + \lambda^I_{fH(e)}(A) + \lambda^F_{fH(e)}(A) \leq 3\}$ on N where:

$\lambda^T_{fH(e)}(A) = \max\{T_{fH(e)}(x): for\ all\ x \in A \subseteq R, e \in E\}$

$\lambda^I_{fH(e)}(A) = \min\{I_{fH(e)}(x): for\ all\ x \in A \subseteq R, e \in E\}$

$\lambda^F_{fH(e)}(A) = \min\{F_{fH(e)}(x): for\ all\ x \in A \subseteq R, e \in E\}$

***Example*(3.4)**

Let $R = (Z_3, +, .)$ be a ring , $E = \{e_1, e_2\}$ and $H = \{(e, \{< x, T_{fH(e)}(x), T_{fH(e)}(x), T_{fH(e)}(x) >: x \in R\})\ : e \in E\}$ a fuzzy neutrosophic soft on R ,

$f_{H(e_1)} = \{< 0, (0.3,0.4,0.7) >, < 1, (0.5,0.6,0.4) >, < 2, (0.6,0.5,0.4) >\}$

$f_{H(e_2)} = \{< 0, (0.6,0.1,0.5) >, < 1, (0.5,0.3,0.9) >, < 2, (0.8,0.5,0.3) >\}$

then $H = \{[e_1, f_{H(e_1)}], [e_2, f_{H(e_2)}]\}$ is an $FNSS$ over (R, E)

Now let $N = \{A, B, C\}$ where $A = \{0\}$, $B = \{1\}$, $C = \{2\}$ and (λ^H, E) where $\lambda^H: E \to FNSS(R)$

$\lambda^T_{fH(e)}(A) = \max\{T_{fH(e)}(x): for\ all\ x \in A \subseteq R, e \in E\}$

$\lambda^I_{fH(e)}(A) = \min\{I_{fH(e)}(x): for\ all\ x \in A \subseteq R, e \in E\}$, $\lambda^F_{fH(e)}(A) = \min\{F_{fH(e)}(x): for\ all\ x \in A \subseteq R, e \in E\}$

$\lambda^T_{fH(e_1)}(A) = 0.3$, $\lambda^I_{fH(e_1)}(A) = 0.4$, $\lambda^F_{fH(e_1)}(A) = 0.7$

$\lambda^T_{fH(e_2)}(A) = 0.6$, $\lambda^I_{fH(e_2)}(A) = 0.1$, $\lambda^F_{fH(e_2)}(A) = 0.5$

$\lambda^T_{fH(e_1)}(B) = 0.5$, $\lambda^I_{fH(e_1)}(B) = 0.6$, $\lambda^F_{fH(e_1)}(B) = 0.4$

$\lambda^T_{fH(e_2)}(B) = 0.5$, $\lambda^I_{fH(e_2)}(B) = 0.3$, $\lambda^F_{fH(e_2)}(B) = 0.9$

$\lambda^T_{fH(e_1)}(C) = 0.6$, $\lambda^I_{fH(e_1)}(C) = 0.5$, $\lambda^F_{fH(e_1)}(C) = 0.4$

$\lambda^T_{fH(e_2)}(C) = 0.8$, $\lambda^I_{fH(e_2)}(C) = 0.5$, $\lambda^F_{fH(e_2)}(C) = 0.3$

$$\lambda^H_{f(e_1)} = \{< A, \lambda^T_{fH(e_1)}(A), \lambda^I_{fH(e_1)}(A), \lambda^F_{fH(e_1)}(A) >: for\ all\ \ A \in N\}\,.$$

$$= \{< A, 0.3,0.4,0.7 >, < B, 0.5,0.6,0.4 >, < C, 0.6,0.5,0.4 >\}$$

$$\lambda^H_{f(e_2)} = \{< A, \lambda^T_{fH(e_2)}(A), \lambda^I_{fH(e_2)}(A), \lambda^F_{fH(e_2)}(A) >: for\ all\ \ A \in N\}$$

$$= \{< A, 0.6,0.1,0.5 >, < B,0.5,0.3,0.9 >, < C, 0.8,0.5,0.3 >\}\,,$$

then $\lambda^H = \{[e_1, \lambda^H_{f(e_1)}], [e_2, \lambda^H_{f(e_2)}]\}$ is fuzzy neutrosophic soft HX subset of HX ring N.

Definition(3.5) Let $H = \{< x, T_{fH(e)}(x), I_{fH(e)}(x), F_{fH(e)}(x) >: x \in R, e \in E\}$ be a fuzzy neutrosophic soft set $(FNSS)$ over a ring R and E be set of parameters and where $T, I, F: R \to [0,1]$ such that $0 \le T_{fH(e)}(x) + T_{fH(e)}(x) + T_{fH(e)}(x) \le 3$. Let $N \subset 2^R - \{\Phi\}$ be a HX ring . A fuzzy neutrosophic soft HX subset $\lambda^H = \{< A, \lambda^T_{fH(e)}(A), \lambda^I_{fH(e)}(A), \lambda^F_{fH(e)}(A) >: A \in N, and\ \ \lambda^T_{f0H(e)}(A) + \lambda^I_{fH(e)}(A) + \lambda^F_{fH(e)}(A) \le 3\}\ of\ N$ is called

1) Fuzzy neutrosophic soft HX subring induced by H if the following condition are satisfied for $A, B \in N$ and $e \in E$

i) $\lambda^T_{fH(e)}(A - B) \ge \min\{\lambda^T_{fH(e)}(A), \lambda^T_{fH(e)}(B)\}$

ii) $\lambda^T_{fH(e)}(AB) \ge \min\{\lambda^T_{fH(e)}(A), \lambda^T_{fH(e)}(B)\}$

iii) $\lambda^I_{fH(e)}(A - B) \le \max\{\lambda^I_{fH(e)}(A), \lambda^I_{fH(e)}(B)\}$

iv) $\lambda^I_{fH(e)}(AB) \le \max\{\lambda^I_{fH(e)}(A), \lambda^I_{fH(e)}(B)\}$

v) $\lambda^F_{fH(e)}(A - B) \le \max\{\lambda^F_{fH(e)}(A), \lambda^F_{fH(e)}(B)\}$

vi) $\lambda^F_{fH(e)}(AB) \leq \max\{\lambda^F_{fH(e)}(A), \lambda^F_{fH(e)}(B)\}$

where $\lambda^T_{fH(e)}(A) = \max\{T_{fH(e)}(x): for\ all\ x \in A \subseteq R, e \in E\}$

$\lambda^I_{fH(e)}(A) = \min\{I_{fH(e)}(x): for\ all\ x \in A \subseteq R, e \in E\}$

$\lambda^F_{fH(e)}(A) = \min\{F_{fH(e)}(x): for\ all\ x \in A \subseteq R, e \in E\}$

