

# Improvement of Proportional Conflict Redistribution Fusion Rules for Levee Characterization

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Levee security assessment is a complex expert assessment process based on several heterogeneous data. In our previous research works, we applied information fusion techniques to characterize flood protection levees. We used the proportional conflict redistribution rule no.6 (PCR6) proposed in DS<sub>m</sub>T (Dezert Smarandache Theory) framework to combine data from geotechnical and geophysical investigation methods. However, in some cases, this rule can generate non satisfactory results. Indeed, the uncertainty between several hypotheses (lithological materials) is overestimated after the fusion process, which is detrimental to decision making in the end. This result occurs because the PCR6 rule does not preserve the neutrality of the vacuous belief assignment, which can be judged as being a counter-intuitive behavior. To overcome this problem we present an improved rule that preserves the neutrality of vacuous belief assignments in the fusion process. Hence, the redistribution of the partial conflict masses using this new rule does not overestimate the masses associated with partial uncertainties. To illustrate the use of this new fusion rule in a levee characterization problematic, we simulate data acquisition. Two geophysical investigation campaigns (electrical resistivity tomography and multi-channel analysis of surface waves methods) and a geotechnical acquisition campaign (core drillings with particle size analysis) are numerically simulated on an earthen structure. The objective is to compare and discuss the fusion results obtained using this new rule with respect to the methodology based on the original PCR6 rule as well as to demonstrate the enhancement of the levee characterization.

**Keywords:** Belief functions, levee, cross-disciplinary approach, natural hazards, fusion rules, risk management, proportional conflict redistribution rule.

## 1. Introduction

This work is part of a problematic of levee characterization for flood protection. Indeed, these hydraulic works are mostly old and heterogeneous and their rupture can lead to disastrous consequences such as human, economic and environmental losses. Since many different materials and construction methods exist, each flood protection embankment is unique, and the nature of its structure goes hand in hand with its environment (Sharp et al., 2013). The structures are more or less subject to breakage in weak areas under specific loads. Reducing the risk of levee rupture requires an improvement of their diagnosis and therefore to enhance their characterization. First, it relies on technical surveys able to determine if specific pathologies that could lead to failure mechanisms are present in the levee structure. Methodologies for the evaluation of these structures usually include geotechnical and geophysical investigation methods. Geophysical methods are mainly non-intrusive and provide physical information on large volumes of subsoil but with

potential significant uncertainties. Geotechnical investigation methods, on the other hand, are intrusive and provide more punctual information spatially, but also more precise. These two sets of methods are complementary. Information fusion is a helpful technique to combine geotechnical and geophysical data in a complex processing for the levee security assessment based on several heterogeneous data. The processing of the data from geophysical and geotechnical investigation methods and their fusion, taking into account their imperfections and associated spatial distributions, is an essential issue for the evaluation of earthen levees. A cross-disciplinary fusion approach for the characterization of lithological materials within the structures has recently been proposed in the mathematical framework of belief functions Dezert et al. (2019).

In this paper, we present a flawed behavior of PCR6 combination rule attributed to the non neutrality of the vacuous BBA (Basic Belief Assignment), and we propose an improvement to this rule (PCR6<sup>+</sup>) in order to ensure the neutrality

property of the vacuous BBA. This improvement helps in reducing the level of uncertainty in fusion results by discarding ignorant sources for each partial conflict. To demonstrate the pertinence and advantages of PCR6<sup>+</sup> over PCR6, we compare the obtained results for i) a simple numerical example and for ii) the fusion of simulated geophysical and geotechnical data on an earthen levee.

## 2. Belief Functions

Based on preliminary works done in Dempster (1967, 1968), Shafer has introduced the belief functions (BF) in Shafer (1976) to model epistemic uncertainty, to reason about uncertainty and to combine uncertain information. The theory of belief functions is also known as Dempster-Shafer Theory (DST) in the literature. We assume that the answer<sup>a</sup> of the problem under concern belongs to a known (or given) finite discrete frame of discernment (FoD)  $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$ , with  $n > 1$ , and where all elements of  $\Theta$  are exhaustive and exclusive<sup>b</sup>. The set of all subsets of  $\Theta$  (including empty set  $\emptyset$ , and  $\Theta$ ) is the power-set of  $\Theta$  denoted by  $2^\Theta$ . The number of elements (i.e. the cardinality) of  $2^\Theta$  is  $2^{|\Theta|}$ . In this section we recall the main definitions related with BF and introduce briefly the conjunctive and Dempster-Shafer rules of combinations.

