DOI: 10.1002/int.22236

RESEARCH ARTICLE

WILEY

Canonical decomposition of dichotomous basic belief assignment

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Abstract

In this paper, we prove that any dichotomous basic belief assignment (BBA) m can be expressed as the combination of two simple belief assignments m_p and m_c called, respectively, the pros and cons BBAs thanks to the proportional conflict redistribution rule no 5 (PCR5). This decomposition always exists and is unique and we call it the canonical decomposition of the BBA m. We also show that canonical decompositions do not exist in general if we use the conjunctive rule, the disjunctive rule, Dempster's rule, Dubois and Prade's or Yager's rules, or even the averaging rule of combination. We give some numerical examples of canonical decompositions and discuss of the potential interest of this canonical decomposition for applications in information fusion.

KEYWORDS

belief functions, canonical decomposition, contra-evidence, PCR5 rule, pro-evidence

1 | INTRODUCTION

The belief functions (BF) introduced by Shafer in the mid of 1970s[1] from Dempster's works are well known and used in the artificial intelligence community to model epistemic uncertainty and to reason with it for information fusion. In Dempster-Shafer theory, the combination of basic belief assignments (BBAs) provided by distinct sources of evidence is done with Dempster's rule of combination which suffers of serious drawbacks in high conflict situation as discussed by Zadeh,[2,3] but also in very low conflict situations.[4] As a matter of fact many rules of combination have been proposed in the literature[5] (Vol. 2), among them the combination of two sources of evidence based on the proportional conflict redistribution principle no5 (PCR5 rule)[6] has been shown successful in applications, and well justified theoretically. However, its complexity remains one of its limitations to prevent its use in large fusion problems.

In this study, we show how the fusion of dichotomous BBAs could be done thanks to their PCR5-based canonical decomposition which is always possible. Such decomposition of dogmatic or nondogmatic BBA has never been presented in the literature so far. Only a canonical decomposition based on conjunctive rule involving improper BBA has been proposed by Smets in 1995[7] and extended later by Denœux[8] to develop the cautious rule of combination. Here the canonical decomposition we present is done differently, and we show that any dichotomous BBA is always the result of the PCR5 fusion of a simple proper pro-evidence BBA m_p with a simple proper contra-evidence BBA m_c, and we show that this decomposition is unique.

This paper is organized as follows. After a brief recall of basics of BF in Section 2, we present the canonical decomposition problem (CDP) in Section 3 and we show the impossibility to realize the CDP of a nondogmatic BBA with conjunctive rule, disjunctive rule, Yager's and Dubois and Prade rules, and even with the averaging rule of combination. In Section 4, we analyze the CDP based on Dempster's rule of combination and we show that it cannot be done for a dogmatic BBA. In Section 5, we prove that the canonical decomposition based on PCR5 rule always exists for all the cases. In Section 6, we present some particular decompositions of a dichotomous BBA (including dogmatic BBA). Some numerical examples are presented in Section 7, and the potential interests of this PCR5-based canonical decomposition are discussed in Section 8. The last section concludes this paper and opens a challenging question for the application of this new approach.

2 | BASICS OF BELIEF FUNCTIONS

BF have been introduced by Shafer[1] to model epistemic uncertainty. We assume that the answer[*] of the problem under concern belongs to a known (or given) finite discrete frame of discernment (FoD) $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$, with $n > 1$, and where all elements of Θ are mutually exclusive.[†] The set of all subsets of Θ (including empty set $\emptyset$ and Θ) is the power-set of Θ denoted by 2^Θ. A proper BBA associated with a given source of evidence is defined[1] as a mapping $m(\cdot) : 2^\Theta \to [0, 1]$ satisfying $m(\emptyset) = 0$ and $\sum_{A \in 2^\Theta} m(A) = 1$. In some BF related frameworks, like in Smets Transferable Belief Model (TBM),[7] $m(\emptyset)$ is allowed to take a positive value. In this case, $m(\cdot)$ is said improper because it doesn't satisfy Shafer's definition.[1] The quantity $m(A)$ is called the mass of A committed by the source of evidence. Belief and plausibility functions are, respectively, defined from a proper BBA $m(\cdot)$ by

$$Bel(A) = \sum_{B \in 2^\Theta | B \subseteq A} m(B) \tag{1}$$

and

$$Pl(A) = \sum_{B \in 2^\Theta | A \cap B \neq \emptyset} m(B) = 1 - Bel(\bar{A}), \tag{2}$$

where $\bar{A}$ is the complement of A in Θ.