2) Anti-fuzzy neutrosophic soft subring induced by H and if the following condition are satisfied for $A, B \in N$ and $e \in E$

i) $\lambda^T_{fH(e)}(A - B) \leq \max\{\lambda^T_{fH(e)}(A), \lambda^T_{fH(e)}(B)\}$

ii) $\lambda^T_{fH(e)}(AB) \leq \max\{\lambda^T_{fH(e)}(A), \lambda^T_{fH(e)}(B)\}$

iii) $\lambda^I_{fH(e)}(A - B) \geq \min\{\lambda^I_{fH(e)}(A), \lambda^I_{fH(e)}(B)\}$

iv) $\lambda^I_{fH(e)}(AB) \geq \min\{\lambda^I_{fH(e)}(A), \lambda^I_{fH(e)}(B)\}$

v) $\lambda^F_{fH(e)}(A - B) \geq \min\{\lambda^F_{fH(e)}(A), \lambda^F_{fH(e)}(B)\}$

vi) $\lambda^F_{fH(e)}(AB) \geq \min\{\lambda^F_{fH(e)}(A), \lambda^F_{fH(e)}(B)\}$

where $\lambda^T_{fH(e)}(A) = \min\{T_{fH(e)}(x): for\ all\ x \in A \subseteq R, e \in E\}$

$\lambda^I_{fH(e)}(A) = \max\{I_{fH(e)}(x): for\ all\ x \in A \subseteq R, e \in E\}$

$\lambda^F_{fH(e)}(A) = \max\{F_{fH(e)}(x): for\ all\ x \in A \subseteq R, e \in E\}$

Example(3.6)

From Example (3.4) we
get fuzzy neutrosophic soft HX subring of a HX ring N.

Remark(3.7)

1) For any fuzzy neutrosophic soft HX subring of HX ring N .Then for all $A, B \in N$

$$\lambda^{T}_{fH(e)}(A) \leq \lambda^{T}_{fH(e)}(Q) \ , \ \lambda^{I}_{fH(e)}(A) \geq \lambda^{I}_{fH(e)}(Q) \ , \ \lambda^{F}_{fH(e)}(A) \geq \lambda^{F}_{fH(e)}(Q)$$

2) For any anti fuzzy neutrosophic soft HX subring of HX ring N .Then for all $A, B \in N$

$$\lambda^{T}_{fH(e)}(A) \geq \lambda^{T}_{fH(e)}(Q) \ , \ \lambda^{I}_{fH(e)}(A) \leq \lambda^{I}_{fH(e)}(Q) \ , \ \lambda^{F}_{fH(e)}(A) \leq \lambda^{F}_{fH(e)}(Q)$$

Theorem(3.8)

1) If H is a fuzzy neutrosophic soft subring of ring R,then the fuzzy neutrosophic soft subset λ^{H} is a fuzzy neutrosophic soft HX subring of a HX ring $N \subset 2^{R} - \{\Phi\}$.

2) If H is anti fuzzy neutrosophic soft subring of ring R,then the fuzzy neutrosophic soft subset λ^{H} is anti fuzzy neutrosophic soft HX subring of a HX ring $N \subset 2^{R} - \{\Phi\}$.

Proof : (1) Let H be a fuzzy neutrosophic soft subring of R such that

$H = \{< x, T_{fH(e)}(x), I_{fH(e)}(x), F_{fH(e)}(x) >: x \in R, e \in E\}$, then

i) $\min\{\lambda^{T}_{fH(e)}(A), \lambda^{T}_{fH(e)}(B)\} = \min\{\max\{T_{fH(e)}(x): \forall x \in A \subseteq R\}, \max\{T_{fH(e)}(y): \forall y \in B \subseteq R\}$

$$= \min\{T_{fH(e)}(x_0), T_{fH(e)}(y_0)\}$$

$$\leq T_{fH(e)}(x_0 - y_0)$$

$$\leq \max\{T_{fH(e)}(x - y): \ \forall (x - y) \in A - B \subseteq R\}$$

$$= \lambda^T_{fH(e)}(A-B)$$

Hence $\lambda^T_{fH(e)}(A-B) \geq \min\{\lambda^T_{fH(e)}(A), \lambda^T_{fH(e)}(B)\}$

ii) $\min\{\lambda^T_{fH(e)}(A), \lambda^T_{fH(e)}(B)\}$

$$= \min\{\max\{T_{fH(e)}(x): \forall x \in A \subseteq R\}, \max\{T_{fH(e)}(y): \forall y \in B \subseteq R\}\}$$

$$= \min\{T_{fH(e)}(x_0), T_{fH(e)}(y_0)\}$$

$$\leq T_{fH(e)}(x_0y_0)$$

$$\leq max\{T_{fH(e)}(xy): \ \forall (xy) \in AB \subseteq R\}$$

$$= \lambda^T_{fH(e)}(AB)$$

Hence $\lambda^T_{fH(e)}(AB) \geq \min\{\lambda^T_{fH(e)}(A), \lambda^T_{fH(e)}(B)\}$

In the same way we can prove that

iii) $\lambda^I_{fH(e)}(A-B) \leq \max\{\lambda^I_{fH(e)}(A), \lambda^I_{fH(e)}(B)\}$

iv) $\lambda^I_{fH(e)}(AB) \leq \max\{\lambda^I_{fH(e)}(A), \lambda^I_{fH(e)}(B)\}$

v) $\lambda^F_{fH(e)}(A-B) \leq \max\{\lambda^F_{fH(e)}(A), \lambda^F_{fH(e)}(B)\}$

vi) $\lambda^F_{fH(e)}(AB) \leq \max\{\lambda^F_{fH(e)}(A), \lambda^F_{fH(e)}(B)\}$

To prove (2) it is clear .

Definition (3.9):

Let (λ^H, E) and (Υ^G, E) be two fuzzy neutrosophic soft HX subsets of the HX ring N then the intersection of (λ^H, E) and (Υ^G, E) defined as $(\lambda^H \cap \Upsilon^G)(A) = \{< A, (\lambda^T_{fH(e)} \cap \Upsilon^T_{fG(e)})(A), (\lambda^I_{fH(e)} \cup \Upsilon^I_{fG(e)})(A), (\lambda^F_{fH(e)} \cup \Upsilon^F_{fG(e)})(A) : \ A \in N \ , e \in E\}$.

$\underline{\boldsymbol{Theorem}(\mathbf{3.10})}$

1) Let H and G be any two fuzzy neutrosophic soft sets on R and λ^H , λ^G any two fuzzy neutrosophic soft HX subring of a HX ring N then their intersection $\lambda^H \cap \lambda^G$ is also a fuzzy neutrosophic soft HX subring of a HX ring N.

2) Let H and G be any two fuzzy neutrosophic soft sets on R and λ^H, γ^G any two anti fuzzy neutrosophic soft HX subring of a HX ring N ,then their intersection $\lambda^H \cap \gamma^G$ is also anti fuzzy neutrosophic soft HX subring of a HX ring N.

