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Abstract

In this study, Tzitzéica curves are taken into consideration in Euclidean 3-space by using the positional adapted frame (PAF). Such curves are characterized according to PAF apparatus. Also, some results are obtained on spherical Tzitzéica curves. The results obtained in this study are new contributions to the field. It is expected that these results will be useful in various application areas of differential geometry in the future.

1. Introduction and Preliminaries

Let the Euclidean 3-space be taken into account with the standard scalar product $\langle \mathbf{E}, \mathbf{F} \rangle = e_1 f_1 + e_2 f_2 + e_3 f_3$ where $\mathbf{E} = (e_1\,, e_2\,, e_3)$, $\mathbf{F} = (f_1\,, f_2\,, f_3)$ are any vectors in space. The norm of $\mathbf{E}$ is given as $\|\mathbf{E}\| = \sqrt{\langle \mathbf{E}\,, \mathbf{E} \rangle}$. If a differentiable curve $\alpha = \alpha(s) : I \subset \mathbb{R} \to E^3$ satisfies $\left\| \frac{d\alpha}{ds} \right\| = 1$ for all $s \in I$, it is called a unit speed curve. In this case, s is said to be arc-length parameter of α. A differentiable curve is called as regular curve if its derivative is never zero along the curve. All regular curves can be reparameterized by the arc-length of itself [1]. Throughout the work, we will show the differentiation with respect to the arc-length parameter s with a dash.

In E^3, suppose that a point particle of constant mass moves on a unit speed curve $\alpha = \alpha(s)$. Let $\{\mathbf{T}(s)\,, \mathbf{N}(s)\,, \mathbf{B}(s)\}$ show the Serret-Frenet frame of $\alpha = \alpha(s)$. $\mathbf{T}(s) = \alpha'(s)$, $\mathbf{N}(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}$ and $\mathbf{B}(s) = \mathbf{T}(s) \wedge \mathbf{N}(s)$ are called the unit tangent, unit principal normal and unit binormal vectors, respectively. Also, the Serret-Frenet formulas are given as follows:

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix} \tag{1.1}$$

where $\kappa(s) = \|\mathbf{T}'(s)\|$ is the curvature function and $\tau(s) = -\langle \mathbf{B}'(s)\,, \mathbf{N}(s) \rangle$ is the torsion function [1]. In the rest of the study, we assume everywhere $\kappa \neq 0$.

Until now, many researchers have developed new moving frames which have a common base vector with the Serret-Frenet frame (see [2, 3, 4] for some examples). One of the newest of them is the study [5] presented by Özen and Tosun. They introduced the Positional Adapted Frame (PAF) for the trajectories with non-vanishing angular momentum in E^3. Also, in [6], they investigated the special trajectories generated by Smarandache curves according to PAF in Euclidean 3-space.

The angular momentum vector of the aforementioned moving particle about the origin has an important place in particle kinematics. It is determined by vector product of the position vector and linear momentum vector of the moving particle and given by $\mathbf{H}^O = m \langle \alpha(s), \mathbf{B}(s) \rangle \left(\frac{ds}{dt} \right) \mathbf{N}(s) - m \langle \alpha(s), \mathbf{N}(s) \rangle \left(\frac{ds}{dt} \right) \mathbf{B}(s)$ where m and t indicate the mass and time, respectively. Assume that this vector does not equal to zero vector along the trajectory $\alpha = \alpha(s)$. This assumption ensures that the functions $\langle \alpha(s), \mathbf{N}(s) \rangle$ and $\langle \alpha(s), \mathbf{B}(s) \rangle$ do not equal to zero at the same time during the motion of the moving particle. Thus, we can say that the tangent line of $\alpha = \alpha(s)$ never passes through the origin. In that case, there exists PAF denoted by $\{\mathbf{T}(s), \mathbf{M}(s), \mathbf{Y}(s)\}$ along $\alpha = \alpha(s)$. Let us take into consideration the vector whose starting point is the foot of the perpendicular (from origin to instantaneous rectifying plane) and endpoint is the foot of the perpendicular (from origin to instantaneous osculating plane). The equivalent of this vector at the point $\alpha(s)$ helps us to determine the vector $\mathbf{Y}(s)$. Hence, $\mathbf{Y}(s)$ is calculated as (see [5] for more details):

$$\mathbf{Y}(s) = \frac{\langle -\alpha(s), \mathbf{N}(s) \rangle}{\sqrt{\langle \alpha(s), \mathbf{N}(s) \rangle^2 + \langle \alpha(s), \mathbf{B}(s) \rangle^2}} \mathbf{N}(s) + \frac{\langle \alpha(s), \mathbf{B}(s) \rangle}{\sqrt{\langle \alpha(s), \mathbf{N}(s) \rangle^2 + \langle \alpha(s), \mathbf{B}(s) \rangle^2}} \mathbf{B}(s).$$