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[*]That is, the solution, or the decision to take.

[†]This is so-called Shafer's model of FoD.[5]

$Bel(A)$ and $Pl(A)$ are usually interpreted, respectively, as lower and upper bounds of an unknown (subjective) probability measure $P(A)$. A is called a focal element (FE) of $m(\cdot)$ if $m(A) > 0$. When all FEs are singletons then $m(\cdot)$ is called a *Bayesian BBA*[1] and its corresponding $Bel(\cdot)$ function is equal to $Pl(\cdot)$ and they are homogeneous to a (subjective) probability measure $P(\cdot)$. The vacuous BBA, or VBBA for short, representing a totally ignorant source is defined as[‡] $m_v(\Theta) = 1$. A dogmatic BBA is a BBA such that $m(\Theta) = 0$. If $m(\Theta) > 0$ the BBA $m(\cdot)$ is nondogmatic. A simple BBA is a BBA that has at most two focal sets and one of them is Θ. A dichotomous nondogmatic mass of belief is a BBA having three FEs A, $\bar{A}$, and $A \cup \bar{A}$ with A and $\bar{A}$ subsets of Θ.

In his Mathematical Theory of Evidence,[1] Shafer proposed to combine $s \geq 2$ distinct sources of evidence represented by BBAs $m_1(.), \ldots, m_s(.)$ over the same FoD Θ with Dempster's rule (ie, the normalized conjunctive rule). The justification and behavior of Dempster's rule have been disputed over the years from many counter-examples involving high and low conflicting sources (from both theoretical and practical standpoints) as reported in [4,9-11].

Many rules of combination exist in the literature,[§] among them we recommend the rule based on the proportional conflict redistribution principle no5 (PCR5 rule)[6] which has been shown successful in applications and well justified theoretically. That is why we analyze it in detail for solving the BF canonical decomposition problem (BF-CDP). PCR5 transfers the conflicting mass only to the elements involved in the conflict and proportionally to their individual masses so that the specificity of the information is entirely preserved in this fusion process. (see [5], Vols. 2 and 3 for full justification and examples). The PCR5 combination of two BBAs m_1 and m_2 defined on the same FoD Θ, denoted by $m_{PCR5} = PCR5(m_1, m_2)$, is mathematically defined as $m_{PCR5}(\emptyset) = 0$ and $\forall\, X \in 2^{\Theta}\setminus\{\emptyset\}$

$$m_{PCR5}(X) = \sum_{\substack{X_1, X_2 \in 2^{\Theta} \\ X_1 \cap X_2 = X}} m_1(X_1)m_2(X_2) + \sum_{\substack{X_2 \in 2^{\Theta} \\ X_2 \cap X = \emptyset}} \left[\frac{m_1(X)^2 m_2(X_2)}{m_1(X) + m_2(X_2)} + \frac{m_2(X)^2 m_1(X_2)}{m_2(X) + m_1(X_2)} \right], \quad (3)$$

where all denominators in (3) are different from zero. If a denominator is zero, that fraction is discarded. The properties of PCR5 can be found in [12]. Extension of PCR5 for combining qualitative BBA's can be found in [5], Vols. 2 and 3. All propositions/sets are in a canonical form. A variant of PCR5, called PCR6 has been proposed by Martin and Osswald in [5], Vol. 2, for combining $s > 2$ sources. The general formulas for PCR5 and PCR6 rules are also given in [5], Vol. 2. PCR6 coincides with PCR5 when one combines two sources. The difference between PCR5 and PCR6 lies in the way the proportional conflict redistribution is done as soon as three (or more) sources are involved in the fusion. From the implementation point of view, PCR6 is simpler to implement than PCR5. For convenience, very basic (not optimized) Matlab implementation of PCR5 and PCR6 fusion rules can be found in [5,13], and from the toolboxes repository on the web (https://bfasociety.org/). In the sequel we work with PCR5 rule because only two BBAs are involved in the canonical decomposition process we present.

