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Effective Number of Parties in A Multi-Party Democracy Under an Entropic Political Equilibrium with Floating Voters

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Abstract

In this short, technical paper we have sought to derive, under a posited formal model of political equilibrium, an expression for the effective number of political parties (ENP) that can contest elections in a multi-party democracy having a plurality voting system(also known as a first-past-the-post voting system). We have postulated a formal definition of political equilibrium borrowed from the financial market equilibrium whereby given the set of utility preferences of all eligible voters as well as of all the candidates, each and every candidate in an electoral fray stands the same objective chance of getting elected. Using an expected information paradigm, we show that under a condition of political equilibrium, the effective number of political parties is given by the reciprocal of the proportion of core electorate (non-floating voters). We have further argued that the formulated index agrees with a party system predicted by Duverger's law.

Key words: Plurality voting, entropic equilibrium, floating voters, Duverger's law

Introduction

Plurality voting systems are currently used in over forty countries worldwide which include some of the largest democracies like USA, Canada, India and UK. Under the basic plurality voting system, a country is divided into territorial single-member constituencies; voters within each constituency cast a single ballot (typically marked by a X) for one candidate; and the candidate with the largest share of votes in each seat is returned to office; and the political party (or a confederation of ideologically similar political parties) with an overall majority of seats forms the government. The fundamental feature of the plurality voting system is that single-member constituencies are based on the size of the electorate. For example, the US is divided into 435 Congressional districts each including roughly equal populations with one House representative per district. Boundaries of constituencies are reviewed at periodic intervals based on the national census to maintain the electorate balance. However the number of voters per constituency varies dramatically across countries e.g. India has 545 representatives for a population of over nine hundred million, so each member of the Lok Sabha (House of the People) serves nearly two million people, while in contrast Ireland has 166 members in the Dial for a population slightly more than three-and-half million or approximately one seat for a little over twenty thousand people.

Under the first-past-the-post voting system candidates only need a simple plurality i.e. at least one more vote than their closest rival to get elected. Hence in three-way electoral contests, the winning candidate can theoretically have less than fifty percent of votes cast in his or her favor e.g. if the vote shares are 35%, 34% and 31%, the candidate with a 35% vote share will get elected. Therefore, although two-thirds of voters support other candidates, the candidate with a simple plurality of votes wins the contest (Norris, 1997).

We define political equilibrium as a condition in which the choices of voters and political parties are all compatible and in which no one group can improve its position by making a different choice. In essence therefore, political equilibrium may be said to exist when, given the set of utility preferences of all eligible voters as well as of all the candidates, each and every candidate in an electoral fray stands the same chance of getting elected. This definition is adequately broad to cover more specific conditional equilibrium models and is based on the principle of efficiency as applied to financial markets. Daniel Sutter (2002) defines political equilibrium as "a balance between demands by citizens on the political system and candidates compete for office". Therefore, translated to a multi-party democracy having a plurality voting system, political equilibrium can be thought to imply a state where perfect balance of power exists between all contesting parties. Methodologically, we build our formal equilibrium model using an expected information approach used in a generalized financial market equilibrium model (Bhattacharya, 2001).

Computing an effective number of political parties

Is there a unique optimum for the number of political parties that have to compete in order to ensure a political equilibrium? If there indeed is such an optimal number then this number necessarily has to be central to any theoretical formalization of political equilibrium as we have defined. Rae (1967) advanced the first formal expression for political fractionalization in a multi-party democracy as follows:

$$F_s = 1 - \sum (s_i)^2$$

Here F_s is known as Rae's index of political fractionalization and s_i is the proportion of seats of the i^{th} political party in the Parliament. Conceptually, Rae's fractionalization index is adapted from the Herfindahl-Hirschman market power concentration index. F is 0 for a single-party system and F tends to 0.50 for a two-party system in equilibrium i.e. when both parties command same proportion of seats in the Parliament. Of course F asymptotically approaches unity as the party system becomes more and more fractionalized. Of course, one may adapt Rae's fractionalization index in terms of the proportion of votes secured in an election instead of seats in Parliament. In that case Rae's index of fractionalization may be represented as follows:

$$F_v = 1 - \sum (v_i)^2$$

Dumont and Caulier (2003) have recognized two major drawbacks of Rae's index. Firstly, the index is not linear for parties that are tied in strength; measured either as proportion of seats or proportion of votes. A two-party system in equilibrium produces an F of 0.50 whereas a four-party system in equilibrium produces 0.75 and a five-party system in equilibrium will have an F of 0.80. Dumont and Caulier (2003) point out that this feature makes the F untenable as an index as the operationalized measure and the phenomenon it measures follow different progression paths. Secondly, Rae's index is, like most other normalized indices of social phenomena, extremely difficult to interpret in objective terms as a unique variable characterizing a party system. The effective number of parties (ENP) measure formulated by Laakso and Taagepera (1979) by improving on Rae's index is now commonly regarded as the classical numerical measure for the comparative analysis of party systems. This ENP formula takes both the number of parties and their relative weights into account when computing a unique variable characterizing a party system thereby making objective interpretation a lot easier as compared to Rae's fractionalization index. The ENP formula is simply stated as the reciprocal of the complement of Rae's fractionalization index i.e.

$$ENP_s = (1 - F_s)^{-1} \text{ and } ENP_v = (1 - F_v)^{-1}$$

In equilibrium, all political parties will command the same strength measured either as proportion of seats or votes and ENP will exactly equal the number of parties in fray. Taagepera and Shugart (1989) have argued that the ENP has become a widely-used index because it "usually tends to agree with our average intuition about the number of serious parties". However Molinar (1991) and Dunleavy and Boucek (2003) have argued that this index produces counter-intuitive and counter-empirical results under a number of circumstances. Taagepera (1999) himself suggested that in cases where one party clearly dominates the political system (commanding more than 50% of the seats), an additional index called the LC (Largest Component) index should be used in conjunction with ENP. The LC is simply the reciprocal of the share of the largest party. When LC is greater than

2 for any party, that party clearly dominates the political system which would however be classified as a multi-party system if only the ENP was the sole classification criterion. Dunleavy and Boucek (2003) have advocated the averaging of ENP index with the LC index to yield a unique classification criterion. Dumont and Caulier (2003) advanced the effective number of relevant parties measure (ENRP) as an improvement over the ENP in a way that their measure yields a unique classification criterion that roughly corresponds to the ENP measure when there are more than two parties that can be considered as major contenders for victory in an electoral contest and collapses to unity if there are only one or two parties that can be seriously considered as a potential winner.

Irrespective of which variant of the ENP index we consider, it is obvious that an intuitive paradigm formalizing political equilibrium in a multi-party democracy having a plurality voting system may be constructed if it can be shown that in equilibrium, all parties in fray are indeed expected to command an equal strength measured either in terms of seats or votes. But such formalization would be considered somewhat limited if it did not take into account the impact of floating voters on electoral outcomes. These are the quintessential fence-sitters who waver between parties during the course of a Parliament, or who don't make up their minds until very close to the election (or even until actually putting their stamps on the ballot paper). The impact of floating voters on electoral outcome is all the more an important issue for large-sized electorates as is the case for very populous countries like India. But none of the ENP indices consider floating voters.