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The second base vector of PAF is obtained by vector product $\mathbf{Y}(s)\wedge\mathbf{T}(s)$ as in the following:

$$\mathbf{M}(s)=\frac{\langle\alpha(s),\mathbf{B}(s)\rangle}{\sqrt{\langle\alpha(s),\mathbf{N}(s)\rangle^2+\langle\alpha(s),\mathbf{B}(s)\rangle^2}}\mathbf{N}(s)+\frac{\langle\alpha(s),\mathbf{N}(s)\rangle}{\sqrt{\langle\alpha(s),\mathbf{N}(s)\rangle^2+\langle\alpha(s),\mathbf{B}(s)\rangle^2}}\mathbf{B}(s).$$

There is a relation between the Serret-Frenet frame and PAF as follows:

$$\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{M}(s)\\ \mathbf{Y}(s)\end{pmatrix}=\begin{pmatrix}1&0&0\\ 0&\cos\Omega(s)&-\sin\Omega(s)\\ 0&\sin\Omega(s)&\cos\Omega(s)\end{pmatrix}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{N}(s)\\ \mathbf{B}(s)\end{pmatrix}$$

where $\Omega(s)$ is the angle between the vectors $\mathbf{B}(s)$ and $\mathbf{Y}(s)$ which is positively oriented from $\mathbf{B}(s)$ to $\mathbf{Y}(s)$(see Figure 1.1). On the other hand, the derivative formulas of PAF are expressed as [5]:

$$\begin{pmatrix}\mathbf{T}'(s)\\ \mathbf{M}'(s)\\ \mathbf{Y}'(s)\end{pmatrix}=\begin{pmatrix}0&k_1(s)&k_2(s)\\ -k_1(s)&0&k_3(s)\\ -k_2(s)&-k_3(s)&0\end{pmatrix}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{M}(s)\\ \mathbf{Y}(s)\end{pmatrix}$$

where

$$\begin{aligned}k_3(s)&=\tau(s)-\Omega'(s)\\ k_1(s)&=\kappa(s)\cos\Omega(s)\\ k_2(s)&=\kappa(s)\sin\Omega(s)\end{aligned}$$

The last two equations yield the followings:

$$\begin{aligned}\frac{k_2(s)}{k_1(s)}&=\tan\Omega(s)\\ k_1(s)&=\sqrt{k_1^2(s)+k_2^2(s)}\cos\Omega(s)\\ k_2(s)&=\sqrt{k_1^2(s)+k_2^2(s)}\sin\Omega(s).\end{aligned}$$

