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Decision-making based on probabilistic linguistic term sets without loss of information

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Abstract

Probabilistic linguistic term set (PLTS) provides a much more effective model to compute with words and to express the uncertainty in the pervasive natural language by probability information. In this paper, to avoid loss of information, we redefine the classical probabilistic linguistic term sets (PLTSs) by multiple probability distributions from an ambiguity perspective and present some basic operations using Archimedean t-(co)norms. Different from the classical PLTSs, the reformulated PLTSs are not necessarily normalized beforehand for further investigations. Moreover, the multiple probability distributions based PLTSs facilitate the incorporation of the different attitudes of the DMs in their score values and the deviation, and thus the comparisons. Then the Decision-Making Trial and Evaluation Laboratory (DEMATEL) method is extended to the reformulated PLTS frame by incorporating probability information. With these newly developed elements in the reformulated PLTSs, a DEMATEL based multiple attributes decision-making is proposed. The illustrative example and comparison analysis show that the method over the reformulated PLTSs is feasible and valid, and has the advantage in processing without any information loss (i.e., without normalization) and fully exploration of the DMs-preference and knowledge.

Keywords Probabilistic linguistic term sets · Triangular (co)norms · Ambiguity · DEMATEL · Multiple-attribute decision-making

Introduction

Decision-making is human-centered and has inborn uncertainty with vague information expressed by natural language. To date, there are many models developed to facilitate the representations of the ill-structured information with uncertainty and vagueness. The first model is the famous fuzzy linguistic model by Zadeh [59], which relies on the membership function to express the non-crisp word or sentences in human language and has been a milestone in qualitative analysis in decision-making. Since then, a formal methodology, known as computing with words (CWW), in reasoning, computing and making decisions with natural language based information has come into being and soon became an eye-catching formulation in various fields. Notably, many linguistic models are introduced to handle the pervasive linguistic qualitative problems and can be fallen into two categories: the single linguistic term-based models and the multiple linguistic term-based models. Among the single linguistic term-based models, there are the symbolic linguistic computing model [56],the virtual linguistic computing model [55], the 2-tuple linguistic model [21] besides the Zadeh's classical fuzzy linguistic variables. These models rely only on a single linguistic term representation of a linguistic variable, which suggests that one has to present his/her opinions without any hesitations. However, it is not the case in practice. Due to the incomplete information or knowledge, people may be uncertain or hesitant on the linguistic expressions and one single linguistic term is not adequate to portray the scenario accurately. Then, the multiple linguistic term-based models are proposed to cover the uncertainties with complex linguistic expressions, such as the hesitant fuzzy linguistic term set

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(HFLTS) [41], which is mathematically reformulated as hesitant fuzzy linguistic element (HFLE) [32]. The HFLTSs use a set of potential linguistic terms to effectively express the vague and complex linguistic statements, such as "between medium and tall", "at least slightly good" and the like. The series of fruitful theoretical and practical results [31] betray that HFLTS or HFLE provides an effective way to capture the uncertainty in natural languages.

To express the preferences of DMs in a more flexible way, HFLTSs are extended to incorporate additional information for differentiating the given linguistic terms. The typical ones are the PLTS [38], equivalent to the proportional linguistic distribution assessment [19,52]), the proportional HFLTSs [7]. Particularly, PLTS proposed by Pang et al. [38] assigns probabilities to the linguistic terms of an HFLTS and is more accurate in describing the vagueness in linguistic statements than HFLTS. Consequently, since the arising of PLTSs, the emerging model attracts extensive attentions from scholars in various fields and substantially advances the progress in CWW [30,37]. The pertaining investigations can be distinguished into three categories. The first is focus on the PLTSs operations and notations [17,28,37,38,54,58,61], which is the basis for CWW and information aggregation; the second is about some PLTS-based methods for decision-making. Many classical methods are tailored to facilitate the PLTSs, such as the PLTS-based versions of MULTIMOORA [51], TODIM [36], ORESTE [49], VIKOR [63], ELECTRE II [33], ELECTRE III [28], LINMAP [29], QUALIFLEX [14], gained and lost dominance score method [50], DEMATEL [9], which bring more convenience for various applications over the PLTSs. The third is about the extensions or variants of PLTSs, such as the probabilistic uncertain linguistic term set [34], probabilistic linguistic vector-term set [60], interval-valued PLTS [3], Uncertain PLTS [23], dual PLTS [53]. The investigations around PLTSs is on-going vigorously. For a more comprehensive and detailed summaries on PLTSs, we recommend the most recent survey contributions [30,37].

As is known, PLTS has the advantage in considering the ignorance in probabilistic information due to the partial information (the summation of probabilities for the given linguistic terms is less than one), then the normalization of PLTSs is inevitably necessary for further investigation. Until now, there are fruitful normalization methods for PLTSs, such as Average assignment [38], full-set assignment [28, 62], power-set assignment [37], envelope assignment [37], attitudes assignment [45], in tackling the incomplete probabilistic information. However, these normalization methods lead to information loss about the ignorance of probability information since the ignorance of probability is eliminated after the normalization. Then the authors in [37] presented an open problem on whether the normalization is necessary. Indeed, the ignorance of probability information is practically inevitable, then it is of vital importance to provide a reasonable process framework without any information loss, which is the very point which drives us to reconsider the process frame for the underlying probability information of the PLTSs in a different perspective.

In general, a PLTS has a set of linguistic terms with a probability distribution with normality, which suggests people have an exact and complete knowledge of probabilistic information. If the normality does not hold, according to [38], then it suggests that people have partial ignorance as a result of incomplete information or knowledge. Indeed, the partial ignorance actually means the 'probabilistic information' for the linguistic terms is non-probabilistic and is referred to a non-additive measure or capacity [16,43]. In other words, for this case a PLTS can be seen as the set of linguistic terms with *a set of probability distributions*. Hence, in this work, we try to reformulate the PLTS from the perspective of ambiguity and provides a new insight to handle the ignorance of probability information through the non-additive probabilities or capacities (alternatively the multiple probability distributions). Furthermore, the classical DEMATEL is extended by the reformulated PLTSs. In detail, the probability information by the PLTSs is incorporated to tackle the uncertainty in the linguistic evaluation of the pair-wise comparisons and consequently used to synthesize the prominence-relation analysis.

Throughout the paper, we focus on the reformulation the classical PLTSs, and introduce a DEMATEL-based multiple attributes decision-making (MADM). In summary, the following aspects in this work can be highlighted.

(1) The redefinition of the PLTS from the perspective of ambiguity. The classical PLTS is equivalently reformulated as a set of potential linguistic terms with multiple probability distributions. Since the lower bound of the accompanying multiple probability distributions is an alternative for the ignorance of probability information, then the reformulation does not lead to any information loss. It is worthy noting that, different from the classical PLTS literature, the reformulated PLTSs can be operated and aggregated using Archimedean t-(co)norms without any probability normalization.
(2) The redefinition of the PLTS by the multiple probability distributions facilitates the modeling of the behavioral attitudes of DMs. Specifically, the score values and the deviations of PLTSs are coupled with an optimism index, through which the behavioral attitudes of the decision-makers can be considered. By this way, we can incorporate the (optimistic, or moderate, or pessimistic) attitudes of the DMs in PLTS-based problem solutions.
(3) The extension of the DEMATEL technology in the reformulated PLTS frame. With the pertaining probability information of the PLTSs, the classical DEMATEL method is extended. A family of initial direct relation matrices are obtained in a lexicographically way

using the linguistic terms in PLTSs for the influence evaluations, and the resultant prominences and relations for each attribute are consequently probabilistically weighted and synthesized.

Here, we focus on the reformulation of the classical PLTS by transforming the ignorance of probabilities into multiple probability distributions in an ambiguity perspective. As a matter of fact, the reformulation is the accurate expression in the probability information, thus takes a step forward in exploring the experts-preference and knowledge. Further, in comparison (see Table 1), the DEMATEL-based aggregation method using the reformulated PLTSs in this work does not necessarily normalize the PLTSs beforehand and can well incorporate the different attitudes of the DMs for decision-making, thus does not result in any information loss in the process. Therefore, the improvement of the information exploration and utilization are the intrinsic starting point and drive for the employment of the reformulated PLTSs in DEMATEL. Compared to DEMATEL method based on the classical PLTS [9], or the linguistic term, or numerical expression, the method with the reformulated PLTS in this work has the advantage in effectively manifesting the experts-preference and knowledge, thus could lead to substantial improvement on the precision of decision-making.

The paper is organized as follows: as a preliminary, the next section provides some justifications on the ambiguity and the ignorance of the probability information about PLTSs; the redefinition and some notations of the PLTS are presented in the subsequent section followed by which the t-(co)norm-based operations over the reformulated PLTSs are focussed upon; then, the DEMATEL technology is extended using the reformulated PLTSs and then a PLTS DEMATEL-based aggregation method for decision-making is proposed; in the penultimate section, the illustrative example for the method is presented. In the end, some concluding remarks are presented.