### 2.1. Main Definitions

A (normal) basic belief assignment (BBA) associated with a given source of evidence is a mapping  $m(\cdot) : 2^\Theta \rightarrow [0, 1]$  satisfying  $m(\emptyset) = 0$  and  $\sum_{A \in 2^\Theta} m(A) = 1$ . The real number  $m(A)$  is called the mass of  $A$  committed by the source of evidence. The subset  $A \in 2^\Theta$  is called a focal element (FE) of the BBA  $m(\cdot)$  if and only if  $m(A) > 0$ . The set of all the focal elements of a BBA  $m(\cdot)$  is denoted  $\mathcal{F}_\Theta(m) = \{X \in 2^\Theta | m(X) > 0\}$ . The set  $\mathcal{F}_\Theta(m)$  has at least one focal element, and at most  $2^{|\Theta|} - 1$  focal elements because one has always  $m(\emptyset) = 0$  by the definition of a normal BBA - see Shafer (1976). Belief and plausibility functions are respectively defined from  $m(\cdot)$  by

$$Bel(A) = \sum_{X \in 2^\Theta | X \subseteq A} m(X) \quad (1)$$

$$Pl(A) = \sum_{X \in 2^\Theta | A \cap X \neq \emptyset} m(X) \quad (2)$$

where  $\bar{A}$  represents the complement of  $A$  in  $\Theta$ , that is  $\bar{A} \triangleq \Theta - \{A\} = \{X | X \in \Theta \text{ and } X \not\subseteq A\}$ .

<sup>a</sup>i.e. the solution, or the decision to take.

<sup>b</sup>This is so-called Shafer's model of FoD Smarandache and Dezert (2004).

The symbol  $\triangleq$  means *equal by definition* and the minus symbol denotes the set difference operator - see Halmos (1974); Li (1999).

$Bel(A)$  and  $Pl(A)$  are usually interpreted respectively as lower and upper bounds of an unknown (subjective) probability measure  $P(A)$ . The width  $Pl(A) - Bel(A)$  of the belief interval  $[Bel(A), Pl(A)]$  is usually called the *uncertainty on  $A$*  but it represents in fact the imprecision on the probability of  $A$  granted by the source of evidence. When all the focal elements of a BBA  $m(\cdot)$  are singletons this BBA is called a *Bayesian BBA* and its corresponding  $Bel(\cdot)$  function is equal to  $Pl(\cdot)$  and they are homogeneous to a (subjective) probability measure  $\bar{P}(\cdot)$ . The vacuous BBA (VBBA for short) representing a totally ignorant source is defined as  $m_v(\Theta) = 1$ .

### 2.2. Conjunctive Combination Rule

We consider  $S \geq 2$  distinct reliable sources of evidence characterized by their BBA  $m_s(\cdot)$  ( $s = 1, \dots, S$ ) defined on the same frame of discernment  $\Theta^c$ . Their conjunctive fusion, denoted  $\text{Conj}(m_1, m_2, \dots, m_S)$ , corresponds to a (non proper) BBA defined for all  $A \in 2^\Theta$  by

$$m_{1,2,\dots,S}^{\text{Conj}}(A) = \sum_{\substack{\mathbf{X}_j \in \mathcal{F}(m_1, \dots, m_S) \\ X_{j1} \cap \dots \cap X_{jS} = A}} \prod_{i=1}^S m_i(X_{ji}) \quad (3)$$

where  $\mathbf{X}_j \triangleq (X_{j1}, X_{j2}, \dots, X_{jS})$  is a possible  $S$ -uple of focal elements, where  $j_1 \in \{1, \dots, \mathcal{F}_1\}$ ,  $j_2 \in \{1, \dots, \mathcal{F}_2\}$ , ..., and  $j_S \in \{1, \dots, \mathcal{F}_S\}$ . The element  $X_{ji}$  is the focal element of the BBA  $m_i(\cdot)$  that makes the  $i$ -th component of the  $j$ -th  $S$ -uple  $\mathbf{X}_j$ . The set  $\mathcal{F}(m_1, \dots, m_S)$  is the set of all possible  $S$ -uples. The cardinality of the set  $\mathcal{F}(m_1, \dots, m_S)$  is noted  $\mathcal{F}$  for convenience. The total conflicting mass, denoted  $m_{1,2,\dots,S}^{\text{Conj}}(\emptyset)$ , is given by

$$m_{1,2,\dots,S}^{\text{Conj}}(\emptyset) = \sum_{\substack{\mathbf{X}_j \in \mathcal{F}(m_1, \dots, m_S) \\ X_{j1} \cap \dots \cap X_{jS} = \emptyset}} \prod_{i=1}^S m_i(X_{ji}) \quad (4)$$