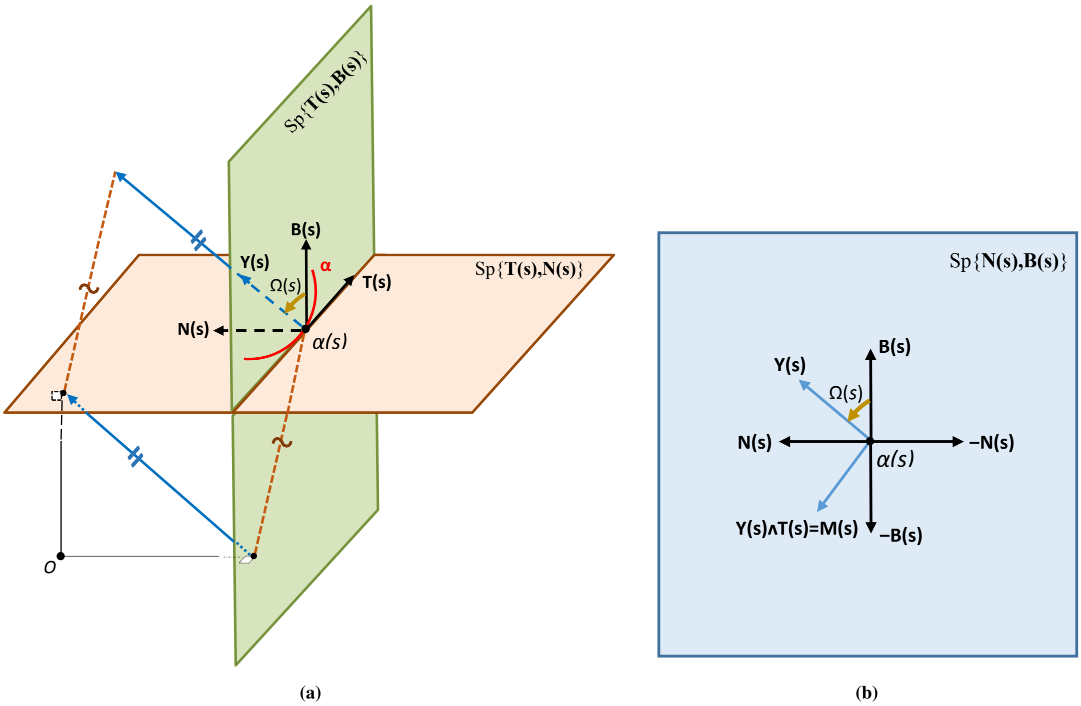

Figure 1.1: An illustration for the Positional Adapted Frame (PAF)

The aforementioned angle $\Omega(s)$ is calculated as in the following:

$$\Omega(s)=\begin{cases}\arctan\left(-\frac{\langle\alpha(s),\mathbf{N}(s)\rangle}{\langle\alpha(s),\mathbf{B}(s)\rangle}\right)\ if\ \langle\alpha(s),\mathbf{B}(s)\rangle>0\\ \arctan\left(-\frac{\langle\alpha(s),\mathbf{N}(s)\rangle}{\langle\alpha(s),\mathbf{B}(s)\rangle}\right)+\pi\ if\ \langle\alpha(s),\mathbf{B}(s)\rangle<0\\ -\frac{\pi}{2}\ if\ \langle\alpha(s),\mathbf{B}(s)\rangle=0,\ \langle\alpha(s),\mathbf{N}(s)\rangle>0\\ \frac{\pi}{2}\ if\ \langle\alpha(s),\mathbf{B}(s)\rangle=0,\ \langle\alpha(s),\mathbf{N}(s)\rangle<0.\end{cases}$$

Any element of the set $\{\mathbf{T}(s), \mathbf{M}(s), \mathbf{Y}(s), k_1(s), k_2(s), k_3(s)\}$ is called PAF apparatus of $\alpha = \alpha(s)$ [5].

A class of curves was introduced by Gheorghe Tzitzéica in 1911. These curves are called as Tzitzéica curves. A Tzitzéica curve in Euclidean 3-space is a curve (with $\kappa > 0$ and $\tau \neq 0$) for which the ratio of τ and the square of the distance from the origin to the osculating plane at an arbitrary point of this curve is a non-zero constant. Hence, the trajectory $\alpha = \alpha(s)$ of the aforesaid point particle is a Tzitzéica curve if

$$\frac{\tau}{\langle \alpha, \mathbf{B} \rangle^2} \tag{1.2}$$

is a constant function other than zero function. In the literature, Tzitzéica curves are studied widely. [7, 8, 9, 10, 11] are some of the studies on Tzitzéica curves.

In the next section, we characterize the Tzitzéica curves in terms of PAF apparatus and obtain some results on the spherical Tzitzéica curves.

2. Application of PAF to Tzitzéica Curves

In this section, we continue to consider any moving point particle satisfying the aforementioned assumption (concerned with the angular momentum) and show the unit speed parameterization of the trajectory with $\alpha = \alpha(s)$.

If the trajectory $\alpha = \alpha(s)$ is a Tzitzéica curve, $\langle \alpha(s), \mathbf{B}(s) \rangle \neq 0$ for all the values s of the parameter. In this case, the angular momentum vector $\mathbf{H}^O$(about origin) of the particle is non-zero along $\alpha = \alpha(s)$ and PAF is well defined during the motion of the particle. Now, we will give a characterization for the trajectory $\alpha = \alpha(s)$ to be a Tzitzéica curve. Also, we will obtain some results on spherical Tzitzéica curves.