Ambiguity and the ignorance of the probability information about PLTSs

In the literature of economics and psychology, the theories of choice have dominantly been the expected utility (EU) theory and the subjective expected utility (SEU) theory. It is assumed that in EU the pertaining probabilities are known while in SEU probabilities of states are subjectively or personally known. Actually, it is unbelievable to think of decision-making in which the probabilities are objectively presented. Thus, SEU is much more widely applied than EU. Nevertheless, there are much empirical evidence against SEU concerning the sharp distinction between whether the probabilities are (un)known. The distinctions are notated by various names, such as the risk vs uncertainty by Knight Frank [25], unambiguous vs ambiguous probabilities [13]. Generally, the term ambiguity refers to the case that the probability is unknown or partially known. Moreover, how much the decision-maker know about the probabilities does substantially influence their decisions [5]. The ignorance of probability, being a vital feature of PLTSs, can be seen as a snapshot of ambiguity. Therefore, ambiguity could be a point of departure to provide some new insights on PLTSs.

In general, a PLTS has the form as

$$L = \left\{ s_{\alpha_i}(p_i) | s_{\alpha_i} \in S, i = 1, 2, \ldots, m, p_i \ge 0, \sum_{i=1}^{m} p_i \le 1 \right\}. \quad (1)$$

For the case $\sum_{i=1}^{m} p_i = 1$, i.e., the PLTS has a set of linguistic terms with a probability distribution, which suggests people have an exact and complete knowledge of probabilistic information; for the case $\sum_{i=1}^{m} p_i < 1$, according to [38], it suggests that people have partial ignorance as a result of incomplete information or knowledge. More precisely, the partial ignorance is referred to a situation that the probabilities are unknown or partially known, which is referred to *ambiguity* argued by Ellsberg [13] in his experimental results known as the Ellsberg paradox. Ellsberg paradox tells us that under ambiguity the decision-makers may rely on non-probabilistic measures which are the so called non-additive probabilities [43], or capacities [16] and are defined to be a set function $\upsilon : \Sigma \to [0, 1]$ satisfying $\upsilon(\emptyset) = 0$ and $\upsilon(S) = 0$, and $\upsilon(E) \le \upsilon(E')$ whenever $E \subseteq E'$. Here Σ is the event algebra of a state space S. It is shown [42] that a capacity υ can be interpreted as a lower envelope on a set core(υ) of considered possible probabilities P, i.e., $\upsilon(E) = \inf_{P \in \text{core}(\upsilon)}\{P(E)\}$. Intuitively, the decision-makers with ambiguity or partial probability entertain multiple probabilities as potential beliefs. Therefore, $\sum_{i=1}^{m} p_i < 1$ in PLTSs actually means the ‘probabilistic information’ for the linguistic terms is non-probabilistic and should be referred to a non-additive measure or capacity. In other words, for this case a PLTS can be seen as the set of linguistic terms with *a family of probability distributions*.

For illustration, we provide the following example which is adapted from [38].

Example 1 In evaluating the overall comfortable degree of a vehicle (e.g., Honda XR-V), one can use a PLTS such as

$$\text{Comfort(Honda XR-V)} = \{\text{slightly high}(0.1), \text{high}(0.65), \text{very high}(0.2)\}. \quad (2)$$

In an ambiguity perspective, 0.1, 0.65 and 0.2 can be seen as the capacity of the underlying linguistic terms. That is $\upsilon(\text{slightly high}) = 0.1$, $\upsilon(\text{high}) = 0.65$, $\upsilon(\text{very high}) = 0.2$. Note that $0.1 + 0.65 + 0.2 < 1$. Since a capacity

Table 1 The comparison of existing methods for PLTS decision-making

Contributions	Methods	Normalizations of PLTSs	Incorporation of attitudes
[38]	Extended TOPSIS	Necessary	No
[38]	Max deviation-based aggregation	Necessary	No
[51]	Extended MULTIMOORA	Necessary	No
[36]	Extended TODIM	Necessary	No
[49]	Extended ORESTE	Necessary	No
[63]	Extended VIKOR	Necessary	No
[33]	Extended ELECTRE II	Necessary	No
[28]	Extended ELECTRE III	Necessary	No
[29]	Extended LINMAP	Necessary	No
[14]	Extended QUALIFLEX	Necessary	No
[9]	DEMATEL by PLTS	Necessary	No
In this work	Extended DEMATEL-based aggregation	Not necessary	Yes

can be interpreted as a lower bound on a set of potential probability distributions. Here we assume two probability distributions P_1 and P_2 for these linguistic terms with P_1(slightly high) $= 0.1$, P_1(high) $= 0.7$, P_1(very high) $= 0.2$; and P_2(slightly high) $= 0.15$, P_2(high) $= 0.65$, P_2(very high) $= 0.2$. Then it holds that
v(slightly high)$=\min\{P_1$(slightly high), P_2(slightly high)$\}$;
v(high) $= \min\{P_1$(high), P_2(high)$\}$;
v(very high) $= \min\{P_1$(very high), P_2(very high)$\}$.
Then the PLTS by (2) can be considered that there are multiple probability distributions (P_1 and P_2) for these linguistic terms and could be expressed equivalently in the multiple distribution-based form: {{slightly high(0.1), high(0.7), very high(0.2)}; {slightly high(0.15), high(0.65) , very high(0.2)}}.

The reformulation of PLTSs and the pertaining notations

By the justifications of the above, a PLTS by Eq. (1) can be redefined as follows.

Definition 1 Let $S = \{s_0, s_1, \ldots, s_\tau.\}$ be a LTS, a PLTS is defined by

$$\mathscr{L}(p) = \left\{L_j(p)|, j = 1, 2, \ldots, k\right\} \quad (3)$$

with $k \geq 1, L_j(p) = \{s_{\alpha_i}(p_{ij})|s_{\alpha_i} \in S, i = 1, 2, \ldots, m, p_{ij} \geq 0, \sum_{i=1}^{m} p_{ij} = 1.\}$ and $p_i = \min_j p_{ij}$ for each i.

By the definition, if $k = 1$, then $\mathscr{L}(p)$ is equivalently reduced to the case $\sum_{i=1}^{m} p_i = 1$ in Eq. (1). In general, people only have partial or incomplete relevant information about the alternatives, and can hardly obtain a precise probability distribution for the potential linguistic terms. Instead, they may choose a set of possible probability distributions for the linguistic terms in compromise. Thus, the redefinition is practically and behaviorally reasonable. Additionally, if $m = 1$, i.e., only one linguistic term is given, the probability is necessary 1 which suggests a complete belief in the specific linguistic term. For this case, we would abbreviate the notation $s_\alpha(1)$ by omitting the probability 1, and use s_α instead. Here, it is assumed the partial belief focuses only on the case with multiple linguistic terms.

With an ambiguity perspective, the reformulation of the PLTS transforms the ignorance of probabilities into multiple probability distributions. Alternatively, the PLTS is a set of linguistic terms with multiple probability distributions. Note that $p_i = \min_j p_{ij}$ implies that $\sum_{i=1}^{m} p_i = \sum_{i=1}^{m} \min_j p_{ij} \leq 1$. Thus, the multiple probability distributions representation of PLTS do not lead to any information loss over the ignorance of probabilities, and provides a smart way to tackle the probability normalization problem. By the reformulation the probability normalization for PLTS is not necessary any more and can be directly skipped for further investigations. In [38], the authors provided a normalization method to handle the ignored probability information and suggested that there may be some alternative methods on probability normalization. However, by these normalization methods the ignored probabilities are assigned to various linguistic term entities, and consequently removed. In other words, the normalization methods on the PLTS bring about much information loss on the ignored probabilities. Thus, Liao et al. [30] adopted a wait-and-see attitude about the normalization and proposed the open problem on whether it is necessary to normalize the PLTS. Here, the reformulation of the PLTS provides an ambiguity perspective into the partial ignorance of probabilistic information. Thus the PLTS considers the partial probability as a set of multiple probability distributions. Therefore, the redefinition of PLTS provides a compromised solution to the open problem [30,37] on PLTS normalization through the multiple probability distribution-based representation of PLTSs. In the rest of the paper, we focus on the PLTSs by (3).

For the normalization of PLTSs having different sets of linguistic terms, we just have to add the unshared linguistic terms and assign zero probabilities to them. We use the dotted symbols $\dot{\mathscr{L}}(p)$ for the normalized PLTSs.

Example 2 Let $\mathscr{L}_1(p)=\{\{s_4(0.2), s_3(0.4), s_2(0.4)\}, \{s_4(0.2), s_3(0.5), s_2(0.3)\}, \{s_4(0.3), s_3(0.4), s_2(0.3)\}\}$, which is equivalent to $L_1(p) = \{s_4(0.2), s_3(0.4), s_2(0.3)\}$; $\mathscr{L}_2(p) = \{\{s_5(0.3), s_4(0.4), s_3(0.3)\}, \{s_5(0.2), s_4(0.6), s_3(0.2)\}, \{s_5(0.3), s_4(0.5), s_3(0.2)\}\}$, which is equivalent to $L_2(p) = \{s_5(0.2), s_4(0.4), s_3(0.2)\}$. For normalization, we can, respectively, add the unshare linguistic terms s_5 and s_2 to $\mathscr{L}_1(p)$ and $\mathscr{L}_2(p)$, and assign zero probabilities for them. Then we can obtain $\dot{\mathscr{L}}_1(p) = \{\{s_5(0), s_4(0.2), s_3(0.4), s_2(0.4)\}, \{s_5(0), s_4(0.2), s_3(0.5), s_2(0.3)\}, \{s_5(0), s_4(0.3), s_3(0.4), s_2(0.3)\}\}$; $\dot{\mathscr{L}}_2(p) = \{\{s_5(0.3), s_4(0.4), s_3(0.3), s_2(0)\}, \{s_5(0.2), s_4(0.6), s_3(0.2), s_2(0)\}, \{s_5(0.3), s_4(0.5), s_3(0.2), s_2(0)\}\}$.