Proof : 1) Let $H = \{< x, T_{fH(e)}(x), I_{fH(e)}(x), F_{fH(e)}(x) >: x \in R, e \in E\}$,

$$G = \{< x, T_{fG(e)}(x), I_{fG(e)}(x), F_{fG(e)}(x) >: x \in R, e \in E\}$$

be any two fuzzy neutrosophic soft sets on a ring R and

$\lambda^H = \{< A, \lambda^T_{fH(e)}(A), \lambda^I_{fH(e)}(A), \lambda^F_{fH(e)}(A) >: A \in N\}$

$\gamma^G = \{< A, \gamma^T_{fG(e)}(A), \gamma^T_{fG(e)}(A), \gamma^T_{fG(e)}(A) >: A \in N\}$

two fuzzy neutrosphic soft HX subring of a HX ring N

$$(\lambda^H \cap \gamma^G)(A) = \{< A, \left(\lambda^T_{fH(e)} \cap \gamma^T_{fG(e)}\right)(A), \left(\lambda^I_{fH(e)} \cup \gamma^I_{fG(e)}\right)(A),$$

$$\left(\lambda^F_{fH(e)} \cup \gamma^F_{fG(e)}\right)(A):\ A \in N\}$$

i) $\left(\lambda^T_{fH(e)} \cap \gamma^T_{fG(e)}\right)(A - B) = \min\{\lambda^T_{fH(e)}(A - B), \gamma^T_{fG(e)}(A - B)\}$

$$\geq \min\{\min\{\lambda^T_{fH(e)}(A), \lambda^T_{fH(e)}(B)\}, \min\{\gamma^T_{fG(e)}(A), \gamma^T_{fG(e)}(B)\}\}$$

$$= \min\{\min\{\lambda^T_{fH(e)}(A), \gamma^T_{fG(e)}(A)\}, \min\{\lambda^T_{fH(e)}(B), \gamma^T_{fG(e)}(B)\}\}$$

$=\min\{(\lambda^T_{fH(e)} \cap \text{ɣ}^T_{fG(e)})(A), (\lambda^T_{fH(e)} \cap \text{ɣ}^T_{fG(e)})(B)\}$

Hence $(\lambda^T_{fH(e)} \cap \text{ɣ}^T_{fG(e)})(A-B) \geq \min\{(\lambda^T_{fH(e)} \cap \text{ɣ}^T_{fG(e)})(A), (\lambda^T_{fH(e)} \cap \text{ɣ}^T_{fG(e)})(B)\}$

ii) $(\lambda^T_{fH(e)} \cap \text{ɣ}^T_{fG(e)})(AB) = \min\{\lambda^T_{fH(e)}(AB), \text{ɣ}^T_{fG(e)}(AB)\}$

$=\min\{\min\{\lambda^T_{fH(e)}(A), \text{ɣ}^T_{fG(e)}(A)\}, \min\{\lambda^T_{fH(e)}(B), \text{ɣ}^T_{fG(e)}(B)\}\}$

$=\min\{(\lambda^T_{fH(e)} \cap \text{ɣ}^T_{fG(e)})(A), (\lambda^T_{fH(e)} \cap \text{ɣ}^T_{fG(e)})(B)\}$

Hence $(\lambda^T_{fH(e)} \cap \text{ɣ}^T_{fG(e)})(AB) \geq \min\{(\lambda^T_{fH(e)} \cap \text{ɣ}^T_{fG(e)})(A), (\lambda^T_{fH(e)} \cap \text{ɣ}^T_{fG(e)})(B)\}$

In the same way we can prove that

iii) $(\lambda^I_{fH(e)} \cup \text{ɣ}^I_{fG(e)})(A-B) \leq \max\{(\lambda^I_{fH(e)} \cup \text{ɣ}^I_{fG(e)})(A), (\lambda^I_{fH(e)} \cup \text{ɣ}^I_{fG(e)})(B)\}$

iv) $(\lambda^I_{fH(e)} \cup \text{ɣ}^I_{fG(e)})(AB) \leq \max\{(\lambda^I_{fH(e)} \cup \text{ɣ}^I_{fG(e)})(A), (\lambda^I_{fH(e)} \cup \text{ɣ}^I_{fG(e)})(B)\}$

v) $(\lambda^F_{fH(e)} \cup \text{ɣ}^F_{fG(e)})(A-B) \leq \max\{(\lambda^F_{fH(e)} \cup \text{ɣ}^F_{fG(e)})(A), (\lambda^F_{fH(e)} \cup \text{ɣ}^F_{fG(e)})(B)\}$

vi) $(\lambda^F_{fH(e)} \cup \text{ɣ}^F_{fG(e)})(AB) \leq \max\{(\lambda^F_{fH(e)} \cup \text{ɣ}^F_{fG(e)})(A), (\lambda^F_{fH(e)} \cup \text{ɣ}^F_{fG(e)})(B)\}$

Hence $\lambda^H \cap \lambda^G$ is a fuzzy neutrosophic soft HX subring of a HX ring N.

2) Let $H = \{< x, T_{fH(e)}(x), I_{fH(e)}(x), F_{fH(e)}(x) >: x \in R, e \in E\}$

$$G = \{< x, T_{fG(e)}(x), I_{fG(e)}(x), F_{fG(e)}(x) >: x \in R, e \in E\}$$

be any two fuzzy neutrosophic soft sets on a ring R

and $\lambda^H = \{< A, \lambda^T_{fH(e)}(A), \lambda^I_{fH(e)}(A), \lambda^F_{fH(e)}(A) >: A \in N\}$

$$\Upsilon^G = \{< A, \Upsilon^T_{fG(e)}(A), \Upsilon^T_{fG(e)}(A), \Upsilon^T_{fG(e)}(A) >: A \in N\}$$

two anti fuzzy neutrosphic soft HX subring of a HX ring N

$$(\lambda^H \cap \Upsilon^G)(A) = \{< A, \left(\lambda^T_{fH(e)} \cap \Upsilon^T_{fG(e)}\right)(A), \left(\lambda^I_{fH(e)} \cup \Upsilon^I_{fG(e)}\right)(A),$$

$$\left(\lambda^F_{fH(e)} \cup \Upsilon^F_{fG(e)}\right)(A):\ A \in N\}$$

i) $\left(\lambda^T_{fH(e)} \cap \Upsilon^T_{fG(e)}\right)(A - B) = \min\{\lambda^T_{fH(e)}(A - B), \Upsilon^T_{fG(e)}(A - B)\}$

$\leq \min\{\max\{\lambda^T_{fH(e)}(A), \lambda^T_{fH(e)}(B)\}, \max\{\Upsilon^T_{fG(e)}(A), \Upsilon^T_{fG(e)}(B)\}\}$

$= \max\{\min\{\lambda^T_{fH(e)}(A), \Upsilon^T_{fG(e)}(A)\}, min\{\lambda^T_{fH(e)}(B), \Upsilon^T_{fG(e)}(B)\}\}$

$$= \max\{\left(\lambda^T_{fH(e)} \cap \Upsilon^T_{fG(e)}\right)(A), \left(\lambda^T_{fH(e)} \cap \Upsilon^T_{fG(e)}\right)(B)\}$$

Hence $\left(\lambda^T_{fH(e)} \cap \Upsilon^T_{fG(e)}\right)(A - B) \leq \max\{\left(\lambda^T_{fH(e)} \cap \Upsilon^T_{fG(e)}\right)(A), \left(\lambda^T_{fH(e)} \cap \Upsilon^T_{fG(e)}\right)(B)\}$

ii) $\left(\lambda^T_{fH(e)} \cap \Upsilon^T_{fG(e)}\right)(AB) = \max\{\lambda^T_{fH(e)}(AB), \Upsilon^T_{fG(e)}(AB)\}$