[‡]The complete ignorance is denoted Θ in Shafer's book.[1]

[§]see [5], Vol. 2 for a detailed list of fusion rules.

3 | THE CANONICAL DECOMPOSITION PROBLEM

We consider a dichotomous (simplest) FoD Θ made of only two exclusive elements A and $\bar{A}$, that is $\Theta = \{A, \bar{A}\}$ and we consider a given proper[¶] BBA $m(\cdot) : 2^\Theta \to [0, 1]$ of the form

$$m(A) = a, \quad m(\bar{A}) = b, \quad m(A \cup \bar{A}) = 1 - a - b \tag{4}$$

with $0 < a < 1$, $0 < b < 1$ and $a + b < 1$.

The conditions $0 < a < 1$ and $0 < b < 1$ mean that A and $\bar{A}$ are FEs of the BBA. The restriction $a + b < 1$ means that the BBA is nondogmatic. This assumption of nondogmaticity of the BBA $m(\cdot)$ can be justified because most (if not all) states of belief, being based on imperfect and not entirely conclusive evidence, should be represented by nondogmatic BF, even if the mass $m(\Theta)$ is very small as argued by Denœux[8] (p. 240). In fact, we can always slightly modify a dogmatic BBA $m(\cdot)$ in a nondogmatic BBA by discounting it with some small discount rate $\epsilon > 0$ and letting ϵ tend toward 0.[7] The case of dogmatic belief, as well as degenerate cases with $a = 0$ and $b = 0$ will be discussed in Section 6. Note that his assumption of nondogmaticity of the BBA $m(\cdot)$ is necessary for Smets canonical decomposition,[7] but it is not essential for our PCR5-based canonical decomposition because it also works with a dogmatic BBA as discussed in Section 6.

The BF-CDP can be expressed as follows:

Given a nondogmatic BBA $m(\cdot)$ as in (4) and a chosen rule of combination, find the two following simple proper BBAs m_p and m_c of the form

$$m_p(A) = x, \quad m_p(A \cup \bar{A}) = 1 - x \tag{5}$$

$$m_c(\bar{A}) = y, \quad m_c(A \cup \bar{A}) = 1 - y \tag{6}$$

with $(x, y) \in [0, 1] \times [0, 1]$, such that $m = Fusion(m_p, m_c)$, for a chosen rule of combination denoted $Fusion(\cdot,\cdot)$.

$m_p(\cdot)$ is called the *pro-BBA* (or pro-evidence) of A, and $m_c(\cdot)$ the *contra-BBA* (or contra-evidence) of A. In the Section 5, we prove that this decomposition is always possible and unique and we call it the (PCR5-based) canonical decomposition of the BBA $m(\cdot)$. The BBA $m_p(\cdot)$ is interpreted as a source of evidence providing uncertain evidence in favor of A, whereas $m_c(\cdot)$ is interpreted as a source of evidence providing uncertain evidence against A. The BBA $m(\cdot)$ can be interpreted as the result of the PCR5 fusion of these two (pros and cons) aspects of evidence about A.

It is worth noting that this BF-CDP must not be confused with CDP addressed by Smets[7] in his TBM framework, which is based on the conjunctive rule of combination and which involves, in general, improper BBAs, called generalized simple BBA (GSBBA) in Smet's terminology.

3.1 | Impossibility of decompositions by some well-known rules

Here we analyze briefly the impossibility of a canonical decomposition for some well-known rules of combination.

[¶]which means that $m(\varnothing) = 0$.

3.1.1 | Conjunctive rule

We consider $x > 0$ and $y > 1$ so that the two BBAs are really informative (otherwise they become vacuous and useless from decision-making standpoint). In this case, we always have a conflict between $m_p(\cdot)$ and $m_c(\cdot)$ resulting of the conjunctive rule of combination. That is

$$m_{\text{conj}}(\varnothing) = m_p(A)m_c(\bar{A}) = x \cdot y > 0. \tag{7}$$

Hence $m_{\text{conj}}(\varnothing) \neq 0$ is incompatible with the constraint $m(\varnothing) = 0$. Therefore, the canonical decomposition of the BBA $m(\cdot)$ expressed as the conjunctive fusion of pros and cons BBAs $m_p(\cdot)$ and $m_c(\cdot)$ is impossible to get in general,[**] but in the very degenerate cases where $a = 0$, or $b = 0$, or $a = 0$ and $b = 0$ which would involve vacuous BBAs in the decomposition and of course will be useless.