Remark 1 Practically, it is statistically feasible [38] to the settle the classical PLTSs in applications. Analogously, how to statistically settle the reformulated PLTSs is vital for applications. Given a classical PLTS, there may be a uncountable number to tackle the ignored probabilities to derive probability distributions. One feasible way is to assign the ignored probabilities to each one of the underlying linguistic terms. For instance, in Example 1 (s_3 = slightly high, s_4 = high, s_5 = very high), we can assign the ignored probability 0.05 to s_3, s_4, s_5, respectively, and then obtain a reformulated PLTS by $\{\{s_3(0.15), s_4(0.65), s_5(0.2)\}, \{s_3(0.1), s_4(0.7), s_5(0.2)\}, \{s_3(0.1), s_4(0.65), s_5(0.25)\}\}$.

To provide a ranking scheme for comparison, we introduce the score values and deviation degrees for the reformulated PLTSs, in which the behavioral (optimism or pessimism) attitudes can be incorporated.

Definition 2 Let $S = \{s_0, s_1, \ldots, s_\tau.\}$ be a LTS, and the PLTS be $\mathscr{L}(p) = \{L_j(p)|j = 1, 2, \ldots, k.\}$ with $L_j(p) = \{s_{\alpha_i}(p_{ij})|s_{\alpha_i} \in S, i = 1, 2, \ldots, m, p_{ij} \geq 0, \sum_{i=1}^m p_{ij} = 1.\}$, then the worse score value of $\mathscr{L}(p)$ is defined to be

$$\text{WSV}[\mathscr{L}(p)] = \min\left\{ s_{\bar{\alpha}_j} | \bar{\alpha}_j = \sum_{i=1}^m \alpha_i p_{ij}, j = 1, 2, \ldots, k. \right\};$$

the best score value of $\mathscr{L}(p)$ is defined to be

$$\text{BSV}[\mathscr{L}(p)] = \max\left\{ s_{\bar{\alpha}_j} | \bar{\alpha}_j = \sum_{i=1}^m \alpha_i p_{ij}, j = 1, 2, \ldots, k. \right\}.$$

Generally, the score value $SV_\pi[\mathscr{L}(p)]$ of $\mathscr{L}(p)$ with the optimism index $\pi \in [0, 1]$ is given by

$$SV_\pi[\mathscr{L}(p)] = \pi \text{BSV}[\mathscr{L}(p)] + (1 - \pi)\text{WSV}[\mathscr{L}(p)].$$

In "Ambiguity and the ignorance of the probability information about PLTSs", the worse case representation of the Choquet integral w.r.t. the capacity is an indication of ambiguity aversion. Also there are some empirical evidences in some recent work showing that people may have behaviors with ambiguity loving or neutrality [40]. Thus, the optimism index for the score value is necessary and suggests the degree of the optimism of a DM. The greater the index, the more optimistic the DM. With different choice of the optimism index, the attitudes of the DM can be demonstrated.

Definition 3 Let $S = \{s_0, s_1, \ldots, s_\tau.\}$ be a LTS, and the PLTS be $\mathscr{L}(p) = \{L_j(p)|j = 1, 2, \ldots, k.\}$ with $L_j(p) = \{s_{\alpha_i}(p_{ij})|s_{\alpha_i} \in S, i = 1, 2, \ldots, m, p_{ij} \geq 0, \sum_{i=1}^m p_{ij} = 1.\}$, then the worse deviation degree of $\mathscr{L}(p)$ is defined to be

$$\sigma_w[\mathscr{L}(p)] = \max\{\sigma_j, j = 1, 2, \ldots, k.\},$$

where $\sigma_j = \sqrt{\sum_{i=1}^m (\alpha_i - \bar{\alpha}_j)^2 p_{ij}}, \bar{\alpha}_j = \sum_{i=1}^m \alpha_i p_{ij}$; the best deviation degree of $\mathscr{L}(p)$ is defined to be

$$\sigma_b[\mathscr{L}(p)] = \min\{\sigma_j, j = 1, 2, \ldots, k.\}.$$

Generally, the deviation degree $\sigma_\pi[\mathscr{L}(p)]$ of $\mathscr{L}(p)$ with the optimism index $\pi \in [0, 1]$ is given by

$$\sigma_\pi[\mathscr{L}(p)] = (1 - \pi)\sigma_w[\mathscr{L}(p)] + \pi\sigma_b[\mathscr{L}(p)].$$

Definition 4 (*Comparison of PLTSs*) Given two PLTSs $\mathscr{L}_1(p)$ and $\mathscr{L}_2(p)$, the ranking scheme with optimism index $\pi \in [0, 1]$ for them is listed as follows.

1. If $SV_\pi[\mathscr{L}_1(p)] > SV_\pi[\mathscr{L}_2(p)]$, then $\mathscr{L}_1(p) \succ \mathscr{L}_2(p)$;
2. If $SV_\pi[\mathscr{L}_1(p)] < SV_\pi[\mathscr{L}_2(p)]$, then $\mathscr{L}_1(p) \prec \mathscr{L}_2(p)$;
3. If $SV_\pi[\mathscr{L}_1(p)] = SV_\pi[\mathscr{L}_2(p)]$, then
   (a) If $\sigma_\pi[\mathscr{L}_1(p)] > \sigma_\pi[\mathscr{L}_2(p)]$, then $\mathscr{L}_1(p) \prec \mathscr{L}_2(p)$;
   (b) If $\sigma_\pi[\mathscr{L}_1(p)] < \sigma_\pi[\mathscr{L}_2(p)]$, then $\mathscr{L}_1(p) \succ \mathscr{L}_2(p)$;
   (c) If $\sigma_\pi[\mathscr{L}_1(p)] = \sigma_\pi[\mathscr{L}_2(p)]$, then $\mathscr{L}_1(p) \sim \mathscr{L}_2(p)$ (indifference).

It can be checked that the normalization of the PLTSs has no influences on their score values and deviation degrees as a result of the assigned zero probabilities, and thus do not change their ranking scheme. But, the optimism index π, which is an indicator of DM's attitudes, is substantial for the ranking of PLTSs. To highlight the point, we provide the following example.

Example 3 Let $\mathscr{L}_1(p) = \{\{s_4(0.2), s_3(0.4), s_2(0.4)\}, \{s_4(0.2), s_3(0.5), s_2(0.3)\}, \{s_4(0.3), s_3(0.4), s_2(0.3)\}\}$, which is equivalent to $L(p) = \{s_4(0.2), s_3(0.4), s_2(0.3)\}$; $\mathscr{L}_2(p) = \{\{s_4(0.4), s_3(0.4), s_1(0.2)\}, \{s_4(0.5), s_3(0.3), s_1(0.2)\}, \{s_4(0.4), s_3(0.25), s_1(0.35)\}\}$, which is equivalent to $L(p) = \{s_4(0.4), s_3(0.25), s_1(0.2)\}$, then $\text{WSV}[\mathscr{L}_1(p)] = \min\{s_{2.8}, s_{2.9}, s_3.\} = s_{2.8}$, $\text{BSV}[\mathscr{L}_1(p)] = s_3$, $\sigma_w[\mathscr{L}_1(p)] = \max\{0.7483, 0.7, 0.7746\} = 0.7746$, $\sigma_b[\mathscr{L}_1(p)] = 0.7$; $\text{WSV}[\mathscr{L}_2(p)] = \min\{s_3, s_{3.1}, s_{2.7}\} = s_{2.7}$, $\text{BSV}[\mathscr{L}_2(p)] = s_{3.1}$, $\sigma_w[\mathscr{L}_2(p)] = \max\{1.0954, 1.1358, 1.3077\} = 1.3077$, $\sigma_b[\mathscr{L}_2(p)] = 1.0954$. Then,

(1) for $\pi = 0$, $\mathscr{L}_1(p) \succ \mathscr{L}_2(p)$ since $\text{WSV}[\mathscr{L}_1(p)] > \text{WSV}[\mathscr{L}_2(p)]$;
(2) for $\pi = 0.5$ (a moderate attitude), $\mathscr{L}_1(p) \succ \mathscr{L}_2(p)$ since $SV_\pi[\mathscr{L}_1(p)] = SV_\pi[\mathscr{L}_2(p)] = 2.9$ and $\sigma_\pi[\mathscr{L}_1(p)] = 0.7373 < \sigma_\pi[\mathscr{L}_2(p)] = 1.2016$;
(3) for $\pi = 1$, $\mathscr{L}_1(p) \prec \mathscr{L}_2(p)$ since $\text{BSV}[\mathscr{L}_1(p)] > \text{BSV}[\mathscr{L}_2(p)]$.