Effective number of political parties with floating voters in entropic equilibrium

Considering a finite fraction of floating voters in any electorate, we may define the following relationship as the (conservative) expected vote share of the i^{th} political party:

$$E(V_i) = [E(S_i)](1 - \lambda_i)$$

Here $E(S_i)$ is the i^{th} candidate's expected vote share as a proportion of the total electorate size and λ_i is the fraction of the i^{th} candidate's vote share that is deemed to come from floating voters. This is the fraction of electorate which is generally supportive of the i^{th} candidate but this support may or may not be translated into actual votes on the day of the election. Thus $E(S_i)$ is the expected proportion of votes to be cast in the i^{th} candidate's favor accepting the existence of floating voters in the electorate. Therefore we may write:

$$\sum_i E(V_i) = \sum_i [E(S_i)](1 - \lambda_i)$$

Let us denote $\sum_i E(V_i)$ as $E(V)$ and $\sum_i E(S_i)$ as $E(S)$. Therefore, re-arranging (5) we get:

$$\sum_i [E(S_i)]\, \lambda_i = E(S) - E(V)$$

In mathematical information theory, entropy or expected information from an event is measured using a logarithmic function borrowed from classical thermodynamics. There are two possible mutually exclusive and exhaustive outcomes for any individual event – either the event occurs or the event does not occur. If there are m candidates in an electoral fray the two events associated with each candidate in fray is that either the particular candidate wins the election or he/she does not win. If p_i is the probability of the i^{th} candidate winning the election, then the expected information content of a message that conveys the outcome of an election with i = 1, 2, …, m candidates is obtained by the classical entropy function as formulated by Shannon (1948) as follows:

$$\psi(p) = (-C')\sum_i(p_i)\log_2(p_i)$$

Here C′ is a positive scale factor (a negentropic counterpart of the Boltzmann constant in thermodynamic entropy). Under an m-party political equilibrium, the long run core (non-floating) vote shares of the i = 1, 2, …, m candidates in electoral fray may be considered as equivalent to their long run winning probabilities. Thus $\psi(p)$ is re-writable as follows:

$$\psi(1 - \lambda) = (-C')\sum_i(1 - \lambda_i)\log_2(1 - \lambda_i)$$

Proposition: If $\psi(1 - \lambda)$ is the expected information from the knowledge of an electoral outcome given the proportion of non-floating voters $(1 - \lambda_i)$ in the vote share of the i^{th} candidate, then the effective number of parties under entropic equilibrium is given as:

$$ENP(\lambda) = (1 - \lambda^*)^{-1}; \text{ where } \lambda^* = 1 - {}^{E(V)}/_{E(S)}$$

Proof: Incorporating the Lagrangian multiplier L the objective function can be written as:

$$Z(1 - \lambda_i, L) = (-C') \sum_i (1 - \lambda_i) \log 2\, (1 - \lambda_i) + L\{1 - \sum_i (1 - \lambda_i)\}$$

Taking partial derivative of Z with respect to $(1 - \lambda_i)$ and setting equal to zero as per the necessary condition of maximization, the following stationary condition is obtained:

$$\partial Z / \partial (1 - \lambda_i) = (-C')\{\log_2 (1 - \lambda_i) + 1\} - L = 0$$

Therefore at the point of maximum entropy one gets $\log_2(1 - \lambda_i) = -({}^{L}/_{C'} + 1)$ i.e. $(1\ \lambda_i)$ becomes a constant value independent of i for all i = 1, 2, …, m candidates in the electoral contest. Since necessarily the $1 - \lambda_i$ values must sum to unity, it implies that at the point of maximum entropy we must have $p_1 = p_2 = \ldots = p_m = (1 - \lambda^*) = {}^{1}/_{m}$.

$$\text{Therefore } m \equiv ENP(\lambda) = (1 - \lambda^*)^{-1}$$

Simplifying the expression for $\sum_i [E(S_i)]\lambda_i = E(S) - E(V)$ under equilibrium we may write:

$$\lambda^* E(S) = E(S) - E(V) \text{ i.e. } \lambda^* = 1 - {}^{E(V)}/_{E(S)} \qquad \text{Q.E.D.}$$

λ^* is simply the total percentage of floating voters under an entropic political equilibrium[1]. Thus $ENP(\lambda)$ is formally obtained (as expected intuitively) as the reciprocal of the equilibrium percentage of non-floating voters in the electorate. The higher the proportion of floating voters within the electorate, the higher is the value of $ENP(\lambda)$. The intuitive reasoning is obvious – with a large number of floating votes to go around, more candidates could stay in the electoral fray than there would be if the electorate consisted of only a very small percentage of floating voters. When $\lambda = 50\%$, $ENP(\lambda) = 2$. If λ goes up to 75%, $ENP(\lambda)$ will go up to 4 i.e. with 25% more floating voters within the electorate, 2 more candidates can stay in electoral fray feeding off the floating votes.