3.1.2 | Disjunctive rule

If we consider the disjunctive rule of combination of $m_p(\cdot)$ and $m_c(\cdot)$ we will always obtain the vacuous BBA because $m_p(A)m_c(\bar{A})$, $m_p(A)m_c(A \cup \bar{A})$, $m_p(A \cup \bar{A})m_c(\bar{A})$, and $m_p(A \cup \bar{A})m_c(A \cup \bar{A})$ will all be committed to the uncertainty $A \cup \bar{A}$. Therefore the combination result is nothing but the vacuous belief assignment m_v, that is $Disj(m_p, m_c) = m_v$. In conclusion, we cannot make a decomposition of the BBA $m(\cdot)$ based on the disjunctive rule in general because if $m(\cdot)$ is informative (eg, not vacuous) one always has $a + b < 1$ so that $m(A \cup \bar{A}) < 1$ whereas the disjunctive rule of $m_p(\cdot)$ and $m_c(\cdot)$ will always provide $m(A \cup \bar{A}) = 1$.

3.1.3 | Yager's and Dubois and Prade rules

Due to the particular simple form of BBAs $m_p(A)$ and $m_c(\cdot)$, Yager's rule[14] and Dubois-Prade rule[15] coincide. Based on these rules we are searching x and y in $[0, 1]$ such that

$$m(A) = a = x(1 - y), \tag{8}$$

$$m(\bar{A}) = b = (1 - x)y, \tag{9}$$

$$m(A \cup \bar{A}) = 1 - a - b = (1 - x)(1 - y) + xy. \tag{10}$$

Because the third equation is dependent of the two first, we have only to solve the following system of equations $x - xy = a$ and $y - xy = b$. Assuming[††] $y < 1$, one gets from the first equation $x = \frac{a}{1-y}$. By replacing x by its expression in the second equation $y - xy = b$ we have to find y in $[0, 1)$ such that (after basic algebraic simplifications)

$$y^2 + (a - b - 1)y + b = 0. \tag{11}$$

[**] that is for any a and b values of mass of FEs A and $\bar{A}$ of the BBA $m(\cdot)$.

[††] taking $y = 1$ would means that $x(1 - y) = 0$ but $m(A) = a$ with $a \neq 0$ in general, so that the choice of $y = 1$ is not possible.

This second-order equation admits one or two real solutions y_1 and y_2 if and only if the discriminant is null or positive, respectively, that is if $(a-b-1)^2-4b\geq 0$. However, this discriminant can become negative depending on the values of a and b. For instance, for $a=0.4$ and $b=0.5$, we have $(a-b-1)^2-4b=-0.79$ which means that there is no real solution for the equation $y^2-1.1\cdot y+0.5=0$. Therefore, in general, the canonical decomposition of the BBA $m(\cdot)$ cannot be accomplished from Yager's and Dubois & Prade rules of combination.

3.1.4 | Averaging rule

Suppose we combine $m_p(\cdot)$ and $m_c(\cdot)$ with the averaging rule. Then we are searching x and y in $[0, 1]$ such that

$$m(A)=a=(x+0)/2, \tag{12}$$

$$m(\bar{A})=b=(0+y)/2, \tag{13}$$

$$m(A\cup\bar{A})=1-a-b=((1-x)+(1-y))/2. \tag{14}$$

This means that $x=2a$ and $y=2b$ with x and y in $[0, 1]$. So, if $a>0.5$ or $b>0.5$ the canonical decomposition is impossible to make with the averaging rule of combination. Therefore, in general, the averaging rule is not able to provide a canonical decomposition of the BBA $m(\cdot)$.