Definition 5 Let $\mathscr{L}_1(p) = \{L_{j_1}(p) \mid j_1 = 1, 2, \ldots, k_1\}$ with $L_{j_1}(p) = \{s_{\alpha_i}(p_{ij_1}) \mid s_{\alpha_i} \in S, i = 1, 2, \ldots, m, p_{ij_1} \geq 0, \sum_{i=1}^{m_1} p_{ij_1} = 1.\}$; and $\mathscr{L}_2(p) = \{L_{j_2}(p) | j_2 = 1, 2, \ldots, k_2.\}$ with $L_{j_2}(p) = \{s_{\alpha_i}(p_{ij_2}) | s_{\alpha_i} \in S, i = 1, 2, \ldots, m, p_{ij_2} \geq 0, \sum_{i=1}^{m_2} p_{ij_2} = 1.\}$ be two PLTSs with the same set of ascending ordered linguistic terms, then the distance between $\mathscr{L}_1(p)$ and $\mathscr{L}_2(p)$ is defined by

$$d[\mathscr{L}_1(p), \mathscr{L}_2(p)] = \max_{1 \leq j_1 \leq k_1} \max_{1 \leq j_2 \leq k_2} \sqrt{\frac{\sum_{i=1}^{m} (\alpha_i p_{ij_1} - \alpha_i p_{ij_2})^2}{m}}. \quad (4)$$

Here, since we can assign zero probabilities to the unshared linguistic terms without influencing their ranking, the two PLTSs are assumed to have the same set of ordered linguistic terms. In addition, we have the following properties for the distance between PLTSs.

Theorem 1 *Let $\mathscr{L}_1(p)$, $\mathscr{L}_2(p)$ be two PLTSs by (3), then it holds that,*

1. $d[\mathscr{L}_1(p), \mathscr{L}_2(p)] = 0$ *if and only if* $\mathscr{L}_1(p) = \mathscr{L}_2(p)$ *in the sense that both of the PLTSs has the same probability distributions (for each* $1 \leq j_1 \leq k_1, 1 \leq j_2 \leq k_2$, $p_{ij_1} = p_{ij_2}$ $(\forall i = 1, 2, \ldots, m)$*)*.
2. $d[\mathscr{L}_1(p), \mathscr{L}_2(p)] = d[\mathscr{L}_2(p), \mathscr{L}_1(p)]$.

The operations over the reformulated PLTSs

In every field, information aggregation relies extensively on the underlying operations. PLTSs are no exceptions, since their introduction, the basic operations have been one of the hot topics in the area. The most recent work by Mi et al. [37] provides an extensively comprehensive summary on the topic. The typical operations are based on the crisp values of the linguistic terms by appropriate transformation, usually the (scaled) subscripts of the linguistic terms, such as the operations in [17,38,50].[1] The pertaining transformation may lead to information loss of the intrinsic linguistic implications, but intuitive and simple for applications. Here, the operations for the reformulated PLTSs are defined over the subscripts and the underlying probabilities in an element-by-element way using one of the important aggregation operators—t-(co)norms [24,35,48], whose notations are provided in Appendix.

Definition 6 (*T-(co)norm-based operations for PLTSs*) Let the linguistic terms set be $S = \{s_i \mid i = 0, 1, \ldots, n.\}$, and $\mathscr{L}(p) = \{L_j(p) | j = 1, 2, \ldots, k\}$ with $L_j(p) = \{s_{\alpha_i}(p_{ij}) | s_{\alpha_i} \in S, i = 1, 2, \ldots, m, p_{ij} \geq 0, \sum_{i=1}^{m} p_{ij} = 1.\}$; $\mathscr{L}_1(p) = \{L_{j_1}(p) | j_1 = 1, 2, \ldots, k_1\}$ with $L_{j_1}(p) = \{s_{\alpha_{i_1}}(p_{i_1 j_1}) | s_{\alpha_{i_1}} \in S, i_1 = 1, 2, \ldots, m_1, p_{i_1 j_1} \geq 0, \sum_{i=1}^{m_1} p_{i_1 j_1} = 1.\}$; and $\mathscr{L}_2(p) = \{L_{j_2}(p) | j_2 = 1, 2, \ldots, k_2.\}$ with $L_{j_2}(p) = \{s_{\alpha_{i_2}}(p_{i_2 j_2}) | s_{\alpha_{i_2}} \in S, i_2 = 1, 2, \ldots, m_2, p_{i_2 j_2} \geq 0, \sum_{i=1}^{m_2} p_{i_2 j_2} = 1.\}$ be three PLTSs, $\mu \geq 0$, $T(C)$ is an Archimedean t-(co)norm with generator $t(c)$, then

(i) $\mathscr{L}_1(p) \oplus \mathscr{L}_2(p) = \{L_{j_1 j_2}(p) | j_1 = 1, 2, \ldots, k_1; j_2 = 1, 2, \ldots, k_2.\}$, where $L_{j_1 j_2}(p) = \{s_\gamma \left(p_{i_1 j_1} p_{i_2 j_2}\right) | \gamma = nC(\frac{\alpha_{i_1}}{n}, \frac{\alpha_{i_2}}{n}) = nc^{-1}(c(\frac{\alpha_{i_1}}{n}) + c(\frac{\alpha_{i_2}}{n})), i_1 = 1, 2, \ldots, m_1; i_2 = 1, 2, \ldots, m_2.\}$;
(ii) $\mu \mathscr{L}(p) = \{L_j(p) | j = 1, 2, \ldots, k.\}$ with $L_j(p) = \{s_\gamma(p_{ij}) | \gamma = nc^{-1}(\mu c(\frac{\alpha_i}{n})), i = 1, 2, \ldots, m\}$;
(iii) $\mathscr{L}_1(p) \odot \mathscr{L}_2(p) = \{L_{j_1 j_2}(p) | j_1 = 1, 2, \ldots, k_1; j_2 = 1, 2, \ldots, k_2.\}$, where $L_{j_1 j_2}(p) = \{s_\gamma \left(p_{i_1 j_1} p_{i_2 j_2}\right) | \gamma = nT(\frac{\alpha_{i_1}}{n}, \frac{\alpha_{i_2}}{n}) = nt^{-1}(t(\frac{\alpha_{i_1}}{n}) + t(\frac{\alpha_{i_2}}{n})), i_1 = 1, 2, \ldots, m_1; i_2 = 1, 2, \ldots, m_2.\}$;
(iv) $\mathscr{L}^\mu(p) = \left\{L_j(p) \middle| j = 1, 2, \ldots, k.\right\}$ with $L_j(p) = \{s_\gamma(p_{ij}) | \gamma = nt^{-1}(\mu t(\frac{\alpha_i}{n})), i = 1, 2, \ldots, m\}$.

By these definitions, the operations are closed in PLTSs by (3). Take $\mathscr{L}_1(p) \oplus \mathscr{L}_2(p)$ as an example, for each of the obtained elements $L_{j_1 j_2}(p)$, it holds that $p_{i_1 j_1} p_{i_2 j_2} \geq 0$, and $\sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} p_{i_1 j_1} p_{i_2 j_2} = \sum_{i_1=1}^{m_1} p_{i_1 j_1} \sum_{i_2=1}^{m_2} p_{i_2 j_2} = 1$, which implies $\mathscr{L}_1(p) \oplus \mathscr{L}_2(p)$ is a reformulated PLTS. In fact, these operations proceed with the two separate operations over the linguistic terms and the probabilities. For the operations $\odot$ and $\oplus$, by the definition, it may lead to repeated linguistic terms, which can be combined by summing up the pertaining probabilities. For illustration, we will only use the Frank and Hamacher t-(co)norms in the further analysis. Moreover, the following properties of these operations, i.e., symmetry, distributivity and associativity, can be

[1] There are some other advanced operations based on the Dempster–Shafer evidence theory [26] and the fuzzy set theory [22].

derived with the properties of the underlying t-(co)norms or their generators.

Theorem 2 *Let $\mathscr{L}(p), \mathscr{L}_i(p)(i = 1, 2, 3)$ be PLTSs by (3), $\mu, \mu_1, \mu_2 \geq 0$, then it holds that,*

(1) $\mathscr{L}_1(p) \oplus \mathscr{L}_2(p) = \mathscr{L}_2(p) \oplus \mathscr{L}_1(p)$, $\mathscr{L}_1(p) \odot \mathscr{L}_2(p) = \mathscr{L}_2(p) \odot \mathscr{L}_1(p)$;

(2) $\mu(\mathscr{L}_1(p) \oplus \mathscr{L}_2(p)) = \mu(\mathscr{L}_1(p)) \oplus \mu(\mathscr{L}_2(p))$; $(\mathscr{L}_1(p) \odot \mathscr{L}_2(p))^{\mu} = (\mathscr{L}_1(p))^{\mu} \odot (\mathscr{L}_2(p))^{\mu}$;

(3) $\mu_1\mathscr{L}(p) \oplus \mu_2\mathscr{L}(p) = (\mu_1 + \mu_2)\mathscr{L}(p)$; $\mathscr{L}^{\mu_1}(p) \oplus \mathscr{L}^{\mu_2}(p) = \mathscr{L}^{\mu_1+\mu_2}(p)$;

(4) $(\mathscr{L}_1(p) \oplus \mathscr{L}_2(p)) \oplus \mathscr{L}_3(p) = \mathscr{L}_1(p) \oplus (\mathscr{L}_2(p) \oplus \mathscr{L}_3(p))$, $(\mathscr{L}_1(p) \odot \mathscr{L}_2(p)) \odot \mathscr{L}_3(p) = \mathscr{L}_1(p) \odot (\mathscr{L}_2(p) \odot \mathscr{L}_3(p))$.

Proof The results can be trivially obtained by Definition 6 and the properties of the underlying t-(co)norms. □

By these operations for PLTSs in Definition 6, we can define some average operators for information aggregations, such as the weighted average and geometric average operators for PLTSs.