$\leq \min\{\max\{\lambda^T_{fH(e)}(A), \lambda^T_{fH(e)}(B)\}, \max\{\Upsilon^T_{fG(e)}(A), \Upsilon^T_{fG(e)}(B)\}\}$

$= \max\{\min\{\lambda^T_{fH(e)}(A), \Upsilon^T_{fG(e)}(A)\}, \min\{\lambda^T_{fH(e)}(B), \Upsilon^T_{fG(e)}(B)\}\}$

$=\max\{(\lambda^T_{fH(e)} \cap \text{Ɣ}^T_{fG(e)})(A), (\lambda^T_{fH(e)} \cap \text{Ɣ}^T_{fG(e)})(B)\}$

Hence $(\lambda^T_{fH(e)} \cap \text{Ɣ}^T_{fG(e)})(AB) \leq \max\{(\lambda^T_{fH(e)} \cap \text{Ɣ}^T_{fG(e)})(A), (\lambda^T_{fH(e)} \cap \text{Ɣ}^T_{fG(e)})(B)\}$.

In the same way we can prove that

iii) $(\lambda^I_{fH(e)} \cup \text{Ɣ}^I_{fG(e)})(A-B) \geq \min\{(\lambda^I_{fH(e)} \cup \text{Ɣ}^I_{fG(e)})(A), (\lambda^I_{fH(e)} \cup \text{Ɣ}^I_{fG(e)})(B)\}$

iv) $(\lambda^I_{fH(e)} \cup \text{Ɣ}^I_{fG(e)})(AB) \geq \min\{(\lambda^I_{fH(e)} \cup \text{Ɣ}^I_{fG(e)})(A), (\lambda^I_{fH(e)} \cup \text{Ɣ}^I_{fG(e)})(B)\}$

v) $(\lambda^F_{fH(e)} \cup \text{Ɣ}^F_{fG(e)})(A-B) \geq \min\{(\lambda^F_{fH(e)} \cup \text{Ɣ}^F_{fG(e)})(A), (\lambda^F_{fH(e)} \cup \text{Ɣ}^F_{fG(e)})(B)\}$

vi) $(\lambda^F_{fH(e)} \cup \text{Ɣ}^F_{fG(e)})(AB) \geq \min\{(\lambda^F_{fH(e)} \cup \text{Ɣ}^F_{fG(e)})(A), (\lambda^F_{fH(e)} \cup \text{Ɣ}^F_{fG(e)})(B)\}$

Hence $\lambda^H \cap \text{Ɣ}^G$ is a fuzzy neutrosophic soft HX subring of a HX ring N .

We will now know the Cartesian product of two fuzzy neutrosophic soft HX subsets of the HX rings

<u>Definition (3.11):</u>

Let (λ^H, E) and $(\text{Ɣ}^G, E)$ be two fuzzy neutrosophic soft subsets of the HX rings N_1 and N_2respectively , then the Cartesian product of $\lambda^H and\ \text{Ɣ}^G$ is defined as:

$\lambda^H \times \text{Ɣ}^G = \{< (A,B), (\lambda^T_{fH(e)} \cap \text{Ɣ}^T_{fG(e)})(A,B), (\lambda^T_{fH(e)} \cup \text{Ɣ}^T_{fG(e)})(A,B), (\lambda^T_{fH(e)} \cup \text{Ɣ}^T_{fG(e)})(A,B) >: (A,B) \in N_1 \times N_2\}$ for every $(A,B) \in N_1 \times N_2$ and for all $e \in E$, where

$\left(\lambda^{T}_{fH(e)} \cap \gamma^{T}_{fG(e)}\right)(A,B) = \min\{\lambda^{T}_{fH(e)}(A), \gamma^{T}_{fG(e)}(B)\}$

$\left(\lambda^{I}_{fH(e)} \cup \gamma^{I}_{fG(e)}\right)(A,B) = \max\{\lambda^{I}_{fH(e)}(A), \gamma^{I}_{fG(e)}(B)\}$

$\left(\lambda^{F}_{fH(e)} \cup \gamma^{F}_{fG(e)}\right)(A,B) = \max\{\lambda^{F}_{fH(e)}(A), \gamma^{F}_{fG(e)}(B)\}$

***Theorem*(3.12)**

Let H and G be any two fuzzy neutrosophic soft sets on R_1 , R_2 respectively and $N_1 \subset 2^{R_1} - \{\Phi\}$, $N_2 \subset 2^{R_2} - \{\Phi\}$ any two HX rings . If λ^{H}, γ^{G} are any two fuzzy neutrosophic soft HX subring of N_1 and N_2 respectively then $\lambda^{H} \times \gamma^{G}$ is also an fuzzy neutrosophic soft HX subring of $N_1 \times N_2$.

***Proof* :** Let $H = \{< x, T_{fH(e)}(x), I_{fH(e)}(x), F_{fH(e)}(x) >: x \in R, e \in E\}$

$G = \{< x, T_{fG(e)}(x), I_{fG(e)}(x), F_{fG(e)}(x) >: x \in R, e \in E\}$

be two fuzzy neutrosophic soft sets of R_1 , R_2 respectively and

$\lambda^{H} = \{< A, \lambda^{T}_{fH(e)}(A), \lambda^{I}_{fH(e)}(A), \lambda^{F}_{fH(e)}(A) >: A \in N_1\}$

$\gamma^{G} = \{< A, \gamma^{T}_{fG(e)}(A), \gamma^{T}_{fG(e)}(A), \gamma^{T}_{fG(e)}(A) >: A \in N_2\}$

be two fuzzy neutrosphic soft HX subring of a HX rings N_1 and N_2 respectively

then $\lambda^{H} \times \gamma^{G} = \{< (A,B), \left(\lambda^{T}_{fH(e)} \cap \gamma^{T}_{fG(e)}\right)(A,B), \left(\lambda^{T}_{fH(e)} \cup \gamma^{T}_{fG(e)}\right)(A,B), \left(\lambda^{T}_{fH(e)} \cup \gamma^{T}_{fG(e)}\right)(A,B) >:\ (A,B) \in N_1 \times N_2\}$

Let $A, B \in N_1 \times N_2$ such that $A = (A_1, A_2)$, $B = (B_1, B_2)$

i) $\left(\lambda^{T}_{fH(e)} \cap \gamma^{T}_{fG(e)}\right)(A - B) = \min\{\lambda^{T}_{fH(e)}(A_1 - B_1), \gamma^{T}_{fG(e)}(A_2 - B_2)\}$

$\geq \min\{\min\{\lambda^{T}_{fH(e)}(A_1), \lambda^{T}_{fH(e)}(B_1)\}, \min\{\gamma^{T}_{fG(e)}(A_2), \gamma^{T}_{fG(e)}(B_2)\}\}$