Theorem 2.1. *Let the trajectory $\alpha = \alpha(s)$ be given. In that case, the equation*

$$\frac{\langle \alpha' \wedge \alpha'', \alpha''' \rangle}{\langle \alpha, \alpha' \wedge \alpha'' \rangle^2} = \frac{k_3 + \Omega'}{\langle \alpha, -\sin\Omega \mathbf{M} + \cos\Omega \mathbf{Y} \rangle^2} \tag{2.1}$$

is satisfied.

Proof. We can easily write

$$\begin{aligned}
\alpha' &= \mathbf{T} \\
\alpha'' &= \mathbf{T}' \\
&= k_1 \mathbf{M} + k_2 \mathbf{Y} \\
\alpha''' &= (k_1 \mathbf{M} + k_2 \mathbf{Y})' \\
&= k_1{}' \mathbf{M} + k_1 (-k_1 \mathbf{T} + k_3 \mathbf{Y}) + k_2{}' \mathbf{Y} + k_2 (-k_2 \mathbf{T} - k_3 \mathbf{M}) \\
&= -\left(k_1{}^2 + k_2{}^2\right) \mathbf{T} + \left(k_1{}' - k_2 k_3\right) \mathbf{M} + \left(k_2{}' + k_1 k_3\right) \mathbf{Y}.
\end{aligned}$$

These equations give us the followings:

$$\begin{aligned}
\alpha' \wedge \alpha'' &= \begin{vmatrix} \mathbf{T} & \mathbf{M} & \mathbf{Y} \\ 1 & 0 & 0 \\ 0 & k_1 & k_2 \end{vmatrix} \\
&= -k_2 \mathbf{M} + k_1 \mathbf{Y} \\
\langle \alpha, \alpha' \wedge \alpha'' \rangle^2 &= \langle \alpha, -k_2 \mathbf{M} + k_1 \mathbf{Y} \rangle^2 \\
&= \left\langle \alpha, -\sqrt{k_1{}^2 + k_2{}^2} \sin\Omega \mathbf{M} + \sqrt{k_1{}^2 + k_2{}^2} \cos\Omega \mathbf{Y} \right\rangle^2 \\
&= \left(\sqrt{k_1{}^2 + k_2{}^2} \langle \alpha, -\sin\Omega \mathbf{M} + \cos\Omega \mathbf{Y} \rangle \right)^2 \\
&= \left(k_1{}^2 + k_2{}^2\right) \langle \alpha, -\sin\Omega \mathbf{M} + \cos\Omega \mathbf{Y} \rangle^2 \\
\langle \alpha' \wedge \alpha'', \alpha''' \rangle &= \left\langle -k_2 \mathbf{M} + k_1 \mathbf{Y}, -\left(k_1{}^2 + k_2{}^2\right) \mathbf{T} + \left(k_1{}' - k_2 k_3\right) \mathbf{M} + \left(k_2{}' + k_1 k_3\right) \mathbf{Y} \right\rangle \\
&= \left(k_1 k_2{}' - k_2 k_1{}'\right) + k_3 \left(k_1{}^2 + k_2{}^2\right).
\end{aligned}$$

In the light of the equations derived above, we get

$$\begin{aligned}
\frac{\langle \alpha' \wedge \alpha'', \alpha''' \rangle}{\langle \alpha, \alpha' \wedge \alpha'' \rangle^2} &= \frac{\left(k_1 k_2{}' - k_2 k_1{}'\right) + k_3 \left(k_1{}^2 + k_2{}^2\right)}{\left(k_1{}^2 + k_2{}^2\right) \langle \alpha, -\sin\Omega \mathbf{M} + \cos\Omega \mathbf{Y} \rangle^2} \\
&= \frac{k_3 + \frac{k_2{}' k_1 - k_1{}' k_2}{k_1{}^2 + k_2{}^2}}{\langle \alpha, -\sin\Omega \mathbf{M} + \cos\Omega \mathbf{Y} \rangle^2} \\
&= \frac{k_3 + \left(\arctan\left(\frac{k_2}{k_1}\right)\right)'}{\langle \alpha, -\sin\Omega \mathbf{M} + \cos\Omega \mathbf{Y} \rangle^2} \\
&= \frac{k_3 + \Omega'}{\langle \alpha, -\sin\Omega \mathbf{M} + \cos\Omega \mathbf{Y} \rangle^2}
\end{aligned}$$

and complete the proof. □

Using the well-known equalities $\tau = \frac{\langle \alpha' \wedge \alpha'', \alpha''' \rangle}{\|\alpha' \wedge \alpha''\|^2}$ and $\mathbf{B} = \frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|}$, one can immediately calculate the following:

$$\frac{\tau}{\langle \alpha, \mathbf{B} \rangle^2} = \frac{\langle \alpha' \wedge \alpha'', \alpha''' \rangle}{\langle \alpha, \alpha' \wedge \alpha'' \rangle^2}.$$

If we consider the last equation, Theorem 2.1 and the condition of being a Tzitzéica curve, a characterization for Tzitzéica curves can be given as in the next corollary.