This fusion rule is commutative and associative, and the vacuous BBA  $m_v$  has a neutral impact, that is  $\text{Conj}(m_1, m_2, \dots, m_S, m_v) = \text{Conj}(m_1, m_2, \dots, m_S)$ . Its main drawback is that it does not generate a proper BBA because  $m_{1,2,\dots,S}^{\text{Conj}}(\emptyset) > 0$  in general. Because the empty

<sup>c</sup>For notation simplicity, we omit  $\Theta$  lower index in the notations of sets of focal elements  $\mathcal{F}_\Theta(m_1), \dots, \mathcal{F}_\Theta(m_S)$ , and their cardinalities are simply written as  $\mathcal{F}_1, \dots, \mathcal{F}_S$ .

set  $\emptyset$  is the absorbing element for the conjunctive operation, this rule generates  $m_{1,2,\dots,S}^{\text{Conj}}(\emptyset)$  that quickly tends to one after only few steps of a sequential fusion processing of the sources which is not very useful for decision-making support. The main interest of this rule is its ability to identify the partial conflicts, and to provide a measure of the total level of conflict  $m_{1,2,\dots,S}^{\text{Conj}}(\emptyset)$  between the sources which can be used to manage (select or discard) the sources in the fusion process if one prefers, see Dezert et al. (2019) for instance.

### 2.3. Dempster-Shafer Combination Rule

Dempster-Shafer (DS) rule of combination is the emblematic rule of combination proposed by Shafer in his Mathematical Theory of Evidence (see Shafer (1976)) which is based on Dempster's early works (see Dempster (1967, 1968)). DS rule is nothing but the normalized version of the conjunctive rule. Hence, DS combination of  $S > 1$  BBAs  $m_s(\cdot)$  ( $s = 1, \dots, S$ ) defined on the same frame of discernment  $\Theta$ , denoted as  $\text{DS}(m_1, m_2, \dots, m_S)$ , is a proper BBA defined by  $m_{1,2,\dots,S}^{\text{DS}}(\emptyset) = 0$ , and for all  $A \in 2^\Theta \setminus \{\emptyset\}$  by

$$m_{1,2,\dots,S}^{\text{DS}}(A) = \frac{m_{1,2,\dots,S}^{\text{Conj}}(A)}{1 - m_{1,2,\dots,S}^{\text{Conj}}(\emptyset)} \quad (5)$$

DS fusion rule is commutative and associative, and the vacuous BBA  $m_v$  has also a neutral impact for this rule, but its justification and behavior have been disputed over the years from many counter-examples involving high or low conflicting sources (from both theoretical and practical standpoints) as reported in Dezert et al. (2012); Tchamova and Dezert (2012); Dezert and Tchamova (2014). In our applications that are related with risk assessment and safety, we do not prefer to use DS rule because of its very serious problems. Actually, many alternative rules of combination exist<sup>d</sup>, and among them we focus on the new interesting rule based on the proportional conflict redistribution no. 6 (PCR6) principle (see Smarandache and Dezert (2004), Vol. 3 for details) which is presented in the next section.

## 3. PCR6 Combination Rule

### 3.1. PCR6 General Formula

The PCR6 rule of combination has been proposed in Martin and Osswald (2006); Martin et al. (2008) as an interesting alternative of original

PCR rule of combination no. 5 (PCR5) proposed in Smarandache and Dezert (2005, 2006). The PCR6 rule coincides with the PCR5 rule when one combines only two sources (i.e. two BBAs defined on the same FoD). The difference between PCR5 and PCR6 rules lies in the way the proportional conflict redistribution is done as soon as three (or more) sources are involved in the fusion. For notation convenience, we define

$$\pi_j(X_{j_1} \cap X_{j_2} \cap \dots \cap X_{j_S}) \triangleq \prod_{i=1}^S m_i(X_{j_i}) \quad (6)$$

If  $X_{j_1} \cap X_{j_2} \cap \dots \cap X_{j_S} = \emptyset$ , then we use the more concise notation  $\pi_j(\emptyset)$  instead of  $\pi_j(X_{j_1} \cap X_{j_2} \cap \dots \cap X_{j_S})$ , and  $\pi_j(\emptyset)$  is called a *conflicting mass product*.

