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Ranking of single-valued neutrosophic numbers through the index of optimism and its reasonable properties

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Abstract
In this paper an innovative method of ranking neutrosophic number based on the notions of value and ambiguity of a single-valued neutrosophic number is being developed. The method is based on the convex combination of value and ambiguity of truth-membership function with the sum of values and ambiguities of indeterminacy-membership and falsity-membership functions. This convex combination is also termed as an index of optimism. The index of optimism, $\lambda = 1$, is termed as optimistic decision-maker as it considers the value and the ambiguity of the truth-membership function, ignoring the contributions from indeterminacy-membership and falsity-membership functions. Similarly, the index of optimism, $\lambda = 0$, is termed as pessimistic decision-maker as it considers the values and the ambiguities of the indeterminacy-membership and falsity-membership functions, ignoring the contribution from truth-membership function. Further, the index of optimism, $\lambda = 0.5$, is termed as moderate decision-maker as it considers the values and the ambiguities of all the membership functions. The approach is a novel as it completely oath to follow the reasonable properties of a ranking method. It is worth to mention that the current approach consistently ranks the single-valued neutrosophic numbers as well as their corresponding images.

Keywords Neutrosophic number · Ranking · Value · Ambiguity · Index of optimism

1 Introduction

Uncertainty due to vagueness is generally handled by the branch of mathematics called fuzzy set theory developed by Zadeh (1965). In such mathematics, the parameters involved are linguistic variables which in turn can be expressed as fuzzy numbers. There are various generalizations of fuzzy numbers, one such generalization is intuitionistic fuzzy number (IFN) developed by Atanassov (1989, 1999, 2000) and octahedron sets developed by Lee et al. (2020). The generalization of fuzzy number to IFN adds more information to the latter as it incorporates non-membership or incomplete information in a fuzzy number.

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Another such generalization of fuzzy numbers, in fact IFNs are neutrosophic numbers, which incorporates indeterminacy-membership apart from the truth-membership and the falsity-membership functions. This generalization was initiated by Smarandache (1998, 1999, 2006). This generalization has added various development in diverse filed, namely, graph theory (Karaaslan and Davvaz 2018) and structure theory (Edalatpanah 2020a), linear equations (Edalatpanah 2020b), etc. This generalization has been used in various fields of decision-making, namely, Ulucay et al. (2018), Karaaslan (2018a), Giri et al. (2018), Deli (2018), etc. Apart from these, various studies are performed by Karaaslan and Hunu (2020), Karaaslan and Hayat (2018), Jana et al. (2020) and Karaaslan (2018b) in multi-criteria group decision making problems. Also, data envelopment analysis under neutrosophic environment are discussed by Yang et al. (2020), Edalatpanah (2020), Edalatpanah and Smarandache (2019) and Mao et al. (2020). Neutrosophic linear programming problems are also being discussed by Edalatpanah (2020). One of the tools in the decision-making process is ranking or ordering of neutrosophic numbers. Single-valued neutrosophic number (SVNN) is a particular type of neutrosophic number developed by Wang et al. (2010). In this work, an attempt to develop a robust method of ranking SVNNs will be made.

A very few works are available in ranking of SVNNs so far. An outranking approach was developed by Peng et al. (2014) and applied in multi-criteria decision-making problems. A outranking approach for multi-criteria decision making problems with neutrosophic multisets was discussed by Ulucay et al. (2019). Ranking of neutrosophic sets based on score function was developed by Nancy and Garg (2016). The notions of the values and the ambiguities of truth-membership, indeterminacy-membership and falsity-membership functions was developed for ranking SVNNs by Deli and Subas (2017). The ranking done on by Deli and Subas (2017) is based on the values; and if the values are equal then the ordering is done by ambiguities, that is, if the $\widetilde{a}$ and $\widetilde{b}$ are SVNNs and ambiguity of $\widetilde{a}$ is numerical greater than $\widetilde{b}$, then $\widetilde{a}$ is ranked to be bigger than $\widetilde{b}$. This ordering is completely irrational, because the SVNN with more ambiguity should be ranked smaller. Aal et al. (2018) concept of ranking SVNNs is similar to that of Deli and Subas (2017), hence their method retains the same drawback as that of Deli and Subas (2017). Evidently, Biswas et al. (2016) rectified the drawbacks of Deli and Subas (2017) and Aal et al. (2018), however in some situations their method fails to rank consistently the corresponding images of the SVNNs. Further, none of the existing method investigated the rationality validation of the methods developed. Intuitively, the existing methods of ranking SVNNs lacks rationality validation. As such these methods are not rich enough to be applied in the decision-making problems. Further, it has been observed that the ranking of SVNNs is in a very premature stage. Motivated by the chronology of the ranking method of SVNNs, it is being observed that a robust method of ranking SVNNs is unavailable. Hence, it is essential to develop a robust and logical methodology of ranking SVNNs for an appropriate decision-making process. In this work, such an attempt will be made to develop a rational and consistent method of ranking SVNNs. It was seen that the existing methods never investigated the ordering of the images of SVNNs. Hence, one objective is to see the consistency in ranking SVNNs with their corresponding images. Further, another objective is to check the robustness of the method by proving the reasonable properties of Wang and Kerre (2001a, 2001b).

The next section discusses various definitions and notations of SVNNs, which will be utilized in discussing the method and its properties. In Sect. 3, the definitions and notions of value and ambiguity of a SVNN are being discussed; and also the proposed method along with its properties are being discussed. In Sect. 4, the method is demonstrated through some numerical examples and compared with some existing methods. Finally, in Sect. 5 conclusions are made and the main features are highlighted.

2 Preliminaries

In this section, a few definitions and notations are being discussed. This discussion will further help in the discussion of the proposed method.

Definition 2.1 A SVNN $\widetilde{a} = \langle \mu_{\widetilde{a}}, \rho_{\widetilde{a}}, \nu_{\widetilde{a}} \rangle$ in the set of real numbers $\mathbb{R}$ with truth-membership function $\mu_{\widetilde{a}}$, indeterminacy-membership function $\rho_{\widetilde{a}}$ and falsity-membership function $\nu_{\widetilde{a}}$ is defined as

$$\mu_{\widetilde{a}}(x) = \begin{cases} f_{\widetilde{a}}(x), & \text{if } a_1 \leq x \leq x_{0,1} \\ 1, & \text{if } x_{0,1} \leq x \leq y_{0,1} \\ g_{\widetilde{a}}(x), & \text{if } y_{0,1} \leq x \leq b_1 \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

,

$$\rho_{\widetilde{a}}(x) = \begin{cases} l_{\widetilde{a}}(x), & \text{if } a_2 \leq x \leq x_{0,2} \\ 0, & \text{if } x_{0,2} \leq x \leq y_{0,2} \\ m_{\widetilde{a}}(x), & \text{if } y_{0,2} \leq x \leq b_2 \\ 1, & \text{otherwise,} \end{cases} \tag{2}$$

and

$$\nu_{\widetilde{a}}(x) = \begin{cases} h_{\widetilde{a}}(x), & \text{if } a_3 \leq x \leq x_{0,3} \\ 0, & \text{if } x_{0,3} \leq x \leq y_{0,3} \\ k_{\widetilde{a}}(x), & \text{if } y_{0,3} \leq x \leq b_3 \\ 1, & \text{otherwise,} \end{cases} \tag{3}$$

respectively, where $0 \leq \mu_{\widetilde{a}}(x) + \rho_{\widetilde{a}}(x) + \nu_{\widetilde{a}}(x) \leq 3$ and $a_i, x_{0,i}, y_{0,i}, b_i \in \mathbb{R}$ such that $a_i \leq x_{0,i} \leq y_{0,i} \leq b_i$, $i = 1, 2, 3$, and the functions $f_{\widetilde{a}}, g_{\widetilde{a}}, l_{\widetilde{a}}, m_{\widetilde{a}}, h_{\widetilde{a}}, k_{\widetilde{a}} : \mathbb{R} \longrightarrow [0, 1]$ are legs of truth-membership function $\mu_{\widetilde{a}}$, indeterminacy-membership function $\rho_{\widetilde{a}}$ and falsity-membership function $\nu_{\widetilde{a}}$. The functions $f_{\widetilde{a}}$, $l_{\widetilde{a}}$ and $k_{\widetilde{a}}$ are non-decreasing continuous functions and the functions $h_{\widetilde{a}}$, $m_{\widetilde{a}}$ and $g_{\widetilde{a}}$ are non-increasing continuous functions. Hence, the SVNN can also be denoted by $\widetilde{a} = \langle (a_1, x_{0,1}, y_{0,1}, b_1), (a_2, x_{0,2}, y_{0,2}, b_2), (a_3, x_{0,3}, y_{0,3}, b_3) \rangle$.

Definition 2.2 Let $\widetilde{a} = \langle (a_1, x_{0,1}, y_{0,1}, b_1), (a_2, x_{0,2}, y_{0,2}, b_2), (a_3, x_{0,3}, y_{0,3}, b_3) \rangle$ be a trapezoidal SVNN where the real numbers are such that $a_i \leq x_{0,i} \leq y_{0,i} \leq b_i$, $i = 1, 2, 3$. Then truth-membership function, indeterminacy-membership function and falsity-membership function are defined as

$$\mu_{\widetilde{a}}(x) = \begin{cases} \frac{x - a_1}{x_{0,1} - a_1}, & \text{if } a_1 \leq x \leq x_{0,1} \\ 1, & \text{if } x_{0,1} \leq x \leq y_{0,1} \\ \frac{b_1 - x}{b_1 - y_{0,1}}, & \text{if } y_{0,1} \leq x \leq b_1 \\ 0, & \text{otherwise,} \end{cases} \tag{4}$$

$$\rho_{\widetilde{a}}(x) = \begin{cases} \frac{x-x_{0,2}}{a_2-x_{0,2}}, & \text{if } a_2 \le x \le x_{0,2} \\ 0, & \text{if } x_{0,2} \le x \le y_{0,2} \\ \frac{x-y_{0,2}}{b_2-y_{0,2}}, & \text{if } y_{0,2} \le x \le b_2 \\ 1, & \text{otherwise,} \end{cases} \tag{5}$$

and

$$\nu_{\widetilde{a}}(x) = \begin{cases} \frac{x-x_{0,3}}{a_3-x_{0,3}}, & \text{if } a_3 \le x \le x_{0,3} \\ 0, & \text{if } x_{0,3} \le x \le y_{0,3} \\ \frac{x-y_{0,3}}{b_3-y_{0,3}}, & \text{if } y_{0,3} \le x \le b_3 \\ 1, & \text{otherwise,} \end{cases} \tag{6}$$

respectively, where $0 \le \mu_{\widetilde{a}}(x) + \rho_{\widetilde{a}}(x) + \nu_{\widetilde{a}}(x) \le 3$.

As like IFN, the cut sets of SVNN can also be defined for truth-membership, indeterminacy-membership and falsity-membership functions. These definitions are being thoroughly discussed by Deli and Subas (2017). These definitions are being adopted in this study.

A $\langle \alpha, \gamma, \beta \rangle$-cut set, of a SVNN $\widetilde{a}$, is a crisp subset of $\mathbb{R}$, which is defined as

$$\widetilde{a}_{\langle \alpha, \gamma, \beta \rangle} = \{x | \mu_{\widetilde{a}}(x) \ge \alpha, \rho_{\widetilde{a}}(x) \le \gamma, \nu_{\widetilde{a}}(x) \le \beta\}$$

where $0 \le \alpha + \gamma + \beta \le 3$, $\mu_{\widetilde{a}}$, $\rho_{\widetilde{a}}$ and $\nu_{\widetilde{a}}$ are truth-membership, indeterminacy-membership and falsity-membership functions of $\widetilde{a}$ respectively.

A α-cut set is a crisp subset of $\mathbb{R}$, which is defined as $\widetilde{a}_\alpha = \{x | \mu_{\widetilde{a}}(x) \ge \alpha\}$ where $0 \le \alpha \le 1$. Further, $\widetilde{a}_\alpha$ represents a closed interval, denoted by $\widetilde{a}_\alpha = \left[L^\mu_{\widetilde{a}}(\alpha), R^\mu_{\widetilde{a}}(\alpha)\right]$. Now, for the truth-membership function defined in Eq. 4, the α-cut set is defined as

$$\widetilde{a}_\alpha = \left[L^\mu_{\widetilde{a}}(\alpha), R^\mu_{\widetilde{a}}(\alpha)\right] = \left[a_1 + \alpha(x_{0,1} - a_1), b_1 - \alpha(b_1 - y_{0,1})\right]. \tag{7}$$

A γ-cut set is also a crisp subset of $\mathbb{R}$, which is defined as $\widetilde{a}_\gamma = \{x | \rho_{\widetilde{a}}(x) \le \gamma\}$, where $0 \le \gamma \le 1$. Further, $\widetilde{a}_\gamma$ represents a closed interval, denoted by $\widetilde{a}_\gamma = \left[L^\rho_{\widetilde{a}}(\gamma), R^\rho_{\widetilde{a}}(\gamma)\right]$. Also, for the falsity-membership function defined in Eq. 5, the γ-cut set is defined as

$$\widetilde{a}_\gamma = \left[L^\rho_{\widetilde{a}}(\gamma), R^\rho_{\widetilde{a}}(\gamma)\right] = \left[x_{0,2} + \beta(a_2 - x_{0,2}), y_{0,2} + \beta(b_2 - y_{0,2})\right]. \tag{8}$$

A β-cut set is again a crisp subset of $\mathbb{R}$, which is defined as $\widetilde{a}_\beta = \{x | \nu_{\widetilde{a}}(x) \le \beta\}$, where $0 \le \beta \le 1$. Further, $\widetilde{a}_\beta$ represents a closed interval, denoted by $\widetilde{a}_\beta = \left[L^\nu_{\widetilde{a}}(\beta), R^\nu_{\widetilde{a}}(\beta)\right]$. Also, for the falsity-membership function defined in Eq. 6, the β-cut set is defined as

$$\widetilde{a}_\beta = \left[L^\nu_{\widetilde{a}}(\beta), R^\nu_{\widetilde{a}}(\beta)\right] = \left[x_{0,3} + \beta(a_3 - x_{0,3}), y_{0,3} + \beta(b_3 - y_{0,3})\right]. \tag{9}$$

Another notion that are necessary for the discussion is the notions of the support of a SVNN. As a SVNN requires three types of functions to represent it. Hence, for each of these functions, the support can be defined. The supports of truth-membership, indeterminacy-membership and falsity-membership functions are denoted and defined as $\text{supp}(\mu_{\widetilde{a}}) = \{x | \mu_{\widetilde{a}}(x) > 0\}$, $\text{supp}(\rho_{\widetilde{a}}) = \{x | \rho_{\widetilde{a}}(x) < 1\}$ and $\text{supp}(\nu_{\widetilde{a}}) = \{x | \nu_{\widetilde{a}}(x) < 1\}$ respectively. Further, the following notations will be used in the further discussion, that is,

$L^{\mu}_{\widetilde{a}}(0) = \inf \mathrm{supp}(\mu_{\widetilde{a}}), R^{\mu}_{\widetilde{a}}(0) = \sup \mathrm{supp}(\mu_{\widetilde{a}})$, $L^{\rho}_{\widetilde{a}}(1) = \inf \mathrm{supp}(\rho_{\widetilde{a}}), R^{\rho}_{\widetilde{a}}(1) = \sup \mathrm{supp}(\rho_{\widetilde{a}})$, $L^{\nu}_{\widetilde{a}}(1) = \inf \mathrm{supp}(\nu_{\widetilde{a}})$ and $R^{\nu}_{\widetilde{a}}(1) = \sup \mathrm{supp}(\nu_{\widetilde{a}})$.

Let $\widetilde{a} = \langle \mu_{\widetilde{a}}, \rho_{\widetilde{a}}, \nu_{\widetilde{a}} \rangle$ be a SVNN, then the image of $\widetilde{a}$ is given by $-\widetilde{a} = \langle \mu_{-\widetilde{a}}, \rho_{-\widetilde{a}}, \nu_{-\widetilde{a}} \rangle$. Thus, if $\widetilde{a}_{\alpha} = \left[L^{\mu}_{\widetilde{a}}(\alpha), R^{\mu}_{\widetilde{a}}(\alpha)\right]$, $\widetilde{a}_{\gamma} = \left[L^{\rho}_{\widetilde{a}}(\gamma), R^{\rho}_{\widetilde{a}}(\gamma)\right]$ and $\widetilde{a}_{\beta} = \left[L^{\nu}_{\widetilde{a}}(\beta), R^{\nu}_{\widetilde{a}}(\beta)\right]$ be the cut sets of $\widetilde{a}$, then the cut set of $-\widetilde{a}$ are $-\widetilde{a}_{\alpha} = \left[-R^{\mu}_{\widetilde{a}}(\alpha), -L^{\mu}_{\widetilde{a}}(\alpha)\right]$, $-\widetilde{a}_{\gamma} = \left[-R^{\rho}_{\widetilde{a}}(\gamma), -L^{\rho}_{\widetilde{a}}(\gamma)\right]$ and $-\widetilde{a}_{\beta} = \left[-R^{\nu}_{\widetilde{a}}(\beta), -L^{\nu}_{\widetilde{a}}(\beta)\right]$. A SVNN $\widetilde{a}$ is symmetric about y-axis, if $-L^{\mu}_{\widetilde{a}}(\alpha) = R^{\mu}_{\widetilde{a}}(\alpha)$, $-L^{\rho}_{\widetilde{a}}(\gamma) = R^{\rho}_{\widetilde{a}}(\gamma)$ and $-L^{\nu}_{\widetilde{a}}(\beta) = R^{\nu}_{\widetilde{a}}(\beta)$.

2.1 Arithmetic of SVNNs

The arithmetic operations of IFNs was extensively studied by Chakraborty et al. (2015) using different methodology, namely, (α, β)-cut method, vertex method and extension principle method. As SVNN is an extension of IFN, these methodology of arithmetic of IFNs can be extended to arithmetic of SVNNs. The arithmetic of SVNNs are also discussed by Biswas et al. (2016) and Deli and Subas (2017) using the (α, γ, β)-cut sets method. In this study, the arithmetic of SVNNs by the (α, γ, β)-cut sets method is adopted. Let $\widetilde{a} = \langle (a_1, x_{0,1}, y_{0,1}, b_1), (a_2, x_{0,2}, y_{0,2}, b_2), (a_3, x_{0,3}, y_{0,3}, b_3) \rangle$ and $\widetilde{b} = \langle (p_1, m_{0,1}, m_{0,1}, q_1), (p_2, m_{0,2}, n_{0,2}, q_2), (p_3, m_{0,3}, n_{0,3}, q_3) \rangle$ be two SVNNs. Let the α-cut, γ-cut and β-cut sets of truth-membership, indeterminacy-membership and falsity-membership functions of $\widetilde{a}$ and $\widetilde{b}$ be $\widetilde{a}_{\alpha} = [L^{\mu}_{\widetilde{a}}(\alpha), R^{\mu}_{\widetilde{a}}(\alpha)]$, $\widetilde{a}_{\gamma} = [L^{\rho}_{\widetilde{a}}(\gamma), R^{\rho}_{\widetilde{a}}(\gamma)]$ and $\widetilde{a}_{\beta} = [L^{\nu}_{\widetilde{a}}(\beta), R^{\nu}_{\widetilde{a}}(\beta)]$, and $\widetilde{b}_{\alpha} = [L^{\mu}_{\widetilde{b}}(\alpha), R^{\mu}_{\widetilde{b}}(\alpha)]$, $\widetilde{b}_{\gamma} = [L^{\rho}_{\widetilde{b}}(\gamma), R^{\rho}_{\widetilde{b}}(\gamma)]$ and $\widetilde{b}_{\beta} = [L^{\nu}_{\widetilde{b}}(\beta), R^{\nu}_{\widetilde{b}}(\beta)]$ respectively. Then the arithmetic operations addition, subtraction and scalar multiplication are defined as

$$
\begin{aligned}
&[\widetilde{a} + \widetilde{b}]_{\alpha} \\
&\quad = [L^{\mu}_{\widetilde{a}}(\alpha) + L^{\mu}_{\widetilde{b}}(\alpha), R^{\mu}_{\widetilde{a}}(\alpha) + R^{\mu}_{\widetilde{b}}(\alpha)], [\widetilde{a} + \widetilde{b}]_{\gamma} = [L^{\rho}_{\widetilde{a}}(\gamma) + L^{\rho}_{\widetilde{b}}(\gamma), R^{\rho}_{\widetilde{a}}(\gamma) + R^{\rho}_{\widetilde{b}}(\gamma)], \\
&\qquad [\widetilde{a} + \widetilde{b}]_{\beta} = [L^{\nu}_{\widetilde{a}}(\beta) + L^{\nu}_{\widetilde{b}}(\beta), R^{\nu}_{\widetilde{a}}(\beta) + R^{\nu}_{\widetilde{b}}(\beta)]; \\
&[\widetilde{a} - \widetilde{b}]_{\alpha} \\
&\quad = [L^{\mu}_{\widetilde{a}}(\alpha) - R^{\mu}_{\widetilde{b}}(\alpha), R^{\mu}_{\widetilde{a}}(\alpha) - R^{\mu}_{\widetilde{b}}(\alpha)], [\widetilde{a} - \widetilde{b}]_{\gamma} = [L^{\rho}_{\widetilde{a}}(\gamma) - R^{\rho}_{\widetilde{b}}(\gamma), R^{\rho}_{\widetilde{a}}(\gamma) - L^{\rho}_{\widetilde{b}}(\gamma)], \\
&\qquad [\widetilde{a} - \widetilde{b}]_{\beta} = [L^{\nu}_{\widetilde{a}}(\beta) - R^{\nu}_{\widetilde{b}}(\beta), R^{\nu}_{\widetilde{a}}(\beta) - L^{\nu}_{\widetilde{b}}(\beta)];
\end{aligned}
$$

and

$$
[\lambda\widetilde{a}]_{\alpha} = \begin{cases} \left[\lambda L^{\mu}_{\widetilde{a}}(\alpha), \lambda R^{\mu}_{\widetilde{a}}(\alpha)\right], & \text{if } \lambda > 0, \\ \left[\lambda R^{\mu}_{\widetilde{a}}(\alpha), \lambda L^{\mu}_{\widetilde{a}}(\alpha)\right], & \text{if } \lambda < 0, \end{cases} ; [\lambda\widetilde{a}]_{\beta} = \begin{cases} \left[\lambda L^{\nu}_{\widetilde{a}}(\beta), \lambda R^{\nu}_{\widetilde{a}}(\beta)\right], & \text{if } \lambda > 0, \\ \left[\lambda R^{\nu}_{\widetilde{a}}(\beta), \lambda L^{\nu}_{\widetilde{a}}(\beta)\right], & \text{if } \lambda < 0, \end{cases} ;
$$

$$
[\lambda\widetilde{a}]_{\beta} = \begin{cases} \left[\lambda L^{\nu}_{\widetilde{a}}(\beta), \lambda R^{\nu}_{\widetilde{a}}(\beta)\right], & \text{if } \lambda > 0, \\ \left[\lambda R^{\nu}_{\widetilde{a}}(\beta), \lambda L^{\nu}_{\widetilde{a}}(\beta)\right], & \text{if } \lambda < 0, \end{cases}
$$

respectively. Eventually, these arithmetic operations on the (α, γ, β)-cut are calculated to obtain the following expressions.

$$
\begin{aligned}
&\widetilde{a} + \widetilde{b} \\
&\quad = \langle (a_1 + p_1, x_{0,1} + m_{0,1}, y_{0,1} + n_{0,1}, b_1 + q_1), \\
&\qquad (a_2 + p_2, x_{0,2} + m_{0,2}, y_{0,2} + n_{0,2}, b_2 + q_2), \\
&\qquad (a_3 + p_3, x_{0,3} + m_{0,3}, y_{0,3} + n_{0,3}, b_3 + q_3) \rangle,
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
&\widetilde{a} - \widetilde{b} \\
&= \langle (a_1 - q_1, x_{0,1} - n_{0,1}, y_{0,1} - m_{0,1}, b_1 - p_1), \\
&\quad (a_2 - q_2, x_{0,2} - m_{0,2}, y_{0,2} - m_{0,2}, b_2 - p_2), \\
&\quad (a_3 - q_3, x_{0,3} - n_{0,3}, y_{0,3} - m_{0,3}, b_3 - p_3) \rangle,
\end{aligned} \tag{11}
$$

$$
\lambda\widetilde{a} = \begin{cases} \langle (\lambda a_1, \lambda x_{0,1}, \lambda y_{0,1}, \lambda b_1), (\lambda a_2, \lambda x_{0,2}, \lambda y_{0,2}, \lambda b_2), (\lambda a_3, \lambda x_{0,3}, \lambda y_{0,3}, \lambda b_3) \rangle, & \text{if } \lambda > 0, \\ \langle (\lambda b_1, \lambda y_{0,1}, \lambda x_{0,1}, \lambda a_1), (\lambda b_2, \lambda y_{0,2}, \lambda x_{0,2}, \lambda a_2), (\lambda b_3, \lambda y_{0,3}, \lambda x_{0,3}, \lambda a_3) \rangle, & \text{if } \lambda < 0, \end{cases} \tag{12}
$$

The collection of the SVNNs that follows the above defined arithmetic operations with bounded supports and convex are denoted by the set $\mathcal{NF}$. The collection of SVNNs means Single-valued Neutrosophic Triangular Number, Single-valued Neutrosophic Trapezoidal Numbers, Single-valued Neutrosophic Polygonal Numbers, etc.