Corollary 2.2. *The trajectory $\alpha = \alpha(s)$ is a Tzitzéica curve iff*

$$\frac{k_3 + \Omega'}{\langle \alpha, -\sin\Omega \mathbf{M} + \cos\Omega \mathbf{Y} \rangle^2}$$

is a constant function other than zero function.

Proposition 2.3. *If the trajectory $\alpha = \alpha(s)$ is a Tzitzéica curve, in this case the equation*

$$\left(k_3 + \Omega'\right)' \langle \alpha, -\sin\Omega \mathbf{M} + \cos\Omega \mathbf{Y} \rangle = -2 \left(k_3 + \Omega'\right)^2 \langle \alpha, \cos\Omega \mathbf{M} + \sin\Omega \mathbf{Y} \rangle$$

is satisfied

Proof. If the trajectory $\alpha = \alpha(s)$ is a Tzitzéica curve,

$$\frac{k_3 + \Omega'}{\langle \alpha, -\sin\Omega \mathbf{M} + \cos\Omega \mathbf{Y} \rangle^2} = c$$

can be easily written where c is a non-zero constant. Differentiating the last equation yields

$$\left(k_3 + \Omega'\right)' \langle \alpha, -\sin\Omega \mathbf{M} + \cos\Omega \mathbf{Y} \rangle + 2 \left(k_3 + \Omega'\right)^2 \langle \alpha, \cos\Omega \mathbf{M} + \sin\Omega \mathbf{Y} \rangle = 0$$

and completes the proof. □

Remark 2.4. *The following derivative formula can be given:*

$$\begin{aligned}
(\cos\Omega \mathbf{M} + \sin\Omega \mathbf{Y})' &= -\Omega' \sin\Omega \mathbf{M} + \cos\Omega \left(-k_1 \mathbf{T} + k_3 \mathbf{Y}\right) + \Omega' \cos\Omega \mathbf{Y} + \sin\Omega \left(-k_2 \mathbf{T} - k_3 \mathbf{M}\right) \\
&= -\left(k_1 \cos\Omega + k_2 \sin\Omega\right) \mathbf{T} + \left(k_3 + \Omega'\right) \left(-\sin\Omega \mathbf{M} + \cos\Omega \mathbf{Y}\right) \\
&= -\left(\sqrt{k_1{}^2 + k_2{}^2} \cos\Omega \cos\Omega + \sqrt{k_1{}^2 + k_2{}^2} \sin\Omega \sin\Omega\right) \mathbf{T} + \left(k_3 + \Omega'\right) \left(-\sin\Omega \mathbf{M} + \cos\Omega \mathbf{Y}\right) \\
&= -\sqrt{k_1{}^2 + k_2{}^2} \left(\cos^2\Omega + \sin^2\Omega\right) \mathbf{T} + \left(k_3 + \Omega'\right) \left(-\sin\Omega \mathbf{M} + \cos\Omega \mathbf{Y}\right) \\
&= -\sqrt{k_1{}^2 + k_2{}^2} \mathbf{T} + \left(k_3 + \Omega'\right) \left(-\sin\Omega \mathbf{M} + \cos\Omega \mathbf{Y}\right).
\end{aligned}$$

Also, one can easily see

$$(-\sin\Omega \mathbf{M} + \cos\Omega \mathbf{Y})' = -\left(k_3 + \Omega'\right) \left(\cos\Omega \mathbf{M} + \sin\Omega \mathbf{Y}\right).$$

Note that these formulas will be used in the proof of the following theorem.

Theorem 2.5. *If the trajectory $\alpha = \alpha(s)$ is a unit speed spherical curve on S_O^2,*

$$\frac{\left(k_3 + \Omega'\right)}{\sqrt{k_1{}^2 + k_2{}^2}} = \left(\frac{\left(\sqrt{k_1{}^2 + k_2{}^2}\right)'}{\left(k_3 + \Omega'\right)\left(k_1{}^2 + k_2{}^2\right)}\right)' \tag{2.2}$$

holds.