The PCR6 fusion of  $S > 2$  BBAs is obtained by  $m_{1,2,\dots,S}^{\text{PCR6}}(\emptyset) = 0$ , and for all  $A \in 2^\Theta \setminus \{\emptyset\}$  by

$$\begin{aligned} m_{1,2,\dots,S}^{\text{PCR6}}(A) = & m_{1,2,\dots,S}^{\text{Conj}}(A) \\ & + \sum_{j \in \{1,\dots,S\} | A \in X_j \wedge \pi_j(\emptyset)} \\ & \left[ \left( \sum_{i \in \{1,\dots,S\} | X_{j_i} = A} m_i(X_{j_i}) \right) \right. \\ & \cdot \left. \frac{\pi_j(\emptyset)}{\sum_{X \in X_j} \left( \sum_{i \in \{1,\dots,S\} | X_{j_i} = X} m_i(X_{j_i}) \right)} \right] \quad (7) \end{aligned}$$

where  $\wedge$  is the logical conjunction<sup>e</sup>.

We use this general PCR6 formula because it is more easy to implement and to improve than the original formula given in Martin and Osswald (2006) and in Martin et al. (2008). The PCR6 rule is quasi-associative and it offers a more refined conflict redistribution than DS rule but it is more complex, and it does not preserve the neutrality of the vacuous BBA. PCR6 is simpler to implement than PCR5. When  $S > 2$ , PCR6 is better than PCR5 for decision-making as shown in Martin and Osswald (2006). Matlab<sup>TM</sup> codes of PCR5 and PCR6 fusion rules can be found in Smarandache and Dezert (2004); Smarandache et al. (2010), and also from the BFAS<sup>f</sup> repository. The PCR5 formula can be obtained from the PCR6 formula by just replacing the two summation operators on  $i \in \{1, \dots, S\} | X_{j_i} = A$  appearing in (7) by the two product operators on  $i \in \{1, \dots, S\} | X_{j_i} = A$ , that is

$$\sum_{i \in \{1,\dots,S\} | X_{j_i} = A} \rightarrow \prod_{i \in \{1,\dots,S\} | X_{j_i} = A}$$

<sup>d</sup>see Smarandache and Dezert (2004), Vol. 2 for a detailed list of many fusion rules.

<sup>e</sup>i.e.  $x \wedge y$  means that conditions  $x$  and  $y$  are both true.

<sup>f</sup>Belief Functions and Applications Society, see <https://www.bfasociety.org/>.

### 3.2. Drawback of PCR6 Rule

The PCR6 (resp. PCR5) rule of combination is not associative which means that the fusion of the BBAs must be done using general formula (7) if one has more than two BBAs to combine, which is not very convenient. Therefore, the sequential PCR6 (resp. PCR5) combination of  $S > 2$  BBAs are not in general equal to the global PCR6 (resp. PCR5) fusion of the  $S$  BBAs altogether because the order of the combination of the sources does matter in the sequential combination. Moreover, the PCR6 rule (resp. PCR5) can become computationally intractable for combining a large number of sources and for working with large FoD. This is a well-known limitation of this rule, but this is the price to pay to get better results than with DS rule. Aside the complexity of this rule, it is worth to mention that the neutral impact property of the vacuous BBA  $m_v$  is lost in general when considering the PCR6 (or PCR5) combination of  $S > 2$  BBAs altogether because of the proportional conflict redistribution principles used in PCR6 (resp. PCR5) rule. The non neutral impact of the vacuous BBA is clearly a drawback because it is naturally expected that the vacuous BBA must not impact the fusion result in the fusion process because the vacuous BBA brings no useful information to exploit. Also a BBA that is close to the vacuous BBA should not have a strong impact on the fusion result because it brings only a very little valuable information. This can be seen as a flaw of the behavior of PCR6 (resp. PCR5) rule of combination. To emphasize clearly this flaw, we give in the example 1 a case where the mass committed to some partial uncertainties can increase more than necessary with PCR6 rule if we have a BBA which is close (or equal) to the vacuous BBA, which is detrimental for the quality of the fusion result and for decision-making (because the result is more uncertain than it should be, and consequently the decision is more difficult to make).

**Example 1:** consider  $\Theta = \{A, B, C, D, E\}$  and the three BBAs listed in Table 1.

Table 1. The three BBAs to combine.

Focal Elements	$m_1(\cdot)$	$m_2(\cdot)$	$m_3(\cdot)$
$B$	0.05	0.05	0
$A \cup B$	0.65	0.05	0
$C \cup D$	0.05	0.50	0
$A \cup B \cup C \cup D$	0.15	0.05	0
$E$	0.10	0.35	0.01
$\Theta$	0	0	0.99

Here  $m_3(\cdot)$  is not equal to the vacuous BBA but it is very close to the vacuous BBA be-

cause  $m_3(\Theta)$  is close to one. The results<sup>§</sup> of the fusion  $\text{PCR6}(m_1, m_2)$ , and the fusion  $\text{PCR6}(m_1, m_2, m_3)$  are given in Table 2.