3 The proposed method of ranking SVNN

The notions of value and ambiguity are enormously discussed in various methodology of ranking fuzzy numbers by Chutia (2017) and Chutia and Chutia (2017). Further, these quantities are also used in ranking IFNs by Chutia and Saikia (2018), and in ranking Z-numbers by Chutia (2020). Although there are various notions of capturing information which are being used in ranking methodologies, yet these two notions are reliable and robust. Hence, these notions are being used in the current methodology of ranking SVNNs. Thus, to move toward the development of the methodology, the following subsection will discuss the notions of value and ambiguity of a SVNN.

3.1 Definitions and notions essential for the discussion

In this subsection, the main definition that the proposed method of ranking SVNNs oath to stand is being discussed. Further, a few properties are also being discussed.

Definition 3.1 Let $\widetilde{a} \in \mathcal{NF}$ and truth-membership function be $\mu_{\widetilde{a}}(x)$, indeterminacy-membership function be $\rho_{\widetilde{a}}(x)$ and falsity-membership function be $\nu_{\widetilde{a}}(x)$ as defined in Definition 2.1. Let $\widetilde{a}_\alpha = [L^{\mu}_{\widetilde{a}}(\alpha), R^{\mu}_{\widetilde{a}}(\alpha)]$ be the α-cut sets of truth-membership function, $\widetilde{a}_\gamma = [L^{\rho}_{\widetilde{a}}(\gamma), R^{\rho}_{\widetilde{a}}(\gamma)]$ be the γ-cut sets of indeterminacy-membership function and $\widetilde{a}_\beta = [L^{\nu}_{\widetilde{a}}(\beta), R^{\nu}_{\widetilde{a}}(\beta)]$ be the β-cut sets of falsity-membership function of $\widetilde{a}$. Then, the quantities values and ambiguities of truth-membership, indeterminacy-membership and falsity-membership functions are denoted as $\mathcal{V}(\mu_{\widetilde{a}})$, $\mathcal{V}(\rho_{\widetilde{a}})$, $\mathcal{V}(\nu_{\widetilde{a}})$ and $\mathcal{A}(\mu_{\widetilde{a}})$, $\mathcal{A}(\rho_{\widetilde{a}})$, $\mathcal{A}(\nu_{\widetilde{a}})$, respectively. Then, these quantities are defined as

$$
\begin{cases} \mathcal{V}(\mu_{\widetilde{a}}) = \int_0^1 (R^{\mu}_{\widetilde{a}}(r) + L^{\mu}_{\widetilde{a}}(r)) f(r) dr, \\ \mathcal{V}(\rho_{\widetilde{a}}) = \int_0^1 (R^{\rho}_{\widetilde{a}}(r) + L^{\rho}_{\widetilde{a}}(r)) g(r) dr, \\ \mathcal{V}(\nu_{\widetilde{a}}) = \int_0^1 (R^{\nu}_{\widetilde{a}}(r) + L^{\nu}_{\widetilde{a}}(r)) g(r) dr, \end{cases} \tag{13}
$$

$$\begin{cases} \mathcal{A}(\mu_{\widetilde{a}}) = \int_0^1 (R^{\mu}_{\widetilde{a}}(r) - L^{\mu}_{\widetilde{a}}(r))f(r)dr, \\ \mathcal{A}(\rho_{\widetilde{a}}) = \int_0^1 (R^{\rho}_{\widetilde{a}}(r) - L^{\rho}_{\widetilde{a}}(r))g(r)dr, \\ \mathcal{A}(\nu_{\widetilde{a}}) = \int_0^1 (R^{\nu}_{\widetilde{a}}(r) - L^{\nu}_{\widetilde{a}}(r))g(r)dr, \end{cases} \tag{14}$$

where, the function $f(\alpha)$ is non-negative and non-decreasing function on the interval [0, 1] with $f(0) = 0$, $f(1) = 1$ and $\int_0^1 f(\alpha)d\alpha = \frac{1}{2}$; the function $g(\beta)$ is a non-negative and non-increasing function on the interval [0, 1] with $g(1) = 0$, $g(0) = 1$ and $\int_0^1 g(\beta)d\beta = \frac{1}{2}$.

Let $\widetilde{a} = \langle (a_1, x_{0,1}, y_{0,1}, b_1), (a_2, x_{0,2}, y_{0,2}, b_2), (a_3, x_{0,3}, y_{0,3}, b_3) \rangle$ be a trapezoidal SVNN defined in Definition 2.2. Let truth-membership, indeterminacy-membership and falsity-membership functions denoted as $\mu_{\widetilde{a}}(x)$, $\rho_{\widetilde{a}}(x)$ and $\nu_{\widetilde{a}}(x)$ as given in Eqs. 4, 5 and 6 , respectively. Let α -cut, γ-cut and β-cut sets of the truth-membership, the indeterminacy-membership and the falsity-membership functions of $\widetilde{a}$ be given by Eqs. 7, 8 and 9 , respectively. Choosing $f(\alpha)$ and $g(\beta)$ as $f(\alpha) = \alpha$ and $g(\beta) = 1 - \beta$, respectively. Then, values and ambiguities of truth-membership function, indeterminacy-membership function, falsity-membership function are $\mathcal{V}(\mu_{\widetilde{a}})$, $\mathcal{V}(\rho_{\widetilde{a}})$, $\mathcal{V}(\nu_{\widetilde{a}})$ and $\mathcal{A}(\mu_{\widetilde{a}})$, $\mathcal{A}(\rho_{\widetilde{a}})$, $\mathcal{A}(\nu_{\widetilde{a}})$ of $\widetilde{a}$ can be derived using the definitions of values and ambiguities defined in the Definition 3.1 as

$$\begin{cases} \mathcal{V}(\mu_{\widetilde{a}}) = \frac{1}{6}[a_1 + 2(x_{0,1} + y_{0,1}) + b_1], \\ \mathcal{V}(\rho_{\widetilde{a}}) = \frac{1}{6}[a_2 + 2(x_{0,2} + y_{0,2}) + b_2], \\ \mathcal{V}(\nu_{\widetilde{a}}) = \frac{1}{6}[a_3 + 2(x_{0,3} + y_{0,3}) + b_3], \end{cases} \tag{15}$$

and

$$\begin{cases} \mathcal{A}(\mu_{\widetilde{a}}) = \frac{1}{6}[b_1 + 2(y_{0,1} - x_{0,1}) - a_1], \\ \mathcal{A}(\rho_{\widetilde{a}}) = \frac{1}{6}[b_2 + 2(y_{0,2} - x_{0,2}) - a_2], \\ \mathcal{A}(\nu_{\widetilde{a}}) = \frac{1}{6}[b_3 + 2(y_{0,3} - x_{0,3}) - a_3], \end{cases} \tag{16}$$

respectively.

Now, a few properties of the quantities values and ambiguities of truth-membership, indeterminacy-membership and falsity-membership functions are being discussed through a few propositions which will be essential for further discussion about the proposed methodology. The above Definition 3.1 of the values and the ambiguities are the basic definitions based on which the proposed method of ranking SVNNs is being formulated.

Proposition 3.1 *Let* $\widetilde{a} \in \mathcal{NF}$. *Then the inequalities* $\sup \mathrm{supp}(\mu_{\widetilde{a}}) \geq \mathcal{V}(\mu_{\widetilde{a}}) \geq \inf \mathrm{supp}(\mu_{\widetilde{a}})$, $\sup \mathrm{supp}(\rho_{\widetilde{a}}) \geq \mathcal{V}(\rho_{\widetilde{a}}) \geq \inf \mathrm{supp}(\mu_{\widetilde{a}})$ *and* $\sup \mathrm{supp}(\nu_{\widetilde{a}}) \geq \mathcal{V}(\nu_{\widetilde{a}}) \geq \inf \mathrm{supp}(\nu_{\widetilde{a}})$ *hold, that is, the value of truth-membership function lies in the support of truth-membership function, the value of indeterminacy-membership function lies in the support of indeterminacy-membership function and the value of falsity-membership function lies in the support of the falsity-membership function.*

Proof Let $\widetilde{a} \in \mathcal{NF}$ and $\widetilde{a}_\alpha = [L^{\mu}_{\widetilde{a}}(\alpha), R^{\mu}_{\widetilde{a}}(\alpha)]$ be the α-cut sets of truth-membership function, $\widetilde{a}_\gamma = [L^{\rho}_{\widetilde{a}}(\gamma), R^{\rho}_{\widetilde{a}}(\gamma)]$ be the γ-cut sets of indeterminacy-membership function and $\widetilde{a}_\beta = [L^{\nu}_{\widetilde{a}}(\beta), R^{\nu}_{\widetilde{a}}(\beta)]$ be the β-cut sets of falsity-membership function. It is true that $[L^{\mu}_{\widetilde{a}}(\alpha), R^{\mu}_{\widetilde{a}}(\alpha)] \subseteq \mathrm{supp}(\mu_{\widetilde{a}}) = [L^{\mu}_{\widetilde{a}}(0), R^{\mu}_{\widetilde{a}}(0)]$. Therefore, it follows that $R^{\mu}_{\widetilde{a}}(0) \geq R^{\mu}_{\widetilde{a}}(\alpha) \geq L^{\mu}_{\widetilde{a}}(\alpha) \geq L^{\mu}_{\widetilde{a}}(0)$, which implies that

$$R^{\mu}_{\widetilde{a}}(0) \geq \frac{1}{2}\left[L^{\mu}_{\widetilde{a}}(\alpha) + R^{\mu}_{\widetilde{a}}(\alpha)\right] \geq L^{\mu}_{\widetilde{a}}(0)$$
$$\text{or}, R^{\mu}_{\widetilde{a}}(0)\int_0^1 f(r)dr \geq \frac{1}{2}\int_0^1 \left(L^{\mu}_{\widetilde{a}}(\alpha) + R^{\mu}_{\widetilde{a}}(\alpha)\right)f(r)dr \geq L^{\mu}_{\widetilde{a}}(0)\int_0^1 f(r)dr.$$

Thus, it implies that $\sup \text{supp}(\mu_{\widetilde{a}}) \geq \mathcal{V}(\mu_{\widetilde{a}}) \geq \inf \text{supp}(\mu_{\widetilde{a}})$. Similarly, $[L^{\rho}_{\widetilde{a}}(\gamma), R^{\rho}_{\widetilde{a}}(\gamma)] \subseteq \text{supp}(\rho_{\widetilde{a}}) = [L^{\rho}_{\widetilde{a}}(1), R^{\rho}_{\widetilde{a}}(1)]$. Then, it follows that $R^{\rho}_{\widetilde{a}}(1) \geq R^{\rho}_{\widetilde{a}}(\gamma) \geq L^{\rho}_{\widetilde{a}}(\gamma) \geq L^{\rho}_{\widetilde{a}}(1)$, which implies that

$$R^{\rho}_{\widetilde{a}}(1) \geq \frac{1}{2}\left[L^{\rho}_{\widetilde{a}}(\gamma) + R^{\rho}_{\widetilde{a}}(\gamma)\right] \geq L^{\rho}_{\widetilde{a}}(1)$$
$$\text{or}, R^{\rho}_{\widetilde{a}}(1)\int_0^1 g(r)dr \geq \frac{1}{2}\int_0^1 \left(L^{\rho}_{\widetilde{a}}(\gamma) + R^{\rho}_{\widetilde{a}}(\gamma)\right)g(r)dr \geq L^{\rho}_{\widetilde{a}}(1)\int_0^1 g(r)dr.$$

Thus, it implies that $\sup \text{supp}(\rho_{\widetilde{a}}) \geq \mathcal{V}(\rho_{\widetilde{a}}) \geq \inf \text{supp}(\rho_{\widetilde{a}})$. Similarly, $[L^{\nu}_{\widetilde{a}}(\beta), R^{\nu}_{\widetilde{a}}(\beta)] \subseteq \text{supp}(\nu_{\widetilde{a}}) = [L^{\nu}_{\widetilde{a}}(1), R^{\nu}_{\widetilde{a}}(1)]$. So, it follows that $R^{\nu}_{\widetilde{a}}(1) \geq R^{\nu}_{\widetilde{a}}(\beta) \geq L^{\nu}_{\widetilde{a}}(\beta) \geq L^{\nu}_{\widetilde{a}}(1)$, which implies that

$$R^{\nu}_{\widetilde{a}}(1) \geq \frac{1}{2}\left[L^{\nu}_{\widetilde{a}}(\beta) + R^{\nu}_{\widetilde{a}}(\beta)\right] \geq L^{\nu}_{\widetilde{a}}(1)$$
$$\text{or}, R^{\nu}_{\widetilde{a}}(1)\int_0^1 g(r)dr \geq \frac{1}{2}\int_0^1 \left(L^{\nu}_{\widetilde{a}}(\beta) + R^{\nu}_{\widetilde{a}}(\beta)\right)g(r)dr \geq L^{\nu}_{\widetilde{a}}(1)\int_0^1 g(r)dr.$$

Hence, it implies that $\sup \text{supp}(\nu_{\widetilde{a}}) \geq \mathcal{V}(\nu_{\widetilde{a}}) \geq \inf \text{supp}(\nu_{\widetilde{a}})$. □

Proposition 3.2 *Let $\widetilde{a}, \widetilde{b} \in \mathcal{NF}$. Then*

$$\mathcal{V}(\mu_{\widetilde{a}+\widetilde{b}}) = \mathcal{V}(\mu_{\widetilde{a}}) + \mathcal{V}(\mu_{\widetilde{b}}),\ \mathcal{V}(\rho_{\widetilde{a}+\widetilde{b}}) = \mathcal{V}(\rho_{\widetilde{a}}) + \mathcal{V}(\rho_{\widetilde{b}}),\ \mathcal{V}(\nu_{\widetilde{a}+\widetilde{b}}) = \mathcal{V}(\nu_{\widetilde{a}}) + \mathcal{V}(\nu_{\widetilde{b}})$$

and

$$\mathcal{V}(\mu_{\widetilde{a}-\widetilde{b}}) = \mathcal{V}(\mu_{\widetilde{a}}) - \mathcal{V}(\mu_{\widetilde{b}}),\ \mathcal{V}(\rho_{\widetilde{a}-\widetilde{b}}) = \mathcal{V}(\rho_{\widetilde{a}}) - \mathcal{V}(\rho_{\widetilde{b}}),\ \mathcal{V}(\nu_{\widetilde{a}-\widetilde{b}}) = \mathcal{V}(\nu_{\widetilde{a}}) - \mathcal{V}(\nu_{\widetilde{b}}).$$

Proof Let $\widetilde{a}, \widetilde{b} \in \mathcal{NF}$, $\widetilde{a}_{\alpha} = [L^{\mu}_{\widetilde{a}}(\alpha), R^{\mu}_{\widetilde{a}}(\alpha)]$ and $\widetilde{b}_{\alpha} = [L^{\mu}_{\widetilde{b}}(\alpha), R^{\mu}_{\widetilde{b}}(\alpha)]$ be the α-cut sets of truth-membership functions of $\widetilde{a}$ and $\widetilde{b}$ respectively, $\widetilde{a}_{\gamma} = [L^{\rho}_{\widetilde{a}}(\gamma), R^{\rho}_{\widetilde{a}}(\gamma)]$ and $\widetilde{b}_{\gamma} = [L^{\rho}_{\widetilde{b}}(\gamma), R^{\rho}_{\widetilde{b}}(\gamma)]$ be the γ-cut sets of indeterminacy-membership functions of $\widetilde{a}$ and $\widetilde{b}$ respectively, $\widetilde{a}_{\beta} = [L^{\nu}_{\widetilde{a}}(\beta), R^{\nu}_{\widetilde{a}}(\beta)]$ and $\widetilde{b}_{\beta} = [L^{\nu}_{\widetilde{b}}(\beta), R^{\nu}_{\widetilde{b}}(\beta)]$ be the β-cut sets of falsity-membership functions of $\widetilde{a}$ and $\widetilde{b}$, respectively. Then, it follows that

$$\begin{aligned}
\mathcal{V}(\mu_{\widetilde{a}+\widetilde{b}}) &= \int_0^1 f(r)\Big[(R^{\mu}_{\widetilde{a}}(r) + R^{\mu}_{\widetilde{b}}(r)) + (L^{\mu}_{\widetilde{a}}(r) + L^{\mu}_{\widetilde{b}}(r))\Big]dr \\
&= \int_0^1 f(r)(R^{\mu}_{\widetilde{a}}(r) + L^{\mu}_{\widetilde{a}}(r))dr + \int_0^1 g(r)(R^{\mu}_{\widetilde{b}}(r) + L^{\mu}_{\widetilde{b}}(r))dr \\
&= \mathcal{V}(\mu_{\widetilde{a}}) + \mathcal{V}(\mu_{\widetilde{b}}), \\
\mathcal{V}(\rho_{\widetilde{a}+\widetilde{b}}) &= \int_0^1 g(r)\Big[(R^{\rho}_{\widetilde{a}}(r) + R^{\rho}_{\widetilde{b}}(r)) + (L^{\rho}_{\widetilde{a}}(r) + L^{\rho}_{\widetilde{b}}(r))\Big]dr \\
&= \int_0^1 g(r)(R^{\rho}_{\widetilde{a}}(r) + L^{\rho}_{\widetilde{a}}(r))dr + \int_0^1 g(r)(R^{\rho}_{\widetilde{b}}(r) + L^{\rho}_{\widetilde{b}}(r))dr \\
&= \mathcal{V}(\rho_{\widetilde{a}}) + \mathcal{V}(\rho_{\widetilde{b}}), \\
\mathcal{V}(\nu_{\widetilde{a}+\widetilde{b}}) &= \int_0^1 g(r)\Big[(R^{\nu}_{\widetilde{a}}(r) + R^{\nu}_{\widetilde{b}}(r)) + (L^{\nu}_{\widetilde{a}}(r) + L^{\nu}_{\widetilde{b}}(r))\Big]dr \\
&= \int_0^1 g(r)(R^{\nu}_{\widetilde{a}}(r) + L^{\nu}_{\widetilde{a}}(r))dr + \int_0^1 g(r)(R^{\nu}_{\widetilde{b}}(r) + L^{\nu}_{\widetilde{b}}(r))dr \\
&= \mathcal{V}(\nu_{\widetilde{a}}) + \mathcal{V}(\nu_{\widetilde{b}}).
\end{aligned}$$

Similarly, it can proved that the equalities $\mathcal{V}(\mu_{\widetilde{a}-\widetilde{b}}) = \mathcal{V}(\mu_{\widetilde{a}}) - \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}-\widetilde{b}}) = \mathcal{V}(\rho_{\widetilde{a}}) - \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}-\widetilde{b}}) = \mathcal{V}(\nu_{\widetilde{a}}) - \mathcal{V}(\nu_{\widetilde{b}})$ hold. □

Proposition 3.3 *Let $\widetilde{a}, \widetilde{b} \in \mathcal{NF}$. Then*

$$\mathcal{A}(\mu_{\widetilde{a}+\widetilde{b}}) = \mathcal{A}(\mu_{\widetilde{a}}) + \mathcal{A}(\mu_{\widetilde{b}}),\ \mathcal{A}(\rho_{\widetilde{a}+\widetilde{b}}) = \mathcal{A}(\rho_{\widetilde{a}}) + \mathcal{A}(\rho_{\widetilde{b}}),\ \mathcal{A}(\nu_{\widetilde{a}+\widetilde{b}}) = \mathcal{A}(\nu_{\widetilde{a}}) + \mathcal{A}(\nu_{\widetilde{b}})$$

and

$$\mathcal{A}(\mu_{\widetilde{a}-\widetilde{b}}) = \mathcal{A}(\mu_{\widetilde{a}}) + \mathcal{A}(\mu_{\widetilde{b}}),\ \mathcal{A}(\rho_{\widetilde{a}-\widetilde{b}}) = \mathcal{A}(\rho_{\widetilde{a}}) + \mathcal{A}(\rho_{\widetilde{b}}),\ \mathcal{A}(\nu_{\widetilde{a}-\widetilde{b}}) = \mathcal{A}(\nu_{\widetilde{a}}) + \mathcal{A}(\nu_{\widetilde{b}}).$$

Proof Let $\widetilde{a}, \widetilde{b} \in \mathcal{NF}$, $\widetilde{a}_{\alpha} = [L^{\mu}_{\widetilde{a}}(\alpha), R^{\mu}_{\widetilde{a}}(\alpha)]$ and $\widetilde{b}_{\alpha} = [L^{\mu}_{\widetilde{b}}(\alpha), R^{\mu}_{\widetilde{b}}(\alpha)]$ be the α-cut sets of truth-membership functions of $\widetilde{a}$ and $\widetilde{b}$, respectively, $\widetilde{a}_{\gamma} = [L^{\rho}_{\widetilde{a}}(\gamma), R^{\rho}_{\widetilde{a}}(\gamma)]$ and $\widetilde{b}_{\gamma} = [L^{\rho}_{\widetilde{b}}(\gamma), R^{\rho}_{\widetilde{b}}(\gamma)]$ be the γ-cut sets of indeterminacy-membership functions of $\widetilde{a}$ and $\widetilde{b}$, respectively and $\widetilde{a}_{\beta} = [L^{\nu}_{\widetilde{a}}(\beta), R^{\nu}_{\widetilde{a}}(\beta)]$ and $\widetilde{b}_{\beta} = [L^{\nu}_{\widetilde{b}}(\beta), R^{\nu}_{\widetilde{b}}(\beta)]$ be the β-cut sets of falsity-membership functions of $\widetilde{a}$ and $\widetilde{b}$, respectively. Then, it follows that

$$\begin{aligned}
\mathcal{A}(\mu_{\widetilde{a}+\widetilde{b}}) &= \int_0^1 f(r)\Big[(R^{\mu}_{\widetilde{a}}(r) + R^{\mu}_{\widetilde{b}}(r)) - (L^{\mu}_{\widetilde{a}}(r) + L^{\mu}_{\widetilde{b}}(r))\Big]dr \\
&= \int_0^1 f(r)(R^{\mu}_{\widetilde{a}}(r) - L^{\mu}_{\widetilde{a}}(r))dr + \int_0^1 g(r)(R^{\mu}_{\widetilde{b}}(r) - L^{\mu}_{\widetilde{b}}(r))dr \\
&= \mathcal{A}(\mu_{\widetilde{a}}) + \mathcal{A}(\mu_{\widetilde{b}}), \\
\mathcal{A}(\rho_{\widetilde{a}+\widetilde{b}}) &= \int_0^1 g(r)\Big[(R^{\rho}_{\widetilde{a}}(r) + R^{\rho}_{\widetilde{b}}(r)) - (L^{\rho}_{\widetilde{a}}(r) + L^{\rho}_{\widetilde{b}}(r))\Big]dr \\
&= \int_0^1 g(r)(R^{\rho}_{\widetilde{a}}(r) - L^{\rho}_{\widetilde{a}}(r))dr + \int_0^1 g(r)(R^{\rho}_{\widetilde{b}}(r) - L^{\rho}_{\widetilde{b}}(r))dr \\
&= \mathcal{A}(\rho_{\widetilde{a}}) + \mathcal{A}(\rho_{\widetilde{b}}) \\
\mathcal{A}(\nu_{\widetilde{a}+\widetilde{b}}) &= \int_0^1 g(r)\Big[(R^{\nu}_{\widetilde{a}}(r) + R^{\nu}_{\widetilde{b}}(r)) - (L^{\nu}_{\widetilde{a}}(r) + L^{\nu}_{\widetilde{b}}(r))\Big]dr \\
&= \int_0^1 g(r)(R^{\nu}_{\widetilde{a}}(r) - L^{\nu}_{\widetilde{a}}(r))dr + \int_0^1 g(r)(R^{\nu}_{\widetilde{b}}(r) - L^{\nu}_{\widetilde{b}}(r))dr \\
&= \mathcal{A}(\nu_{\widetilde{a}}) + \mathcal{A}(\nu_{\widetilde{b}}).
\end{aligned}$$

Similarly, it can proved that the equalities $\mathcal{A}(\mu_{\widetilde{a}-\widetilde{b}}) = \mathcal{A}(\mu_{\widetilde{a}}) + \mathcal{A}(\mu_{\widetilde{b}})$, $\mathcal{A}(\rho_{\widetilde{a}-\widetilde{b}}) = \mathcal{A}(\rho_{\widetilde{a}}) + \mathcal{A}(\rho_{\widetilde{b}})$ and $\mathcal{A}(\nu_{\widetilde{a}-\widetilde{b}}) = \mathcal{A}(\nu_{\widetilde{a}}) + \mathcal{A}(\nu_{\widetilde{b}})$ hold. □