4 | DECOMPOSITION BASED ON DEMPSTER'S RULE

Let consider a nondogmatic BBA $m(A)=a$, $m(\bar{A})=b$ and $m(A\cup\bar{A})=1-a-b$ with $0\leq a,b\leq 1$ and $1-a-b>0$, and let's see if a decomposition of $m(\cdot)$ is possible based on Dempster's rule of combination.[1] For this, we are searching x and y in $[0, 1]$ such that $xy\neq 1$ and

$$m(A)=a=\frac{x(1-y)}{1-xy}, \tag{15}$$

$$m(\bar{A})=b=\frac{y(1-x)}{1-xy}, \tag{16}$$

$$m(A\cup\bar{A})=1-a-b=\frac{(1-x)(1-y)}{1-xy}. \tag{17}$$

Because the third equality is redundant with the two first, we just have to solve the system of two equations expressed as

$$(1-xy)a=x(1-y), \tag{18}$$

$$(1-xy)b=y(1-x). \tag{19}$$

That is, one should have

$$x-xy+axy=a, \tag{20}$$

$$y-xy+bxy=b, \tag{21}$$

with the constraints $0 < x < 1$ and $0 < y < 1$. So one must have

$$x = \frac{a}{1 - y + ay}, \quad y \neq \frac{1}{1 - a} \tag{22}$$

and solve the equation $y - xy + bxy = b$ with x expressed as a function of y as above. We get the equation for $a \neq 1$

$$(a - 1)y^2 + (1 + b - a)y - b = 0 \tag{23}$$

whose solutions have the form

$$y = \frac{-(1 + b - a) \pm \sqrt{\Delta}}{2(a - 1)}, \tag{24}$$

where the discriminant Δ is given by

$$\begin{aligned}\Delta &= (1 + b - a)^2 - 4(1 - a)b = 1 + b^2 + a^2 + 2b - 2a - 2ab + 4ab - 4b \\ &= a^2 + b^2 + 1 - 2b + 2ab - 2a = (a + b - 1)^2 = (1 - a - b)^2.\end{aligned}$$

One sees that Δ is strictly positive because $a + b < 1$ (m being a nondogmatic BBA). So, there exist two real solutions y_1 and y_2 of (23) of the form

$$y_1 = \frac{-(1 + b - a) + \sqrt{\Delta}}{2(a - 1)} = \frac{b}{1 - a}, \tag{25}$$

$$y_2 = \frac{-(1 + b - a) - \sqrt{\Delta}}{2(a - 1)} = \frac{1 - a}{1 - a} = 1. \tag{26}$$

For the case $a \neq 1$, the second "solution" $y_2 = 1$ implies $x = \frac{a}{1 - y_2 + ay_2} = \frac{a}{a} = 1$ which is not an acceptable solution[‡‡] because one must have $xy \neq 1$. The solution (x, y) of the decomposition problem for $a \neq 1$ is actually given by the first solution y_1, that is

$$y = y_1 = \frac{b}{1 - a} \in [0, 1), \tag{27}$$

$$x = \frac{a}{1 - y + ay} = \frac{a}{1 - b} \in [0, 1). \tag{28}$$

The case $a = 1$ corresponding to the dogmatic BBA given by $m(A) = a = 1$, $m(\bar{A}) = b = 0$, $m(A \cup \bar{A}) = 1 - a - b = 0$ is analyzed in detail in Section 6—see lemma right after Theorem 4.

In summary, the unique solution of decomposition of a nondogmatic BBA with $0 < a < 1$, $0 < b < 1$ and $a + b < 1$ using Dempster's rule is $x = \frac{a}{1 - b}$ and $y = \frac{b}{1 - a}$.

[‡‡]otherwise the denominators of Equations (15) to (17) will be equal to zero.

Example 1. Consider $m(A) = a = 0.6$, $m(\bar{A}) = b = 0.2$ and $m(A \cup \bar{A}) = 1 - a - b = 0.2$. The solution (x, y) of the decomposition of $m(\cdot)$ based on Dempster's rule is $x = \frac{a}{1-b} = \frac{0.6}{1-0.2} = 0.75$ and $y = \frac{b}{1-a} = \frac{0.2}{1-0.6} = 0.5$. Therefore, $m_p(A) = x = 0.75$, $m_p(A \cup \bar{A}) = 1 - x = 0.25$ and $m_c(\bar{A}) = y = 0.5$, $m_c(A \cup \bar{A}) = 1 - y = 0.5$. It can be verified that $m_p \oplus m_c = m$, where $\oplus$ represents symbolically Dempster's rule of combination.[1]