Table 2.  $m_{1,2}^{\text{PCR6}}(\cdot)$  and  $m_{1,2,3}^{\text{PCR6}}(\cdot)$  results.

Focal Elements	$m_{1,2}^{\text{PCR6}}(\cdot)$	$m_{1,2,3}^{\text{PCR6}}(\cdot)$
$B$	0.054877	0.048939
$A \cup B$	0.406987	0.247656
$C \cup D$	0.312886	0.204005
$A \cup B \cup C \cup D$	0.024917	0.013439
$E$	0.200333	0.101731
$\Theta$	0	0.384230

One sees that combining the BBAs  $m_1, m_2$  with the BBA  $m_3$  (where  $m_3$  is close to vacuous BBA, and therefore  $m_3$  is almost non-informative) generates a big increase of the belief of the uncertainty in the resulting BBA. This behavior is clearly counter-intuitive because if the source is almost vacuous, only a small degradation of the uncertainty is expected and in the limit case when  $m_3$  is the vacuous BBA no impact of  $m_3$  on the fusion result should occur. Because of this flawed behavior, we propose in the next section an improvement of PCR6 rule (called  $\text{PCR6}^+$  fusion rule) in order to preserve the neutrality of the vacuous BBA.

### 4. Improvement of PCR6 Rule

The very simple and basic idea to improve the PCR6 conflict redistribution principle is to discard the elements that contain the other elements implied in the conflict mass product  $\pi_j(\emptyset)$  calculation. Indeed, the elements discarded are regarded as non informative and not useful for making the conflict redistribution. To illustrate clearly this point, let's consider again Example 1 and the conflicting product

$$\pi_{16}(\emptyset) = m_1(A \cup B)m_2(C \cup D)m_3(\Theta).$$

With PCR6, the redistribution of  $\pi_{16}(\emptyset)$  follows

$$\begin{aligned} \frac{x_{16}(A \cup B)}{m_1(A \cup B)} &= \frac{x_{16}(C \cup D)}{m_2(C \cup D)} = \frac{x_{16}(\Theta)}{m_3(\Theta)} \\ &= \frac{\pi_{16}(\emptyset)}{m_1(A \cup B) + m_2(C \cup D) + m_3(\Theta)} \end{aligned}$$

which is not very efficient because  $\Theta$  is not the source of conflict in this case since  $A \cup B \subseteq \Theta$  and  $C \cup D \subseteq \Theta$ . The conflict exists only because  $(A \cup B) \cap (C \cup D) = \emptyset$ . In the improved version of PCR6 rule, denoted  $\text{PCR6}^+$ , the conflicting

<sup>§</sup>The numerical values have been rounded to their sixth digit.

product  $\pi_{16}(\emptyset)$  will be redistributed only to  $A \cup B$  and to  $C \cup D$  but not to  $\Theta$ . With PCR6<sup>+</sup> rule we will make the new (simpler) redistribution of  $\pi_{16}(\emptyset)$  according to

$$\begin{aligned} \frac{x_{16}(A \cup B)}{m_1(A \cup B)} &= \frac{x_{16}(C \cup D)}{m_2(C \cup D)} = \\ &= \frac{\pi_{16}(\emptyset)}{m_1(A \cup B) + m_2(C \cup D)} \end{aligned}$$

#### 4.1. PCR6<sup>+</sup> general formula

The general expression of PCR6<sup>+</sup> (and also PCR5<sup>+</sup>) is presented in details, with many examples and Matlab<sup>TM</sup> codes in Dezert et al. (2021). Here, due to space limitation, we just recall its expression for convenience. Actually, PCR6<sup>+</sup> fusion rule is the proper modification of PCR6 formula (7) taking into account the selection of focal elements on which the proportional redistribution must apply thanks to the value of their keeping-index. More precisely, the PCR6<sup>+</sup> fusion of  $S > 2$  BBAs is obtained by  $m_{1,2,\dots,S}^{\text{PCR6}^+}(\emptyset) = 0$ , and for all  $A \in 2^\Theta \setminus \{\emptyset\}$  by

$$\begin{aligned} m_{1,2,\dots,S}^{\text{PCR6}^+}(A) &= m_{1,2,\dots,S}^{\text{Conj}}(A) \\ &+ \sum_{j \in \{1,\dots,S\} | A \in \mathbf{X}_j \wedge \pi_j(\emptyset)} \\ &\left[ (\kappa_j(A) \sum_{i \in \{1,\dots,S\} | X_{j_i}=A} m_i(X_{j_i})) \cdot \right. \\ &\left. \frac{\pi_j(\emptyset)}{\sum_{X \in \mathbf{X}_j} (\kappa_j(X) \sum_{i \in \{1,\dots,S\} | X_{j_i}=X} m_i(X_{j_i}))} \right] \quad (8) \end{aligned}$$