Proposition 3.4 *Let $\widetilde{a} \in \mathcal{NF}$, $k \in \mathbb{R} - \{0\}$ be any real number. Then the values and the ambiguities of truth-membership, indeterminacy-membership and falsity-membership functions hold the following equalities, that is,* $\mathcal{V}(\mu_{k\widetilde{a}}) = k\mathcal{V}(\mu_{\widetilde{a}})$, $\mathcal{V}(\rho_{k\widetilde{a}}) = k\mathcal{V}(\rho_{\widetilde{a}})$, $\mathcal{V}(\nu_{k\widetilde{a}}) = k\mathcal{V}(\nu_{\widetilde{a}})$ *and*

$$\mathcal{A}(\mu_{k\widetilde{a}}) = \begin{cases} k\mathcal{A}(\mu_{\widetilde{a}}) & \text{if } k > 0 \\ -k\mathcal{A}(\mu_{\widetilde{a}}) & \text{if } k < 0 \end{cases},$$

$$\mathcal{A}(\nu_{k\widetilde{a}}) = \begin{cases} k\mathcal{A}(\nu_{\widetilde{a}}) & \text{if } k > 0 \\ -k\mathcal{A}(\nu_{\widetilde{a}}) & \text{if } k < 0 \end{cases},$$

$$\mathcal{A}(\nu_{k\widetilde{a}}) = \begin{cases} k\mathcal{A}(\nu_{\widetilde{a}}) & \text{if } k > 0 \\ -k\mathcal{A}(\nu_{\widetilde{a}}) & \text{if } k < 0 \end{cases}.$$

Proof Let $\widetilde{a} \in \mathcal{NF}$ and $\widetilde{a}_\alpha = [L^{\mu}_{\widetilde{a}}(\alpha), R^{\mu}_{\widetilde{a}}(\alpha)]$ be the α-cut sets of truth-membership function, $\widetilde{a}_\gamma = [L^{\rho}_{\widetilde{a}}(\gamma), R^{\rho}_{\widetilde{a}}(\gamma)]$ be the γ-cut sets of indeterminacy-membership function and $\widetilde{a}_\beta = [L^{\nu}_{\widetilde{a}}(\beta), R^{\nu}_{\widetilde{a}}(\beta)]$ be the β-cut sets of falsity-membership function. Now, for $k(>0) \in \mathbb{R} - \{0\}$ it follows immediately from the Definition 3.1 and the definition of scalar multiplication that $\mathcal{V}(\mu_{k\widetilde{a}}) = k\mathcal{V}(\mu_{\widetilde{a}})$, $\mathcal{V}(\rho_{k\widetilde{a}}) = k\mathcal{V}(\rho_{\widetilde{a}})$, $\mathcal{V}(\nu_{k\widetilde{a}}) = k\mathcal{V}(\nu_{\widetilde{a}})$ and $\mathcal{A}(\mu_{k\widetilde{a}}) = k\mathcal{A}(\mu_{\widetilde{a}})$, $\mathcal{A}(\rho_{k\widetilde{a}}) = k\mathcal{A}(\rho_{\widetilde{a}})$, $\mathcal{A}(\nu_{k\widetilde{a}}) = k\mathcal{A}(\nu_{\widetilde{a}})$. Let $k < 0$. Assume $k = -m < 0$ Then it follows the Definition 3.1 and the definition of scalar multiplication that $\mathcal{V}(\mu_{-m\widetilde{a}}) = -m\mathcal{V}(\mu_{\widetilde{a}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) = -m\mathcal{A}(\rho_{\widetilde{a}})$, $\mathcal{V}(\nu_{-m\widetilde{a}}) = -m\mathcal{V}(\nu_{\widetilde{a}})$ and $\mathcal{A}(\mu_{-m\widetilde{a}}) = m\mathcal{A}(\mu_{\widetilde{a}})$, $\mathcal{A}(\rho_{-m\widetilde{a}}) = m\mathcal{A}(\rho_{\widetilde{a}})$, $\mathcal{A}(\nu_{-m\widetilde{a}}) = m\mathcal{A}(\nu_{\widetilde{a}})$. Hence, the proposition holds. □

Proposition 3.5 *Let $\widetilde{a} \in \mathcal{NF}$, $-\widetilde{a} \in \mathcal{NF}$ be its image. Then* $\mathcal{V}(\mu_{-\widetilde{a}}) = -\mathcal{V}(\mu_{\widetilde{a}})$, $\mathcal{V}(\rho_{-\widetilde{a}}) = -\mathcal{V}(\rho_{\widetilde{a}})$, $\mathcal{V}(\nu_{-\widetilde{a}}) = -\mathcal{V}(\nu_{\widetilde{a}})$ *and* $\mathcal{A}(\mu_{-\widetilde{a}}) = \mathcal{A}(\mu_{\widetilde{a}})$, $\mathcal{A}(\rho_{-\widetilde{a}}) = \mathcal{A}(\rho_{\widetilde{a}})$, $\mathcal{A}(\nu_{-\widetilde{a}}) = \mathcal{A}(\nu_{\widetilde{a}})$.

Proof The proof is very trivial, as this proposition is a particular case of the above Proposition 3.4. Thus, the proof follows immediately taking $k = -1$ in its proof. $\square$

Proposition 3.6 *Let* $\widetilde{a}, \widetilde{b} \in \mathcal{NF}$, *such that* $\inf \operatorname{supp}(\mu_{\widetilde{a}}) > \sup \operatorname{supp}(\mu_{\widetilde{b}})$, $\inf \operatorname{supp}(\rho_{\widetilde{a}}) > \sup \operatorname{supp}(\rho_{\widetilde{b}})$ *and* $\inf \operatorname{supp}(\nu_{\widetilde{a}}) > \sup \operatorname{supp}(\nu_{\widetilde{b}})$, *then* $\mathcal{V}(\mu_{\widetilde{a}}) > \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) > \mathcal{V}(\rho_{\widetilde{b}})$ *and* $\mathcal{V}(\nu_{\widetilde{a}}) > \mathcal{V}(\nu_{\widetilde{b}})$, *respectively.*

Proof Let $\widetilde{a}, \widetilde{b} \in \mathcal{NF}$, $\widetilde{a}_\alpha = [L^\mu_{\widetilde{a}}(\alpha), R^\mu_{\widetilde{a}}(\alpha)]$ and $\widetilde{b}_\alpha = [L^\mu_{\widetilde{b}}(\alpha), R^\mu_{\widetilde{b}}(\alpha)]$ be the α-cut sets of truth-membership functions of $\widetilde{a}$ and $\widetilde{b}$, respectively, $\widetilde{a}_\gamma = [L^\rho_{\widetilde{a}}(\gamma), R^\rho_{\widetilde{a}}(\gamma)]$ and $\widetilde{b}_\gamma = [L^\rho_{\widetilde{b}}(\gamma), R^\rho_{\widetilde{b}}(\gamma)]$ be the γ-cut sets of indeterminacy-membership functions of $\widetilde{a}$ and $\widetilde{b}$, respectively and $\widetilde{a}_\beta = [L^\nu_{\widetilde{a}}(\beta), R^\nu_{\widetilde{a}}(\beta)]$ and $\widetilde{b}_\beta = [L^\nu_{\widetilde{b}}(\beta), R^\nu_{\widetilde{b}}(\beta)]$ be the β-cut sets of falsity-membership functions of $\widetilde{a}$ and $\widetilde{b}$, respectively. Now, if $\inf \operatorname{supp}(\mu_{\widetilde{a}}) > \sup \operatorname{supp}(\mu_{\widetilde{b}})$, $\inf \operatorname{supp}(\rho_{\widetilde{a}}) > \sup \operatorname{supp}(\rho_{\widetilde{b}})$ and $\inf \operatorname{supp}(\nu_{\widetilde{a}}) > \sup \operatorname{supp}(\nu_{\widetilde{b}})$, then $L^\mu_{\widetilde{a}}(\alpha) > R^\mu_{\widetilde{b}}(\alpha)$, $L^\rho_{\widetilde{a}}(\gamma) > R^\rho_{\widetilde{b}}(\gamma)$and $L^\nu_{\widetilde{a}}(\alpha) > R^\nu_{\widetilde{b}}(\alpha)$. Thus, it implies that $R^\mu_{\widetilde{a}}(\alpha) \geq L^\mu_{\widetilde{a}}(\alpha) > R^\mu_{\widetilde{b}}(\alpha) \geq L^\mu_{\widetilde{b}}(\alpha)$, $R^\rho_{\widetilde{a}}(\gamma) \geq L^\rho_{\widetilde{a}}(\gamma) > R^\rho_{\widetilde{b}}(\gamma) \geq L^\rho_{\widetilde{b}}(\gamma)$ and $R^\nu_{\widetilde{a}}(\beta) \geq L^\nu_{\widetilde{a}}(\beta) > R^\nu_{\widetilde{b}}(\beta) \geq L^\nu_{\widetilde{b}}(\beta)$. So, it follows immediately that

$$\begin{aligned} & R^\mu_{\widetilde{a}}(\alpha) + L^\mu_{\widetilde{a}}(\alpha) > R^\mu_{\widetilde{b}}(\alpha) + L^\mu_{\widetilde{b}}(\alpha) \\ \text{or, } & \int_0^1 f(r)(R^\mu_{\widetilde{a}}(r) + L^\mu_{\widetilde{a}}(r))dr > \int_0^1 f(r)(R^\mu_{\widetilde{b}}(r) + L^\mu_{\widetilde{b}}(r))dr \end{aligned} \tag{17}$$

,

$$\begin{aligned} & R^\rho_{\widetilde{a}}(\gamma) + L^\rho_{\widetilde{a}}(\gamma) > R^\rho_{\widetilde{b}}(\gamma) + L^\rho_{\widetilde{b}}(\gamma) \\ \text{or, } & \int_0^1 g(r)(R^\rho_{\widetilde{a}}(r) + L^\rho_{\widetilde{a}}(r))dr > \int_0^1 g(r)(R^\rho_{\widetilde{b}}(r) + L^\rho_{\widetilde{b}}(r))dr \end{aligned} \tag{18}$$

and

$$\begin{aligned} & R^\nu_{\widetilde{a}}(\beta) + L^\nu_{\widetilde{a}}(\beta) > R^\nu_{\widetilde{b}}(\beta) + L^\nu_{\widetilde{b}}(\beta) \\ \text{or, } & \int_0^1 g(r)(R^\nu_{\widetilde{a}}(r) + L^\nu_{\widetilde{a}}(r))dr > \int_0^1 g(r)(R^\nu_{\widetilde{b}}(r) + L^\nu_{\widetilde{b}}(r))dr \end{aligned} \tag{19}$$

Hence, from the inequalities 17, 18 and 19 , the result follows immediately. $\square$

Proposition 3.7 *If* $\widetilde{a} \in \mathcal{NF}$ *be a SVNN such that it is symmetric about the y-axis, then* $\mathcal{V}(\mu_{\widetilde{a}}) = 0$, $\mathcal{V}(\rho_{\widetilde{a}}) = 0$ *and* $\mathcal{V}(\nu_{\widetilde{a}}) = 0$.

Proof Let $\widetilde{a} \in \mathcal{NF}$ and $\widetilde{a}_\alpha = [L^\mu_{\widetilde{a}}(\alpha), R^\mu_{\widetilde{a}}(\alpha)]$ be the α-cut sets of truth-membership function, $\widetilde{a}_\gamma = [L^\rho_{\widetilde{a}}(\gamma), R^\rho_{\widetilde{a}}(\gamma)]$ be the γ-cut sets of indeterminacy-membership function and $\widetilde{a}_\beta = [L^\nu_{\widetilde{a}}(\beta), R^\nu_{\widetilde{a}}(\beta)]$ be the β-cut sets of falsity-membership function. Since, $\widetilde{a}$ is symmetric about the y-axis, it follows that $-L^\mu_{\widetilde{a}}(\alpha) = R^\mu_{\widetilde{a}}(\alpha)$, $-L^\rho_{\widetilde{a}}(\gamma) = R^\rho_{\widetilde{a}}(\gamma)$ and $-L^\nu_{\widetilde{a}}(\beta) = R^\nu_{\widetilde{a}}(\beta)$. Then, it is evident that $\mathcal{V}(\mu_{\widetilde{a}}) = 0$, $\mathcal{V}(\rho_{\widetilde{a}}) = 0$ and $\mathcal{V}(\nu_{\widetilde{a}}) = 0$. $\square$

Proposition 3.8 *For an arbitrary SVNN* $\widetilde{a} \in \mathcal{NF}$, $\mathcal{A}(\mu_{\widetilde{a}}) \geq 0$, $\mathcal{A}(\rho_{\widetilde{a}}) \geq 0$ *and* $\mathcal{A}(\nu_{\widetilde{a}}) \geq 0$.

Proof Let $\widetilde{a} \in \mathcal{NF}$. Then the α-cut sets, γ-cut sets and the β-cut sets of truth-membership, indeterminacy-membership and falsity-membership functions of $\widetilde{a}$ be $\widetilde{a}_\alpha = [L^\mu_{\widetilde{a}}(\alpha), R^\mu_{\widetilde{a}}(\alpha)]$, $\widetilde{a}_\gamma = [L^\rho_{\widetilde{a}}(\gamma), R^\rho_{\widetilde{a}}(\gamma)]$ and $\widetilde{a}_\beta = [L^\nu_{\widetilde{a}}(\beta), R^\nu_{\widetilde{a}}(\beta)]$, respectively. As $R^\mu_{\widetilde{a}}(\alpha) - L^\mu_{\widetilde{a}}(\alpha) \geq 0$, $R^\rho_{\widetilde{a}}(\gamma) - L^\rho_{\widetilde{a}}(\gamma) \geq 0$ and $R^\nu_{\widetilde{a}}(\alpha) - L^\nu_{\widetilde{a}}(\alpha) \geq 0$, it follows that $\int_0^1 f(r)(R^\mu_{\widetilde{a}}(r) - L^\mu_{\widetilde{a}}(r))dr \geq 0$, $\int_0^q g(r)(R^\rho_{\widetilde{a}}(r) - L^\rho_{\widetilde{a}}(r))dr \geq 0$ and $\int_0^1 g(r)(R^\nu_{\widetilde{a}}(r) - L^\nu_{\widetilde{a}}(r))dr \geq 0$. Hence, the result $\mathcal{A}(\mu_{\widetilde{a}}) \geq 0$, $\mathcal{A}(\rho_{\widetilde{a}}) \geq 0$ and $\mathcal{A}(\nu_{\widetilde{a}}) \geq 0$. □

3.2 The proposed method

Let $\widetilde{a}, \widetilde{b} \in \mathcal{NF}$, $\widetilde{a}_\alpha = [L^\mu_{\widetilde{a}}(\alpha), R^\mu_{\widetilde{a}}(\alpha)]$ and $\widetilde{b}_\alpha = [L^\mu_{\widetilde{b}}(\alpha), R^\mu_{\widetilde{b}}(\alpha)]$ be the α-cut sets of truth-membership functions of $\widetilde{a}$ and $\widetilde{b}$, respectively, $\widetilde{a}_\gamma = [L^\rho_{\widetilde{a}}(\gamma), R^\rho_{\widetilde{a}}(\gamma)]$ and $\widetilde{b}_\gamma = [L^\rho_{\widetilde{b}}(\gamma), R^\rho_{\widetilde{b}}(\gamma)]$ be the γ-cut sets of indeterminacy-membership functions of $\widetilde{a}$ and $\widetilde{b}$, respectively, $\widetilde{a}_\beta = [L^\nu_{\widetilde{a}}(\beta), R^\nu_{\widetilde{a}}(\beta)]$ and $\widetilde{b}_\beta = [L^\nu_{\widetilde{b}}(\beta), R^\nu_{\widetilde{b}}(\beta)]$ be the β-cut sets of falsity-membership functions of $\widetilde{a}$ and $\widetilde{b}$, respectively. Let, $\mathcal{V}(\mu_{\widetilde{a}})$, $\mathcal{V}(\rho_{\widetilde{a}})$, $\mathcal{V}(\nu_{\widetilde{a}})$ and $\mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{b}})$, $\mathcal{V}(\nu_{\widetilde{b}})$ be the values of truth-membership, indeterminacy-membership and falsity-membership functions of $\widetilde{a}$ and $\widetilde{b}$, respectively; and $\mathcal{A}(\mu_{\widetilde{a}})$, $\mathcal{A}(\rho_{\widetilde{a}})$, $\mathcal{A}(\nu_{\widetilde{a}})$ and $\mathcal{A}(\mu_{\widetilde{b}})$, $\mathcal{A}(\rho_{\widetilde{b}})$, $\mathcal{A}(\nu_{\widetilde{b}})$ be the ambiguities of truth-membership, indeterminacy-membership, falsity-membership functions of $\widetilde{a}$ and $\widetilde{b}$, respectively. Let $\lambda \in [0, 1]$ be the index of optimism. Then the ranking index $\mathcal{R}_\lambda$ is defined as

$$\mathcal{R}_\lambda(\widetilde{a}, \theta_1, \theta_2, \theta_3) = \lambda\{\mathcal{V}(\mu_{\widetilde{a}}) + \theta_1\mathcal{A}(\mu_{\widetilde{a}})\} + (1-\lambda)\{\mathcal{V}(\rho_{\widetilde{a}}) + \theta_2\mathcal{A}(\rho_{\widetilde{a}}) + \mathcal{V}(\nu_{\widetilde{a}}) + \theta_3\mathcal{A}(\nu_{\widetilde{a}})\} \tag{20}$$

where $\theta_1, \theta_2, \theta_2 : \mathcal{NF} \to \{0, -1, 1\}$ are the ambiguity inclusion function of truth-membership, indeterminacy-membership, falsity-membership functions such that

$$\theta_1 = \begin{cases} 0, & \text{if } \mathcal{V}(\mu_{\widetilde{a}}) \neq \mathcal{V}(\mu_{\widetilde{b}}) \\ -1, & \text{if } \mathcal{V}(\mu_{\widetilde{a}}) = \mathcal{V}(\mu_{\widetilde{b}}) \text{ and } t_{\theta_1} \geq 0 \\ 1, & \text{if } \mathcal{V}(\mu_{\widetilde{a}}) = \mathcal{V}(\mu_{\widetilde{b}}) \text{ and } t_{\theta_1} < 0 \end{cases}$$

$$\theta_2 = \begin{cases} 0, & \text{if } \mathcal{V}(\rho_{\widetilde{a}}) \neq \mathcal{V}(\rho_{\widetilde{b}}) \\ -1, & \text{if } \mathcal{V}(\rho_{\widetilde{a}}) = \mathcal{V}(\rho_{\widetilde{b}}) \text{ and } t_{\theta_2} \geq 0 \\ 1, & \text{if } \mathcal{V}(\rho_{\widetilde{a}}) = \mathcal{V}(\rho_{\widetilde{b}}) \text{ and } t_{\theta_2} < 0 \end{cases}$$

$$\theta_3 = \begin{cases} 0, & \text{if } \mathcal{V}(\nu_{\widetilde{a}}) \neq \mathcal{V}(\nu_{\widetilde{b}}) \\ -1, & \text{if } \mathcal{V}(\nu_{\widetilde{a}}) = \mathcal{V}(\nu_{\widetilde{b}}) \text{ and } t_{\theta_3} \geq 0 \\ 1, & \text{if } \mathcal{V}(\nu_{\widetilde{a}}) = \mathcal{V}(\nu_{\widetilde{b}}) \text{ and } t_{\theta_3} < 0 \end{cases}$$

where $t_{\theta_1} = \frac{1}{2}[L^\mu_{\widetilde{a}}(0) + R^\mu_{\widetilde{a}}(0)]$ or $t_{\theta_1} = \frac{1}{2}[L^\mu_{\widetilde{b}}(0) + R^\mu_{\widetilde{b}}(0)]$ and $t_{\theta_2} = \frac{1}{2}[L^\rho_{\widetilde{a}}(1) + R^\rho_{\widetilde{a}}(1)]$ or $t_{\theta_2} = \frac{1}{2}[L^\rho_{\widetilde{b}}(1) + R^\rho_{\widetilde{b}}(1)]$ and $t_{\theta_3} = \frac{1}{2}[L^\nu_{\widetilde{a}}(1) + R^\nu_{\widetilde{a}}(1)]$ or $t_{\theta_3} = \frac{1}{2}[L^\nu_{\widetilde{b}}(1) + R^\nu_{\widetilde{b}}(1)]$.

The ordering of SVNNs, $\widetilde{a}, \widetilde{b} \in \mathcal{NF}$, based on the ranking index $\mathcal{R}_\lambda$ for $0 \leq \lambda \leq 1$ is defined by relations $\succ$, $\prec$ and $\sim$ as;

- $\widetilde{a} \succ \widetilde{b}$ if, and only if, $\mathcal{R}_\lambda(\widetilde{a}, \theta_1, \theta_2, \theta_3) > \mathcal{R}_\lambda(\widetilde{b}, \theta_1, \theta_2, \theta_3)$;
- $\widetilde{a} \prec \widetilde{b}$ if, and only if, $\mathcal{R}_\lambda(\widetilde{a}, \theta_1, \theta_2, \theta_3) < \mathcal{R}_\lambda(\widetilde{b}, \theta_1, \theta_2, \theta_3)$;
- $\widetilde{a} \sim \widetilde{b}$ if, and only if, $\mathcal{R}_\lambda(\widetilde{a}, \theta_1, \theta_2, \theta_3) = \mathcal{R}_\lambda(\widetilde{b}, \theta_1, \theta_2, \theta_3)$.

The order relations $\succeq$ and $\preceq$ are formulated as

- $\widetilde{a} \succeq \widetilde{b}$ if, and only if, $\widetilde{a} \succ \widetilde{b}$ or $\widetilde{a} \sim \widetilde{b}$.
- $\widetilde{a} \preceq \widetilde{b}$ if, and only if, $\widetilde{a} \prec \widetilde{b}$ or $\widetilde{a} \sim \widetilde{b}$.

The index of optimism $\lambda (0 \leq \lambda \leq 1)$ represents the decision-maker's attitude towards the uncertainty. An optimistic decision-maker ($\lambda = 1$) ranks the SVNNs based on truth-membership function without taking into account of indeterminacy-membership and falsity-membership functions. A pessimistic decision-maker ($\lambda = 0$) ranks the SVNNs based on indeterminacy-membership and falsity-membership functions without taking into account of the truth-membership function. Finally, moderate decision-maker ($\lambda = 0.5$) ranks the SVNNs taking into account of all the membership functions. Further, the θ_i's take care of the ranking index by deciding whether and how to include the ambiguities into the ranking index. If values are unequal, then the decision is based on values, in which case, $\theta_i = 0$. If values are equal, then the decision is based on ambiguities, in which case, $\theta_i = \pm 1$ depending upon positivity or negativity of t_{θ_i}'s.

Next theorem discusses the linearity property of the ranking index $\mathcal{R}_\lambda$. This linearity property will be further helpful in discussing the properties of the current method of ordering SVNNs.

Theorem 3.1 *Let* $\widetilde{a}, \widetilde{b} \in \mathcal{NF}$. *Then*

$$\mathcal{R}_\lambda(\widetilde{a} + \widetilde{b}, \theta_1, \theta_2, \theta_3) = \mathcal{R}_\lambda(\widetilde{a}, \theta_1, \theta_2, \theta_3) + \mathcal{R}_\lambda(\widetilde{b}, \theta_1, \theta_2, \theta_3).$$

Hence, it follows that

$$\mathcal{R}_\lambda(\widetilde{a} - \widetilde{b}, \theta_1, \theta_2, \theta_3) = \mathcal{R}_\lambda(\widetilde{a}, \theta_1, \theta_2, \theta_3) + \mathcal{R}_\lambda(-\widetilde{b}, \theta_1, \theta_2, \theta_3).$$

Proof Let $\widetilde{a}, \widetilde{b} \in \mathcal{NF}$. Then it follows from the Propositions 3.2 and 3.3 that

$$\mathcal{V}(\mu_{\widetilde{a}+\widetilde{b}}) = \mathcal{V}(\mu_{\widetilde{a}}) + \mathcal{V}(\mu_{\widetilde{b}}),\ \mathcal{V}(\rho_{\widetilde{a}+\widetilde{b}}) = \mathcal{V}(\rho_{\widetilde{a}}) + \mathcal{V}(\rho_{\widetilde{b}}),\ \mathcal{V}(\nu_{\widetilde{a}+\widetilde{b}}) = \mathcal{V}(\nu_{\widetilde{a}}) + \mathcal{V}(\nu_{\widetilde{b}}) \tag{21}$$

and

$$\mathcal{A}(\mu_{\widetilde{a}+\widetilde{b}}) = \mathcal{A}(\mu_{\widetilde{a}}) + \mathcal{A}(\mu_{\widetilde{b}}),\ \mathcal{A}(\rho_{\widetilde{a}+\widetilde{b}}) = \mathcal{A}(\rho_{\widetilde{a}}) + \mathcal{A}(\rho_{\widetilde{b}}),\ \mathcal{A}(\nu_{\widetilde{a}+\widetilde{b}}) = \mathcal{A}(\nu_{\widetilde{a}}) + \mathcal{A}(\nu_{\widetilde{b}}). \tag{22}$$

Thus, the results follows as

$$\begin{aligned}\mathcal{R}_\lambda(\widetilde{a} + \widetilde{b}, \theta_1, \theta_2, \theta_3) =& \lambda\{\mathcal{V}(\mu_{\widetilde{a}+\widetilde{b}}) + \theta_1 \mathcal{A}(\mu_{\widetilde{a}+\widetilde{b}})\} \\ &+ (1-\lambda)\{\mathcal{V}(\rho_{\widetilde{a}+\widetilde{b}}) + \theta_2 \mathcal{A}(\rho_{\widetilde{a}+\widetilde{b}}) + \mathcal{V}(\nu_{\widetilde{a}+\widetilde{b}}) + \theta_3 \mathcal{A}(\nu_{\widetilde{a}+\widetilde{b}})\}\end{aligned}$$

So, using the Eqs. 21 and 22 in the above equality, it can be derived easily that $\mathcal{R}_\lambda(\widetilde{a} + \widetilde{b}, \theta_1, \theta_2, \theta_3) = \mathcal{R}_\lambda(\widetilde{a}, \theta_1, \theta_2, \theta_3) + \mathcal{R}_\lambda(\widetilde{b}, \theta_1, \theta_2, \theta_3)$. Eventually, it is true that $\mathcal{R}_\lambda(\widetilde{a} - \widetilde{b}, \theta_1, \theta_2, \theta_3) = \mathcal{R}_\lambda(\widetilde{a} + (-\widetilde{b}), \theta_1, \theta_2, \theta_3) = \mathcal{R}_\lambda(\widetilde{a}, \theta_1, \theta_2, \theta_3) + \mathcal{R}_\lambda(-\widetilde{b}, \theta_1, \theta_2, \theta_3)$. □

Next a few theorems are being discussed. Eventually, from these discussion it will be evident that the current ranking index abide by the reasonable properties of Wang and Kerre (2001a, 2001b). Further, these theorems will give some light to newer properties of ranking fuzzy numbers as well as SVNNs.