$=\min\{\min\{\{\lambda^T_{fH(e)}(A_1), \text{ɣ}^T_{fG(e)}(A_2)\}, \min\{\lambda^T_{fH(e)}(B_1), \text{ɣ}^T_{fG(e)}(B_2)\}\}$

$$=\min\{\left(\lambda^T_{fH(e)} \cap \text{ɣ}^T_{fG(e)}\right)(A_1, A_2), \left(\lambda^T_{fH(e)} \cap \text{ɣ}^T_{fG(e)}\right)(B_1, B_2)\}$$

Hence $\left(\lambda^T_{fH(e)} \cap \text{ɣ}^T_{fG(e)}\right)(A-B) \geq \min\{\left(\lambda^T_{fH(e)} \cap \text{ɣ}^T_{fG(e)}\right)(A), \left(\lambda^T_{fH(e)} \cap \text{ɣ}^T_{fG(e)}\right)(B)\}$

ii) $\left(\lambda^T_{fH(e)} \cap \text{ɣ}^T_{fG(e)}\right)(AB) = \min\{\lambda^T_{fH(e)}(A_1B_1), \text{ɣ}^T_{fG(e)}(A_2B_2)\}$

$\geq \min\{\min\{\lambda^T_{fH(e)}(A_1), \lambda^T_{fH(e)}(B_1)\}, \min\{\text{ɣ}^T_{fG(e)}(A_2, \text{ɣ}^T_{fG(e)}(B_2)\}\}$

$$= \min\{\min\{\lambda^T_{fH(e)}(A_1), \text{ɣ}^T_{fG(e)}(A_2)\}, \min\{\lambda^T_{fH(e)}(B_1), \text{ɣ}^T_{fG(e)}(B_2)\}\}$$

$$=\min\{\left(\lambda^T_{fH(e)} \cap \text{ɣ}^T_{fG(e)}\right)(A_1, A_2), \left(\lambda^T_{fH(e)} \cap \text{ɣ}^T_{fG(e)}\right)(B_1, B_2)\}$$

Hence $\left(\lambda^T_{fH(e)} \cap \text{ɣ}^T_{fG(e)}\right)(AB) \geq \min\{\left(\lambda^T_{fH(e)} \cap \text{ɣ}^T_{fG(e)}\right)(A), \left(\lambda^T_{fH(e)} \cap \text{ɣ}^T_{fG(e)}\right)(B)\}$

In the same way we can prove that

iii) $\left(\lambda^I_{fH(e)} \cup \text{ɣ}^I_{fG(e)}\right)(A-B) \leq \max\{\left(\lambda^I_{fH(e)} \cup \text{ɣ}^I_{fG(e)}\right)(A), \left(\lambda^I_{fH(e)} \cup \text{ɣ}^I_{fG(e)}\right)(B)\}$

iv) $\left(\lambda^I_{fH(e)} \cup \text{ɣ}^I_{fG(e)}\right)(AB) \leq \max\{\left(\lambda^I_{fH(e)} \cup \text{ɣ}^I_{fG(e)}\right)(A), \left(\lambda^I_{fH(e)} \cup \text{ɣ}^I_{fG(e)}\right)(B)\}$

v) $\left(\lambda^F_{fH(e)} \cup \text{ɣ}^F_{fG(e)}\right)(A-B) \leq \max\{\left(\lambda^F_{fH(e)} \cup \text{ɣ}^F_{fG(e)}\right)(A), \left(\lambda^F_{fH(e)} \cup \text{ɣ}^F_{fG(e)}\right)(B)\}$

vi) $\left(\lambda_{fH(e)}^{F} \cup \text{Ɣ}_{fG(e)}^{F}\right)(AB) \leq \max\{\left(\lambda_{fH(e)}^{F} \cup \text{Ɣ}_{fG(e)}^{F}\right)(A), \left(\lambda_{fH(e)}^{F} \cup \text{Ɣ}_{fG(e)}^{F}\right)(B)\}$

Theorem(3.13)

Let H and G be any two fuzzy neutrosophic soft sets on R_1 , R_2 respectively,

$N_1 \subset 2^{R_1} - \{\Phi\}$ and $N_2 \subset 2^{R_2} - \{\Phi\}$ any two HX rings , and $\lambda^H, \text{Ɣ}^G$ are any two fuzzy neutrosophic soft HX subrings of N_1 and N_2 respectively suppose that Q_1, Q_2 are identity elements of $N_1 and\ N_2$respectively . If $\lambda^H \times \text{Ɣ}^G$ is a fuzzy neutrosophic soft HX ring of $N_1 \times N_2$, then at least one of the following statement is hold

i) $\text{Ɣ}_{fG(e)}^{T}(Q_2) \geq \lambda_{fH(e)}^{T}(A)$, $\text{Ɣ}_{fG(e)}^{I}(Q_2) \leq \lambda_{fH(e)}^{I}(A)$, $\text{Ɣ}_{fG(e)}^{F}(Q_2) \leq \lambda_{fH(e)}^{F}(A)$

ii) $\lambda_{fH(e)}^{T}(Q_1) \geq \text{Ɣ}_{fG(e)}^{T}(B), \lambda_{fH(e)}^{I}(Q_1) \leq \text{Ɣ}_{fG(e)}^{I}(B)$, $\lambda_{fH(e)}^{F}(Q_1) \leq \text{Ɣ}_{fG(e)}^{F}(B)$

for all $A \in N_1 \quad and \quad B \in N_2$

Proof : Let $H = \{< x, T_{fH(e)}(x), I_{fH(e)}(x), F_{fH(e)}(x) >: x \in R, e \in E\}$

$$G = \{< x, T_{fG(e)}(x), I_{fG(e)}(x), F_{fG(e)}(x) >: x \in R, e \in E\}$$

are two fuzzy neutrosophic soft sets of R_1 , R_2 respectively and

$\lambda^H = \{< A, \lambda_{fH(e)}^{T}(A), \lambda_{fH(e)}^{I}(A), \lambda_{fH(e)}^{F}(A) >: A \in N_1\}$

$\text{Ɣ}^G = \{< A, \text{Ɣ}_{fG(e)}^{T}(A), \text{Ɣ}_{fG(e)}^{T}(A), \text{Ɣ}_{fG(e)}^{T}(A) >: A \in N_2\}$

two fuzzy neutrosphic soft HX ring of N_1 and N_2 respectively,

then $\lambda^H \times \text{Ɣ}^G = \{< (A,B), \left(\lambda_{fH(e)}^{T} \cap \text{Ɣ}_{fG(e)}^{T}\right)(A,B), \left(\lambda_{fH(e)}^{I} \cup \text{Ɣ}_{fG(e)}^{I}\right)(A,B), \left(\lambda_{fH(e)}^{F} \cup \text{Ɣ}_{fG(e)}^{F}\right)(A,B) >: \ (A,B) \in N_1 \times N_2\}$

Let $(A,B) \in N_1 \times N_2$, suppose that the statements (i) and (ii) are not holds , then we can find $A \in N_1 \ and \ B \in N_2$ such that