Proof. Let the trajectory $\alpha = \alpha(s)$ be a unit speed spherical curve on S_O^2. Then, we have

$$\langle \alpha, \alpha \rangle = r^2$$

where r is the radius of sphere. By differentiating this equation with respect to arc-length parameter s, we obtain

$$\langle \alpha, \mathbf{T} \rangle = 0 \tag{2.3}$$

The equation (2.3) gives us the following

$$\begin{aligned}
0 &= \left\langle \alpha', \mathbf{T} \right\rangle + \left\langle \alpha, \mathbf{T}' \right\rangle \\
&= 1 + \langle \alpha, k_1 \mathbf{M} + k_2 \mathbf{Y} \rangle \\
&= 1 + \left\langle \alpha, \sqrt{k_1{}^2 + k_2{}^2} \cos\Omega \mathbf{M} + \sqrt{k_1{}^2 + k_2{}^2} \sin\Omega \mathbf{Y} \right\rangle \\
&= 1 + \left\langle \alpha, \sqrt{k_1{}^2 + k_2{}^2} \left(\cos\Omega \mathbf{M} + \sin\Omega \mathbf{Y}\right) \right\rangle.
\end{aligned}$$

Hence we find

$$\left\langle \alpha,\, \sqrt{{k_1}^2+{k_2}^2}\,(\cos\Omega\mathbf{M}+\sin\Omega\mathbf{Y})\right\rangle = -1 \tag{2.4}$$

Differentiating (2.4) yields

$$\begin{aligned}
0 &= \left\langle \alpha,\, \left(\sqrt{{k_1}^2+{k_2}^2}\right)'(\cos\Omega\mathbf{M}+\sin\Omega\mathbf{Y})+\sqrt{{k_1}^2+{k_2}^2}(\cos\Omega\mathbf{M}+\sin\Omega\mathbf{Y})'\right\rangle \\
&= \left\langle \alpha,\, \left(\sqrt{{k_1}^2+{k_2}^2}\right)'(\cos\Omega\mathbf{M}+\sin\Omega\mathbf{Y})\right\rangle + \left\langle \alpha,\, \sqrt{{k_1}^2+{k_2}^2}\left(-\sqrt{{k_1}^2+{k_2}^2}\mathbf{T}+(k_3+\Omega')\,(-\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y})\right)\right\rangle \\
&= -\left({k_1}^2+{k_2}^2\right)\langle\alpha,\mathbf{T}\rangle+\left(\sqrt{{k_1}^2+{k_2}^2}\right)'\langle\alpha,\,(\cos\Omega\mathbf{M}+\sin\Omega\mathbf{Y})\rangle+\sqrt{{k_1}^2+{k_2}^2}\,(k_3+\Omega')\,\langle\alpha,\,(-\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y})\rangle \\
&= \left(\sqrt{{k_1}^2+{k_2}^2}\right)'\langle\alpha,\,(\cos\Omega\mathbf{M}+\sin\Omega\mathbf{Y})\rangle+\sqrt{{k_1}^2+{k_2}^2}\,(k_3+\Omega')\,\langle\alpha,\,(-\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y})\rangle\,.
\end{aligned}$$

If we derive the equation

$$\langle\alpha,\,\cos\Omega\mathbf{M}+\sin\Omega\mathbf{Y}\rangle = -\frac{1}{\sqrt{{k_1}^2+{k_2}^2}} \tag{2.5}$$

from (2.4) and substitute this into the last equation, we find

$$\frac{-\left(\sqrt{{k_1}^2+{k_2}^2}\right)'}{\sqrt{{k_1}^2+{k_2}^2}}+\left(\sqrt{{k_1}^2+{k_2}^2}\right)(k_3+\Omega')\,\langle\alpha,\,-\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y}\rangle=0$$

and so

$$\frac{\left(\sqrt{{k_1}^2+{k_2}^2}\right)'}{{k_1}^2+{k_2}^2}=(k_3+\Omega')\,\langle\alpha,\,-\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y}\rangle\,. \tag{2.6}$$