Theorem 3.2 *Let $\widetilde{a}, \widetilde{b}, \widetilde{c} \in \mathcal{NF}$. Then the order relations $\succ$ and $\sim$ satisfy the following properties:*

1. *The order relation is reflexive, that is, $\widetilde{a} \succeq \widetilde{a}$.*
2. *The order relation is transitive, that is, if $\widetilde{a} \succ \widetilde{b}$ and $\widetilde{b} \succ \widetilde{c}$, then $\widetilde{a} \succ \widetilde{c}$. The same holds for the order relation $\succeq$.*
3. *The order relation follows the law of trichotomy, that is, $\widetilde{a} \succ \widetilde{b}$ or $\widetilde{b} \succeq \widetilde{a}$.*
4. *$\widetilde{a} = \widetilde{b}$ if and only if $\widetilde{a} \sim \widetilde{b}$.*

The detailed proof of this theorem is available in Appendix A.1. This theorem establishes the reflexivity, transivity and the trichotomy properties of the current method.

Theorem 3.3 *Let $\widetilde{a}, \widetilde{b} \in \mathcal{NF}$ and $\inf \operatorname{supp}(\mu_{\widetilde{a}}) > \sup \operatorname{supp}(\mu_{\widetilde{b}})$, $\inf \operatorname{supp}(\rho_{\widetilde{a}}) > \sup \operatorname{supp}(\rho_{\widetilde{b}})$ and $\inf \operatorname{supp}(\nu_{\widetilde{a}}) > \sup \operatorname{supp}(\nu_{\widetilde{b}})$. Then $\widetilde{a} \succeq \widetilde{b}$.*

Proof Let, $\inf \operatorname{supp}(\mu_{\widetilde{a}}) > \sup \operatorname{supp}(\mu_{\widetilde{b}})$, $\inf \operatorname{supp}(\rho_{\widetilde{a}}) > \sup \operatorname{supp}(\rho_{\widetilde{b}})$ and $\inf \operatorname{supp}(\nu_{\widetilde{a}}) > \sup \operatorname{supp}(\nu_{\widetilde{b}})$. Then by the Propositions 3.1 and 3.6 , it is evident that $\mathcal{V}(\mu_{\widetilde{a}}) > \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) > \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) > \mathcal{V}(\nu_{\widetilde{b}})$. Thus, $\theta_i = 0$, which implies that $\mathcal{R}_\lambda(\widetilde{a}, 0, 0, 0) > \mathcal{R}_\lambda(\widetilde{b}, 0, 0, 0)$. So, it follows that $\widetilde{a} \succ \widetilde{b}$, in fact by definition of $\succeq$, $\widetilde{a} \succeq \widetilde{b}$. □

Theorem 3.4 *Let $\widetilde{a}, \widetilde{b} \in \mathcal{NF}$ and $\inf \operatorname{supp}(\mu_{\widetilde{a}}) > \sup \operatorname{supp}(\mu_{\widetilde{b}})$, $\inf \operatorname{supp}(\rho_{\widetilde{a}}) > \sup \operatorname{supp}(\rho_{\widetilde{b}})$ and $\inf \operatorname{supp}(\nu_{\widetilde{a}}) > \sup \operatorname{supp}(\nu_{\widetilde{b}})$. Then $\widetilde{a} \succ \widetilde{b}$.*

Proof Let, $\inf \operatorname{supp}(\mu_{\widetilde{a}}) > \sup \operatorname{supp}(\mu_{\widetilde{b}})$, $\inf \operatorname{supp}(\rho_{\widetilde{a}}) > \sup \operatorname{supp}(\rho_{\widetilde{b}})$ and $\inf \operatorname{supp}(\nu_{\widetilde{a}}) > \sup \operatorname{supp}(\nu_{\widetilde{b}})$. Then by the Propositions 3.1 and 3.6 , trivially it follows that $\mathcal{V}(\mu_{\widetilde{a}}) > \mathcal{V}(\mu_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) > \mathcal{V}(\nu_{\widetilde{b}})$. Thus, $\theta_i = 0$, which implies that $\mathcal{R}_\lambda(\widetilde{a}, 0, 0, 0) > \mathcal{R}_\lambda(\widetilde{b}, 0, 0, 0)$. So, it follows that $\widetilde{a} \succ \widetilde{b}$. □

Theorem 3.5 *Let $\widetilde{a}, \widetilde{b}, \widetilde{c} \in \mathcal{NF}$. If $\widetilde{a} \succeq \widetilde{b}$, then $\widetilde{a} + \widetilde{c} \succeq \widetilde{b} + \widetilde{c}$.*

The detailed proof of this theorem is available in Appendix A.2.

Theorem 3.6 *Let $\widetilde{a}, \widetilde{b}, \widetilde{c} \in \mathcal{NF}$. If $\widetilde{a} + \widetilde{c} \succeq \widetilde{b} + \widetilde{c}$, then $\widetilde{a} \succeq \widetilde{b}$.*

Proof Let $\widetilde{a}, \widetilde{b}, \widetilde{c} \in \mathcal{NF}$ and $\widetilde{a} + \widetilde{c} \succeq \widetilde{b} + \widetilde{c}$. Then, it follows that $\mathcal{R}_\lambda(\widetilde{a} + \widetilde{c}, \theta_1, \theta_2, \theta_3) \geq \mathcal{R}_\lambda(\widetilde{b} + \widetilde{c}, \theta_1, \theta_2, \theta_3)$. Thus, by the Theorem 3.1, it follows that $\mathcal{R}_\lambda(\widetilde{a}, \theta_1, \theta_2, \theta_3) + \mathcal{R}_\lambda(\widetilde{c}, \theta_1, \theta_2, \theta_3) \geq \mathcal{R}_\lambda(\widetilde{b}, \theta_1, \theta_2, \theta_3) + \mathcal{R}_\lambda(\widetilde{c}, \theta_1, \theta_2, \theta_3)$. Eventually, it leads to $\mathcal{R}_\lambda(\widetilde{a}, \theta_1, \theta_2, \theta_3) \geq \mathcal{R}_\lambda(\widetilde{b}, \theta_1, \theta_2, \theta_3)$. Hence, the result follows immediately. □

Theorem 3.7 *Let $\widetilde{a}, \widetilde{b}, \widetilde{c} \in \mathcal{NF}$. If $\widetilde{a} \succ \widetilde{b}$, then $\widetilde{a} + \widetilde{c} \succ \widetilde{b} + \widetilde{c}$.*

Proof The proof is very trivial by taking into account '$\succ$' in the proof of the Theorem 3.5. □

Theorem 3.8 *Let $\widetilde{a}, \widetilde{b}, \widetilde{c} \in \mathcal{NF}$. If $\widetilde{a} + \widetilde{c} \succ \widetilde{b} + \widetilde{c}$, then $\widetilde{a} \succ \widetilde{b}$.*

Proof The proof is very trivial by taking into account '$\succ$' in the proof of the Theorem 3.6. $\square$

Theorem 3.9 *Let* $\widetilde{a}, \widetilde{b} \in \mathcal{NF}$ *and* $k \in \mathbb{R} - \{0\}$*. If* $\widetilde{a} \succeq \widetilde{b}$*, then* $k\widetilde{a} \succeq k\widetilde{b}$ *if* $k > 0$*, and* $k\widetilde{a} \preceq k\widetilde{b}$ *if* $k < 0$*.*

The detailed proof of this theorem is available in Appendix A.3.

Theorem 3.10 *Let* $\widetilde{a}, \widetilde{b} \in \mathcal{NF}$ *and* $k \in \mathbb{R} - \{0\}$*. If* $k\widetilde{a} \succeq k\widetilde{b}$*, then* $\widetilde{a} \succeq \widetilde{b}$ *if* $k > 0$*, and* $\widetilde{a} \preceq \widetilde{b}$ *if* $k < 0$*.*

The detailed proof of this theorem is available in Appendix A.4.

Theorem 3.11 *Let* $\widetilde{a}, \widetilde{b} \in \mathcal{NF}$ *and* $k \in \mathbb{R} - \{0\}$*. If* $\widetilde{a} \succ \widetilde{b}$*, then* $k\widetilde{a} \succ k\widetilde{b}$ *if* $k > 0$*, and* $k\widetilde{a} \prec k\widetilde{b}$ *if* $k < 0$*.*

Proof The proof is very trivial by taking into account '$\succ$' in the proof of the Theorem 3.9. $\square$

Theorem 3.12 *Let* $\widetilde{a}, \widetilde{b} \in \mathcal{NF}$ *and* $k \in \mathbb{R} - \{0\}$*. If* $k\widetilde{a} \succ k\widetilde{b}$*, then* $\widetilde{a} \succ \widetilde{b}$ *if* $k > 0$*, and* $\widetilde{a} \prec \widetilde{b}$ *if* $k < 0$*.*

Proof The proof is very trivial by taking into account '$\succ$' in the proof of the Theorem 3.10. $\square$

Theorem 3.13 *Let* $\widetilde{a}, \widetilde{b}, \widetilde{c} \in \mathcal{NF}$*. If* $\widetilde{a} \succeq \widetilde{b}$*, then* $\widetilde{a} - \widetilde{c} \succeq \widetilde{b} - \widetilde{c}$*.*

The detailed proof of this theorem is available in Appendix A.5.

Theorem 3.14 *Let* $\widetilde{a}, \widetilde{b}, \widetilde{c} \in \mathcal{NF}$*. If* $\widetilde{a} \succ \widetilde{b}$*, then* $\widetilde{a} - \widetilde{c} \succ \widetilde{b} - \widetilde{c}$*.*

Proof Taking into account the proof of the Theorem 3.13 and the definition of $\succeq$, the result follows immediately. $\square$

Theorem 3.15 *Let* $\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d} \in \mathcal{NF}$*. If* $\widetilde{a} \succ \widetilde{b}$ *and* $\widetilde{c} \succ \widetilde{d}$*, then* $\widetilde{a} + \widetilde{c} \succ \widetilde{b} + \widetilde{d}$*.*

The detailed proof of this theorem is available in Appendix A.6.

Theorem 3.16 *Let* $\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d} \in \mathcal{NF}$*. If* $\widetilde{a} \succeq \widetilde{b}$ *and* $\widetilde{c} \succeq \widetilde{d}$*, then* $\widetilde{a} + \widetilde{c} \succeq \widetilde{b} + \widetilde{d}$*.*

Proof Taking into account the proof of the Theorem 3.14 and the definition of $\succeq$, the result follows immediately. $\square$

Theorem 3.17 *Let* $\widetilde{a}, \widetilde{b} \in \mathcal{NF}$ *such that* $\widetilde{a}$ *and* $\widetilde{b}$ *are not symmetric about y-axis. If* $\widetilde{a} \succeq \widetilde{b}$*, then* $-\widetilde{a} \preceq -\widetilde{b}$*.*

Proof The proof follows immediately, taking $k = -1$ in the Theorem 3.10. $\square$

Theorem 3.18 *Let $\widetilde{a}, \widetilde{b} \in \mathcal{NF}$ such that $\widetilde{a}$ and $\widetilde{b}$ are not symmetric about y-axis. If $\widetilde{a} \succ \widetilde{b}$, then $-\widetilde{a} \prec -\widetilde{b}$.*

Proof Taking into account the proof of the Theorem 3.17 and the definition of $\succeq$, the result follows immediately. □

Theorem 3.19 *Let $\widetilde{a}, \widetilde{b} \in \mathcal{NF}$ and symmetric about y-axis. If $\widetilde{a} \succeq \widetilde{b}$, then $-\widetilde{a} \succeq -\widetilde{b}$.*

Proof If $\widetilde{a}, \widetilde{b} \in \mathcal{NF}$ be two symmetric SVNNs about the y-axis, then from the Proposition 3.7 it implies $\mathcal{V}(\mu_{\widetilde{a}}) = 0 = \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) = 0 = \mathcal{V}(\rho_{\widetilde{b}})$ and also $\mathcal{V}(\nu_{\widetilde{a}}) = 0 = \mathcal{V}(\nu_{\widetilde{b}})$. Since, $\widetilde{a} \succeq \widetilde{b}$, then $\theta_i = -1$. Thus, it follows that $\mathcal{R}_\lambda(\widetilde{a}, -1, -1, -1) \geq \mathcal{R}_\lambda(\widetilde{b}, -1, -1, -1)$. This inequality leads to the fact $-\mathcal{A}(\mu_{\widetilde{a}}) \geq -\mathcal{A}(\mu_{\widetilde{b}})$, $-\mathcal{A}(\rho_{\widetilde{a}}) \geq -\mathcal{A}(\rho_{\widetilde{b}})$ and $-\mathcal{A}(\nu_{\widetilde{a}}) \geq -\mathcal{A}(\nu_{\widetilde{b}})$. Equivalently, $-\mathcal{A}(\mu_{-\widetilde{a}}) \geq -\mathcal{A}(\mu_{-\widetilde{b}})$, $-\mathcal{A}(\rho_{-\widetilde{a}}) \geq -\mathcal{A}(\rho_{-\widetilde{b}})$ and $-\mathcal{A}(\nu_{-\widetilde{a}}) \geq -\mathcal{A}(\nu_{-\widetilde{b}})$. So, it is true that $\mathcal{R}_\lambda(-\widetilde{a}, -1, -1, -1) \geq \mathcal{R}_\lambda(-\widetilde{b}, -1, -1, -1)$. Hence, the result follows immediately. □

Theorem 3.20 *Let $\widetilde{a}, \widetilde{b} \in \mathcal{F}$ and symmetric about y-axis. If $\widetilde{a} \succ \widetilde{b}$, then $-\widetilde{a} \succ -\widetilde{b}$.*

Proof Taking into account the proof of the Theorem 3.19 and the definition of $\succeq$, the result follows immediately. □

3.3 Properties and validation of the proposed method

In this subsection, the properties that the present method follow are being stated. The properties includes the reasonable properties of Wang and Kerre (2001a, 2001b). Further, newer properties are also stated which can be considered as reasonable in developing a ranking method. The properties are as follows. Let $\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d} \in \mathcal{NF}$. Then the order relation $\succeq$ satisfies the following properties.

- $\mathbb{A}_1$: $\widetilde{a} \succeq \widetilde{a}$
- $\mathbb{A}_2$: If $\widetilde{a} \succeq \widetilde{b}$ and $\widetilde{a} \preceq \widetilde{b}$, then $\widetilde{a} \sim \widetilde{b}$.
- $\mathbb{A}_3$: If $\widetilde{a} \succeq \widetilde{b}$ and $\widetilde{b} \succeq \widetilde{c}$, then $\widetilde{a} \succeq \widetilde{c}$.
- $\mathbb{A}_4$: If $\inf \operatorname{supp}(\mu_{\widetilde{a}}) > \sup \operatorname{supp}(\mu_{\widetilde{b}})$, $\inf \operatorname{supp}(\rho_{\widetilde{a}}) > \sup \operatorname{supp}(\rho_{\widetilde{b}})$ and $\inf \operatorname{supp}(\nu_{\widetilde{a}}) > \sup \operatorname{supp}(\nu_{\widetilde{b}})$, then $\widetilde{a} \succeq \widetilde{b}$.
- $\mathbb{A}'_4$: If $\inf \operatorname{supp}(\mu_{\widetilde{a}}) > \sup \operatorname{supp}(\mu_{\widetilde{b}})$, $\inf \operatorname{supp}(\rho_{\widetilde{a}}) > \sup \operatorname{supp}(\rho_{\widetilde{b}})$ and $\inf \operatorname{supp}(\nu_{\widetilde{a}}) > \sup \operatorname{supp}(\nu_{\widetilde{b}})$, then $\widetilde{a} \succ \widetilde{b}$.
- $\mathbb{A}_5$: Let $\mathcal{NF}$ and $\mathcal{NF}'$ be two arbitrary finite sets of fuzzy quantities in which $\mathcal{R}_\lambda$ can be applied and $\widetilde{a}$ and $\widetilde{b}$ are in $\mathcal{NF} \cap \mathcal{NF}'$, then the ranking order $\widetilde{a} \succ \widetilde{b}$ by $\mathcal{R}_\lambda$ on $\mathcal{NF}'$ if and only if $\widetilde{a} \succ \widetilde{b}$ by $\mathcal{R}_\lambda$ on $\mathcal{NF}$.
- $\mathbb{A}_6$: If $\widetilde{a} \succeq \widetilde{b}$, then $\widetilde{a} + \widetilde{c} \succeq \widetilde{b} + \widetilde{c}$.
- $\mathbb{B}_6$: If $\widetilde{a} + \widetilde{c} \succeq \widetilde{b} + \widetilde{c}$, then $\widetilde{a} \succeq \widetilde{b}$.
- $\mathbb{A}'_6$: If $\widetilde{a} \succ \widetilde{b}$, then $\widetilde{a} + \widetilde{c} \succ \widetilde{b} + \widetilde{c}$.
- $\mathbb{B}'_6$: If $\widetilde{a} + \widetilde{c} \succ \widetilde{b} + \widetilde{c}$, then $\widetilde{a} \succ \widetilde{b}$.
- $\mathbb{A}_7$: Let $k \in \mathbb{R} - \{0\}$. If $\widetilde{a} \succeq \widetilde{b}$, then $k\widetilde{a} \succeq k\widetilde{b}$ if $k > 0$, and $k\widetilde{a} \preceq k\widetilde{b}$ if $k < 0$.
- $\mathbb{B}_7$: Let $k \in \mathbb{R} - \{0\}$. If $k\widetilde{a} \succeq k\widetilde{b}$, then $\widetilde{a} \succeq \widetilde{b}$ if $k > 0$, and $\widetilde{a} \preceq \widetilde{b}$ if $k < 0$.
- $\mathbb{A}'_7$: Let $k \in \mathbb{R} - \{0\}$. If $\widetilde{a} \succ \widetilde{b}$, then $k\widetilde{a} \succ k\widetilde{b}$ if $k > 0$, and $k\widetilde{a} \prec k\widetilde{b}$ if $k < 0$.
- $\mathbb{B}'_7$: Let $k \in \mathbb{R} - \{0\}$. If $k\widetilde{a} \succ k\widetilde{b}$, then $\widetilde{a} \succ \widetilde{b}$ if $k > 0$, and $\widetilde{a} \prec \widetilde{b}$ if $k < 0$.

$\mathbb{B}_8$: If $\widetilde{a} \succeq \widetilde{b}$, then $\widetilde{a} - \widetilde{c} \succeq \widetilde{b} - \widetilde{c}$.
$\mathbb{B}'_8$: If $\widetilde{a} \succ \widetilde{b}$, then $\widetilde{a} - \widetilde{c} \succ \widetilde{b} - \widetilde{c}$.
$\mathbb{B}_9$: If $\widetilde{a} \succeq \widetilde{b}$ and $\widetilde{c} \succeq \widetilde{d}$, then $\widetilde{a} + \widetilde{c} \succeq \widetilde{b} + \widetilde{d}$.
$\mathbb{B}'_9$: If $\widetilde{a} \succ \widetilde{b}$ and $\widetilde{c} \succ \widetilde{d}$, then $\widetilde{a} + \widetilde{c} \succ \widetilde{b} + \widetilde{d}$.
$\mathbb{B}_{10}$: If $\widetilde{a} \succeq \widetilde{b}$, then $-\widetilde{a} \preceq -\widetilde{b}$, provided $\widetilde{a}$ and $\widetilde{b}$ are not symmetric about y-axis.
$\mathbb{B}'_{10}$: If $\widetilde{a} \succ \widetilde{b}$, then $-\widetilde{a} \prec -\widetilde{b}$, provided $\widetilde{a}$ and $\widetilde{b}$ are not symmetric about y-axis.
$\mathbb{B}_{11}$: Let $\widetilde{a}, \widetilde{b} \in \mathcal{NF}$ and symmetric about y-axis; if $\widetilde{a} \succeq \widetilde{b}$, then $-\widetilde{a} \succeq -\widetilde{b}$.
$\mathbb{B}'_{11}$: Let $\widetilde{a}, \widetilde{b} \in \mathcal{NF}$ and symmetric about y-axis; if $\widetilde{a} \succ \widetilde{b}$, then $-\widetilde{a} \succ -\widetilde{b}$.

The proofs of the theorems stated and proved in the Sect. 3.2 depicts that the present method follows all these reasonable properties of a ranking method. Hence, it is claimed that the current method is reasonable and logical. Further, the consistency in ordering the images with the corresponding SVNNs is also depicted through these properties. However, it is to mentioned that the property $\mathbb{A}_7$ is a particular case of the property $\mathbb{A}_7$ of Wang and Kerre (2001a). This property $\mathbb{A}_7$ of Wang and Kerre (2001a) is not obeyed by the proposed method as $\mathcal{V}(\mu_{\widetilde{a}}\mu_{\widetilde{b}}) \neq \mathcal{V}(\mu_{\widetilde{a}})\mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}\rho_{\widetilde{b}}) \neq \mathcal{V}(\rho_{\widetilde{a}})\mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}\nu_{\widetilde{b}}) \neq \mathcal{V}(\nu_{\widetilde{a}})\mathcal{V}(\nu_{\widetilde{b}})$, and $\mathcal{A}(\mu_{\widetilde{a}}\mu_{\widetilde{b}}) \neq \mathcal{A}(\mu_{\widetilde{a}})\mathcal{A}(\mu_{\widetilde{b}})$, $\mathcal{A}(\rho_{\widetilde{a}}\rho_{\widetilde{b}}) \neq \mathcal{A}(\rho_{\widetilde{a}})\mathcal{A}(\rho_{\widetilde{b}})$ and $\mathcal{A}(\nu_{\widetilde{a}}\nu_{\widetilde{b}}) \neq \mathcal{A}(\nu_{\widetilde{a}})\mathcal{A}(\nu_{\widetilde{b}})$.

4 Numerical examples

In this section, the method is demonstrated by two numerical examples, which highlight its robustness.