Thus $ENP(\lambda)$ (the formula for which is structurally quite similar to Laakso and Taagepera's ENP index) is a generalized measure of ENP based on the entropic formalization of political equilibrium accepting the very real existence of floating voters.

Entropic political equilibrium and Duverger's law

Duverger (1951) stated that the electoral contest in a single-seat electoral constituency following a plurality voting system tends to converge to a two-party system. *Duverger's law* basically stems from the premise of *strategic voting*. Palfrey (1989) has showed that in large electorates, equilibrium voting behavior implies that a voter will always vote for the most preferred candidate of the two frontrunners. For a given electorate of size n, Palfrey's model is stated in terms of the following inequality:

$$u_k > u_j \, [(\textstyle\sum_{i \neq j}(p^n{}_{ij}/p^n{}_{kl}) \,/\, (\sum_{h \neq k}(p^n{}_{kh}/p^n{}_{kl})] + \sum_{i \neq j,k} u_i \, [\{(p^n{}_{ki} - p^n{}_{ij})/p^n{}_{kl}\}/\sum_{h \neq k}(p^n{}_{kh}/p^n{}_{kl})]$$

In this model, u_k denotes the voter's utility of his/her first choice among the two frontrunners and u_l denotes the voter's utility for his/her second choice among the frontrunners so that $u_k > u_l$. Also j is any other candidate from among the i = 1, 2, …, m candidates. The notation $p^n{}_{ij}$ stands for the probability that the candidate i and candidate j are tied for the most votes and the interpretation is similar for notations $p^n{}_{kh}$ and p^{nkl}. In the limiting case, the likelihood ratio $p^n{}_{kh}/p^{nkl}$ tends to zero for all ij ≠ kl. Thus the right-hand side of the inequality converges to u_l irrespective of j; thereby mathematically establishing Duverger's law. Apart from Palfrey's theoretical formalization, Cox and Amorem Neto (1997) and Benoit (1998) and Schneider (2004) have provided empirical evidence generally supportive of Duverger's law.

It therefore seems rather appropriate that an intuitive model of political equilibrium in a multi-party democracy that follows a plurality voting system should at least take Duverger's law into consideration if not actually have it embedded in some form within its formal structure. This is true for our entropic model, because as m increases $(1 - \lambda^*) = {}^1/_m$ becomes smaller and smaller, thereby implying that for multi-party democracies that follow a plurality voting system, the political equilibrium most likely to prevail in the long run will tend to occur at the highest possible value of $(1 - \lambda^*) = 50\%$. In other words, although some relatively new democracies may start off with a number of political parties contesting elections and a very large percentage of floating voters in the electorate, the likelihood is very low that a very high proportion (exceeding 50%) of the electorate will be composed of floating voters in the long run which implies that in the long run, "mature" multi-party democracies having plurality voting systems will tend to have only two parties as serious contenders for victory in an election; corresponding to a two-party system as stated by Duverger's law.

Conclusion

We have proposed and mathematically derived a formula for the effective number of political parties that can be in electoral fray under a condition of political equilibrium in a multi-party democracy following a plurality voting system. We have posited the expected information approach to formalize the concept of political equilibrium in a parliamentary democracy. Our advocated model aims to improve upon existing ENP indices by incorporating the very realistic consideration of the impact of floating voters on elections. Of course, ours has been an entirely theoretical exercise and a potentially rewarding direction of future research would be to empirically investigate the veracity of ENP(λ) possibly in conjunction with a suitable classification model to distinguish floating voters.

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