Definition 7 Let $\mathscr{L}_i(p)(i = 1, 2, \ldots, k)$ be PLTSs, then the PLTS weighted average(PLTSWA) and geometric average (PLTSWGA) operators are given, respectively, as follows:

$$\text{PLTSWA}(\mathscr{L}_1(p), \mathscr{L}_2(p), \ldots, \mathscr{L}_k(p)) = \bigoplus_{i=1}^{k} w_i \mathscr{L}_i(p), \quad (5)$$

$$\text{PLTSWGA}(\mathscr{L}_1(p), \mathscr{L}_2(p), \ldots, \mathscr{L}_k(p)) = \bigodot_{i=1}^{k} \mathscr{L}_i(p)^{w_i} \quad (6)$$

where

$$\sum_{i=1}^{k} w_i = 1, \quad w_i \in [0, 1]. \quad (7)$$

MADM using the reformulated PLTSs

In this part, to illustrate the feasibility of the PLTS based method in applications, we use the reformulated PLTSs to provide a PLTS DEMATEL-based frameworks for MADM.

As usual, we assume $\mathbf{X} = \{x_1, x_2, \ldots, x_m\}$ is a set of alternatives, $\mathbf{A} = \{a_1, a_2, \ldots, a_n\}$ is a set of attributes with the attribute weighting vector $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$. The DM provides the evaluation of the alternative x_i on attribute a_j using the PLTS $\mathscr{L}_{ij}(p)$, then the decision matrix can be presented as follows.

$$\mathbf{R} = \begin{pmatrix} \mathscr{L}_{11}(p) & \mathscr{L}_{12}(p) & \ldots & \mathscr{L}_{1n}(p) \\ \mathscr{L}_{21}(p) & \mathscr{L}_{22}(p) & \ldots & \mathscr{L}_{2n}(p) \\ \vdots & \vdots & & \vdots \\ \mathscr{L}_{m1}(p) & \mathscr{L}_{m2}(p) & \ldots & \mathscr{L}_{mn}(p) \end{pmatrix}_{m\times n}, \quad (8)$$

where $\mathscr{L}_{ij}(p) = \{L_{ij}^{(j_1)} | j_1 = 1, 2, \ldots, k_1\}$ with $L_{ij}^{(j_1)} = \{s_{\alpha_i}(p_{ij}^{(i_1 j_1)}) | s_{\alpha_i} \in S, i = 1, 2, \ldots, m_{ij}, p_{ij}^{(i_1 j_1)} \geq 0, \sum_{i=1}^{m_{ij}} p_{ij}^{(i_1 j_1)} = 1\}$. In the following, we present the DEMATEL-based aggregation method for decision-making.

Some essentials on the classical DEMATEL

DEMATEL was introduced in 1970s [15,44] to provide solutions to the interrelated and complicated problems with causal relationships between complex attributes. The method is usually used to visualize the complex causal relationships by separating the attributes into cause and effect groups, and present the influence degrees of each attributes. Thus, the DEMATEL method is also employed to prioritize alternatives for decision-making.

The linguistic terms scale for pair-wise comparison is assumed to be linguistic terms, such as 'None influence', 'Low influence', 'Moderate influence', 'High influence', and 'Extreme high influence', which are denoted by 0, 1, 2, 3, 4, 5. To facilitate the operations over matrices, here we omit the prefix s for the linguistic terms and directly use the numerical values. The initial direct relation matrix Z over n attributes is a matrix with order n and can be obtained by pair-wise comparison in terms of influences and directions between attributes. The elements z_{ij} is perceived to be the degree to which the attribute a_i affects attribute a_j and the diagonal elements $z_{ii} = 0$. That is

$$Z = \begin{matrix} \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{matrix} \begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ 0 & z_{21} & \ldots & z_{n1} \\ z_{12} & 0 & \ldots & z_{n2} \\ \vdots & \vdots & & \vdots \\ z_{1n} & z_{2n} & \ldots & 0 \end{pmatrix}. \quad (9)$$

By the initial direct relation matrix Z, we can obtain the normalized direct relation matrix

$$X = \frac{1}{\max_{1\leq i\leq n}\{\sum_{j=1}^{n} z_{ij}\}} Z,$$

such that

- $\lim_{t\to\infty} X^t = O$ (the null matrix);
- $\lim_{t\to\infty}(E + X + X^2 + \cdots + X^t) = (E - X)^{-1}$, where E is the identity matrix.

In general, the normalized direct relation matrix can be seen as a sub-stochastic matrix obtained from an absorbing Markov chain matrix by deleting all rows and columns associated with the absorbing states. The total relation matrix T can be calculated by the following equation:

$$T = \lim_{t \to \infty} (X + X^2 + \cdots + X^t) = X(E - X)^{-1}.$$

By the equation, it can be seen that the element τ_{ij} of T represents the overall degree to which the attribute a_i affects attribute a_j. Thus, $c_i = \sum_{j=1}^{m} \tau_{ij}$ is the overall degree to which the attribute a_i affects the other attribute; $h_j = \sum_{i=1}^{m} \tau_{ij}$ is the overall degree to which the attribute a_j is affected by the other attributes. The prominence P_i of the attribute a_i is defined by $P_i = c_i + h_i$, which is an indicator of the importance degree of the attribute a_i. The larger the individual prominence p_i, the more important the attribute a_i. The relation r_i of the criterion C_i is defined by $R_i = c_i - h_i$, which is used to distinguish cause or effect attribute. If $R_i > 0$, then the attribute a_i belongs to the cause group; If $R_i < 0$, then the attribute a_i belong to the effect group. A cause diagram is obtained by mapping the ordered pairs of (P_i, R_i) in the prominence-relation space, can be used to visualize the complicated causal relationships among criteria in a visible way to facilitate reasonable decision-making.

DEMATEL has the capacity in exploring the overall effect (influence) of factors, has been successfully employed in various problems in different fields. In the recent, the methodology about DEMATEL has been deepen or strengthen by the facilitating of various scenarios [2,4,6,8,18,20,27,46] and the combinations with the pertaining methodologies such as the analytic network process (ANP) [47], Association Rule Mining (ARM) [1], and so on. A newly emerging direction is about the DEMATEL with both subjectivity and objectivity [10,11]. In [11], with the basic belief assignment function and Dezert–Smarandache theory in deriving the group initial direct relation matrix, group DEMATEL with the combination of the subjective (experts') and the objective (pair-factors') assessments can reach a prescribed consensus level with some feedback mechanism. Notably, the linguistic DEMATEL [8,20,46] methods are shown to be much effective and reasonable in applications.

PLTS-based DEMATEL method

As an important direction of PLTS method [30], DEMATEL under PLTS is necessary. Here we introduce the reformulated PLTS-based DEMATEL method on a step-by-step basis.

STEP 1. Formulation of the PLTS-based initial direct relation matrix. It is assumed that, in the pair-wise comparison of attributes PLTSs are used. We can have a PLTS-based initial direct relation matrix

$$\mathbf{Z} = \begin{pmatrix} \mathscr{L}_{11}(p) & \mathscr{L}_{12}(p) & \ldots & \mathscr{L}_{1m}(p) \\ \mathscr{L}_{21}(p) & \mathscr{L}_{22}(p) & \ldots & \mathscr{L}_{2m}(p) \\ \vdots & \vdots & & \vdots \\ \mathscr{L}_{m1}(p) & \mathscr{L}_{m2}(p) & \ldots & \mathscr{L}_{mm}(p) \end{pmatrix},$$

where $\mathscr{L}_{ij}(p) = \{L_{k_{ij}}^{(ij)} | k_{ij} = 1, 2, \ldots, l_{ij}.\}$ with $L_{k_{ij}}^{(ij)} = \{\alpha_{t_{ij}}^{(ij)}(p_{k_{ij}t_{ij}}^{(ij)}) | t_{ij} = 1, 2, \ldots, \tau_{ij}\}$ and in particular $\mathscr{L}_{ii}(p) = \{0\}$, i.e., $l_{ii} = \tau_{ii} = 1$, $p_{k_{ii}t_{ii}}^{(ii)} = 1$ and $\alpha_{t_{ii}}^{(ii)} = 0$.

STEP 2. Derivation of the segmental initial direct relation matrices with probability. For each $t_{ij} \in \{1, 2, \ldots, \tau_{ij}\}$ $(i, j = 1, 2, \ldots, m)$, we can get a series of segmental initial direct relation matrices with probability

$$\prod_{i=1}^{m} \prod_{j=1}^{m} \left(\frac{1}{l_{ij}} \sum_{k_{ij}=1}^{l_{ij}} p_{k_{ij}t_{ij}}^{(ij)} \right),$$

which are lexicographically structured by the linguistic terms $\alpha_{t_{ij}}^{(ij)}$ from the PLTSs $\mathscr{L}_{ij}(p)$ and have the form as follows.

$$Z_p^{(I)} = \begin{pmatrix} \alpha_{t_{11}}^{(11)} & \alpha_{t_{12}}^{(12)} & \ldots & \alpha_{t_{1m}}^{(1m)} \\ \alpha_{t_{21}}^{(21)} & \alpha_{t_{22}}^{(22)} & \ldots & \alpha_{t_{2m}}^{(2m)} \\ \vdots & \vdots & & \vdots \\ \alpha_{t_{m1}}^{(m1)} & \alpha_{t_{m2}}^{(m2)} & \ldots & \alpha_{t_{mm}}^{(mm)} \end{pmatrix}_{p^{(I)}},$$

where $\alpha_{t_{ii}}^{(ii)} = 0$ and the subscript $p^{(I)} = \prod_{i=1}^{m} \prod_{j=1}^{m} \left(\frac{1}{l_{ij}} \sum_{k_{ij}=1}^{l_{ij}} p_{k_{ij}t_{ij}}^{(ij)} \right)$ is the potential probabilistic information,

$$I = \left\{ \begin{matrix} t_{11} & t_{12} & \ldots & t_{1m} \\ t_{21} & t_{22} & \ldots & t_{2m} \\ \vdots & \vdots & & \vdots \\ t_{m1} & t_{m2} & \ldots & t_{mm} \end{matrix} \right\}$$ is an index notation.