Example 4.1 Consider the SVNNs $\widetilde{a} = \langle(2,4,4,5),(0,1,4,7),(1,4,4,6)\rangle$ and $\widetilde{b} = \langle(2,3,3,5),(0,1,4,7),(1,3,3,6)\rangle$. Firstly, the values of truth-membership, indeterminacy-membership and falsity-membership of $\widetilde{a}$ and $\widetilde{b}$ are obtained as $\mathcal{V}(\mu_{\widetilde{a}}) = 3.8333$, $\mathcal{V}(\rho_{\widetilde{a}}) = 2.8333$ and $\mathcal{V}(\nu_{\widetilde{a}}) = 3.8333$, and $\mathcal{V}(\mu_{\widetilde{b}}) = 3.1667$, $\mathcal{V}(\rho_{\widetilde{b}}) = 2.8333$ and $\mathcal{V}(\nu_{\widetilde{b}}) = 3.1667$ respectively. Further, the ambiguities of truth-membership, indeterminacy-membership and falsity-membership of $\widetilde{a}$ and $\widetilde{b}$ are obtained as $\mathcal{A}(\mu_{\widetilde{a}}) = 0.5000 = \mathcal{A}(\mu_{\widetilde{b}})$, $\mathcal{A}(\rho_{\widetilde{a}}) = 2.1667 = \mathcal{A}(\rho_{\widetilde{b}})$ and $\mathcal{A}(\nu_{\widetilde{a}}) = 0.8333 = \mathcal{A}(\nu_{\widetilde{b}})$, respectively. Now, it is seen that $\mathcal{V}(\mu_{\widetilde{a}}) \neq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) = \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \neq \mathcal{V}(\nu_{\widetilde{b}})$. Then, $\theta_1 = 0$ and $\theta_3 = 0$, however $\theta_2 = -1$. Thus,

$$\begin{aligned}
\mathcal{R}_\lambda(\widetilde{a},0,-1,0) &= \lambda\{\mathcal{V}(\mu_{\widetilde{a}}) + 0\cdot\mathcal{A}(\mu_{\widetilde{a}})\} + (1-\lambda)\{\mathcal{V}(\rho_{\widetilde{a}}) - \mathcal{A}(\rho_{\widetilde{a}}) + \mathcal{V}(\nu_{\widetilde{a}}) + 0\cdot\mathcal{A}(\nu_{\widetilde{a}})\}\\
&= \lambda\{3.8333\} + (1-\lambda)\{2.8333 - 2.1666 + 3.8333\}\\
&= 4.5000 - 0.6667\lambda,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{R}_\lambda(\widetilde{b},0,-1,0) &= \lambda\{\mathcal{V}(\mu_{\widetilde{b}}) + 0\cdot\mathcal{A}(\mu_{\widetilde{b}})\} + (1-\lambda)\{\mathcal{V}(\rho_{\widetilde{b}}) - \mathcal{A}(\rho_{\widetilde{b}}) + \mathcal{V}(\nu_{\widetilde{b}}) + 0\cdot\mathcal{A}(\nu_{\widetilde{b}})\}\\
&= \lambda\{3.1667\} + (1-\lambda)\{2.8333 - 2.1666 + 3.1667\}\\
&= 3.8000 - 0.6667\lambda.
\end{aligned}$$

So, for all decision-makers $\widetilde{a} \succ \widetilde{b}$. Consider the images of $\widetilde{a}$ and $\widetilde{b}$. Now, the values of truth-membership, indeterminacy-membership and falsity-membership of $-\widetilde{a}$ and $-\widetilde{b}$ are obtained as $\mathcal{V}(\mu_{-\widetilde{a}}) = -3.8333$, $\mathcal{V}(\rho_{-\widetilde{a}}) = -2.8333$ and $\mathcal{V}(\nu_{-\widetilde{a}}) = -3.8333$, and $\mathcal{V}(\mu_{-\widetilde{b}}) = -3.1667$, $\mathcal{V}(\rho_{-\widetilde{b}}) = -2.8333$ and $\mathcal{V}(\nu_{-\widetilde{b}}) = -3.1667$, respectively by

Proposition 3.5. Further, the ambiguities of truth-membership, indeterminacy-membership and falsity-membership of $-\widetilde{a}$ and $-\widetilde{b}$ are obtained as $\mathcal{A}(\mu_{-\widetilde{a}}) = 0.5000 = \mathcal{A}(\mu_{-\widetilde{b}})$, $\mathcal{A}(\rho_{-\widetilde{a}}) = 2.1667 = \mathcal{A}(\rho_{-\widetilde{b}})$ and $\mathcal{A}(\nu_{-\widetilde{a}}) = 0.8333 = \mathcal{A}(\nu_{-\widetilde{b}})$ respectively by Proposition 3.5. Now, it is seen that $\mathcal{V}(\mu_{-\widetilde{a}}) \neq \mathcal{V}(\mu_{-\widetilde{b}})$, $\mathcal{V}(\rho_{-\widetilde{a}}) = \mathcal{V}(\rho_{-\widetilde{b}})$ and $\mathcal{V}(\nu_{-\widetilde{a}}) \neq \mathcal{V}(\nu_{-\widetilde{b}})$. Then, $\theta_1 = 0$ and $\theta_3 = 0$, however $\theta_2 = 1$. Thus,

$$\begin{aligned}
&\mathcal{R}_\lambda(-\widetilde{a}, 0, 1, 0)\\
&\quad = \lambda\{\mathcal{V}(\mu_{-\widetilde{a}}) + 0 \cdot \mathcal{A}(\mu_{-\widetilde{a}})\} + (1-\lambda)\{\mathcal{V}(\rho_{-\widetilde{a}}) + \mathcal{A}(\rho_{-\widetilde{a}}) + \mathcal{V}(\nu_{-\widetilde{a}}) + 0 \cdot \mathcal{A}(\nu_{-\widetilde{a}})\}\\
&\quad = \lambda\{-3.8333\} + (1-\lambda)\{-2.8333 + 2.1666 - 3.8333\}\\
&\quad = -4.5000 + 0.6667\lambda,
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{R}_\lambda(-\widetilde{b}, 0, 1, 0)\\
&\quad = \lambda\{\mathcal{V}(\mu_{-\widetilde{b}}) + 0 \cdot \mathcal{A}(\mu_{-\widetilde{b}})\} + (1-\lambda)\{\mathcal{V}(\rho_{-\widetilde{b}}) + \mathcal{A}(\rho_{-\widetilde{b}}) + \mathcal{V}(\nu_{-\widetilde{b}}) + 0 \cdot \mathcal{A}(\nu_{-\widetilde{b}})\}\\
&\quad = \lambda\{-3.1667\} + (1-\lambda)\{-2.8333 + 2.1666 - 3.1667\}\\
&\quad = -3.8000 + 0.6667\lambda.
\end{aligned}$$

So, for all decision-makers $-\widetilde{a} \prec -\widetilde{b}$. This numerical example depicts that the current method consistently and logically ranks the SVNNs as well as their corresponding images.

For a comparative study, the current method is compared with the methods of Deli and Subas (2017) and Biswas et al. (2016). The results depicted in Table 1 of the methods by Deli and Subas (2017) and Biswas et al. (2016) are the value index. The current method tallies with the methods by Deli and Subas (2017) and Biswas et al. (2016).

Example 4.2 Consider the SVNNs $\widetilde{a} = \langle(-1,0,0,1),(-1,0,0,1),(-2,0,0,2)\rangle$ and $\widetilde{b} = \langle(-2,0,0,2),(-2,0,0,2),(-3,0,0,3)\rangle$ such that they are symmetric about the y-axis.

Table 1 Ranking of SVNNs in Examples 4.1

Methods	$\widetilde{a}$	$\widetilde{b}$	$-\widetilde{a}$	$-\widetilde{b}$	Decision result
Deli and Subas (2017)'s value					
Optimistic $\alpha = 1.0$	3.8333	3.1667	−3.8333	−3.1667	$\widetilde{a} \succ \widetilde{b}, -\widetilde{a} \prec -\widetilde{b}$
Moderate $\alpha = 0.5$	5.2500	4.5833	−5.2500	−4.5833	$\widetilde{a} \succ \widetilde{b}, -\widetilde{a} \prec -\widetilde{b}$
Pessimistic $\alpha = 0.0$	6.6667	6.0000	−6.6667	−6.0000	$\widetilde{a} \succ \widetilde{b}, -\widetilde{a} \prec -\widetilde{b}$
Biswas et al. (2016)'s value					
Optimistic $\alpha = 1.0$	3.8333	3.1667	−3.8333	−3.1667	$\widetilde{a} \succ \widetilde{b}, -\widetilde{a} \prec -\widetilde{b}$
Moderate $\alpha = 0.5$	5.2500	4.5833	−5.2500	−4.5833	$\widetilde{a} \succ \widetilde{b}, -\widetilde{a} \prec -\widetilde{b}$
Pessimistic $\alpha = 0.0$	6.6667	6.0000	−6.6667	−6.0000	$\widetilde{a} \succ \widetilde{b}, -\widetilde{a} \prec -\widetilde{b}$
Current method					
Optimistic $\alpha = 1.0$	3.8333	3.1667	−3.8333	−3.1667	$\widetilde{a} \succ \widetilde{b}, -\widetilde{a} \prec -\widetilde{b}$
Moderate $\alpha = 0.5$	4.1667	3.5000	−4.1667	−3.5000	$\widetilde{a} \succ \widetilde{b}, -\widetilde{a} \prec -\widetilde{b}$
Pessimistic $\alpha = 0.0$	4.5000	3.8333	−4.5000	−3.8333	$\widetilde{a} \succ \widetilde{b}, -\widetilde{a} \prec -\widetilde{b}$

Then, by the Proposition 3.7, $\mathcal{V}(\mu_{\widetilde{a}}) = 0 = \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) = 0 = \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) = 0 = \mathcal{V}(\nu_{\widetilde{b}})$. Thus, $\theta_i = -1$. Further, the ambiguities of truth-membership, indeterminacy-membership and falsity-membership of $\widetilde{a}$ and $\widetilde{b}$ are obtained as $\mathcal{A}(\mu_{\widetilde{a}}) = 0.3333$, $\mathcal{A}(\rho_{\widetilde{a}}) = 0.3333$ and $\mathcal{A}(\nu_{\widetilde{a}}) = 0.6667$, and $\mathcal{A}(\mu_{\widetilde{a}}) = 0.6667$, $\mathcal{A}(\rho_{\widetilde{a}}) = 0.6667$ and $\mathcal{A}(\nu_{\widetilde{a}}) = 1.0000$, respectively. So,

$$\begin{aligned}&\mathcal{R}_\lambda(\widetilde{a}, -1, -1, -1)\\&\quad = \lambda\{\mathcal{V}(\mu_{\widetilde{a}}) - \mathcal{A}(\mu_{\widetilde{a}})\} + (1-\lambda)\{\mathcal{V}(\rho_{\widetilde{a}}) - \mathcal{A}(\rho_{\widetilde{a}}) + \mathcal{V}(\nu_{\widetilde{a}}) - \mathcal{A}(\nu_{\widetilde{a}})\}\\&\quad = \lambda\{-0.3333\} + (1-\lambda)\{-0.3333 - 0.6667\}\\&\quad = 1.0000 - 1.3333\lambda\end{aligned}$$

and

$$\begin{aligned}&\mathcal{R}_\lambda(\widetilde{b}, -1, -1, -1)\\&\quad = \lambda\{\mathcal{V}(\mu_{\widetilde{b}}) - \mathcal{A}(\mu_{\widetilde{b}})\} + (1-\lambda)\{\mathcal{V}(\rho_{\widetilde{b}}) - \mathcal{A}(\rho_{\widetilde{b}}) + \mathcal{V}(\nu_{\widetilde{b}}) - \mathcal{A}(\nu_{\widetilde{b}})\}\\&\quad = \lambda\{-0.6667\} + (1-\lambda)\{-0.6667 - 1.0000\}\\&\quad = -1.6667 + \lambda.\end{aligned}$$

Hence, it can be concluded that $\widetilde{a} \prec \widetilde{b}$ for all decision-makers. Consider the images of $\widetilde{a}$ and $\widetilde{b}$, it can be seen that $-\widetilde{a} = \widetilde{a}$ and $-\widetilde{b} = \widetilde{b}$. Therefore, by Theorem 3.20, it can be concluded that $-\widetilde{a} \prec -\widetilde{b}$ for all decision-makers.

For a comparative study, the current method is compared with the methods of Deli and Subas (2017) and Biswas et al. (2016). The results depicted in Table 2 of the methods by Deli and Subas (2017) and Biswas et al. (2016) are the ambiguity index as the value index are equal. The current method tallies with the methods by Biswas et al. (2016). However, Deli and Subas (2017) depicts irrational results.

Table 2 Ranking of SVNNs in Examples 4.2

Methods	$\widetilde{a}$	$\widetilde{b}$	$-\widetilde{a}$	$-\widetilde{b}$	Decision result
Deli and Subas (2017)'s ambiguity					
Optimistic $\alpha = 1.0$	0.3333	0.6667	0.3333	0.6667	$\widetilde{a} \prec \widetilde{b}, -\widetilde{a} \prec -\widetilde{b}$
Moderate $\alpha = 0.5$	0.6667	1.0000	0.6667	1.6667	$\widetilde{a} \prec \widetilde{b}, -\widetilde{a} \prec -\widetilde{b}$
Pessimistic $\alpha = 0.0$	1.0000	1.3333	1.0000	1.3333	$\widetilde{a} \prec \widetilde{b}, -\widetilde{a} \prec -\widetilde{b}$
Biswas et al. (2016)'s ambiguity					
Optimistic $\alpha = 1.0$	0.3333	0.6667	0.3333	0.6667	$\widetilde{a} \succ \widetilde{b}, -\widetilde{a} \succ -\widetilde{b}$
Moderate $\alpha = 0.5$	0.6667	1.0000	0.6667	1.6667	$\widetilde{a} \succ \widetilde{b}, -\widetilde{a} \succ -\widetilde{b}$
Pessimistic $\alpha = 0.0$	1.0000	1.3333	1.0000	1.3333	$\widetilde{a} \succ \widetilde{b}, -\widetilde{a} \succ -\widetilde{b}$
Current method					
Optimistic $\alpha = 1.0$	−0.3333	−0.6667	−0.3333	−0.6667	$\widetilde{a} \succ \widetilde{b}, -\widetilde{a} \succ -\widetilde{b}$
Moderate $\alpha = 0.5$	−0.6667	−1.1667	−0.6667	−1.1667	$\widetilde{a} \succ \widetilde{b}, -\widetilde{a} \succ -\widetilde{b}$
Pessimistic $\alpha = 0.0$	−1.0000	−1.6667	−1.0000	−1.6667	$\widetilde{a} \succ \widetilde{b}, -\widetilde{a} \succ -\widetilde{b}$

Example 4.3 Consider the SVNNs $\widetilde{a} = \langle (1,4,4,7),(0,4,4,8),(1,4,4,7)\rangle$ and $\widetilde{b} = \langle (2,4,4,6),(1,4,4,7),(2,4,4,6)\rangle$. For comparison the results of ranking index of the methods by Deli and Subas (2017), Biswas et al. (2016) and the current method are depicted in Table 3. Biswas et al. (2016) ordering of the SVNNs are logical as the ranking are based on the ambiguity index. The SVNN with low ambiguity is chosen to be greater in their approach. However, the ordering of the images of SVNNs in their approach is illogical. That is, their method depicts inconsistency in ordering the images of SVNNs in some situations. Deli and Subas (2017) method is illogical as the SVNN with low ambiguity is smaller. Further, it is to be mentioned and also evident from the Table 3 that the existing methods could not rank the corresponding images of the SVNNs consistently. The current approach is logical; further its rank consistently the corresponding images of the SVNNs.

The above numerical examples highlight the fact that the current method is more robust and reasonable. Thus, this methodology of ranking SVNNs will be reasonable to apply in various decision-making problems.

5 Discussions and conclusions

In this paper, an innovative method of ranking SVNNs has been developed based on the concept of values and ambiguities of truth-membership, indeterminacy-membership and falsity-membership functions. The index of optimism is also utilized which reflects the decision-makers attitude towards the uncertainty. That is, the convex combination of value and ambiguity of truth-membership function with the sum of values and ambiguities of indeterminacy-membership and falsity-membership functions. The parameters θ_i's decides inclusion or exclusion of ambiguities in the decision-making process. An optimistic decision-maker ($\lambda = 1$) considers the value and θ_1 multiple of the ambiguity of truth-membership function. A pessimistic decision-maker ($\lambda = 0$) considers the values and θ_2, θ_3

Table 3 Ranking of SVNNs in Examples 4.3

Methods	$\widetilde{a}$	$\widetilde{b}$	$-\widetilde{a}$	$-\widetilde{b}$	Decision result
Deli and Subas (2017)'s ambiguity					
Optimistic $\alpha = 1.0$	1.0000	0.6667	1.0000	0.6667	$\widetilde{a} \succ \widetilde{b}, -\widetilde{a} \succ -\widetilde{b}$
Moderate $\alpha = 0.5$	1.6667	1.3333	1.6667	1.3333	$\widetilde{a} \succ \widetilde{b}, -\widetilde{a} \succ -\widetilde{b}$
Pessimistic $\alpha = 0.0$	2.3333	2.0000	2.3333	2.0000	$\widetilde{a} \succ \widetilde{b}, -\widetilde{a} \succ -\widetilde{b}$
Biswas et al. (2016)'s ambiguity					
Optimistic $\alpha = 1.0$	1.0000	0.6667	1.0000	0.6667	$\widetilde{a} \prec \widetilde{b}, -\widetilde{a} \prec -\widetilde{b}$
Moderate $\alpha = 0.5$	1.6667	1.3333	1.6667	1.3333	$\widetilde{a} \prec \widetilde{b}, -\widetilde{a} \prec -\widetilde{b}$
Pessimistic $\alpha = 0.0$	2.3333	2.0000	2.3333	2.0000	$\widetilde{a} \prec \widetilde{b}, -\widetilde{a} \prec -\widetilde{b}$
Current method					
Optimistic $\alpha = 1.0$	3.0000	3.3333	−3.0000	−3.3333	$\widetilde{a} \prec \widetilde{b}, -\widetilde{a} \succ -\widetilde{b}$
Moderate $\alpha = 0.5$	4.3333	4.8333	−4.3333	−4.8333	$\widetilde{a} \prec \widetilde{b}, -\widetilde{a} \succ -\widetilde{b}$
Pessimistic $\alpha = 0.0$	5.6667	6.3333	−5.6667	−6.3333	$\widetilde{a} \prec \widetilde{b}, -\widetilde{a} \succ -\widetilde{b}$

multiples of the ambiguities of indeterminacy-membership and falsity-membership functions respectively. Further, the moderate decision-maker considers the contributions from all the membership functions. It should be mentioned that the proofs of the Theorems 3.2 and 3.15 are cut short. The proofs are very simple but very lengthy as it involves a discussion of 8×8 cases, which will make this work lengthy. A shorter and logical proof can be a future study.

An attractive feature of the current method is that it completely comply with the reasonable properties of Wang and Kerre (2001a) which were never investigated in the existing methods. This establishes the rationality validity of the current approach. Apart from it, newer properties are also be investigated in this study. Another way to establish the rationality validity of a ranking method is to investigate the consistency in ordering the corresponding images of the SVNNs. Apparently, the properties $\mathbb{B}_{10} - \mathbb{B}'_{10}$ establish this fact. It is to mentioned that the property $\mathbb{A}_7$ is a particular case of the property $\mathbb{A}_7$ of Wang and Kerre (2001a). This property $\mathbb{A}_7$ of Wang and Kerre (2001a) is not obeyed by the proposed method as $\mathcal{V}(\mu_{\tilde{a}}\mu_{\tilde{b}}) \neq \mathcal{V}(\mu_{\tilde{a}})\mathcal{V}(\mu_{\tilde{b}})$, $\mathcal{V}(\rho_{\tilde{a}}\rho_{\tilde{b}}) \neq \mathcal{V}(\rho_{\tilde{a}})\mathcal{V}(\rho_{\tilde{b}})$ and $\mathcal{V}(\nu_{\tilde{a}}\nu_{\tilde{b}}) \neq \mathcal{V}(\nu_{\tilde{a}})\mathcal{V}(\nu_{\tilde{b}})$, and $\mathcal{A}(\mu_{\tilde{a}}\mu_{\tilde{b}}) \neq \mathcal{A}(\mu_{\tilde{a}})\mathcal{A}(\mu_{\tilde{b}})$, $\mathcal{A}(\rho_{\tilde{a}}\rho_{\tilde{b}}) \neq \mathcal{A}(\rho_{\tilde{a}})\mathcal{A}(\rho_{\tilde{b}})$ and $\mathcal{A}(\nu_{\tilde{a}}\nu_{\tilde{b}}) \neq \mathcal{A}(\nu_{\tilde{a}})\mathcal{A}(\nu_{\tilde{b}})$.

Proofs of the theorems

Proof of the Theorem 3.2

The proof of the above statements are as follows.

1. The proof of this statement is followed immediately.
2. Consider the cases when $\tilde{a} \succ \tilde{b}$ happens, that is,

$$\tilde{a} \succ \tilde{b} \text{ happens for} \begin{cases} \mathcal{R}_\lambda(\tilde{a},0,0,0) > \mathcal{R}_\lambda(\tilde{b},0,0,0) \\ \mathcal{R}_\lambda(\tilde{a},0,0,\pm1) > \mathcal{R}_\lambda(\tilde{b},0,0,\pm1) \\ \mathcal{R}_\lambda(\tilde{a},0,\pm1,0) > \mathcal{R}_\lambda(\tilde{b},0,\pm1,0) \\ \mathcal{R}_\lambda(\tilde{a},0,\pm1,\pm1) > \mathcal{R}_\lambda(\tilde{b},0,\pm1,\pm1) \\ \mathcal{R}_\lambda(\tilde{a},\pm1,0,0) > \mathcal{R}_\lambda(\tilde{b},\pm1,0,0) \\ \mathcal{R}_\lambda(\tilde{a},\pm1,0,\pm1) > \mathcal{R}_\lambda(\tilde{b},\pm1,0,\pm1) \\ \mathcal{R}_\lambda(\tilde{a},\pm1,\pm1,0) > \mathcal{R}_\lambda(\tilde{b},\pm1,\pm1,0) \\ \mathcal{R}_\lambda(\tilde{a},\pm1,\pm1,\pm1) > \mathcal{R}_\lambda(\tilde{b},\pm1,\pm1,\pm1) \end{cases}.$$

Consider the cases when $\tilde{b} \succ \tilde{c}$ happens, that is,

$$\tilde{b} \succ \tilde{c} \text{ happens for} \begin{cases} \mathcal{R}_\lambda(\tilde{b},0,0,0) > \mathcal{R}_\lambda(\tilde{c},0,0,0) \\ \mathcal{R}_\lambda(\tilde{b},0,0,\pm1) > \mathcal{R}_\lambda(\tilde{c},0,0,\pm1) \\ \mathcal{R}_\lambda(\tilde{b},0,\pm1,0) > \mathcal{R}_\lambda(\tilde{c},0,\pm1,0) \\ \mathcal{R}_\lambda(\tilde{b},0,\pm1,\pm1) > \mathcal{R}_\lambda(\tilde{c},0,\pm1,\pm1) \\ \mathcal{R}_\lambda(\tilde{b},\pm1,0,0) > \mathcal{R}_\lambda(\tilde{c},\pm1,0,0) \\ \mathcal{R}_\lambda(\tilde{b},\pm1,0,\pm1) > \mathcal{R}_\lambda(\tilde{c},\pm1,0,\pm1) \\ \mathcal{R}_\lambda(\tilde{b},\pm1,\pm1,0) > \mathcal{R}_\lambda(\tilde{c},\pm1,\pm1,0) \\ \mathcal{R}_\lambda(\tilde{b},\pm1,\pm1,\pm1) > \mathcal{R}_\lambda(\tilde{c},\pm1,\pm1,\pm1) \end{cases}$$

Now, to proof this property, it need to discuss these 8×8 cases, which will make the proof tedious. However, one can see the proof trivially, if following claims can be established. **Claim 1:** Let $\theta_i = 0$ in ordering $\widetilde{a}$ and $\widetilde{b}$, and $\theta_i = 0$ in ordering $\widetilde{b}$ and $\widetilde{c}$. Then $\theta_i = 0$ in ordering $\widetilde{a}$ and $\widetilde{c}$. The proof of this claim is as follows. Let $\theta_i = 0$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Then $\mathcal{V}(\mu_{\widetilde{a}}) \neq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \neq \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \neq \mathcal{V}(\nu_{\widetilde{b}})$. Similarly, if $\theta_i = 0$ in ordering $\widetilde{b}$ and $\widetilde{c}$, then $\mathcal{V}(\mu_{\widetilde{b}}) \neq \mathcal{V}(\mu_{\widetilde{c}})$, $\mathcal{V}(\rho_{\widetilde{b}}) \neq \mathcal{V}(\rho_{\widetilde{c}})$ and $\mathcal{V}(\nu_{\widetilde{b}}) \neq \mathcal{V}(\nu_{\widetilde{c}})$. Thus, it follows that $\mathcal{V}(\mu_{\widetilde{a}}) \neq \mathcal{V}(\mu_{\widetilde{c}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \neq \mathcal{V}(\rho_{\widetilde{c}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \neq \mathcal{V}(\nu_{\widetilde{c}})$. So, $\theta_i = 0$ in ordering $\widetilde{a}$ and $\widetilde{c}$. **Claim 2:** Let $\theta_i = \pm 1$ in ordering $\widetilde{a}$ and $\widetilde{b}$, and $\theta_i = \pm 1$ in ordering $\widetilde{b}$ and $\widetilde{c}$. Then $\theta_i = \pm 1$ in ordering $\widetilde{a}$ and $\widetilde{c}$. The proof of this claim is as follows. Let $\theta_i = \pm 1$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Then $\mathcal{V}(\mu_{\widetilde{a}}) = \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) = \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) = \mathcal{V}(\nu_{\widetilde{b}})$. Similarly, if $\theta_i = \pm 1$ in ordering $\widetilde{b}$ and $\widetilde{c}$, then $\mathcal{V}(\mu_{\widetilde{b}}) = \mathcal{V}(\mu_{\widetilde{c}})$, $\mathcal{V}(\rho_{\widetilde{b}}) = \mathcal{V}(\rho_{\widetilde{c}})$ and $\mathcal{V}(\nu_{\widetilde{b}}) = \mathcal{V}(\nu_{\widetilde{c}})$. Thus, it follows that $\mathcal{V}(\mu_{\widetilde{a}}) = \mathcal{V}(\mu_{\widetilde{c}})$, $\mathcal{V}(\rho_{\widetilde{a}}) = \mathcal{V}(\rho_{\widetilde{c}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) = \mathcal{V}(\nu_{\widetilde{c}})$. So, $\theta_i = \pm 1$ in ordering $\widetilde{a}$ and $\widetilde{c}$. **Claim 3:** Let $\theta_i = 0$ in ordering $\widetilde{a}$ and $\widetilde{b}$, and $\theta_i = \pm 1$ in ordering $\widetilde{b}$ and $\widetilde{c}$. then $\theta_i = 0$ in ordering $\widetilde{a}$ and $\widetilde{c}$. The proof of this claim is as follows. Let $\theta_i = 0$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Then $\mathcal{V}(\mu_{\widetilde{a}}) \neq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \neq \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \neq \mathcal{V}(\nu_{\widetilde{b}})$. Similarly, if $\theta_i = \pm 1$ in ordering $\widetilde{b}$ and $\widetilde{c}$, then $\mathcal{V}(\mu_{\widetilde{b}}) = \mathcal{V}(\mu_{\widetilde{c}})$, $\mathcal{V}(\rho_{\widetilde{b}}) = \mathcal{V}(\rho_{\widetilde{c}})$ and $\mathcal{V}(\nu_{\widetilde{b}}) = \mathcal{V}(\nu_{\widetilde{c}})$. Thus, it follows that $\mathcal{V}(\mu_{\widetilde{a}}) \neq \mathcal{V}(\mu_{\widetilde{c}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \neq \mathcal{V}(\rho_{\widetilde{c}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \neq \mathcal{V}(\nu_{\widetilde{c}})$. So, $\theta_i = 0$ in ordering $\widetilde{a}$ and $\widetilde{c}$. **Claim 4:** Let $\theta_i = \pm 1$ in ordering $\widetilde{a}$ and $\widetilde{b}$, and $\theta_i = 0$ in ordering $\widetilde{b}$ and $\widetilde{c}$. Then $\theta_i = 0$ in ordering $\widetilde{a}$ and $\widetilde{c}$. The proof of this claim is as follows. Let $\theta_i = \pm 1$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Then $\mathcal{V}(\mu_{\widetilde{a}}) = \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) = \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) = \mathcal{V}(\nu_{\widetilde{b}})$. Similarly, if $\theta_i = 0$ in ordering $\widetilde{b}$ and $\widetilde{c}$, then $\mathcal{V}(\mu_{\widetilde{b}}) \neq \mathcal{V}(\mu_{\widetilde{c}})$, $\mathcal{V}(\rho_{\widetilde{b}}) \neq \mathcal{V}(\rho_{\widetilde{c}})$ and $\mathcal{V}(\nu_{\widetilde{b}}) \neq \mathcal{V}(\nu_{\widetilde{c}})$. Thus, it follows that $\mathcal{V}(\mu_{\widetilde{a}}) \neq \mathcal{V}(\mu_{\widetilde{c}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \neq \mathcal{V}(\rho_{\widetilde{c}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \neq \mathcal{V}(\nu_{\widetilde{c}})$. So, $\theta_i = 0$ in ordering $\widetilde{a}$ and $\widetilde{c}$. From these four claims, it is trivial enough to show that if $\widetilde{a} \succ \widetilde{b}$ and $\widetilde{b} \succ \widetilde{c}$, then $\widetilde{a} \succ \widetilde{c}$. Further, from the definition of $\succeq$, it follows that transitivity also holds for the order relation $\succeq$.