The last equation gives us the following:

$$\begin{aligned}
\left(\frac{\left(\sqrt{{k_1}^2+{k_2}^2}\right)'}{{k_1}^2+{k_2}^2}\right)' &= (k_3+\Omega')'\,\langle\alpha,-\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y}\rangle+(k_3+\Omega')\,\langle\alpha,-\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y}\rangle' \\
&= (k_3+\Omega')'\,\langle\alpha,-\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y}\rangle+(k_3+\Omega')\,[\langle\mathbf{T},-\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y}\rangle+\langle\alpha,(-\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y})'\rangle] \\
&= (k_3+\Omega')'\,\langle\alpha,-\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y}\rangle+(k_3+\Omega')\,\langle\alpha,\,-(k_3+\Omega')\,(\cos\Omega\mathbf{M}+\sin\Omega\mathbf{Y})\rangle\,.
\end{aligned}$$

Hence we obtain

$$\left(\frac{\left(\sqrt{{k_1}^2+{k_2}^2}\right)'}{{k_1}^2+{k_2}^2}\right)'=(k_3+\Omega')'\,\langle\alpha,-\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y}\rangle-(k_3+\Omega')^2\,\langle\alpha,\,(\cos\Omega\mathbf{M}+\sin\Omega\mathbf{Y})\rangle\,. \tag{2.7}$$

From (2.6),

$$\langle\alpha,\,-\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y}\rangle=\frac{\left(\sqrt{{k_1}^2+{k_2}^2}\right)'}{({k_1}^2+{k_2}^2)\,(k_3+\Omega')} \tag{2.8}$$

can be immediately written. By substituting (2.5) and (2.8) into (2.7), we get

$$\left(\frac{\left(\sqrt{{k_1}^2+{k_2}^2}\right)'}{{k_1}^2+{k_2}^2}\right)'=\frac{(k_3+\Omega')'\left(\sqrt{{k_1}^2+{k_2}^2}\right)'}{({k_1}^2+{k_2}^2)\,(k_3+\Omega')}+\frac{(k_3+\Omega')^2}{\sqrt{{k_1}^2+{k_2}^2}}.$$

This equation yields

$$\frac{(k_3+\Omega')^2}{\sqrt{{k_1}^2+{k_2}^2}}=\left(\frac{\left(\sqrt{{k_1}^2+{k_2}^2}\right)'}{{k_1}^2+{k_2}^2}\right)'-\frac{(k_3+\Omega')'\left(\sqrt{{k_1}^2+{k_2}^2}\right)'}{({k_1}^2+{k_2}^2)\,(k_3+\Omega')}.$$

In that case, we can write

$$\frac{(k_3+\Omega')^2}{\sqrt{{k_1}^2+{k_2}^2}}=\frac{\left(\sqrt{{k_1}^2+{k_2}^2}\right)''(k_3+\Omega')\,({k_1}^2+{k_2}^2)-2\,(k_3+\Omega')\sqrt{{k_1}^2+{k_2}^2}\left(\left(\sqrt{{k_1}^2+{k_2}^2}\right)'\right)^2-({k_1}^2+{k_2}^2)\left(\sqrt{{k_1}^2+{k_2}^2}\right)'(k_3+\Omega')'}{(k_3+\Omega')\,({k_1}^2+{k_2}^2)^2}.$$

Eventually, we find

$$\begin{aligned}\frac{(k_3+\Omega')}{\sqrt{{k_1}^2+{k_2}^2}} &= \frac{\left(\sqrt{{k_1}^2+{k_2}^2}\right)''(k_3+\Omega')\left({k_1}^2+{k_2}^2\right)-2\left(k_3+\Omega'\right)\sqrt{{k_1}^2+{k_2}^2}\left(\left(\sqrt{{k_1}^2+{k_2}^2}\right)'\right)^2-\left({k_1}^2+{k_2}^2\right)\left(\sqrt{{k_1}^2+{k_2}^2}\right)'(k_3+\Omega')'}{(k_3+\Omega')^2\left({k_1}^2+{k_2}^2\right)^2} \\ &= \left(\frac{\left(\sqrt{{k_1}^2+{k_2}^2}\right)'}{(k_3+\Omega')\left({k_1}^2+{k_2}^2\right)}\right)'\end{aligned}$$

and complete the proof. □

Theorem 2.6. *Let the trajectory $\alpha=\alpha(s)$ be a unit speed spherical curve on S^2_O. If it is a Tzitzéica curve,*

$$\frac{(k_3+\Omega')'}{2(k_3+\Omega')^3}=\frac{\sqrt{{k_1}^2+{k_2}^2}}{\left(\sqrt{{k_1}^2+{k_2}^2}\right)'} \tag{2.9}$$

holds.