$\Upsilon^T_{fG(e)}(Q_2) < \lambda^T_{fH(e)}(A)$, $\Upsilon^I_{fG(e)}(Q_2) > \lambda^I_{fH(e)}(A)$, $\Upsilon^F_{fG(e)}(Q_2) > \lambda^F_{fH(e)}(A)$ $\lambda^T_{fH(e)}(Q_1) < \Upsilon^T_{fG(e)}(B)$, $\lambda^I_{fH(e)}(Q_1) > \Upsilon^I_{fG(e)}(B)$, $\lambda^F_{fH(e)}(Q_1) > \Upsilon^F_{fG(e)}(B)$

i) $$\begin{aligned}\left(\lambda^T_{fH(e)} \cap \Upsilon^T_{fG(e)}\right)(A,B) &= \min\{\lambda^T_{fH(e)}(A), \Upsilon^T_{fG(e)}(B)\}\\ &> \min\{\Upsilon^T_{fG(e)}(Q_2), \lambda^T_{fH(e)}(Q_1)\}\\ &= \min\{\lambda^T_{fH(e)}(Q_1), \Upsilon^T_{fG(e)}(Q_2)\}\\ &= \left(\lambda^T_{fH(e)} \cap \Upsilon^T_{fG(e)}\right)(Q_1,Q_2)\end{aligned}$$

Hence $\left(\lambda^T_{fH(e)} \cap \Upsilon^T_{fG(e)}\right)(A,B) > \left(\lambda^T_{fH(e)} \cap \Upsilon^T_{fG(e)}\right)(Q_1,Q_2)$

ii) $$\begin{aligned}\left(\lambda^I_{fH(e)} \cup \Upsilon^I_{fG(e)}\right)(A,B) &= \max\{\lambda^I_{fH(e)}(A), \Upsilon^I_{fG(e)}(B)\}\\ &< \max\{\Upsilon^I_{fG(e)}(Q_2), \lambda^I_{fH(e)}(Q_1)\}\\ &= \max\{\lambda^I_{fH(e)}(Q_1), \Upsilon^I_{fG(e)}(Q_2)\}\\ &= \left(\lambda^I_{fH(e)} \cup \Upsilon^I_{fG(e)}\right)(Q_1,Q_2)\end{aligned}$$

Hence $\left(\lambda^I_{fH(e)} \cup \Upsilon^I_{fG(e)}\right)(A,B) < \left(\lambda^I_{fH(e)} \cup \Upsilon^I_{fG(e)}\right)(Q_1,Q_2)$

In the same way we can prove that

$$\left(\lambda^F_{fH(e)} \cup \Upsilon^F_{fG(e)}\right)(A,B) < \left(\lambda^F_{fH(e)} \cup \Upsilon^F_{fG(e)}\right)(Q_1,Q_2)\ ,$$

this a contradiction for $\lambda^H \times \Upsilon^G$ is a fuzzy neutrosophic soft HX ring of $N_1 \times N_2$.

Then i) $\Upsilon^T_{fG(e)}(Q_2) \geq \lambda^T_{fH(e)}(A)$,
$\Upsilon^I_{fG(e)}(Q_2) \leq \lambda^I_{fH(e)}(A)$, $\Upsilon^F_{fG(e)}(Q_2) \leq \lambda^F_{fH(e)}(A)$

ii) $\lambda^{T}_{fH(e)}(Q_1) \geq Ɣ^{T}_{fG(e)}(B)$, $\lambda^{I}_{fH(e)}(Q_1) \leq Ɣ^{I}_{fG(e)}(B)$, $\lambda^{F}_{fH(e)}(Q_1) \leq Ɣ^{F}_{fG(e)}(B)$

for all $A \in N_1$ *and* $B \in N_2$

Theorem(3.14)

Let H and G be two fuzzy neutrosophic soft sets on R_1 , R_2 respectively ,

$N_1 \subset 2^{R_1} - \{\Phi\}$ and $N_2 \subset 2^{R_2} - \{\Phi\}$ any two HX rings and $\lambda^H, Ɣ^G$ are any two fuzzy neutrosophic soft HX sets of N_1 and N_2 respectively such that $Ɣ^{T}_{fG(e)}(Q_2) \geq \lambda^{T}_{fH(e)}(A)$, $Ɣ^{I}_{fG(e)}(Q_2) \leq \lambda^{I}_{fH(e)}(A)$, $Ɣ^{F}_{fG(e)}(Q_2) \leq \lambda^{F}_{fH(e)}(A)$, Q_2 being identity elements of N_2 . If $\lambda^H \times Ɣ^G$ is a fuzzy neutrosophic soft HX ring of $N_1 \times N_2$,then λ^H is a fuzzy neutrosophic soft HX ring of N_1.

Proof :

Let $H = \{< x, T_{fH(e)}(x), I_{fH(e)}(x), F_{fH(e)}(x) >: x \in R, e \in E\}$

$G = \{< x, T_{fG(e)}(x), I_{fG(e)}(x), F_{fG(e)}(x) >: x \in R, e \in E\}$

are two fuzzy neutrosophic soft sets of R_1 , R_2 respectively and

$\lambda^H = \{< A, \lambda^{T}_{fH(e)}(A), \lambda^{I}_{fH(e)}(A), \lambda^{F}_{fH(e)}(A) >: A \in N_1\}$

$Ɣ^G = \{< A, Ɣ^{T}_{fG(e)}(A), Ɣ^{T}_{fG(e)}(A), Ɣ^{T}_{fG(e)}(A) >: A \in N_2\}$

be two fuzzy neutrosphic soft HX ring of N_1 and N_2 respectively , then $\lambda^H \times Ɣ^G = \{< (A,B), (\lambda^{T}_{fH(e)} \cap Ɣ^{T}_{fG(e)})(A,B), (\lambda^{I}_{fH(e)} \cup Ɣ^{I}_{fG(e)})(A,B), (\lambda^{F}_{fH(e)} \cup Ɣ^{F}_{fG(e)})(A,B) >: (A,B) \in N_1 \times N_2\}$

Let $A, B \in N_1$ then $(A, Q_2), (B, Q_2) \in N_1 \times N_2$, we have

$\gamma^T_{fG(e)}(Q_2) \geq \lambda^T_{fH(e)}(A)$, $\gamma^I_{fG(e)}(Q_2) \leq \lambda^I_{fH(e)}(A)$, $\gamma^F_{fG(e)}(Q_2) \leq \lambda^F_{fH(e)}(A)$ for all $A \in N_1$.