3. This statement is followed immediately, as the order relations $\succ$ and $\sim$ particularly based on order relation $>$ and $=$ of real numbers.
4. If $\widetilde{a} = \widetilde{b}$, then $\mathcal{R}_\lambda(\widetilde{a}, \theta_1, \theta_2, \theta_3) = \mathcal{R}_\lambda(\widetilde{b}, \theta_1, \theta_2, \theta_3)$. Thus, the statement is followed.

Proof of the Theorem 3.5

The proof of this theorem, follows immediately if the invariance of θ_i in ordering $\widetilde{a}$, $\widetilde{b}$ and $\widetilde{a} + \widetilde{c}$, $\widetilde{b} + \widetilde{c}$ can be established. Hence, a claim has to be made. The claim is as follows.

Claim : The value of θ_i in ordering $\widetilde{a}$ and $\widetilde{b}$ are invariant in ordering $\widetilde{a} + \widetilde{c}$ and $\widetilde{b} + \widetilde{c}$. The proof of the claim follows from the following eight cases:

Case 1: Let $\theta_i = 0$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Then, $\mathcal{V}(\mu_{\widetilde{a}}) \neq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \neq \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \neq \mathcal{V}(\nu_{\widetilde{b}})$. Thus, it follows that $\mathcal{V}(\mu_{\widetilde{a}+\widetilde{c}}) \neq \mathcal{V}(\mu_{\widetilde{b}+\widetilde{c}})$, $\mathcal{V}(\rho_{\widetilde{a}+\widetilde{c}}) \neq \mathcal{V}(\rho_{\widetilde{b}+\widetilde{c}})$ and $\mathcal{V}(\nu_{\widetilde{a}+\widetilde{c}}) \neq \mathcal{V}(\nu_{\widetilde{b}+\widetilde{c}})$. So, $\theta_i = 0$ in ordering $\widetilde{a} + \widetilde{c}$ and $\widetilde{b} + \widetilde{c}$.

Case 2: Let $\theta_1 = 0$, $\theta_2 = 0$ and $\theta_3 = \pm 1$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Then, $\mathcal{V}(\mu_{\widetilde{a}}) \neq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \neq \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) = \mathcal{V}(\nu_{\widetilde{b}})$. Thus, it follows that $\mathcal{V}(\mu_{\widetilde{a}+\widetilde{c}}) \neq \mathcal{V}(\mu_{\widetilde{b}+\widetilde{c}})$, $\mathcal{V}(\rho_{\widetilde{a}+\widetilde{c}}) \neq \mathcal{V}(\rho_{\widetilde{b}+\widetilde{c}})$ and $\mathcal{V}(\nu_{\widetilde{a}+\widetilde{c}}) = \mathcal{V}(\nu_{\widetilde{b}+\widetilde{c}})$. So, $\theta_1 = 0$, $\theta_2 = 0$ and $\theta_3 = \pm 1$ in ordering $\widetilde{a} + \widetilde{c}$ and $\widetilde{b} + \widetilde{c}$.

Case 3: Let $\theta_1 = 0$, $\theta_2 = \pm 1$ and $\theta_3 = 0$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Then, $\mathcal{V}(\mu_{\widetilde{a}}) \neq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) = \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \neq \mathcal{V}(\nu_{\widetilde{b}})$. Thus, it follows that $\mathcal{V}(\mu_{\widetilde{a}+\widetilde{c}}) \neq \mathcal{V}(\mu_{\widetilde{b}+\widetilde{c}})$, $\mathcal{V}(\rho_{\widetilde{a}+\widetilde{c}}) = \mathcal{V}(\rho_{\widetilde{b}+\widetilde{c}})$ and $\mathcal{V}(\nu_{\widetilde{a}+\widetilde{c}}) \neq \mathcal{V}(\nu_{\widetilde{b}+\widetilde{c}})$. So, $\theta_1 = 0$, $\theta_2 = \pm 1$ and $\theta_3 = 0$ in ordering $\widetilde{a} + \widetilde{c}$ and $\widetilde{b} + \widetilde{c}$.

Case 4: Let $\theta_1 = 0$, $\theta_2 = \pm 1$ and $\theta_3 = \pm 1$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Then, $\mathcal{V}(\mu_{\widetilde{a}}) \neq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) = \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) = \mathcal{V}(\nu_{\widetilde{b}})$. Thus, it follows that $\mathcal{V}(\mu_{\widetilde{a}+\widetilde{c}}) \neq \mathcal{V}(\mu_{\widetilde{b}+\widetilde{c}})$, $\mathcal{V}(\rho_{\widetilde{a}+\widetilde{c}}) = \mathcal{V}(\rho_{\widetilde{b}+\widetilde{c}})$ and $\mathcal{V}(\nu_{\widetilde{a}+\widetilde{c}}) = \mathcal{V}(\nu_{\widetilde{b}+\widetilde{c}})$. So, $\theta_1 = 0$, $\theta_2 = \pm 1$ and $\theta_3 = \pm 1$ in ordering $\widetilde{a} + \widetilde{c}$ and $\widetilde{b} + \widetilde{c}$.

Case 5: Let $\theta_1 = \pm 1$, $\theta_2 = 0$ and $\theta_3 = 0$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Then, $\mathcal{V}(\mu_{\widetilde{a}}) = \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \neq \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \neq \mathcal{V}(\nu_{\widetilde{b}})$. Thus, it follows that $\mathcal{V}(\mu_{\widetilde{a}+\widetilde{c}}) = \mathcal{V}(\mu_{\widetilde{b}+\widetilde{c}})$, $\mathcal{V}(\rho_{\widetilde{a}+\widetilde{c}}) \neq \mathcal{V}(\rho_{\widetilde{b}+\widetilde{c}})$ and $\mathcal{V}(\nu_{\widetilde{a}+\widetilde{c}}) \neq \mathcal{V}(\nu_{\widetilde{b}+\widetilde{c}})$. So, $\theta_1 = \pm 1$, $\theta_2 = 0$ and $\theta_3 = 0$ in ordering $\widetilde{a} + \widetilde{c}$ and $\widetilde{b} + \widetilde{c}$.

Case 6: Let $\theta_1 = \pm 1$, $\theta_2 = 0$ and $\theta_3 = \pm 1$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Then, $\mathcal{V}(\mu_{\widetilde{a}}) = \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \neq \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) = \mathcal{V}(\nu_{\widetilde{b}})$. Thus, it follows that $\mathcal{V}(\mu_{\widetilde{a}+\widetilde{c}}) = \mathcal{V}(\mu_{\widetilde{b}+\widetilde{c}})$, $\mathcal{V}(\rho_{\widetilde{a}+\widetilde{c}}) \neq \mathcal{V}(\rho_{\widetilde{b}+\widetilde{c}})$ and $\mathcal{V}(\nu_{\widetilde{a}+\widetilde{c}}) = \mathcal{V}(\nu_{\widetilde{b}+\widetilde{c}})$. So, $\theta_1 = \pm 1$, $\theta_2 = 0$ and $\theta_3 = \pm 1$ in ordering $\widetilde{a} + \widetilde{c}$ and $\widetilde{b} + \widetilde{c}$.

Case 7: Let $\theta_1 = \pm 1$, $\theta_2 = \pm 1$ and $\theta_3 = 0$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Then, $\mathcal{V}(\mu_{\widetilde{a}}) = \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) = \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \neq \mathcal{V}(\nu_{\widetilde{b}})$. Thus, it follows that $\mathcal{V}(\mu_{\widetilde{a}+\widetilde{c}}) = \mathcal{V}(\mu_{\widetilde{b}+\widetilde{c}})$, $\mathcal{V}(\rho_{\widetilde{a}+\widetilde{c}}) = \mathcal{V}(\rho_{\widetilde{b}+\widetilde{c}})$ and $\mathcal{V}(\nu_{\widetilde{a}+\widetilde{c}}) \neq \mathcal{V}(\nu_{\widetilde{b}+\widetilde{c}})$. So, $\theta_1 = \pm 1$, $\theta_2 = \pm 1$ and $\theta_3 = 0$ in ordering $\widetilde{a} + \widetilde{c}$ and $\widetilde{b} + \widetilde{c}$.

Case 8: Let $\theta_i = \pm 1$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Then, $\mathcal{V}(\mu_{\widetilde{a}}) = \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) = \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) = \mathcal{V}(\nu_{\widetilde{b}})$. Thus, it follows that $\mathcal{V}(\mu_{\widetilde{a}+\widetilde{c}}) = \mathcal{V}(\mu_{\widetilde{b}\pm\widetilde{c}})$, $\mathcal{V}(\rho_{\widetilde{a}+\widetilde{c}}) = \mathcal{V}(\rho_{\widetilde{b}+\widetilde{c}})$ and $\mathcal{V}(\nu_{\widetilde{a}+\widetilde{c}}) = \mathcal{V}(\nu_{\widetilde{b}+\widetilde{c}})$. So, $\theta_i = \pm 1$ in ordering $\widetilde{a} + \widetilde{c}$ and $\widetilde{b} + \widetilde{c}$.

The above eight cases suggest that θ_1 and θ_2 are invariant in ordering $\widetilde{a}, \widetilde{b}$ and $\widetilde{a} + \widetilde{c}, \widetilde{b} + \widetilde{c}$. Hence, the claim.

Now, by the Theorem 3.1 it follows that

$$\mathcal{R}_\lambda(\widetilde{a} + \widetilde{c}, \theta_1, \theta_2, \theta_3) = \mathcal{R}_\lambda(\widetilde{a}, \theta_1, \theta_2, \theta_3) + \mathcal{R}_\lambda(\widetilde{c}, \theta_1, \theta_2, \theta_3),$$

and

$$\mathcal{R}_\lambda(\widetilde{b} + \widetilde{c}, \theta_1, \theta_2, \theta_3) = \mathcal{R}_\lambda(\widetilde{b}, \theta_1, \theta_2, \theta_3) + \mathcal{R}_\lambda(\widetilde{c}, \theta_1, \theta_2, \theta_3),$$

Hence, if $\widetilde{a} \geq \widetilde{b}$, then it is obvious that $\mathcal{R}_\lambda(\widetilde{a}, \theta_1, \theta_2, \theta_3) \geq \mathcal{R}_\lambda(\widetilde{b}, \theta_1, \theta_2, \theta_3)$. This leads to the inequality $\mathcal{R}_\lambda(\widetilde{a}, \theta_1, \theta_2, \theta_3) + \mathcal{R}_\lambda(\widetilde{c}, \theta_1, \theta_2, \theta_3) \geq \mathcal{R}_\lambda(\widetilde{b}, \theta_1, \theta_2, \theta_3) + \mathcal{R}_\lambda(\widetilde{c}, \theta_1, \theta_2, \theta_3)$, which evidently follows the inequality $\mathcal{R}_\lambda(\widetilde{a} + \widetilde{c}, \theta_1, \theta_2, \theta_3) \geq \mathcal{R}_\lambda(\widetilde{b} + \widetilde{c}, \theta_1, \theta_2, \theta_3)$. Thus, the result follows immediately.

Proof of the Theorem 3.9

Let $\widetilde{a} \geq \widetilde{b}$. Then $\mathcal{R}_\lambda(\widetilde{a}, \theta_1, \theta_2, \theta_3) \geq \mathcal{R}_\lambda(\widetilde{b}, \theta_1, \theta_2, \theta_3)$. Let $k > 0$. Then using the Proposition 3.4, it follows that

$$\begin{aligned}\mathcal{R}_\lambda(k\widetilde{a}, \theta_1, \theta_2, \theta_3) =& \lambda\{\mathcal{V}(\mu_{k\widetilde{a}}) + \theta_1 \mathcal{A}(\mu_{k\widetilde{a}})\} \\ & + (1-\lambda)\{\mathcal{V}(\rho_{k\widetilde{a}}) + \theta_2 \mathcal{A}(\rho_{k\widetilde{a}}) + \mathcal{V}(\nu_{k\widetilde{a}}) + \theta_3 \mathcal{A}(\nu_{k\widetilde{a}})\} \\ =& k\lambda\{\mathcal{V}(\mu_{\widetilde{a}}) + \theta_1 \mathcal{A}(\mu_{\widetilde{a}})\} \\ & + k(1-\lambda)\{\mathcal{V}(\rho_{\widetilde{a}}) + \theta_2 \mathcal{A}(\rho_{\widetilde{a}}) + \mathcal{V}(\nu_{\widetilde{a}}) + \theta_3 \mathcal{A}(\nu_{\widetilde{a}})\} \\ =& k\mathcal{R}_\lambda(\widetilde{a}, \theta_1, \theta_2, \theta_3).\end{aligned}$$

Thus, when $\widetilde{a} \succeq \widetilde{b}$, it follows that $\mathcal{R}_\lambda(\widetilde{a}, \theta_1, \theta_2, \theta_3) \geq \mathcal{R}_\lambda(\widetilde{b}, \theta_1, \theta_2, \theta_3)$. Equivalently, it follows that $k\mathcal{R}_\lambda(\widetilde{a}, \theta_1, \theta_2, \theta_3) \geq k\mathcal{R}_\lambda(\widetilde{b}, \theta_1, \theta_2, \theta_3)$, which can be trivially expressed as $\mathcal{R}_\lambda(k\widetilde{a}, \theta_1, \theta_2, \theta_3) \geq \mathcal{R}_\lambda(k\widetilde{b}, \theta_1, \theta_2, \theta_3)$. So, the result, $k\widetilde{a} \succeq k\widetilde{b}$, follows immediately.

Let $k < 0$, assume $k = -m < 0$, then the following cases arise.

Case 1: Let $\widetilde{a} \succeq \widetilde{b}$ for $\theta_i = 0$. Then $\mathcal{V}(\mu_{\widetilde{a}}) \neq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \neq \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \neq \mathcal{V}(\nu_{\widetilde{b}})$ and $\mathcal{R}_\lambda(\widetilde{a}, 0, 0) \geq \mathcal{R}_\lambda(\widetilde{b}, 0, 0)$. Now, as $\widetilde{a} \succeq \widetilde{b}$ it follows that $\mathcal{V}(\mu_{\widetilde{a}}) \geq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \geq \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \geq \mathcal{V}(\nu_{\widetilde{b}})$. Clearly, $\mathcal{V}(\mu_{-m\widetilde{a}}) \neq \mathcal{V}(\mu_{-m\widetilde{b}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) \neq \mathcal{V}(\rho_{-m\widetilde{b}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) \neq \mathcal{V}(\nu_{-m\widetilde{b}})$. Thus, $\theta_i = 0$ in ordering $-m\widetilde{a}$ and $-m\widetilde{b}$. Further, it follows that $\mathcal{V}(\mu_{-m\widetilde{a}}) \leq \mathcal{V}(\mu_{-m\widetilde{b}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) \leq \mathcal{V}(\rho_{-m\widetilde{b}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) \leq \mathcal{V}(\nu_{-m\widetilde{b}})$. So, $\mathcal{R}_\lambda(-m\widetilde{a}, 0, 0, 0) \leq \mathcal{R}_\lambda(-m\widetilde{b}, 0, 0, 0)$. Hence, the result $-m\widetilde{a} \preceq -m\widetilde{b}$ follows immediately.

Case 2: Let $\widetilde{a} \succeq \widetilde{b}$ for $\theta_1 = 0$, $\theta_2 = 0$ and $\theta_3 = \pm 1$. Then $\mathcal{V}(\mu_{\widetilde{a}}) \neq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \neq \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) = \mathcal{V}(\nu_{\widetilde{b}})$ and $\mathcal{R}_\lambda(\widetilde{a}, 0, 0, \pm 1) \geq \mathcal{R}_\lambda(\widetilde{b}, 0, 0, \pm 1)$. Now, as $\widetilde{a} \succeq \widetilde{b}$ it follows that $\mathcal{V}(\mu_{\widetilde{a}}) \geq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \geq \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \pm \mathcal{A}(\nu_{\widetilde{a}}) \geq \mathcal{V}(\nu_{\widetilde{b}}) \pm \mathcal{A}(\nu_{\widetilde{b}})$. Clearly, $\mathcal{V}(\mu_{-m\widetilde{a}}) \neq \mathcal{V}(\mu_{-m\widetilde{b}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) \neq \mathcal{V}(\rho_{-m\widetilde{b}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) = \mathcal{V}(\nu_{-m\widetilde{b}})$. Thus, $\theta_1 = 0$, $\theta_2 = 0$ and $\theta_3 = \mp 1$ in ordering $-m\widetilde{a}$ and $-m\widetilde{b}$. Further, it follows that $\mathcal{V}(\mu_{-m\widetilde{a}}) \leq \mathcal{V}(\mu_{-m\widetilde{b}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) \leq \mathcal{V}(\rho_{-m\widetilde{b}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) \mp \mathcal{A}(\nu_{-m\widetilde{a}}) \leq \mathcal{V}(\nu_{-m\widetilde{b}}) \mp \mathcal{A}(\nu_{-m\widetilde{a}})$. So, $\mathcal{R}_\lambda(-m\widetilde{a}, 0, 0, \mp 1) \leq \mathcal{R}_\lambda(-m\widetilde{b}, 0, 0, \mp 1)$. Hence, the result $-m\widetilde{a} \preceq -m\widetilde{b}$ follows immediately.

Case 3: Let $\widetilde{a} \succeq \widetilde{b}$ for $\theta_1 = 0$, $\theta_2 = \pm 1$ and $\theta_3 = 0$. Then $\mathcal{V}(\mu_{\widetilde{a}}) \neq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) = \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \neq \mathcal{V}(\nu_{\widetilde{b}})$ and $\mathcal{R}_\lambda(\widetilde{a}, 0, \pm 1, 0) \geq \mathcal{R}_\lambda(\widetilde{b}, 0, \pm 1, 0)$. Now, as $\widetilde{a} \succeq \widetilde{b}$ it follows that $\mathcal{V}(\mu_{\widetilde{a}}) \geq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \pm \mathcal{A}(\rho_{\widetilde{a}}) \geq \mathcal{V}(\rho_{\widetilde{b}}) \pm \mathcal{A}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \geq \mathcal{V}(\nu_{\widetilde{b}})$. Clearly, $\mathcal{V}(\mu_{-m\widetilde{a}}) \neq \mathcal{V}(\mu_{-m\widetilde{b}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) = \mathcal{V}(\rho_{-m\widetilde{b}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) \neq \mathcal{V}(\nu_{-m\widetilde{b}})$. Thus, $\theta_1 = 0$, $\theta_2 = \mp 1$ and $\theta_3 = 0$ in ordering $-m\widetilde{a}$ and $-m\widetilde{b}$. Further, it follows that $\mathcal{V}(\mu_{-m\widetilde{a}}) \leq \mathcal{V}(\mu_{-m\widetilde{b}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) \mp \mathcal{A}(\rho_{-m\widetilde{a}}) \leq \mathcal{V}(\rho_{-m\widetilde{b}}) \mp \mathcal{A}(\rho_{-m\widetilde{b}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) \leq \mathcal{V}(\nu_{-m\widetilde{b}})$. So, $\mathcal{R}_\lambda(-m\widetilde{a}, 0, \mp 1, 0) \leq \mathcal{R}_\lambda(-m\widetilde{b}, 0, \mp 1, 0)$. Hence, the result $-m\widetilde{a} \preceq -m\widetilde{b}$ follows immediately.

Case 4: Let $\widetilde{a} \succeq \widetilde{b}$ for $\theta_1 = 0$, $\theta_2 = \pm 1$ and $\theta_3 = \pm 1$. Then $\mathcal{V}(\mu_{\widetilde{a}}) \neq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) = \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) = \mathcal{V}(\nu_{\widetilde{b}})$ and $\mathcal{R}_\lambda(\widetilde{a}, 0, \pm 1, \pm 1) \geq \mathcal{R}_\lambda(\widetilde{b}, 0, \pm 1, \pm 1)$. Now, as $\widetilde{a} \succeq \widetilde{b}$ it follows that $\mathcal{V}(\mu_{\widetilde{a}}) \geq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \pm \mathcal{A}(\rho_{\widetilde{a}}) \geq \mathcal{V}(\rho_{\widetilde{b}}) \pm \mathcal{A}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \pm \mathcal{A}(\nu_{\widetilde{a}}) \geq \mathcal{V}(\nu_{\widetilde{b}}) \pm \mathcal{A}(\nu_{\widetilde{b}})$. Clearly, $\mathcal{V}(\mu_{-m\widetilde{a}}) \neq \mathcal{V}(\mu_{-m\widetilde{b}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) = \mathcal{V}(\rho_{-m\widetilde{b}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) = \mathcal{V}(\nu_{-m\widetilde{b}})$. Thus, $\theta_1 = 0$, $\theta_2 = \mp 1$ and $\theta_3 = \mp 1$ in ordering $-m\widetilde{a}$ and $-m\widetilde{b}$. Further, it follows that $\mathcal{V}(\mu_{-m\widetilde{a}}) \leq \mathcal{V}(\mu_{-m\widetilde{b}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) \mp \mathcal{A}(\rho_{-m\widetilde{a}}) \leq \mathcal{V}(\rho_{-m\widetilde{b}}) \mp \mathcal{A}(\rho_{-m\widetilde{b}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) \mp \mathcal{A}(\nu_{-m\widetilde{a}}) \leq \mathcal{V}(\nu_{-m\widetilde{b}}) \mp \mathcal{A}(\nu_{-m\widetilde{b}})$. So, $\mathcal{R}_\lambda(-m\widetilde{a}, 0, \mp 1, \mp 1) \leq \mathcal{R}_\lambda(-m\widetilde{b}, 0, \mp 1, \mp 1)$. Hence, the result $-m\widetilde{a} \preceq -m\widetilde{b}$ follows immediately.