where  $\kappa_j(A)$  and  $\kappa_j(X)$  are respectively the keeping indexes of elements  $A$  and  $X$  involved in the conflicting product  $\pi_j(\emptyset)$ , that are calculated by the formula

$$\kappa_j(X_{j_i}) \triangleq 1 - \prod_{\substack{X_{l'}, X_l \in \mathbf{X}_j | X_{l'} \neq X_l \\ |X_{j_i}| \leq |X_l| \\ |X_{l'}| \leq |X_l|}} \delta_j(X_{l'}, X_l) \quad (9)$$

$\mathbf{X}_j = \{X_1, \dots, X_{s_j}, s_j \leq S\}$  is the set of all distinct components of the  $S$ -uple  $\mathbf{X}_j$  related with the conflicting product  $\pi_j(\emptyset)$ . The term  $\delta_j(X_{l'}, X_l)$  is the binary *containing indicator* of  $X_l$  with respect to  $X_{l'} \in \mathbf{X}_j$  that characterizes if  $X_l$  contains (includes)  $X_{l'}$  in wide sense, or not. More precisely,  $\delta_j(X_{l'}, X_l)$  is defined by

$$\delta_j(X_{l'}, X_l) \triangleq \begin{cases} 1 & \text{if } X_{l'} \subseteq X_l, \\ 0 & \text{if } X_{l'} \not\subseteq X_l. \end{cases} \quad (10)$$

The value  $\kappa_j(X_{j_i}) = 1$  stipulates that the focal element  $X_{j_i} \in \mathbf{X}_j$  must receive some proportional redistribution from the conflicting mass  $\pi_j(\emptyset)$ , and  $\kappa_j(X_{j_i}) = 0$  indicates that  $X_{j_i} \in \mathbf{X}_j$  will not be involved in the proportional redistribution of  $\pi_j(\emptyset)$ . Note that  $\kappa_j(\Theta) = 0$  if  $\Theta \in \mathbf{X}_j$  because  $\Theta$  always includes all other focal elements of  $\mathbf{X}_j$  and  $\Theta$  has the highest cardinality. For a given FoD and a given number of BBAs to combine, it is always possible to calculate offline the values of the keeping-indexes of focal elements for all combinations leading to conflicting products  $\pi_j(\emptyset) > 0$ . We can verify that formula (8) is consistent with PCR6 formula (7) when all keeping indexes are equal to one. The fusion rule (8) is commutative and non associative, and the vacuous BBA  $m_v$  has a neutral impact in PCR6<sup>+</sup> rule - see proof in Dezert et al. (2021).

#### 4.2. Example 1 revisited with PCR6<sup>+</sup>

Consider the example 1 with the three BBAs given in table 1. If we combine the BBAs  $m_1$  and  $m_2$ , we have  $\text{PCR6}^+(m_1, m_2) = \text{PCR6}(m_1, m_2)$  because these rules coincide when combining two BBAs. If we make the PCR6<sup>+</sup> fusion of the three BBAs altogether we obtain different results which is normal, because for  $S > 2$  one has in general  $\text{PCR6}^+(m_1, \dots, m_S) \neq \text{PCR6}(m_1, \dots, m_S)$ . For this example we get results shown in Table 3.

Table 3.  $m_{1,2}^{\text{PCR6}^+}(\cdot)$  and  $m_{1,2,3}^{\text{PCR6}^+}(\cdot)$  results.

Focal Elements	$m_{1,2}^{\text{PCR6}^+}(\cdot)$	$m_{1,2,3}^{\text{PCR6}^+}(\cdot)$
$B$	0.054877	0.054485
$A \cup B$	0.406987	0.407174
$C \cup D$	0.312886	0.312660
$A \cup B \cup C \cup D$	0.024917	0.025232
$E$	0.200333	0.200449
$\Theta$	0	0

We can verify that the result obtained by PCR6<sup>+</sup> fusion rule is more judicious than with PCR6 rule because the fusion of the almost vacuous BBA  $m_3(\cdot)$  has a very little impact in the fusion result as we intuitively expect. This is because the PCR6<sup>+</sup> combination rule discards the ignorant (or almost ignorant) information. With  $m_{1,2,3}^{\text{PCR6}^+}(\cdot)$ , the largest mass is allocated to  $A \cup B$  as with  $m_{1,2}^{\text{PCR6}^+}(\cdot)$ , and contrariwise to  $m_{1,2,3}^{\text{PCR6}}(\cdot)$  when using the PCR6 fusion rule - see results in Table 2.