i) $\lambda^T_{fH(e)}(A-B) = \min\{\lambda^T_{fH(e)}(A-B), \gamma^T_{fG(e)}(Q_2-Q_2)\}$

$$= \left(\lambda^T_{fH(e)} \cap \gamma^T_{fG(e)}\right)(A-B, Q_2-Q_2)$$

$$= \left(\lambda^T_{fH(e)} \cap \gamma^T_{fG(e)}\right)((A,Q_2)-(B,Q_2))$$

$$\geq \min\{\left(\lambda^T_{fH(e)} \cap \gamma^T_{fG(e)}\right)(A,Q_2), \left(\lambda^T_{fH(e)} \cap \gamma^T_{fG(e)}\right)(B,Q_2)\}$$

$$= \min\{\min\{\lambda^T_{fH(e)}(A), \gamma^T_{fG(e)}(Q_2)\}, \min\{\lambda^T_{fH(e)}(B), \gamma^T_{fG(e)}(Q_2)\}\}$$

$$= \min\{\lambda^T_{fH(e)}(A), \lambda^T_{fH(e)}(B)\}$$

Hence $\lambda^T_{fH(e)}(A-B) \geq \min\{\lambda^T_{fH(e)}(A), \lambda^T_{fH(e)}(B)\}$

ii) $\lambda^T_{fH(e)}(AB) = \min\{\lambda^T_{fH(e)}(AB), \gamma^T_{fG(e)}(Q_2Q_2)\}$

$$= \left(\lambda^T_{fH(e)} \cap \gamma^T_{fG(e)}\right)(AB, Q_2Q_2)$$

$$= \left(\lambda^T_{fH(e)} \cap \gamma^T_{fG(e)}\right)((A,Q_2).(B,Q_2))$$

$$\geq \min\{\left(\lambda^T_{fH(e)} \cap \gamma^T_{fG(e)}\right)(A,Q_2), \left(\lambda^T_{fH(e)} \cap \gamma^T_{fG(e)}\right)(B,Q_2)\}$$

$$= \min\{\min\{\lambda^T_{fH(e)}(A), \gamma^T_{fG(e)}(Q_2)\}, \min\{\lambda^T_{fH(e)}(B), \gamma^T_{fG(e)}(Q_2)\}\}$$

$$= \min\{\lambda^T_{fH(e)}(A), \lambda^T_{fH(e)}(B)\}$$

Hence $\lambda^T_{fH(e)}(AB) \geq \min\{\lambda^T_{fH(e)}(A), \lambda^T_{fH(e)}(B)\}$

iii) $\lambda^I_{fH(e)}(A-B) = \max\{\lambda^I_{fH(e)}(A-B), \gamma^I_{fG(e)}(Q_1-Q_1)\}$

$$= \left(\lambda^I_{fH(e)} \cup \gamma^I_{fG(e)}\right)(A-B, Q_1-Q_1)$$

$$= \left(\lambda^I_{fH(e)} \cup \gamma^I_{fG(e)}\right)((A,Q_2)-(B,Q_2))$$

$$\leq \max\{(\lambda^{I}_{fH(e)} \cup \Upsilon^{I}_{fG(e)})(A, Q_2), (\lambda^{I}_{fH(e)} \cup \Upsilon^{I}_{fG(e)})(B, Q_2)\}$$

$$= \max\{\max\{\lambda^{I}_{fH(e)}(A), \Upsilon^{I}_{fG(e)}(Q_2)\}, \max\{\lambda^{I}_{fH(e)}(B), \Upsilon^{I}_{fG(e)}(Q_2)\}\}$$

$$= \max\{\lambda^{I}_{fH(e)}(A), \lambda^{I}_{fH(e)}(B)\}$$

Hence $\lambda^{I}_{fH(e)}(A)(A - B) \leq \max\{\lambda^{I}_{fH(e)}(A), \lambda^{I}_{fH(e)}(B)\}$

iv) $\lambda^{I}_{fH(e)}(AB) = \max\{\lambda^{I}_{fH(e)}(AB), \Upsilon^{I}_{fG(e)}(Q_1 Q_1)\}$

$$= (\lambda^{I}_{fH(e)} \cup \Upsilon^{I}_{fG(e)})(AB, Q_1 Q_1)$$

$$= (\lambda^{I}_{fH(e)} \cup \Upsilon^{I}_{fG(e)})((A, Q_2).(B, Q_2))$$

$$\leq \max\{(\lambda^{I}_{fH(e)} \cup \Upsilon^{I}_{fG(e)})(A, Q_2), (\lambda^{I}_{fH(e)} \cup \Upsilon^{I}_{fG(e)})(B, Q_2)\}$$

$$= \max\{\max\{\lambda^{I}_{fH(e)}(A), \Upsilon^{I}_{fG(e)}(Q_2)\}, \max\{\lambda^{I}_{fH(e)}(B), \Upsilon^{I}_{fG(e)}(Q_2)\}\}$$

$$= \max\{\lambda^{I}_{fH(e)}(A), \lambda^{I}_{fH(e)}(B)\}$$

Hence $\lambda^{I}_{fH(e)}(AB) \leq \max\{\lambda^{I}_{fH(e)}(A), \lambda^{I}_{fH(e)}(B)\}$

In the same way we can prove that

v) $\lambda^{F}_{fH(e)}(A - B) \leq \max\{\lambda^{F}_{fH(e)}(A), \lambda^{F}_{fH(e)}(B)\}$

vi) $\lambda^{F}_{fH(e)}(AB) \leq \max\{\lambda^{F}_{fH(e)}(A), \lambda^{F}_{fH(e)}(B)\}$

Theorem(3.15)

Let H and G be two fuzzy neutrosophic soft sets on R_1 , R_2 respectively,

$N_1 \subset 2^{R_1} - \{\Phi\}$ and $N_2 \subset 2^{R_2} - \{\Phi\}$ any two HX rings and $\lambda^H, \text{ɣ}^G$ are any two fuzzy neutrosophic soft HX sets of N_1 and N_2 respectively such that $\lambda^T_{fH(e)}(Q_1) \geq \text{ɣ}^T_{fG(e)}(B)$, $\lambda^I_{fH(e)}(Q_1) \leq \text{ɣ}^I_{fG(e)}(B)$, $\lambda^F_{fH(e)}(Q_1) \leq \text{ɣ}^F_{fG(e)}(B)$, Q_1 being identity elements of N_1 . If $\lambda^H \times \text{ɣ}^G$ is a fuzzy neutrosophic soft HX ring of $N_1 \times N_2$,then
ɣ^G is a fuzzy neutrosophic soft HX ring of N_2.

Proof : Let $H = \{< x, T_{fH(e)}(x), I_{fH(e)}(x), F_{fH(e)}$ $G = \{< x, T_{fG(e)}(x), I_{fG(e)}(x), F_{fG(e)}(x) >: x \in R, e \in E\}$

are two fuzzy neutrosophic soft sets of R_1 , R_2 respectively and

$$\lambda^H = \{< A, \lambda^T_{fH(e)}(A), \lambda^I_{fH(e)}(A), \lambda^F_{fH(e)}(A) >: A \in N_1\}$$

$$\text{ɣ}^G = \{< A, \text{ɣ}^T_{fG(e)}(A), \text{ɣ}^T_{fG(e)}(A), \text{ɣ}^T_{fG(e)}(A) >: A \in N_2\}$$

be two fuzzy neutrosphic soft HX ring of N_1 and N_2 respectively,then

$$\lambda^H \times \text{ɣ}^G = \{< (A,B), (\lambda^T_{fH(e)} \cap \text{ɣ}^T_{fG(e)})(A,B), (\lambda^I_{fH(e)} \cup \text{ɣ}^I_{fG(e)})(A,B), (\lambda^F_{fH(e)} \cup \text{ɣ}^F_{fG(e)})(A,B) >: (A,B) \in N_1 \times N_2\}$$

Let $A, B \in N_2$ $then$ $(Q_1, A), (Q_1, B) \in N_1 \times N_2$.