Proof. Let the trajectory $\alpha=\alpha(s)$ be a unit speed spherical curve on S^2_O. Assume that $\alpha=\alpha(s)$ is a Tzitzéica curve. In that case

$$\left(k_3+\Omega'\right)'\langle\alpha,-\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y}\rangle=-2\left(k_3+\Omega'\right)^2\langle\alpha,\cos\Omega\mathbf{M}+\sin\Omega\mathbf{Y}\rangle$$

can be written thanks to the Proposition 2.3. Using (2.5) and (2.8) in the last equation gives us the desired result. □

Corollary 2.7. *Let the trajectory $\alpha=\alpha(s)$ be a unit speed spherical Tzitzéica curve on S^2_O. In this case, the equation*

$$(k_3+\Omega')=\sqrt{\frac{\left(\sqrt{{k_1}^2+{k_2}^2}\right)''\sqrt{{k_1}^2+{k_2}^2}-2\left(\left(\sqrt{{k_1}^2+{k_2}^2}\right)'\right)^2}{3\left({k_1}^2+{k_2}^2\right)}}$$

is satisfied.

Proof. Suppose that the trajectory $\alpha=\alpha(s)$ is a unit speed spherical Tzitzéica curve on S^2_O. Then, we get

$$\frac{(k_3+\Omega')}{\sqrt{{k_1}^2+{k_2}^2}}=\frac{\left(\sqrt{{k_1}^2+{k_2}^2}\right)''(k_3+\Omega')\left({k_1}^2+{k_2}^2\right)-2\left(k_3+\Omega'\right)\sqrt{{k_1}^2+{k_2}^2}\left(\left(\sqrt{{k_1}^2+{k_2}^2}\right)'\right)^2-\left({k_1}^2+{k_2}^2\right)\left(\sqrt{{k_1}^2+{k_2}^2}\right)'(k_3+\Omega')'}{(k_3+\Omega')^2\left({k_1}^2+{k_2}^2\right)^2} \tag{2.10}$$

by means of (2.2). On the other hand, we can write

$$\left(k_3+\Omega'\right)'=2\left(k_3+\Omega'\right)^3\frac{\sqrt{{k_1}^2+{k_2}^2}}{\left(\sqrt{{k_1}^2+{k_2}^2}\right)'} \tag{2.11}$$

thanks to (2.9). Then, the desired result is obtained by substituting (2.11) into (2.10). □

Corollary 2.8. *Suppose that the trajectory $\alpha=\alpha(s)$ is a unit speed spherical Tzitzéica curve on S^2_O. Let the functions $k_3+\Omega'$ and $\sqrt{{k_1}^2+{k_2}^2}$ be constant function and non-constant function, respectively. Then the equation*

$$\sqrt{{k_1}^2+{k_2}^2}=\frac{\sqrt{3}\left(k_3+\Omega'\right)}{\lambda_1\sin\left(\sqrt{3}\left(k_3+\Omega'\right)s\right)-\lambda_2\cos\left(\sqrt{3}\left(k_3+\Omega'\right)s\right)}$$

is satisfied where λ_1 and λ_2 are integral constants.

Proof. Let the trajectory $\alpha=\alpha(s)$ be a unit speed spherical Tzitzéica curve on S^2_O. Assume that the function $k_3+\Omega'$ is a constant function and the function $\sqrt{{k_1}^2+{k_2}^2}$ is not a constant function. From Corollary 2.7, we have the following equation.

$$(k_3+\Omega')=\sqrt{\frac{\left(\sqrt{{k_1}^2+{k_2}^2}\right)''\sqrt{{k_1}^2+{k_2}^2}-2\left(\left(\sqrt{{k_1}^2+{k_2}^2}\right)'\right)^2}{3\left({k_1}^2+{k_2}^2\right)}}$$

which gives us the differential equation

$$\sqrt{{k_1}^2+{k_2}^2}\left(\sqrt{{k_1}^2+{k_2}^2}\right)''-2\left(\left(\sqrt{{k_1}^2+{k_2}^2}\right)'\right)^2-3\left({k_1}^2+{k_2}^2\right)\left(k_3+\Omega'\right)^2=0. \tag{2.12}$$

The solution of (2.12) corresponds to the desired result. □

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