STEP 3. Derivation of the segmental prominences and the segmental relations. By excluding the probabilistic information, the segmental initial direct relation matrices are reduced to the classical initial direct relation matrices by Eq. (9). By the classical DEMATEL method, the corresponding total relation matrix $T_p^{(I)} = (\tau_{ij})_{p^{(I)}}$, called segmental total relation matrix hereafter, can be obtained for each segmental initial direct relation matrix. Note the segmental total relation matrix inherit the probabilistic information of the corresponding segmental initial direct relation matrix. Then the segmental prominence $P_{i,p}^{(I)} = \sum_j \tau_{ij} + \sum_j \tau_{ji}$ and the segmental relation $R_{i,p}^{(I)} = \sum_j \tau_{ij} - \sum_j \tau_{ji}$ with probability

$p^{(I)}$ for attribute C_i can be generated using these segmental total relation matrices.

STEP 4. The synthesization of the segmental prominences and the segmental relations. By the underlying probability information, we can get the synthesized prominence and the synthesized relation, respectively, by $P_i = \sum_I P_{i,p}^{(I)} p^{(I)}$ and $R_i = \sum_I R_{i,p}^{(I)} p^{(I)}$. The synthesized prominence can be seen as the comprehensive degree of an attribute to influence the others and be influenced by the others, then it provides reliable information about weighting.

STEP 5. The determination of the attribute weights. Here, the weight w_i for a_i can be calculated by

$$w_i = \frac{P_i}{\sum_{i=1}^{n} P_i}. \tag{10}$$

Note that, equivalently, we can also firstly derive a synthesized total relation matrix T by probabilistically weighting average of the segmental total relation matrices. That is

$$T = \sum_I p^{(I)} T_p^{(I)}, \tag{11}$$

from which the synthesized prominence can also be obtained.

Example 4 Assume the PLTS-based initial direct relation matrix is given by

$$\mathbf{Z} = \begin{pmatrix} 0 & \{\{3(0.5), 4(0.5)\}, \{3(0.4), 4(0.6)\}\} & 3 \\ 4 & 0 & 4 \\ \{\{4(0.3), 5(0.7)\}, \{4(0.4), 5(0.6)\}\} & 2 & 0 \end{pmatrix} \tag{12}$$

then the segmental initial direct relation matrices can be obtained as follows.

$$Z_p^{(1)} = \begin{pmatrix} 0 & 3 & 3 \\ 4 & 0 & 4 \\ 4 & 2 & 0 \end{pmatrix}_{0.45\times 0.35}, Z_p^{(2)} = \begin{pmatrix} 0 & 4 & 3 \\ 4 & 0 & 4 \\ 4 & 2 & 0 \end{pmatrix}_{0.55\times 0.35},$$

$$Z_p^{(3)} = \begin{pmatrix} 0 & 3 & 3 \\ 4 & 0 & 4 \\ 5 & 2 & 0 \end{pmatrix}_{0.45\times 0.65}, Z_p^{(4)} = \begin{pmatrix} 0 & 4 & 3 \\ 4 & 0 & 4 \\ 5 & 2 & 0 \end{pmatrix}_{0.55\times 0.65}.$$

By the classical DEMATEL method, we can get the segmental total relation matrix for each segmental initial direct relation matrix, i.e.,

$$T_p^{(1)} = \begin{pmatrix} 1.4348 & 1.3043 & 1.5652 \\ 2.0870 & 1.2609 & 1.9130 \\ 1.7391 & 1.2174 & 1.2609 \end{pmatrix}_{0.45\times 0.35}$$

$$T_p^{(2)} = \begin{pmatrix} 2.2941 & 2.2353 & 2.3529 \\ 2.8235 & 2.0588 & 2.5882 \\ 2.3529 & 1.8824 & 1.8235 \end{pmatrix}_{0.55\times 0.35}$$

$$T_p^{(3)} = \begin{pmatrix} 2.0270 & 1.6216 & 1.9459 \\ 2.8108 & 1.6486 & 2.3784 \\ 2.5946 & 1.6757 & 1.8108 \end{pmatrix}_{0.45\times 0.65}$$

$$T_p^{(4)} = \begin{pmatrix} 3.6667 & 3.1667 & 3.3333 \\ 4.3333 & 3.0833 & 3.6667 \\ 4.0000 & 3.0000 & 3.0000 \end{pmatrix}_{0.55\times 0.65}.$$

Table 2 The segmental and synthesized prominence for each attributes with probability

Probability attributes	a_1	a_2	a_3
0.45 × 0.35	9.5652	9.0435	8.9565
0.55 × 0.35	14.3528	13.6470	12.8234
0.45 × 0.65	13.0269	11.7837	12.2162
0.55 × 0.65	22.1667	20.3333	20.0000
The synthesized prominence	16.0045	14.7673	14.6024

The obtained segmental and synthesized prominence for each attributes are listed in Table 2. The synthesized prominence can also be obtained from the synthesized total relation matrix $T = \begin{pmatrix} 2.5713 & 2.2421 & 2.4603 \\ 3.2436 & 2.1794 & 2.8060 \\ 2.9158 & 2.1167 & 2.1518 \end{pmatrix}$ by (11).

Then by (10), we can obtain the weights for a_1, a_2, a_3 are $w_1 = 0.3521$, $w_2 = 0.3255$, $w_3 = 0.3224$.

The DEMATEL-based MADM method

With the above DEMATEL method, a PLTS-based MADM method can be obtained as follows.

STEP 1: The DMs or a group of experts provide the decision matrix, where each element by the reformulated PLTS is the evaluation of an alternative on some attribute, and the initial direct relation matrix **R** by (8) using the reformulated PLTSs.

STEP 2: Obtain the attribute weights $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ by the above DEMATEL method based on the reformulated PLTSs.

STEP 3: Aggregate the overall attribute values $Z_i(w)$ for each alternative x_i with the obtained attribute weights $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$. That is $Z_i(w) = \mathrm{PLTSWA}(\mathscr{L}_{i1}(p), \mathscr{L}_{i2}(p), \ldots, \mathscr{L}_{in}(p)) = \oplus_{j=1}^{n} w_j \mathscr{L}_{ij}(p)$, or $Z_i(w) = \mathrm{PLTSWGA}(\mathscr{L}_{i1}(p), \mathscr{L}_{i2}(p), \ldots, \mathscr{L}_{in}(p)) = \odot_{j=1}^{n} \mathscr{L}_{ij}^{w_j}(p)$. Using the ranking scheme based on the score values and deviation degrees of the overall attribute values by Definition 4 $Z_i(w)$, we can obtain the rank of the alternatives x_is. Then the best alternative can be obtained by the ranking according to the attitudes of the DMs by the optimism index.

Illustrative case analysis

In this part, we provide an illustrative example, which is adapted from [38,39] on the feasibility and the validity of the reformulated PLTS-based applications.

Table 3 The PLTS decision matrix by the director board

	a_1	a_2	a_3	a_4
x_1	$\{s_3(0.4), s_4(0.6)\}$	$\{s_2(0.2), s_4(0.8)\}$	$\{s_3(0.2), s_4(0.8)\}$	$\{s_3(0.4), s_5(0.6)\}$
x_2	$\{s_3(0.8, s_5(0.2))\}$	$\{\{s_2(0.2), s_3(0.6), s_4(0.2)\}, \{s_2(0.3), s_3(0.4), s_4(0.3)\}\}$	$\{\{s_1(0.2), s_2(0.6), s_3(0.2)\}, \{s_1(0.25), s_2(0.4), s_3(0.35)\}\}$	$\{s_3(0.8), s_4(0.2)\}$
x_3	$\{s_3(0.6), s_4(0.4)\}$	$\{\{s_3(0.6), s_4(0.4)\}, \{s_3(0.8), s_4(0.2)\}\}$	$\{\{s_3(0.2), s_4(0.4), s_5(0.4)\}, \{s_3(0.4), s_4(0.2), s_5(0.4)\}, \{s_3(0.4), s_4(0.4), s_5(0.2)\}\}$	$\{s_4(0.8), s_6(0.2)\}$

Table 4 The PLTS initial direct relation matrix for the attributes by the director board[a]

	a_1	a_2	a_3	a_4
a_1	0	$\{\{2(0.8), 3(0.2)\}, \{2(0.6), 3(0.4)\}\}$	$\{\{2(0.5), 3(0.5)\}, \{2(0.4), 3(0.6)\}\}$	$\{3(0.6), 4(0.4)\}$
a_2	$\{\{3(0.5), 4(0.5)\}, \{3(0.4), 4(0.6)\}\}$	0	$\{\{3(0.6), 4(0.4)\}, \{3(0.7), 4(0.3)\}\}$	$\{2(0.8), 3(0.2)\}$
a_3	$\{4(0.5), 5(0.5)\}$	$\{\{3(0.3), 4(0.7)\}, \{3(0.4), 4(0.6)\}\}$	0	$\{3(0.8), 4(0.2)\}$
a_4	$\{\{2(0.3), 3(0.7)\}, \{2(0.2), 3(0.8)\}\}$	5	$\{\{4(0.5), 5(0.5)\}, \{4(0.6), 5(0.4)\}\}$	0

[a] Here 0, 1, 2, 3, 4, 5, respectively, represent 'None influence', 'Low influence', 'Moderate influence', 'High influence', and 'Extreme high influence'

Case study

The example is about the selection of strategic projects which are evaluated from the attributes in investment position (a_1), profitability (a_2), investment risk (a_3) and social responsibility (a_4). A group of members from the director board are asked to present their evaluations on three possible projects $x_j (j = 1, 2, 3)$. By the DEMATEL-based MADM method, we can proceed with the selection as follows.