We have $\lambda^T_{fH(e)}(Q_1) \geq \text{ɣ}^T_{fG(e)}(B)$, $\lambda^I_{fH(e)}(Q_1) \leq \text{ɣ}^I_{fG(e)}(B)$, $\lambda^F_{fH(e)}(Q_1) \leq \text{ɣ}^F_{fG(e)}(B)$

i) $\text{ɣ}^T_{fG(e)}(A - B) = \min\{\lambda^T_{fH(e)}(Q_1 - Q_1), \text{ɣ}^T_{fG(e)}(A - B)\}$

$$= (\lambda^T_{fH(e)} \cap \text{ɣ}^T_{fG(e)})(Q_1 - Q_1, A - B)$$

$$= (\lambda^T_{fH(e)} \cap \text{ɣ}^T_{fG(e)})((Q_1, A) - (Q_1, B))$$

$$\geq \min\{(\lambda^T_{fH(e)} \cap \text{ɣ}^T_{fG(e)})(Q_1, A), (\lambda^T_{fH(e)} \cap \text{ɣ}^T_{fG(e)})(Q_1, B)\}$$

$$= \min\{\min\{\lambda^T_{fH(e)}(Q_1), \text{ɣ}^T_{fG(e)}(A)\}, \min\{\lambda^T_{fH(e)}(Q_1), \text{ɣ}^T_{fG(e)}(B)\}\}$$

$$= \min\{, \Upsilon^T_{fG(e)}(A), \Upsilon^T_{fG(e)}(B)\}$$

Hence $\Upsilon^T_{fG(e)}(A - B) \geq \min\{, \Upsilon^T_{fG(e)}(A), \Upsilon^T_{fG(e)}(B)\}$

ii) $\Upsilon^T_{fG(e)}(AB) = \min\{\lambda^T_{fH(e)}(Q_1Q_1), \Upsilon^T_{fG(e)}(AB)\}$

$$= \left(\lambda^T_{fH(e)} \cap \Upsilon^T_{fG(e)}\right)(Q_1Q_1, AB)$$

$$= \left(\lambda^T_{fH(e)} \cap \Upsilon^T_{fG(e)}\right)((Q_1, A).(Q_1, B))$$

$$\geq \min\{\left(\lambda^T_{fH(e)} \cap \Upsilon^T_{fG(e)}\right)(Q_1, A), \left(\lambda^T_{fH(e)} \cap \Upsilon^T_{fG(e)}\right)(Q_1, B)\}$$

$$= \min\{\min\{\lambda^T_{fH(e)}(Q_1), \Upsilon^T_{fG(e)}(A)\}, \min\{\lambda^T_{fH(e)}(Q_1), \Upsilon^T_{fG(e)}(B)\}\}$$

$$= \min\{, \Upsilon^T_{fG(e)}(A), \Upsilon^T_{fG(e)}(B)\}$$

Hence $\Upsilon^T_{fG(e)}(A - B) \geq \min\{, \Upsilon^T_{fG(e)}(A), \Upsilon^T_{fG(e)}(B)\}$

iii) $\Upsilon^I_{fG(e)}(A - B) = \max\{\lambda^I_{fH(e)}(Q_1 - Q_1), \Upsilon^I_{fG(e)}(A - B)\}$

$$= \left(\lambda^I_{fH(e)} \cup \Upsilon^I_{fG(e)}\right)(Q_1 - Q_1, A - B)$$

$$= \left(\lambda^I_{fH(e)} \cup \Upsilon^I_{fG(e)}\right)((Q_1, A) - (Q_1, B))$$

$$\leq \max\{\left(\lambda^I_{fH(e)} \cup \Upsilon^I_{fG(e)}\right)(Q_1, A), \left(\lambda^I_{fH(e)} \cup \Upsilon^I_{fG(e)}\right)(Q_1, B)\}$$

$$= \max\{\max\{\lambda^I_{fH(e)}(Q_1), \Upsilon^I_{fG(e)}(A)\}, \max\{\lambda^I_{fH(e)}(Q_1), \Upsilon^I_{fG(e)}(B)\}\}$$

$$= \max\{\Upsilon^I_{fG(e)}(A), \Upsilon^I_{fG(e)}(B)\}$$

Hence $\Upsilon^I_{fG(e)}(A - B) \leq \max\{, \Upsilon^I_{fG(e)}(A), \Upsilon^I_{fG(e)}(B)\}$

iv) $\Upsilon^I_{fG(e)}(AB) = \max\{\lambda^I_{fH(e)}(Q_1Q_1), \Upsilon^I_{fG(e)}(AB)\}$

$$= \left(\lambda^{I}_{fH(e)} \cup Ɣ^{I}_{fG(e)}\right)(Q_1Q_1, AB)$$

$$= \left(\lambda^{I}_{fH(e)} \cup Ɣ^{I}_{fG(e)}\right)((Q_1, A).(Q_1, B))$$

$$\leq \max\{\left(\lambda^{I}_{fH(e)} \cup Ɣ^{I}_{fG(e)}\right)(Q_1, A), \left(\lambda^{I}_{fH(e)} \cup Ɣ^{I}_{fG(e)}\right)(Q_1, B)\}$$

$$= \max\{\max\{\lambda^{I}_{fH(e)}(Q_1), Ɣ^{I}_{fG(e)}(A)\}, \max\{\lambda^{I}_{fH(e)}(Q_1), Ɣ^{I}_{fG(e)}(B)\}\}$$

$$= \max\{Ɣ^{I}_{fG(e)}(A), Ɣ^{I}_{fG(e)}(B)\}$$

Hence $Ɣ^{I}_{fG(e)}(AB) \leq \max\{Ɣ^{I}_{fG(e)}(A), Ɣ^{I}_{fG(e)}(B)\}$

In the same way we can prove that

v) $Ɣ^{F}_{fG(e)}(A - B) \leq \max\{ Ɣ^{F}_{fG(e)}(A), Ɣ^{F}_{fG(e)}(B)\}$

vi) $Ɣ^{F}_{fG(e)}(AB) \leq \max\{ Ɣ^{I}_{fG(e)}(A), Ɣ^{I}_{fG(e)}(B)\}$

Hence $Ɣ^{G}$ is a fuzzy neutrosophic soft HX ring of N_2.

Corollary(3.16)

Let H and G be any two fuzzy neutrosophic soft sets on R_1 , R_2 respectively,

$N_1 \subset 2^{R_1} - \{\Phi\}$ and $N_2 \subset 2^{R_2} - \{\Phi\}$ any two HX rings and $\lambda^{H}, Ɣ^{G}$ are any two fuzzy neutrosophic soft HX sets of N_1 and N_2 respectively . If $\lambda^{H} \times Ɣ^{G}$ is a fuzzy neutrosophic soft HX ring of $N_1 \times N_2$,then either λ^{H} is a fuzzy neutrosophic soft HX ring of N_1 or $Ɣ^{G}$ is a fuzzy neutrosophic soft HX ring of N_2.

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