Case 5: Let $\widetilde{a} \succeq \widetilde{b}$ for $\theta_1 = \pm 1$, $\theta_2 = 0$ and $\theta_3 = 0$. Then $\mathcal{V}(\mu_{\widetilde{a}}) = \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \neq \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \neq \mathcal{V}(\nu_{\widetilde{b}})$ and $\mathcal{R}_\lambda(\widetilde{a}, \pm 1, 0, 0) \geq \mathcal{R}_\lambda(\widetilde{b}, \pm 1, 0, 0)$. Now, as $\widetilde{a} \succeq \widetilde{b}$ it follows that $\mathcal{V}(\mu_{\widetilde{a}}) \pm \mathcal{A}(\mu_{\widetilde{a}}) \geq \mathcal{V}(\mu_{\widetilde{b}}) \pm \mathcal{A}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \geq \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \geq \mathcal{V}(\nu_{\widetilde{b}})$. Clearly, $\mathcal{V}(\mu_{-m\widetilde{a}}) = \mathcal{V}(\mu_{-m\widetilde{b}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) \neq \mathcal{V}(\rho_{-m\widetilde{b}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) \neq \mathcal{V}(\nu_{-m\widetilde{b}})$. Thus, $\theta_1 = \mp 1$, $\theta_2 = 0$ and $\theta_3 = 0$ in ordering $-m\widetilde{a}$ and $-m\widetilde{b}$. Further, it follows that $\mathcal{V}(\mu_{-m\widetilde{a}}) \mp \mathcal{A}(\mu_{-m\widetilde{a}}) \leq \mathcal{V}(\mu_{-m\widetilde{b}}) \mp \mathcal{A}(\mu_{-m\widetilde{b}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) \leq \mathcal{V}(\rho_{-m\widetilde{b}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) \leq \mathcal{V}(\nu_{-m\widetilde{b}})$. So, $\mathcal{R}_\lambda(-m\widetilde{a}, \mp 1, 0, 0) \leq \mathcal{R}_\lambda(-m\widetilde{b}, \mp 1, 0, 0)$. Hence, the result $-m\widetilde{a} \preceq -m\widetilde{b}$ follows immediately.

Case 6: Let $\widetilde{a} \succeq \widetilde{b}$ for $\theta_1 = \pm 1$, $\theta_2 = 0$ and $\theta_3 = \pm 1$. Then $\mathcal{V}(\mu_{\widetilde{a}}) = \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \neq \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) = \mathcal{V}(\nu_{\widetilde{b}})$ and $\mathcal{R}_\lambda(\widetilde{a}, \pm 1, 0, \pm 1) \geq \mathcal{R}_\lambda(\widetilde{b}, \pm 1, 0, \pm 1)$. Now, as $\widetilde{a} \succeq \widetilde{b}$ it follows that $\mathcal{V}(\mu_{\widetilde{a}}) \pm \mathcal{A}(\mu_{\widetilde{a}}) \geq \mathcal{V}(\mu_{\widetilde{b}}) \pm \mathcal{A}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \geq \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \pm \mathcal{A}(\nu_{\widetilde{a}}) \geq \mathcal{V}(\nu_{\widetilde{b}}) \pm \mathcal{A}(\nu_{\widetilde{b}})$. Clearly, $\mathcal{V}(\mu_{-m\widetilde{a}}) = \mathcal{V}(\mu_{-m\widetilde{b}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) \neq \mathcal{V}(\rho_{-m\widetilde{b}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) = \mathcal{V}(\nu_{-m\widetilde{b}})$. Thus, $\theta_1 = \mp 1$, $\theta_2 = 0$ and $\theta_3 = \mp 1$ in ordering $-m\widetilde{a}$ and $-m\widetilde{b}$. Further, it follows that $\mathcal{V}(\mu_{-m\widetilde{a}}) \mp \mathcal{A}(\mu_{-m\widetilde{a}}) \leq \mathcal{V}(\mu_{-m\widetilde{b}}) \mp \mathcal{A}(\mu_{-m\widetilde{b}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) \leq \mathcal{V}(\rho_{-m\widetilde{b}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) \mp \mathcal{A}(\nu_{-m\widetilde{a}}) \leq \mathcal{V}(\nu_{-m\widetilde{b}}) \mp \mathcal{A}(\nu_{-m\widetilde{b}})$. So, $\mathcal{R}_\lambda(-m\widetilde{a}, \mp 1, 0, \mp 1) \leq \mathcal{R}_\lambda(-m\widetilde{b}, \mp 1, 0, \mp 1)$. Hence, the result $-m\widetilde{a} \preceq -m\widetilde{b}$ follows immediately.

Case 7: Let $\widetilde{a} \succeq \widetilde{b}$ for $\theta_1 = \pm 1$, $\theta_2 = \pm 1$ and $\theta_3 = 0$. Then $\mathcal{V}(\mu_{\widetilde{a}}) = \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) = \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \neq \mathcal{V}(\nu_{\widetilde{b}})$ and $\mathcal{R}_\lambda(\widetilde{a}, \pm 1, \pm 1, 0) \geq \mathcal{R}_\lambda(\widetilde{b}, \pm 1, \pm 1, 0)$. Now, as $\widetilde{a} \succeq \widetilde{b}$ it follows that $\mathcal{V}(\mu_{\widetilde{a}}) \pm \mathcal{A}(\mu_{\widetilde{a}}) \geq \mathcal{V}(\mu_{\widetilde{b}}) \pm \mathcal{A}(\mu_{\widetilde{a}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \pm \mathcal{A}(\rho_{\widetilde{a}}) \geq \mathcal{V}(\rho_{\widetilde{b}}) \pm \mathcal{A}(\rho_{\widetilde{a}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \geq \mathcal{V}(\nu_{\widetilde{b}})$. Clearly, $\mathcal{V}(\mu_{-m\widetilde{a}}) = \mathcal{V}(\mu_{-m\widetilde{b}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) = \mathcal{V}(\rho_{-m\widetilde{b}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) \neq \mathcal{V}(\nu_{-m\widetilde{b}})$. Thus, $\theta_1 = \mp 1$, $\theta_2 = \mp 1$ and $\theta_3 = 0$ in ordering $-m\widetilde{a}$ and $-m\widetilde{b}$. Further, it follows that $\mathcal{V}(\mu_{-m\widetilde{a}}) \mp \mathcal{A}(\mu_{-m\widetilde{a}}) \leq \mathcal{V}(\mu_{-m\widetilde{b}}) \mp \mathcal{A}(\mu_{-m\widetilde{a}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) \mp \mathcal{A}(\rho_{-m\widetilde{a}}) \leq \mathcal{V}(\rho_{-m\widetilde{b}}) \mp \mathcal{A}(\rho_{-m\widetilde{a}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) \leq \mathcal{V}(\nu_{-m\widetilde{b}})$. So, $\mathcal{R}_\lambda(-m\widetilde{a}, \mp 1, \mp 1, 0) \leq \mathcal{R}_\lambda(-m\widetilde{b}, \mp 1, \mp 1, 0)$. Hence, the result $-m\widetilde{a} \preceq -m\widetilde{b}$ follows immediately.

Case 8: Let $\widetilde{a} \succeq \widetilde{b}$ for $\theta_i = \pm 1$. Then $\mathcal{V}(\mu_{\widetilde{a}}) = \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) = \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) = \mathcal{V}(\nu_{\widetilde{b}})$ and $\mathcal{R}_\lambda(\widetilde{a}, \pm 1, \pm 1, \pm 1) \geq \mathcal{R}_\lambda(\widetilde{b}, \pm 1, \pm 1, \pm 1)$. Now, as $\widetilde{a} \succeq \widetilde{b}$ it follows that $\mathcal{V}(\mu_{\widetilde{a}}) \pm \mathcal{A}(\mu_{\widetilde{a}}) \geq \mathcal{V}(\mu_{\widetilde{b}}) \pm \mathcal{A}(\mu_{\widetilde{a}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \pm \mathcal{A}(\rho_{\widetilde{a}}) \geq \mathcal{V}(\rho_{\widetilde{b}}) \pm \mathcal{A}(\rho_{\widetilde{a}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \pm \mathcal{A}(\nu_{\widetilde{a}}) \geq \mathcal{V}(\nu_{\widetilde{b}}) \pm \mathcal{A}(\nu_{\widetilde{b}})$. Clearly, $\mathcal{V}(\mu_{-m\widetilde{a}}) = \mathcal{V}(\mu_{-m\widetilde{b}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) = \mathcal{V}(\rho_{-m\widetilde{b}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) = \mathcal{V}(\nu_{-m\widetilde{b}})$. Thus, $\theta_i = \mp 1$ in ordering $-m\widetilde{a}$ and $-m\widetilde{b}$. Further, it follows that $\mathcal{V}(\mu_{-m\widetilde{a}}) \mp \mathcal{A}(\mu_{-m\widetilde{a}}) \leq \mathcal{V}(\mu_{-m\widetilde{b}}) \mp \mathcal{A}(\mu_{-m\widetilde{a}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) \mp \mathcal{A}(\rho_{-m\widetilde{a}}) \leq \mathcal{V}(\rho_{-m\widetilde{b}}) \mp \mathcal{A}(\rho_{-m\widetilde{a}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) \mp \mathcal{A}(\nu_{-m\widetilde{a}}) \leq \mathcal{V}(\nu_{-m\widetilde{b}}) \mp \mathcal{A}(\nu_{-m\widetilde{b}})$. So, $\mathcal{R}_\lambda(-m\widetilde{a}, \mp 1, \mp 1, \mp 1) \leq \mathcal{R}_\lambda(-m\widetilde{b}, \mp 1, \mp 1, \mp 1)$. Hence, the result $-m\widetilde{a} \preceq -m\widetilde{b}$ follows immediately.

Proof of the Theorem 3.10

Let $k > 0$ and $k\widetilde{a} \succeq k\widetilde{b}$. Then $\mathcal{R}_\lambda(k\widetilde{a}, \theta_1, \theta_2, \theta_3) \geq \mathcal{R}_\lambda(k\widetilde{b}, \theta_1, \theta_2, \theta_3)$. However, by Proposition 3.4, it follows that $k\mathcal{R}_\lambda(\widetilde{a}, \theta_1, \theta_2, \theta_3) \geq k\mathcal{R}_\lambda(\widetilde{b}, \theta_1, \theta_2, \theta_3)$. Thus, the result follows immediately. If $k < 0$, let $k = -m < 0$, then $-m\widetilde{a} \succeq -m\widetilde{b}$ implies that $\mathcal{R}_\lambda(-m\widetilde{a}, \theta_1, \theta_2, \theta_3) \geq \mathcal{R}_\lambda(-m\widetilde{b}, \theta_1, \theta_2, \theta_3)$. Now, eight cases arise.

Case 1: Let $-m\widetilde{a} \succeq -m\widetilde{b}$ for $\theta_i = 0$. Then $\mathcal{V}(\mu_{-m\widetilde{a}}) \neq \mathcal{V}(\mu_{-m\widetilde{b}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) \neq \mathcal{V}(\rho_{-m\widetilde{b}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) \neq \mathcal{V}(\nu_{-m\widetilde{b}})$ and $\mathcal{R}_\lambda(-m\widetilde{a}, 0, 0, 0) \geq \mathcal{R}_\lambda(-m\widetilde{b}, 0, 0, 0)$. Now, as $-m\widetilde{a} \succeq -m\widetilde{b}$ it follows that $\mathcal{V}(\mu_{-m\widetilde{a}}) \geq \mathcal{V}(\mu_{-m\widetilde{b}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) \geq \mathcal{V}(\rho_{-m\widetilde{b}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) \geq \mathcal{V}(\nu_{-m\widetilde{b}})$. Clearly, $\mathcal{V}(\mu_{\widetilde{a}}) \neq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \neq \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \neq \mathcal{V}(\nu_{\widetilde{b}})$. Thus, $\theta_i = 0$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Further, it follows that $\mathcal{V}(\mu_{\widetilde{a}}) \leq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \leq \mathcal{V}(\rho_{\widetilde{b}})$and $\mathcal{V}(\nu_{\widetilde{a}}) \leq \mathcal{V}(\nu_{\widetilde{b}})$. So, $\mathcal{R}_\lambda(\widetilde{a}, 0, 0, 0) \leq \mathcal{R}_\lambda(\widetilde{b}, 0, 0, 0)$. Hence, the result $\widetilde{a} \preceq \widetilde{b}$ follows immediately.

Case 2: Let $-m\widetilde{a} \succeq -m\widetilde{b}$ for $\theta_1 = 0$, $\theta_2 = 0$ and $\theta_3 = \pm 1$ then $\mathcal{V}(\mu_{-m\widetilde{a}}) \neq \mathcal{V}(\mu_{-m\widetilde{b}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) \neq \mathcal{V}(\rho_{-m\widetilde{b}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) = \mathcal{V}(\nu_{-m\widetilde{b}})$ and $\mathcal{R}_\lambda(-m\widetilde{a}, 0, 0, \pm 1) \geq \mathcal{R}_\lambda(-m\widetilde{b}, 0, 0, \pm 1)$. Now, as $-m\widetilde{a} \succeq -m\widetilde{b}$ it follows that $\mathcal{V}(\mu_{-m\widetilde{a}}) \geq \mathcal{V}(\mu_{-m\widetilde{b}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) \geq \mathcal{V}(\rho_{-m\widetilde{b}})$ and

$\mathcal{V}(\nu_{-m\widetilde{a}}) \pm \mathcal{A}(\nu_{-m\widetilde{a}}) \geq \mathcal{V}(\nu_{-m\widetilde{b}}) \pm \mathcal{A}(\nu_{-m\widetilde{b}})$. Clearly, $\mathcal{V}(\mu_{\widetilde{a}}) \neq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \neq \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) = \mathcal{V}(\nu_{\widetilde{b}})$. Thus, $\theta_1 = 0$, $\theta_2 = 0$ and $\theta_3 = \mp 1$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Further, it follows that $\mathcal{V}(\mu_{\widetilde{a}}) \leq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \leq \mathcal{V}(\rho_{\widetilde{b}})$and $\mathcal{V}(\nu_{\widetilde{a}}) \mp \mathcal{A}(\nu_{\widetilde{a}}) \leq \mathcal{V}(\nu_{\widetilde{b}}) \mp \mathcal{A}(\nu_{\widetilde{b}})$. So, $\mathcal{R}_\lambda(\widetilde{a}, 0, 0, \mp 1) \leq \mathcal{R}_\lambda(\widetilde{b}, 0, 0, \mp 1)$. Hence, the result $\widetilde{a} \preceq \widetilde{b}$ follows immediately.

Case 3: Let $-m\widetilde{a} \succeq -m\widetilde{b}$ for $\theta_1 = 0$, $\theta_2 = \pm 1$ and $\theta_3 = 0$. Then $\mathcal{V}(\mu_{-m\widetilde{a}}) \neq \mathcal{V}(\mu_{-m\widetilde{b}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) = \mathcal{V}(\rho_{-m\widetilde{b}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) \neq \mathcal{V}(\nu_{-m\widetilde{b}})$ and $\mathcal{R}_\lambda(-m\widetilde{a}, 0, \pm 1, 0) \geq \mathcal{R}_\lambda(-m\widetilde{b}, 0, \pm 1, 0)$. Now, as $-m\widetilde{a} \succeq -m\widetilde{b}$ it follows that $\mathcal{V}(\mu_{-m\widetilde{a}}) \geq \mathcal{V}(\mu_{-m\widetilde{b}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) \pm \mathcal{A}(\rho_{-m\widetilde{a}}) \geq \mathcal{V}(\rho_{-m\widetilde{b}}) \pm \mathcal{A}(\rho_{-m\widetilde{b}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) \geq \mathcal{V}(\nu_{-m\widetilde{b}})$. Clearly, $\mathcal{V}(\mu_{\widetilde{a}}) \neq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) = \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \neq \mathcal{V}(\nu_{\widetilde{b}})$. Thus, $\theta_1 = 0$, $\theta_2 = \mp 1$ and $\theta_3 = 0$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Further, it follows that $\mathcal{V}(\mu_{\widetilde{a}}) \leq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \mp \mathcal{A}(\rho_{\widetilde{a}}) \leq \mathcal{V}(\rho_{\widetilde{b}}) \mp \mathcal{A}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \leq \mathcal{V}(\nu_{\widetilde{b}})$. So, $\mathcal{R}_\lambda(\widetilde{a}, 0, \mp 1, 0) \leq \mathcal{R}_\lambda(\widetilde{b}, 0, \mp 1, 0)$. Hence, the result $\widetilde{a} \preceq \widetilde{b}$ follows immediately.

Case 4: Let $-m\widetilde{a} \succeq -m\widetilde{b}$ for $\theta_1 = 0$, $\theta_2 = \pm 1$ and $\theta_3 = \pm 1$. Then $\mathcal{V}(\mu_{-m\widetilde{a}}) \neq \mathcal{V}(\mu_{-m\widetilde{b}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) = \mathcal{V}(\rho_{-m\widetilde{b}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) = \mathcal{V}(\nu_{-m\widetilde{b}})$ and $\mathcal{R}_\lambda(-m\widetilde{a}, 0, \pm 1, \pm 1) \geq \mathcal{R}_\lambda(-m\widetilde{b}, 0, \pm 1, \pm 1)$. Now, as $-m\widetilde{a} \succeq -m\widetilde{b}$ it follows that $\mathcal{V}(\mu_{-m\widetilde{a}}) \geq \mathcal{V}(\mu_{-m\widetilde{b}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) \pm \mathcal{A}(\rho_{-m\widetilde{a}}) \geq \mathcal{V}(\rho_{-m\widetilde{b}}) \pm \mathcal{A}(\rho_{-m\widetilde{a}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) \pm \mathcal{A}(\nu_{-m\widetilde{a}}) \geq \mathcal{V}(\nu_{-m\widetilde{b}}) \pm \mathcal{A}(\nu_{-m\widetilde{b}})$. Clearly, $\mathcal{V}(\mu_{\widetilde{a}}) \neq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) = \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) = \mathcal{V}(\nu_{\widetilde{b}})$. Thus, $\theta_1 = 0$, $\theta_2 = \mp 1$ and $\theta_3 = \mp 1$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Further, it follows that $\mathcal{V}(\mu_{\widetilde{a}}) \leq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \mp \mathcal{A}(\rho_{\widetilde{a}}) \leq \mathcal{V}(\rho_{\widetilde{b}}) \mp \mathcal{A}(\rho_{\widetilde{a}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \mp \mathcal{A}(\nu_{\widetilde{a}}) \leq \mathcal{V}(\nu_{\widetilde{b}}) \mp \mathcal{A}(\nu_{\widetilde{b}})$. So, $\mathcal{R}_\lambda(\widetilde{a}, 0, \mp 1, \mp 1) \leq \mathcal{R}_\lambda(\widetilde{b}, 0, \mp 1, \mp 1)$. Hence, the result $\widetilde{a} \preceq \widetilde{b}$ follows immediately.

Case 5: Let $-m\widetilde{a} \succeq -m\widetilde{b}$ for $\theta_1 = \pm 1$, $\theta_2 = 0$ and $\theta_3 = 0$. Then $\mathcal{V}(\mu_{-m\widetilde{a}}) = \mathcal{V}(\mu_{-m\widetilde{b}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) \neq \mathcal{V}(\rho_{-m\widetilde{b}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) \neq \mathcal{V}(\nu_{-m\widetilde{b}})$ and $\mathcal{R}_\lambda(-m\widetilde{a}, \pm 1, 0, 0) \geq \mathcal{R}_\lambda(-m\widetilde{b}, \pm 1, 0, 0)$. Now, as $-m\widetilde{a} \succeq -m\widetilde{b}$ it follows that $\mathcal{V}(\mu_{-m\widetilde{a}}) \pm \mathcal{A}(\mu_{-m\widetilde{a}}) \geq \mathcal{V}(\mu_{-m\widetilde{b}}) \pm \mathcal{A}(\mu_{-m\widetilde{b}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) \geq \mathcal{V}(\rho_{-m\widetilde{b}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) \geq \mathcal{V}(\nu_{-m\widetilde{b}})$. Clearly, $\mathcal{V}(\mu_{\widetilde{a}}) = \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \neq \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \neq \mathcal{V}(\nu_{\widetilde{b}})$. Thus, $\theta_1 = \mp 1$, $\theta_2 = 0$ and $\theta_3 = 0$, in ordering $\widetilde{a}$ and $\widetilde{b}$. Further, it follows that $\mathcal{V}(\mu_{\widetilde{a}}) \mp \mathcal{A}(\mu_{\widetilde{a}}) \leq \mathcal{V}(\mu_{\widetilde{b}}) \mp \mathcal{A}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \leq \mathcal{V}(\rho_{\widetilde{b}})$and $\mathcal{V}(\nu_{\widetilde{a}}) \leq \mathcal{V}(\nu_{\widetilde{b}})$. So, $\mathcal{R}_\lambda(\widetilde{a}, \mp 1, 0, 0) \leq \mathcal{R}_\lambda(\widetilde{b}, \mp 1, 0, 0)$. Hence, the result $\widetilde{a} \preceq \widetilde{b}$ follows immediately.

Case 6: Let $-m\widetilde{a} \succeq -m\widetilde{b}$ for $\theta_1 = \pm 1$, $\theta_2 = 0$ and $\theta_3 = \pm 1$. Then $\mathcal{V}(\mu_{-m\widetilde{a}}) = \mathcal{V}(\mu_{-m\widetilde{b}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) \neq \mathcal{V}(\rho_{-m\widetilde{b}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) = \mathcal{V}(\nu_{-m\widetilde{b}})$ and $\mathcal{R}_\lambda(-m\widetilde{a}, \pm 1, 0, \pm 1) \geq \mathcal{R}_\lambda(-m\widetilde{b}, \pm 1, 0, \pm 1)$. Now, as $-m\widetilde{a} \succeq -m\widetilde{b}$ it follows that $\mathcal{V}(\mu_{-m\widetilde{a}}) \pm \mathcal{A}(\mu_{-m\widetilde{a}}) \geq \mathcal{V}(\mu_{-m\widetilde{b}}) \pm \mathcal{A}(\mu_{-m\widetilde{a}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) \geq \mathcal{V}(\rho_{-m\widetilde{b}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) \pm \mathcal{A}(\nu_{-m\widetilde{a}}) \geq \mathcal{V}(\nu_{-m\widetilde{b}}) \pm \mathcal{A}(\nu_{-m\widetilde{b}})$. Clearly, $\mathcal{V}(\mu_{\widetilde{a}}) = \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \neq \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) = \mathcal{V}(\nu_{\widetilde{b}})$. Thus, $\theta_1 = \mp 1$, $\theta_2 = 0$ and $\theta_3 = \mp 1$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Further, it follows that $\mathcal{V}(\mu_{\widetilde{a}}) \mp \mathcal{A}(\mu_{\widetilde{a}}) \leq \mathcal{V}(\mu_{\widetilde{b}}) \mp \mathcal{A}(\mu_{\widetilde{a}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \leq \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \mp \mathcal{A}(\nu_{\widetilde{a}}) \leq \mathcal{V}(\nu_{\widetilde{b}}) \mp \mathcal{A}(\nu_{\widetilde{b}})$. So, $\mathcal{R}_\lambda(\widetilde{a}, \mp 1, 0, \mp 1) \leq \mathcal{R}_\lambda(\widetilde{b}, \mp 1, 0, \mp 1)$. Hence, the result $\widetilde{a} \preceq \widetilde{b}$ follows immediately.

Case 7: Let $-m\widetilde{a} \succeq -m\widetilde{b}$ for $\theta_1 = \pm 1$, $\theta_2 = \pm 1$ and $\theta_3 = 0$. Then $\mathcal{V}(\mu_{-m\widetilde{a}}) = \mathcal{V}(\mu_{-m\widetilde{b}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) = \mathcal{V}(\rho_{-m\widetilde{b}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) \neq \mathcal{V}(\nu_{-m\widetilde{b}})$ and $\mathcal{R}_\lambda(-m\widetilde{a}, \pm 1, \pm 1, 0) \geq \mathcal{R}_\lambda(-m\widetilde{b}, \pm 1, \pm 1, 0)$. Now, as $-m\widetilde{a} \succeq -m\widetilde{b}$ it follows that $\mathcal{V}(\mu_{-m\widetilde{a}}) \pm \mathcal{A}(\mu_{-m\widetilde{a}}) \geq \mathcal{V}(\mu_{-m\widetilde{b}}) \pm \mathcal{A}(\mu_{-m\widetilde{a}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) \pm \mathcal{A}(\rho_{-m\widetilde{a}}) \geq \mathcal{V}(\rho_{-m\widetilde{b}}) \pm \mathcal{A}(\rho_{-m\widetilde{a}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) \geq \mathcal{V}(\nu_{-m\widetilde{b}})$. Clearly, $\mathcal{V}(\mu_{\widetilde{a}}) = \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) = \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \neq \mathcal{V}(\nu_{\widetilde{b}})$. Thus, $\theta_1 = \mp 1$, $\theta_2 = \mp 1$ and $\theta_3 = 0$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Further, it follows that $\mathcal{V}(\mu_{\widetilde{a}}) \mp \mathcal{A}(\mu_{\widetilde{a}}) \leq \mathcal{V}(\mu_{\widetilde{b}}) \mp \mathcal{A}(\mu_{\widetilde{a}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \mp \mathcal{A}(\rho_{\widetilde{a}}) \leq \mathcal{V}(\rho_{\widetilde{b}}) \mp \mathcal{A}(\rho_{\widetilde{a}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \leq \mathcal{V}(\nu_{\widetilde{b}})$. So, $\mathcal{R}_\lambda(\widetilde{a}, \mp 1, \mp 1, 0) \leq \mathcal{R}_\lambda(\widetilde{b}, \mp 1, \mp 1, 0)$. Hence, the result $\widetilde{a} \preceq \widetilde{b}$ follows immediately.