<sup>h</sup>We recall that one always has  $m_{1,2}^{\text{PCR6}^+}(\cdot) = m_{1,2}^{\text{PCR6}}(\cdot)$ .



## 5. Application to Levee Characterization

We now present the advantages of the new PCR6<sup>+</sup> rule for an application on a numerical case study representing a levee section. To do so, we use the geophysical and geotechnical information fusion methodology introduced in Dezert et al. (2019).

### 5.1. Model and Information Sources

The figure 1 displays the structure of the levee, the location of the different layers and the representation of the study levee section.

The area is a lengthwise (parallel to the river) vertical section composed of two lithological materials: i) compact clays (*C* hypothesis) and ii) soft sands (*S* hypothesis). The sands are present over 6 meters thick on the first 125 meters of the section and over 10 meters thick after. Clayey materials are positioned below. A small electrically conductive anomaly is located near the surface in the center of the model. Thus, the FoD is defined such that  $\Theta = \{C, S, O\}$ . As required by the fusion method, *O* is an additional hypothesis standing for any other material different from the other two known. For this case study, two geophysical methods are used: the Electrical Resistivity Tomography (ERT) and the Multi-channel Analysis of Surface Waves (MASW). Two geotechnical boreholes providing information on the lithology are also considered in this study.

### 5.2. BBA Distribution for Each Source

#### 5.2.1. Electrical Resistivity Tomography

The basic principle of DC-resistivity methods consists in injecting an electric current of known intensity [*A*] by means of two "current" electrodes and measuring a voltage [*V*] between two "potential" electrodes. Such measurements are acquired for several positions of the current and the potential electrodes. Apparent resistivity values can then be computed and inverted to reconstruct a complete section of electrical resistivity [ $\Omega \cdot m$ ]. From these electrical resistivity data, the fusion methodology (Dezert et al., 2019) enables the BBA distribution depicted in Figure 2. The ERT characterization is disturbed by the conductive electrical artifact. Thus, clays are locally characterized in the center of the section while we know that sands are actually present. Also, the interface between clays and sands are not correctly defined.

#### 5.2.2. Multichannel Analysis of Surface Waves

The MASW method consists in studying the surface waves dispersion (waveform deformation) to determine the shear waves velocity [ $m \cdot s^{-1}$ ]. A seismic source is generated at various locations and geophones are aligned on the ground surface to record the seismic waves arrival times. The use

of this method comprises three stages: (i) the data acquisition, (ii) the determination of the Rayleigh dispersion curve, and (iii) the inversion process with the determination of the shear velocities. In this work, the seismic acquisition is carried out from  $x = 212$  m to  $x = 428$  m. From the shear wave velocity data, the associated BBA distribution is displayed in Figure 3. The MASW characterization is not disturbed by the electrical artifact. Thus, the method characterizes correctly the two lithological materials as well as the lithological interface position. However,  $\Theta$  is characterized in most part of the section (in black, Figure 3.a), where no data is available.

#### 5.2.3. Core Drillings

Two core drillings with particle size analysis are simulated at  $x = 80$  m and  $x = 350$  m from the surface to 20 m depth. From the simulated geotechnical data, the associated BBA distribution is displayed in Figure 4. The lithological materials are correctly characterized but an important area of uncertainty remains between 6 and 10 m depth. Indeed, since two different materials are identified in both boreholes at such depths, the section is poorly defined between them (Dezert et al., 2019).

### 5.3. PCR6 and PCR6<sup>+</sup> Fusion Results

The fusion results using PCR6 and PCR6<sup>+</sup> rules are respectively depicted in Figures 5.a-b and Figures 5.c-d. These results highlight the lack of characterization at the center of the model using PCR6 rule (in the red boxes, Figure 5.a). Indeed,  $\Theta$  is characterized while PCR6<sup>+</sup> rule enables to correctly characterize sands.

For PCR6, this area is difficult to define since the ERT suggests the presence of clays, the MASW suggests the presence of sands and the geotechnical source of information is ignorant. However, PCR6<sup>+</sup> rule manages to allocate the conflictual masses on the individual hypothesis instead of  $\Theta$ . Furthermore, the global belief mass values are greater with PCR6<sup>+</sup> rule (Figure 5.d) than with PCR6 (Figure 5.c).

This improvement in the results could be valuable in the context of an investigation campaign on a real earthen structure. Indeed, knowing the nature of the materials as well as their location is crucial to achieve a good diagnosis and limit the risk of breakage. Since many investigation methods can be ignorant or partially ignorant in the context of levee characterization, this new combination rule would be of great operational interest to give credit to the most informative source and to avoid uncharacterized areas inside the earthen structure.