STEP 1: The director board works out the decision matrix using the reformulated PLTSs, which is shown in Table 3. In addition, since the weight information for these attributes are completely unknown, then to determine the attributes weighting vector the director board has to provide the initial direct relation matrix (see Table 4) in PLTS form to facilitate the proposed DEMATEL method.

STEP 2: The determination of the attribute weights by the PLTS-based DEMATEL method. First, with the PLTSs based initial direct relation matrix we can obtain 2^{11} segmental initial direct relation matrices[2] coupled with probabilities, from which the corresponding segmental total relation matrices can be calculated. By calculation, the synthesized total relation matrix $T = \begin{pmatrix} 1.0068 & 1.1717 & 1.1410 & 1.0807 \\ 1.2957 & 1.0593 & 1.2313 & 1.0671 \\ 1.5327 & 1.4826 & 1.2006 & 1.2832 \\ 1.5300 & 1.6363 & 1.5523 & 1.1397 \end{pmatrix}$. Then the synthesized prominences for the attributes a_1, a_2, a_3, a_4 are, respectively, 9.7654, 10.0033, 10.6243, 10.4290. Thus, the weighting vector for these attributes is $w = (0.2392, 0.2450, 0.2603, 0.2555)^{\mathrm{T}}$.

STEP 3: With the obtained weighting vector, we can use the PLTSWA or PLTSWGWA operator to get the overall attribute value $Z_i(w)$ for each alternative a_i. Here we can choose Frank or Hamacher t-(co)norms with different parameters for these underlying operations, and we can also use different optimism indices $\pi \in [0, 1]$ in the ranking of the evaluations. The evaluating results by the method with the parameterized Frank t-(co)norms under different optimism indices are listed in Table 5.

Remarks on the results

For the PLTSWA-based results, the ranking results are robust to the parameters of Frank t-(co)norm and even to the optimism indices, which suggests a strong compensation effect of the PLTSWA operator analogous to the classical weighted average. The compensation effect is the very point to determine the choice of PLTSWA or PLTSGWA. The PLTSWA works much better when compensation is allowable. In a situation where extreme results may lead to tremendous losses and compensation is unreasonable, it is recommended to use the PLTSGWA. Here the problem is to consider the selection of strategic projects in the long run, thus the results with PLTSGWA are more reliable.

For the PLTSGWA-based results, we can see the optimism indices affect the ranking results while the parameters for the Frank t-(co)norms do not. For the optimistic DM, the best alternative is a_3, while for the less optimistic or pessimistic, the best selection is a_1. In both cases, the parameter for the Frank t-(co)norms have little influence on the results, thus from the angle of the complexity of the calculation, it is

[2] Here, we can see there would be a disastrously large number of segmental initial direct relation matrices in a large sized attributes, leading to computational complexity. Therefore, the method is much more practically reasonable in problems with a small number of attributes.

Table 5 The evaluating results by the method with the θ-parameterized Frank t-(co)norms under different optimism indices π

	PLTSWA				PLTSGWA			
	a_1	a_2	a_3	The ranking results	a_1	a_2	a_3	The ranking results
$\pi = 1$								
$\theta = 0.2$	3.9434	3.0738	4.2943	$a_3 \succ a_1 \succ a_2$	3.7086	2.7475	3.7617	$a_3 \succ a_1 \succ a_2$
$\theta = 0.6$	3.9215	3.0413	4.2806	$a_3 \succ a_1 \succ a_2$	3.733	2.7796	3.7886	$a_3 \succ a_1 \succ a_2$
$\theta = 1$[a]	3.9119	3.0274	4.2748	$a_3 \succ a_1 \succ a_2$	3.743	2.7934	3.7997	$a_3 \succ a_1 \succ a_2$
$\theta = 2$	3.8997	3.01	4.2673	$a_3 \succ a_1 \succ a_2$	3.7551	2.8106	3.813	$a_3 \succ a_1 \succ a_2$
$\pi = 0.5$								
$\theta = 0.2$	3.9434	3.058	4.2252	$a_3 \succ a_1 \succ a_2$	3.7086	2.7315	3.6869	$a_1 \succ a_3 \succ a_2$
$\theta = 0.6$	3.9215	3.026	4.2124	$a_3 \succ a_1 \succ a_2$	3.733	2.7645	3.713	$a_1 \succ a_3 \succ a_2$
$\theta = 1$[a]	3.9119	3.0124	4.207	$a_3 \succ a_1 \succ a_2$	3.743	2.7786	3.7238	$a_1 \succ a_3 \succ a_2$
$\theta = 2$	3.8997	2.9953	4.2	$a_3 \succ a_1 \succ a_2$	3.7551	2.7962	3.7368	$a_1 \succ a_3 \succ a_2$
$\pi = 0$								
$\theta = 0.2$	3.9434	3.0421	4.1561	$a_3 \succ a_1 \succ a_2$	3.7086	2.7155	3.6122	$a_1 \succ a_3 \succ a_2$
$\theta = 0.6$	3.9215	3.0108	4.1442	$a_3 \succ a_1 \succ a_2$	3.733	2.7493	3.6373	$a_1 \succ a_3 \succ a_2$
$\theta = 1$[a]	3.9119	2.9974	4.1392	$a_3 \succ a_1 \succ a_2$	3.743	2.7638	3.6478	$a_1 \succ a_3 \succ a_2$
$\theta = 2$	3.8997	2.9806	4.1328	$a_3 \succ a_1 \succ a_2$	3.7551	2.7818	3.6606	$a_1 \succ a_3 \succ a_2$

[a] The corresponding t-(co)norm is the product t-(co)norm

recommend to use $\theta = 1$, for which case the t-(co)norm is the product t-(co)norm. We can get the similar results (see Table 6) with the Hamacher t-(co)norms, which include the Einstein t-(co)norm ($\delta = 0$). Practically, it is enough to employ some typical t-(co)norms such as the product or Einstein t-(co)norms.

Note that, for the alternative a_1, since in the decision matrix the evaluations in each attributes have a complete belief or knowledge of the probabilities, thus the score values for the overall attributes evaluations are free of the DMs' attitudes. Meanwhile, for the alternatives a_2 and a_3 with partial probabilities, the score values for their overall evaluations are subject to changes in the DMs' attitudes modeled by the optimism index. Therefore, the reformulated PLTSs in the multiple probability distributions has the advantage in facilitating to incorporate the attitudes of the DM for problem solutions.

Comparison analysis

In this part, we provide some comparison results of the case study. Since the reformulated PLTSs by (3) are employed in the work, to compare with the methods by the classical PLTSs by Pang et al. [38], we would equivalently transform the reformulated PLTSs in the dataset into the classical ones, which then may be subject to the normalization, and then proceed with the analysis with some corresponding methods to obtain the ranking results of the example. By the above justifications, we refer to the results using the product t-(co)norm-based PLTSGWA operator in the comparison (see Item 5 in Table 7).

In [38], the extended TOPSIS to rank the alternatives is dependent on their the closeness coefficients and the maximum deviation-based aggregation method relies on the aggregation operator PLWA where the attribute weight is gotten by the maximum deviation method. The ranking results by the two methods are all $a_1 \succ a_3 \succ a_2$ (see Items 1 and 2 in Table 7). The results are consistent with those results of the work with the optimism index $\pi = 1$ and 0.5.

Also, we can use some existing basic PLTS operations in the maximum deviation method [38]. If we use the novel operations on the basis of scaled subscripts of linguistic by Gou and Xu [17], then the ranking result is $a_3 \succ a_1 \succ a_2$ (see Item 3 in Table 7), which is consistent with the result of the work with the optimism index $\pi = 0$. In [35], Liu and Teng developed the Archimedean t-(co)norm-based operations of PLTSs, on the basis of which the Muirhead mean operations (PLAMM) are introduced. If we use the product t-(co)norm-based PLAMM with parameter vector $P = (1, 1, 0, 0)$ (i.e., the probabilistic linguistic Bonferroni mean operator), then it can be obtained that $a_1 \succ a_3 \succ a_2$ (see Item 4 in Table 7).