Case 8: Let $-m\widetilde{a} \succeq -m\widetilde{b}$ for $\theta_i = \pm 1$. Then $\mathcal{V}(\mu_{-m\widetilde{a}}) = \mathcal{V}(\mu_{-m\widetilde{b}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) = \mathcal{V}(\rho_{-m\widetilde{b}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) = \mathcal{V}(\nu_{-m\widetilde{b}})$ and $\mathcal{R}_\lambda(-m\widetilde{a}, \pm 1, \pm 1, \pm 1) \geq \mathcal{R}_\lambda(-m\widetilde{b}, \pm 1, \pm 1, \pm 1)$. Now, as $-m\widetilde{a} \succeq -m\widetilde{b}$ it follows that $\mathcal{V}(\mu_{-m\widetilde{a}}) \pm \mathcal{A}(\mu_{-m\widetilde{a}}) \geq \mathcal{V}(\mu_{-m\widetilde{b}}) \pm \mathcal{A}(\mu_{-m\widetilde{a}})$, $\mathcal{V}(\rho_{-m\widetilde{a}}) \pm \mathcal{A}(\rho_{-m\widetilde{a}}) \geq \mathcal{V}(\rho_{-m\widetilde{b}}) \pm \mathcal{A}(\rho_{-m\widetilde{a}})$ and $\mathcal{V}(\nu_{-m\widetilde{a}}) \pm \mathcal{A}(\nu_{-m\widetilde{a}}) \geq \mathcal{V}(\nu_{-m\widetilde{b}}) \pm \mathcal{A}(\nu_{-m\widetilde{b}})$. Clearly, $\mathcal{V}(\mu_{\widetilde{a}}) = \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) = \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) = \mathcal{V}(\nu_{\widetilde{b}})$. Thus, $\theta_i = \mp 1$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Further, it follows that $\mathcal{V}(\mu_{\widetilde{a}}) \mp \mathcal{A}(\mu_{\widetilde{a}}) \leq \mathcal{V}(\mu_{\widetilde{b}}) \mp \mathcal{A}(\mu_{\widetilde{a}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \mp \mathcal{A}(\rho_{\widetilde{a}}) \leq \mathcal{V}(\rho_{\widetilde{b}}) \mp \mathcal{A}(\rho_{\widetilde{a}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \mp \mathcal{A}(\nu_{\widetilde{a}}) \leq \mathcal{V}(\nu_{\widetilde{b}}) \mp \mathcal{A}(\nu_{\widetilde{b}})$. So, $\mathcal{R}_\lambda(\widetilde{a}, \mp 1, \mp 1, \mp 1) \leq \mathcal{R}_\lambda(\widetilde{b}, \mp 1, \mp 1, \mp 1)$. Hence, the result $\widetilde{a} \preceq \widetilde{b}$ follows immediately.

Proof of the Theorem 3.13

The proof of this theorem, follows immediately if the invariance of θ_1 and θ_2 in ordering $\widetilde{a}$, $\widetilde{b}$ and $\widetilde{a} - \widetilde{c}$, $\widetilde{b} - \widetilde{c}$ can be established. Hence, a claim has to be made. The claim is as follows.

Claim : The value of θ_1 and θ_2 in ordering $\widetilde{a}$ and $\widetilde{b}$ are invariant in ordering $\widetilde{a} - \widetilde{c}$ and $\widetilde{b} - \widetilde{c}$. The proof of the claim follows from the following eight cases:

Case 1: Let $\theta_i = 0$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Then, $\mathcal{V}(\mu_{\widetilde{a}}) \neq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \neq \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \neq \mathcal{V}(\nu_{\widetilde{b}})$. Thus, it follows that $\mathcal{V}(\mu_{\widetilde{a}-\widetilde{c}}) \neq \mathcal{V}(\mu_{\widetilde{b}-\widetilde{c}})$, $\mathcal{V}(\rho_{\widetilde{a}-\widetilde{c}}) \neq \mathcal{V}(\rho_{\widetilde{b}-\widetilde{c}})$ and $\mathcal{V}(\nu_{\widetilde{a}-\widetilde{c}}) \neq \mathcal{V}(\nu_{\widetilde{b}-\widetilde{c}})$. So, $\theta_i = 0$ in ordering $\widetilde{a} - \widetilde{c}$ and $\widetilde{b} - \widetilde{c}$.

Case 2: Let $\theta_1 = 0$, $\theta_2 = 0$ and $\theta_3 = \pm 1$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Then, $\mathcal{V}(\mu_{\widetilde{a}}) \neq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \neq \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) = \mathcal{V}(\nu_{\widetilde{b}})$. Thus, it follows that $\mathcal{V}(\mu_{\widetilde{a}-\widetilde{c}}) \neq \mathcal{V}(\mu_{\widetilde{b}-\widetilde{c}})$, $\mathcal{V}(\rho_{\widetilde{a}-\widetilde{c}}) \neq \mathcal{V}(\rho_{\widetilde{b}-\widetilde{c}})$ and $\mathcal{V}(\nu_{\widetilde{a}-\widetilde{c}}) = \mathcal{V}(\nu_{\widetilde{b}-\widetilde{c}})$. So, $\theta_1 = 0$, $\theta_2 = 0$ and $\theta_3 = \pm 1$ in ordering $\widetilde{a} - \widetilde{c}$ and $\widetilde{b} - \widetilde{c}$.

Case 3: Let $\theta_1 = 0$, $\theta_2 = \pm 1$ and $\theta_3 = 0$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Then, $\mathcal{V}(\mu_{\widetilde{a}}) \neq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) = \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \neq \mathcal{V}(\nu_{\widetilde{b}})$. Thus, it follows that $\mathcal{V}(\mu_{\widetilde{a}-\widetilde{c}}) \neq \mathcal{V}(\mu_{\widetilde{b}-\widetilde{c}})$, $\mathcal{V}(\rho_{\widetilde{a}-\widetilde{c}}) = \mathcal{V}(\rho_{\widetilde{b}-\widetilde{c}})$ and $\mathcal{V}(\nu_{\widetilde{a}-\widetilde{c}}) \neq \mathcal{V}(\nu_{\widetilde{b}-\widetilde{c}})$. So, $\theta_1 = 0$, $\theta_2 = \pm 1$ and $\theta_3 = 0$ in ordering $\widetilde{a} - \widetilde{c}$ and $\widetilde{b} - \widetilde{c}$.

Case 4: Let $\theta_1 = 0$, $\theta_2 = \pm 1$ and $\theta_3 = \pm 1$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Then, $\mathcal{V}(\mu_{\widetilde{a}}) \neq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) = \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) = \mathcal{V}(\nu_{\widetilde{b}})$. Thus, it follows that $\mathcal{V}(\mu_{\widetilde{a}-\widetilde{c}}) \neq \mathcal{V}(\mu_{\widetilde{b}-\widetilde{c}})$, $\mathcal{V}(\rho_{\widetilde{a}-\widetilde{c}}) = \mathcal{V}(\rho_{\widetilde{b}-\widetilde{c}})$ and $\mathcal{V}(\nu_{\widetilde{a}-\widetilde{c}}) = \mathcal{V}(\nu_{\widetilde{b}-\widetilde{c}})$. So, $\theta_1 = 0$, $\theta_2 = \pm 1$ and $\theta_3 = \pm 1$ in ordering $\widetilde{a} - \widetilde{c}$ and $\widetilde{b} - \widetilde{c}$.

Case 5: Let $\theta_1 = \pm 1$, $\theta_2 = 0$ and $\theta_3 = 0$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Then, $\mathcal{V}(\mu_{\widetilde{a}}) = \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \neq \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \neq \mathcal{V}(\nu_{\widetilde{b}})$. Thus, it follows that $\mathcal{V}(\mu_{\widetilde{a}-\widetilde{c}}) = \mathcal{V}(\mu_{\widetilde{b}-\widetilde{c}})$, $\mathcal{V}(\rho_{\widetilde{a}-\widetilde{c}}) \neq \mathcal{V}(\rho_{\widetilde{b}-\widetilde{c}})$ and $\mathcal{V}(\nu_{\widetilde{a}-\widetilde{c}}) \neq \mathcal{V}(\nu_{\widetilde{b}-\widetilde{c}})$. So, $\theta_1 = \pm 1$, $\theta_2 = 0$ and $\theta_3 = 0$ in ordering $\widetilde{a} - \widetilde{c}$ and $\widetilde{b} - \widetilde{c}$.

Case 6: Let $\theta_1 = \pm 1$, $\theta_2 = 0$ and $\theta_3 = \pm 1$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Then, $\mathcal{V}(\mu_{\widetilde{a}}) = \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \neq \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) = \mathcal{V}(\nu_{\widetilde{b}})$. Thus, it follows that $\mathcal{V}(\mu_{\widetilde{a}-\widetilde{c}}) = \mathcal{V}(\mu_{\widetilde{b}-\widetilde{c}})$, $\mathcal{V}(\rho_{\widetilde{a}-\widetilde{c}}) \neq \mathcal{V}(\rho_{\widetilde{b}-\widetilde{c}})$ and $\mathcal{V}(\nu_{\widetilde{a}-\widetilde{c}}) = \mathcal{V}(\nu_{\widetilde{b}-\widetilde{c}})$. So, $\theta_1 = \pm 1$, $\theta_2 = 0$ and $\theta_3 = \pm 1$ in ordering $\widetilde{a} - \widetilde{c}$ and $\widetilde{b} - \widetilde{c}$.

Case 7: Let $\theta_1 = \pm 1$, $\theta_2 = \pm 1$ and $\theta_3 = 0$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Then, $\mathcal{V}(\mu_{\widetilde{a}}) = \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) = \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \neq \mathcal{V}(\nu_{\widetilde{b}})$. Thus, it follows that $\mathcal{V}(\mu_{\widetilde{a}-\widetilde{c}}) = \mathcal{V}(\mu_{\widetilde{b}-\widetilde{c}})$, $\mathcal{V}(\rho_{\widetilde{a}-\widetilde{c}}) = \mathcal{V}(\rho_{\widetilde{b}-\widetilde{c}})$ and $\mathcal{V}(\nu_{\widetilde{a}-\widetilde{c}}) \neq \mathcal{V}(\nu_{\widetilde{b}-\widetilde{c}})$. So, $\theta_1 = \pm 1$, $\theta_2 = \pm 1$ and $\theta_3 = 0$ in ordering $\widetilde{a} - \widetilde{c}$ and $\widetilde{b} - \widetilde{c}$.

Case 8: Let $\theta_i = \pm 1$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Then, $\mathcal{V}(\mu_{\widetilde{a}}) = \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) = \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) = \mathcal{V}(\nu_{\widetilde{b}})$. Thus, it follows that $\mathcal{V}(\mu_{\widetilde{a}-\widetilde{c}}) = \mathcal{V}(\mu_{\widetilde{b}-\widetilde{c}})$, $\mathcal{V}(\rho_{\widetilde{a}-\widetilde{c}}) = \mathcal{V}(\rho_{\widetilde{b}-\widetilde{c}})$ and $\mathcal{V}(\nu_{\widetilde{a}-\widetilde{c}}) = \mathcal{V}(\nu_{\widetilde{b}-\widetilde{c}})$. So, $\theta_i = \pm 1$ in ordering $\widetilde{a} - \widetilde{c}$ and $\widetilde{b} - \widetilde{c}$.

The above eight cases suggest that θ_1 and θ_2 are invariant in ordering $\widetilde{a}, \widetilde{b}$ and $\widetilde{a}-\widetilde{c}, \widetilde{b}-\widetilde{c}$. Hence, the claim.

Now, by the Theorem 3.1 it follows that it follows that

$$\mathcal{R}_\lambda(\widetilde{a}-\widetilde{c}, \theta_1, \theta_2, \theta_3) = \mathcal{R}_\lambda(\widetilde{a}, \theta_1, \theta_2, \theta_3) + \mathcal{R}_\lambda(-\widetilde{c}, \theta_1, \theta_2, \theta_3),$$

and

$$\mathcal{R}_\lambda(\widetilde{b}-\widetilde{c}, \theta_1, \theta_2, \theta_3) = \mathcal{R}_\lambda(\widetilde{b}, \theta_1, \theta_2, \theta_3) + \mathcal{R}_\lambda(-\widetilde{c}, \theta_1, \theta_2, \theta_3),$$

Then, if $\widetilde{a} \succeq \widetilde{b}$, then it is obvious that $\mathcal{R}_\lambda(\widetilde{a}, \theta_1, \theta_2, \theta_3) \geq \mathcal{R}_\lambda(\widetilde{b}, \theta_1, \theta_2, \theta_3)$. Eventually, it leads to the inequality $\mathcal{R}_\lambda(\widetilde{a}, \theta_1, \theta_2, \theta_3) + \mathcal{R}_\lambda(-\widetilde{c}, \theta_1, \theta_2, \theta_3) \geq \mathcal{R}_\lambda(\widetilde{b}, \theta_1, \theta_2, \theta_3) + \mathcal{R}_\lambda(-\widetilde{c}, \theta_1, \theta_2, \theta_3)$. Thus, evidently it follows that $\mathcal{R}_\lambda(\widetilde{a}+(-\widetilde{c}), \theta_1, \theta_2, \theta_3) \geq \mathcal{R}_\lambda(\widetilde{b}+(-\widetilde{c}), \theta_1, \theta_2, \theta_3)$. So, the result follows immediately.

Proof of the Theorem 3.15

Consider the cases when $\widetilde{a} \succ \widetilde{b}$ happens, that is,

$$\widetilde{a} \succ \widetilde{b} \text{ happens for} \begin{cases} \mathcal{R}_\lambda(\widetilde{a},0,0,0) > \mathcal{R}_\lambda(\widetilde{b},0,0,0) \\ \mathcal{R}_\lambda(\widetilde{a},0,0,\pm1) > \mathcal{R}_\lambda(\widetilde{b},0,0,\pm1) \\ \mathcal{R}_\lambda(\widetilde{a},0,\pm1,0) > \mathcal{R}_\lambda(\widetilde{b},0,\pm1,0) \\ \mathcal{R}_\lambda(\widetilde{a},0,\pm1,\pm1) > \mathcal{R}_\lambda(\widetilde{b},0,\pm1,\pm1) \\ \mathcal{R}_\lambda(\widetilde{a},\pm1,0,0) > \mathcal{R}_\lambda(\widetilde{b},\pm1,0,0) \\ \mathcal{R}_\lambda(\widetilde{a},\pm1,0,\pm1) > \mathcal{R}_\lambda(\widetilde{b},\pm1,0,\pm1) \\ \mathcal{R}_\lambda(\widetilde{a},\pm1,\pm1,0) > \mathcal{R}_\lambda(\widetilde{b},\pm1,\pm1,0) \\ \mathcal{R}_\lambda(\widetilde{a},\pm1,\pm1,\pm1) > \mathcal{R}_\lambda(\widetilde{b},\pm1,\pm1,\pm1) \end{cases}.$$

Consider the cases when $\widetilde{c} \succ \widetilde{d}$ happens, that is,

$$\widetilde{c} \succ \widetilde{d} \text{ happens for} \begin{cases} \mathcal{R}_\lambda(\widetilde{c},0,0,0) > \mathcal{R}_\lambda(\widetilde{d},0,0,0) \\ \mathcal{R}_\lambda(\widetilde{c},0,0,\pm1) > \mathcal{R}_\lambda(\widetilde{d},0,0,\pm1) \\ \mathcal{R}_\lambda(\widetilde{c},0,\pm1,0) > \mathcal{R}_\lambda(\widetilde{d},0,\pm1,0) \\ \mathcal{R}_\lambda(\widetilde{c},0,\pm1,\pm1) > \mathcal{R}_\lambda(\widetilde{d},0,\pm1,\pm1) \\ \mathcal{R}_\lambda(\widetilde{c},\pm1,0,0) > \mathcal{R}_\lambda(\widetilde{d},\pm1,0,0) \\ \mathcal{R}_\lambda(\widetilde{c},\pm1,0,\pm1) > \mathcal{R}_\lambda(\widetilde{d},\pm1,0,\pm1) \\ \mathcal{R}_\lambda(\widetilde{c},\pm1,\pm1,0) > \mathcal{R}_\lambda(\widetilde{d},\pm1,\pm1,0) \\ \mathcal{R}_\lambda(\widetilde{c},\pm1,\pm1,\pm1) > \mathcal{R}_\lambda(\widetilde{d},\pm1,\pm1,\pm1) \end{cases}$$

Now, to proof this property, it need to discuss these 8×8 cases, which will make the proof tedious. However, one can see the proof trivially, if following claims can be established.

Claim 1: Let $\theta_i = 0$ in ordering $\widetilde{a}$ and $\widetilde{b}$, and $\theta_i = 0$ in ordering $\widetilde{c}$ and $\widetilde{d}$. Then $\theta_i = 0$ in ordering $\widetilde{a}+\widetilde{c}$ and $\widetilde{b}+\widetilde{d}$. The proof of this claim is as follows. Let $\theta_i = 0$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Then $\mathcal{V}(\mu_{\widetilde{a}}) \neq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \neq \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \neq \mathcal{V}(\nu_{\widetilde{b}})$. Similarly, if $\theta_i = 0$ in ordering $\widetilde{c}$ and $\widetilde{d}$. Then $\mathcal{V}(\mu_{\widetilde{c}}) \neq \mathcal{V}(\mu_{\widetilde{d}})$, $\mathcal{V}(\rho_{\widetilde{c}}) \neq \mathcal{V}(\rho_{\widetilde{d}})$ and $\mathcal{V}(\nu_{\widetilde{c}}) \neq \mathcal{V}(\nu_{\widetilde{d}})$. Thus, it follows that $\mathcal{V}(\mu_{\widetilde{a}+\widetilde{c}}) \neq \mathcal{V}(\mu_{\widetilde{b}+\widetilde{d}})$, $\mathcal{V}(\rho_{\widetilde{a}+\widetilde{c}}) \neq \mathcal{V}(\rho_{\widetilde{b}+\widetilde{d}})$ and $\mathcal{V}(\nu_{\widetilde{a}+\widetilde{c}}) \neq \mathcal{V}(\nu_{\widetilde{b}+\widetilde{d}})$. So, $\theta_i = 0$ in ordering $\widetilde{a}+\widetilde{c}$ and $\widetilde{b}+\widetilde{d}$.

Claim 2: Let $\theta_i = \pm 1$ in ordering $\widetilde{a}$ and $\widetilde{b}$, and $\theta_i = \pm 1$ in ordering $\widetilde{c}$ and $\widetilde{d}$. Then $\theta_i = \pm 1$ in ordering $\widetilde{a} + \widetilde{c}$ and $\widetilde{b} + \widetilde{d}$. The proof of this claim is as follows. Let $\theta_i = \pm 1$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Then $\mathcal{V}(\mu_{\widetilde{a}}) = \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) = \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) = \mathcal{V}(\nu_{\widetilde{b}})$. Similarly, if $\theta_i = \pm 1$ in ordering $\widetilde{c}$ and $\widetilde{d}$, then $\mathcal{V}(\mu_{\widetilde{c}}) = \mathcal{V}(\mu_{\widetilde{d}})$, $\mathcal{V}(\rho_{\widetilde{c}}) = \mathcal{V}(\rho_{\widetilde{d}})$ and $\mathcal{V}(\nu_{\widetilde{c}}) = \mathcal{V}(\nu_{\widetilde{d}})$. Thus, it follows that $\mathcal{V}(\mu_{\widetilde{a}+\widetilde{c}}) = \mathcal{V}(\mu_{\widetilde{b}+\widetilde{d}})$, $\mathcal{V}(\rho_{\widetilde{a}+\widetilde{c}}) = \mathcal{V}(\rho_{\widetilde{b}+\widetilde{d}})$ and $\mathcal{V}(\nu_{\widetilde{a}+\widetilde{c}}) = \mathcal{V}(\nu_{\widetilde{b}+\widetilde{d}})$. So, $\theta_i = \pm 1$ in ordering $\widetilde{a} + \widetilde{c}$ and $\widetilde{b} + \widetilde{d}$.

Claim 3: Let $\theta_i = 0$ in ordering $\widetilde{a}$ and $\widetilde{b}$, and $\theta_i = \pm 1$ in ordering $\widetilde{b}$ and $\widetilde{c}$. Then $\theta_i = 0$ in ordering $\widetilde{a} + \widetilde{c}$ and $\widetilde{b} + \widetilde{d}$. The proof of this claim is as follows. Let $\theta_i = 0$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Then $\mathcal{V}(\mu_{\widetilde{a}}) \neq \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) \neq \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) \neq \mathcal{V}(\nu_{\widetilde{b}})$. Similarly, if $\theta_i = \pm 1$ in ordering $\widetilde{c}$ and $\widetilde{d}$, then $\mathcal{V}(\mu_{\widetilde{c}}) = \mathcal{V}(\mu_{\widetilde{d}})$, $\mathcal{V}(\rho_{\widetilde{c}}) = \mathcal{V}(\rho_{\widetilde{d}})$ and $\mathcal{V}(\nu_{\widetilde{c}}) = \mathcal{V}(\nu_{\widetilde{d}})$. Thus, it follows that $\mathcal{V}(\mu_{\widetilde{a}+\widetilde{c}}) \neq \mathcal{V}(\mu_{\widetilde{b}+\widetilde{d}})$, $\mathcal{V}(\rho_{\widetilde{a}+\widetilde{c}}) \neq \mathcal{V}(\rho_{\widetilde{b}+\widetilde{d}})$ and $\mathcal{V}(\nu_{\widetilde{a}+\widetilde{c}}) \neq \mathcal{V}(\nu_{\widetilde{b}+\widetilde{d}})$. So, $\theta_i = 0$ in ordering $\widetilde{a} + \widetilde{c}$ and $\widetilde{b} + \widetilde{d}$.

Claim 4: Let $\theta_i = \pm 1$ in ordering $\widetilde{a}$ and $\widetilde{b}$, and $\theta_i = 0$ in ordering $\widetilde{c}$ and $\widetilde{d}$. Then $\theta_i = 0$ in ordering $\widetilde{a} + \widetilde{c}$ and $\widetilde{b} + \widetilde{d}$. The proof of this claim is as follows. Let $\theta_i = \pm 1$ in ordering $\widetilde{a}$ and $\widetilde{b}$. Then $\mathcal{V}(\mu_{\widetilde{a}}) = \mathcal{V}(\mu_{\widetilde{b}})$, $\mathcal{V}(\rho_{\widetilde{a}}) = \mathcal{V}(\rho_{\widetilde{b}})$ and $\mathcal{V}(\nu_{\widetilde{a}}) = \mathcal{V}(\nu_{\widetilde{b}})$. Similarly, if $\theta_i = 0$ in ordering $\widetilde{c}$ and $\widetilde{d}$, then $\mathcal{V}(\mu_{\widetilde{c}}) \neq \mathcal{V}(\mu_{\widetilde{d}})$, $\mathcal{V}(\rho_{\widetilde{c}}) \neq \mathcal{V}(\rho_{\widetilde{d}})$ and $\mathcal{V}(\nu_{\widetilde{c}}) \neq \mathcal{V}(\nu_{\widetilde{d}})$. Thus, it follows that $\mathcal{V}(\mu_{\widetilde{a}+\widetilde{c}}) \neq \mathcal{V}(\mu_{\widetilde{b}+\widetilde{d}})$, $\mathcal{V}(\rho_{\widetilde{a}+\widetilde{c}}) \neq \mathcal{V}(\rho_{\widetilde{b}+\widetilde{d}})$ and $\mathcal{V}(\nu_{\widetilde{a}+\widetilde{c}}) \neq \mathcal{V}(\nu_{\widetilde{b}+\widetilde{d}})$. So, $\theta_i = 0$ in ordering $\widetilde{a} + \widetilde{c}$ and $\widetilde{b} + \widetilde{d}$.

From these four claims, it is trivial enough to show that if $\widetilde{a} \succ \widetilde{b}$ and $\widetilde{b} \succ \widetilde{c}$, then $\widetilde{a} + \widetilde{c} \succ \widetilde{b} + \widetilde{d}$.

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