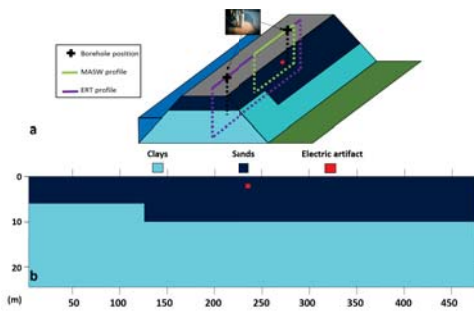


Fig. 1. a) Levee with position of investigation methods and b) materials in the section of interest.

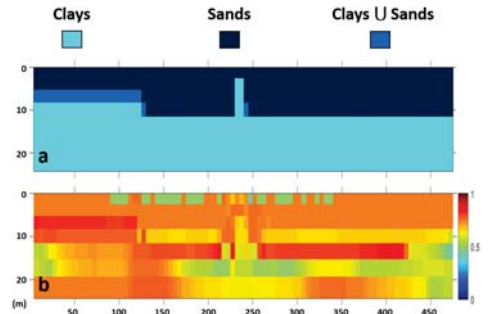


Fig. 2. a) Material with highest mass (from electrical resistivity data) and b) their mass values.

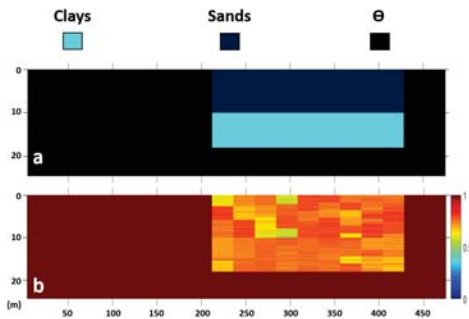


Fig. 3. a) Material with highest mass (from shear wave velocity data) and b) their mass values.

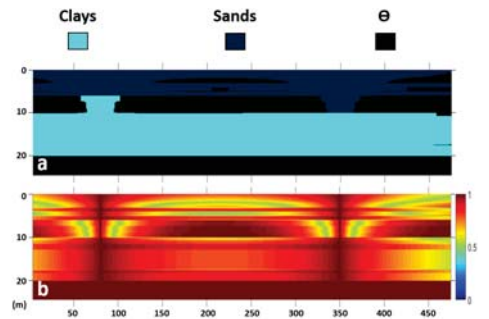


Fig. 4. a) Material with highest mass (from two borehole data) and b) their mass values.

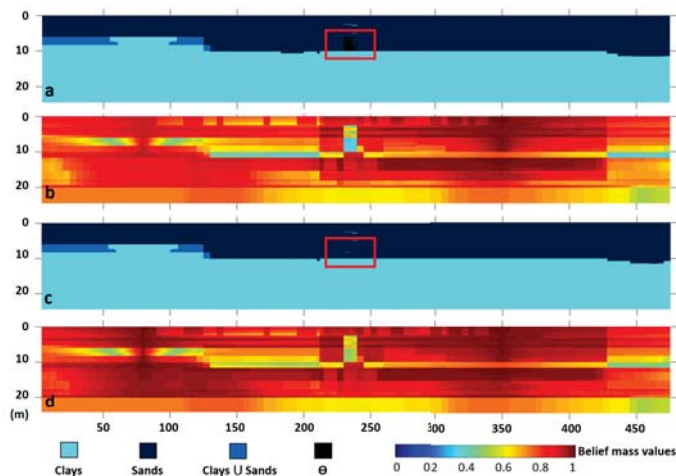


Fig. 5. Material with highest mass (from ERT, MASW and core drillings), using PCR6 (a) and PCR6<sup>+</sup> (c) rules, with area of interest in red box. b) and d) mass values associated with the hypothesis depicted in a) and c).

## 6. Conclusions

In this work, after having introduced the belief functions as well as conjunctive, DS and PCR6 rules of combination, we presented the flawed behavior of PCR6 rule. We then described improvements to correct these behaviors, introducing a new PCR6<sup>+</sup> rule. The computation of a keeping index, making it possible to discard ignorant information sources for the calculation of each partial conflict, was detailed. This keeping index has been integrated into the original formulation of PCR6 in order to ensure the neutrality property of the vacuous BBA. The interest of such combination rule has finally been demonstrated for an application on a numerical levee section with simulated geophysical and geotechnical acquisitions. As a following perspective, we wish to apply this new PCR6<sup>+</sup> rule to risk analysis issues with data fusion acquired from real investigation campaigns.

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