It can be seen that the results by the method in this work are consistent with those of the existing methods. But it is worthy noting that, normalizations are routinely undertaken over the classical PLTSs before the operations, which is necessarily leads to information loss (i.e., the partial probabilistic information or the ignorance of probability, which are the vital feature of PLTSs, is eliminated). By contrast, through the representation by multiple probabilities over the poten-

Table 6 The evaluating results by the method with the δ-parameterized Hamacher t-(co)norms under different optimism indices π

	PLTSWA				PLTSGWA			
	a_1	a_2	a_3	The ranking results	a_1	a_2	a_3	The ranking results
$\pi=1$								
$\delta=0$[a]	4.022	3.1473	4.3435	$a_3 \succ a_1 \succ a_2$	3.6784	2.6641	3.7387	$a_3 \succ a_1 \succ a_2$
$\delta=0.2$	3.9727	3.1012	4.3128	$a_3 \succ a_1 \succ a_2$	3.6985	2.7162	3.7554	$a_3 \succ a_1 \succ a_2$
$\delta=0.6$	3.9315	3.0539	4.2869	$a_3 \succ a_1 \succ a_2$	3.7252	2.7666	3.7808	$a_3 \succ a_1 \succ a_2$
$\delta=1$[b]	3.9119	3.0274	4.2748	$a_3 \succ a_1 \succ a_2$	3.743	2.7934	3.7997	$a_3 \succ a_1 \succ a_2$
$\delta=2$	3.8893	2.9917	4.2611	$a_3 \succ a_1 \succ a_2$	3.77	2.8277	3.832	$a_3 \succ a_1 \succ a_2$
$\pi=0.5$								
$\delta=0$[a]	4.022	3.1308	4.2685	$a_3 \succ a_1 \succ a_2$	3.6784	2.649	3.6642	$a_1 \succ a_3 \succ a_2$
$\delta=0.2$	3.9727	3.0851	4.2416	$a_3 \succ a_1 \succ a_2$	3.6985	2.7005	3.6805	$a_1 \succ a_3 \succ a_2$
$\delta=0.6$	3.9315	3.0384	4.2182	$a_3 \succ a_1 \succ a_2$	3.7252	2.7513	3.7053	$a_1 \succ a_3 \succ a_2$
$\delta=1$[b]	3.9119	3.0124	4.207	$a_3 \succ a_1 \succ a_2$	3.743	2.7786	3.7238	$a_1 \succ a_3 \succ a_2$
$\delta=2$	3.8893	2.9774	4.1942	$a_3 \succ a_1 \succ a_2$	3.77	2.8139	3.7553	$a_1 \succ a_3 \succ a_2$
$\pi=0$								
$\delta=0$[a]	4.022	3.1144	4.1934	$a_3 \succ a_1 \succ a_2$	3.6784	2.634	3.5897	$a_1 \succ a_3 \succ a_2$
$\delta=0.2$	3.9727	3.069	4.1704	$a_3 \succ a_1 \succ a_2$	3.6985	2.6847	3.6057	$a_1 \succ a_3 \succ a_2$
$\delta=0.6$	3.9315	3.023	4.1495	$a_3 \succ a_1 \succ a_2$	3.7252	2.7359	3.6299	$a_1 \succ a_3 \succ a_2$
$\delta=1$[b]	3.9119	2.9974	4.1392	$a_3 \succ a_1 \succ a_2$	3.743	2.7638	3.6478	$a_1 \succ a_3 \succ a_2$
$\delta=2$	3.8893	2.9631	4.1272	$a_3 \succ a_1 \succ a_2$	3.77	2.8	3.6786	$a_1 \succ a_3 \succ a_2$

[a] The corresponding t-(co)norm is the Einstein t-(co)norm
[b] The corresponding t-(co)norm is the product t-(co)norm

Table 7 Some results by the existing methods or operations for the case study

Items	Based operations	Methods	The ranking results
1	[38]	Extended TOPSIS [38]	$a_1 \succ a_3 \succ a_2$
2	[38]	Max deviation-based aggregation [38]	$a_1 \succ a_3 \succ a_2$
3	[17]	Max deviation-based aggregation [38]	$a_3 \succ a_1 \succ a_2$
4	[35]	Max deviation-based aggregation [38]	$a_1 \succ a_3 \succ a_2$
5	In this work	DEMATEL-based MADM[a]	$a_1 \succ a_3 \succ a_2 (\pi=1)$
			$a_1 \succ a_3 \succ a_2 (\pi=0.5)$
			$a_3 \succ a_1 \succ a_2 (\pi=0)$

[a] $\pi=1, \pi=0.5, \pi=0$, respectively, means the extreme optimistic, the moderate, the extreme pessimistic attitude

tial linguistic terms, we need not carry out the normalization on the reformulated PLTS. Therefore, the method using the reformulated PTLSs has the advantage in probability information utilization without any normalization. Moreover, we can incorporate the (optimistic, or moderate, or pessimistic) DMs' attitudes in the ranking scheme, thus consequently the obtained results can be compatible with the DMs' attitudes. However, in the proposed method by the reformulated PTLSs we have to face the complexity of the underlying calculations as a result of the dimension disaster.

Concluding remarks

Probabilistic linguistic term set (PLTS) provides a much more effective model to compute with words and to express the uncertainty in the pervasive natural language by probability information. In this paper, to avoid loss of information, we redefine the classical PLTSs by multiple probability distributions from an ambiguity perspective and present some basic operations using Archimedean t-(co)norms. Different from the classical PLTSs, the reformulated PLTSs are not necessarily normalized beforehand for further investigations. Moreover, the multiple probability distributions based PLTSs

Table 8 Families of t-(co)norms and the related generators

	Formulas for t-(co)norms	Additive generators
Hamacher t-conorm	$C_\delta^H(x,y)=\begin{cases}1, & \delta=0,\ x=y=1;\\ \frac{x+y+(\delta-2)xy}{1+(\delta-1)xy}, & \text{else},\end{cases}$	$c_\delta^H(x)=\begin{cases}\frac{x}{1-x}, & \delta=0\\ \ln\frac{\delta+(1-\delta)(1-x)}{1-x}, & \delta>0;\end{cases}$
Hamacher t-norm	$T_\delta^H(x,y)=\begin{cases}0, & \delta=x=y=0;\\ \frac{xy}{\delta+(1-\delta)(x+y-xy)}, & \text{else}.\end{cases}$	$t_\delta^H(x)=\begin{cases}\frac{1-x}{x}, & \delta=0\\ \ln\frac{\delta+(1-\delta)x}{x}, & \delta>0;\end{cases}$
Frank t-conorm	$C_\theta^F(x,y)=1-\log_\theta\left(1+\frac{(\theta^{1-x}-1)(\theta^{1-y}-1)}{\theta-1}\right)$	$c_\theta^F(x)=\begin{cases}-\ln(1-x), & \theta=1\\ -\ln\frac{\theta-1}{\theta^{1-x}-1}, & \theta\neq 1;\end{cases}$
Frank t-norm	$T_\theta^F(x,y)=\log_\theta\left(1+\frac{(\theta^{x}-1)(\theta^{y}-1)}{\theta-1}\right)$	$t_\theta^F(x)=\begin{cases}-\ln x, & \theta=1\\ -\ln\frac{\theta-1}{\theta^{x}-1}, & \theta\neq 1.\end{cases}$

facilitate the incorporation of the different attitudes of the DMs in their score values and the deviation, and thus the comparisons. In particular, the reformulated PLTSs can be incorporated in DEMATEL. In fact, the reformulation is the accurate expression in the probability information, thus takes a step forward in exploring the experts-preference and knowledge. Further, the reformulated PLTS requires no pre-stage normalization, thus does not result in any information loss in the process. Therefore, comparing to the existing DEMATEL methods under the classical PLTSs or numerical expression, the DEMATEL based on the reformulated PLTSs takes a step forward to exploring the experts' probabilistic knowledge in a more accurate way without information loss.

From an ambiguity perspective, the paper concentrates on the reformulation of the classical PLTSs, which contributes a new insight into the PLTS-based methods and applications. With the reformulated PLTSs, we can provide some new thought to eliminate the pre-stage normalization and proceed with further investigations without any information loss. Thus, some known methods (see Table 1) for PLTS decision-making and some smart extensions of PLTSs [3,23,34,53,60] could be also restructured by the reformulated PLTSs in this work. In the future work, we would attempt to investigate the extended or improved DEMATEL methods [1,10–12,47] using the reformulated PLTSs.

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Declarations

Conflict of interest The author declares no conflicts of interest.

Appendix: Introduction on triangular (co) norms

Triangular (co)norms (T-(co)norms) [24,57] are a class of symmetric, monotonic and associative operations on the unit square with neutral element 1(0) in fuzzy theory and information aggregation. A t-norm T and a t-conorm C is dual in the sense that $C(x,y)=1-T(1-x,1-y)$. In particular, Archimedean t-norms can be additively generated by their generators t, i.e., $T(x,y)=t^{-1}(\min(t(x)+t(y),t(0)))$ where the generator t is a $[0,1]\to[0,\infty]$ a continuous strictly decreasing function. Similarly, Archimedean t-conorms can also be additively generated by their generators C, i.e., $C(x,y)=c^{-1}(\min(c(x)+c(y),c(1)))$ where the generator c is a $[0,1]\to[0,\infty]$ a continuous strictly increasing function. Here we recall the notations on the Hamacher and Frank t-norms and t-conorms [24] (Table 8).

In particular, for the Hamacher t-(co)norm, if $\delta=0$, then $T_\delta^H(C_\delta^H)$ reduces to the Einstein t-(co)norm, i.e., $T_E(C_E)$; if $\delta=1$, then $T_\delta^H(C_\delta^H)$ reduces to the product t-(co)norm, i.e., $T_P(C_P)$, which is also the case for the case $\theta=1$ in the Frank t-(co)norm.

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