5 | DECOMPOSITION BASED ON PCR5 RULE

In this section, we prove that the decomposition of a dichotomous nondogmatic BBA $m(\cdot)$ based on PCR5 rule of combination is always possible and unique. Suppose we combine $m_p(\cdot)$ and $m_c(\cdot)$ with the PCR5 rule of combination. Then we are searching $(x, y) \in [0, 1]^2$ satisfying

$$m(A) = a = x(1-y) + \frac{x^2 y}{x+y} = \frac{x^2 + xy - xy^2}{x+y}, \tag{29}$$

$$m(\bar{A}) = b = (1-x)y + \frac{xy^2}{x+y} = \frac{y^2 + xy - x^2 y}{x+y}, \tag{30}$$

$$m(A \cup \bar{A}) = 1 - a - b = 1 - x - y + xy, \tag{31}$$

under the constraints $(a, b) \in [0, 1]^2$, and $0 < a + b < 1$.

Equations (29) and (30) can be rewritten as

$$x - \frac{xy^2}{x+y} = a, \tag{32}$$

$$y - \frac{x^2 y}{x+y} = b, \tag{33}$$

from which (31) is redundant because (29) + (30) gives

$$x + y - xy = a + b. \tag{34}$$

Therefore $(1-x)(1-y) = 1 - (a+b)$ and that is why the constraint $a + b \leq 1$ is necessary[§§] for the existence of the solution (x, y).

With x and y in $[0, 1]$ the solutions of (32) and (33) verify

$$x \geq a, \tag{35}$$

$$y \geq b. \tag{36}$$

Moreover, the equality (34) implies

$$x(1-y) = a + b - y \quad \Rightarrow \quad y \leq a + b, \tag{37}$$

$$y(1-x) = a + b - x \quad \Rightarrow \quad x \leq a + b. \tag{38}$$

[§§] In fact we use the constraint $a + b < 1$ because in this Section we consider only nondogmatic BBA. The canonical decomposition of a dichotomous dogmatic BBA will be analyzed in Section 6.

For $x \neq 1$, from (34) one gets $y = \frac{a+b-x}{1-x}$ and from (32) one has

$$x^2 + xy - xy^2 = ax + ay. \tag{39}$$

Putting this expression of y in (39), yields the equation

$$x^2 + (x-a)\frac{a+b-x}{1-x} - x\left(\frac{a+b-x}{1-x}\right)^2 - ax = 0, \tag{40}$$

which can be expressed after elementary algebraic calculation as

$$x^4 + (-a-2)x^3 + (2a+b)x^2 + (a+b-ab-b^2)x + (-a^2-ab) = 0. \tag{41}$$

This equation of degree 4 has at most four real solutions. We have to take only the solution x from the open interval $(0, 1)$ and $y = (a+b-x)/(1-x)$ with $y \in [0, 1]$.

The general expression of the solutions of this quartic equation[16] is very complicate to obtain analytically even with modern symbolic computing systems like Maple, or Mathematica, but the solutions can be easily calculated numerically by these computing systems, and even with Matlab system (thanks to the *fsolve* command) as soon as the numerical values are committed to a and to b. Another method to make the decomposition consists to solve numerically the system of equations $\frac{x^2+xy-xy^2}{x+y} = a$ and $\frac{y^2+xy-x^2y}{x+y} = b$ for numerical values committed to a and b thanks to Mathematica, Maple, or Matlab computing systems for instance. Of course, the solutions provided by the two methods are the same.

Example 2. Let consider $m(A) = 0.6$, $m(\bar{A}) = 0.3$ and $m(A \cup \bar{A}) = 0.1$, therefore $a = 0.6$ and $b = 0.3$. The quartic Equation (41) becomes

$$x^4 - 2.6x^3 + 1.5x^2 + 0.63x - 0.54 = 0. \tag{42}$$

The four solutions of this quartic equation provided by the computing system[¶¶] are approximately

$$\begin{aligned} x_1 &\approx 0.7774780438 \\ x_2 &\approx 0.9297589637 \\ x_3 &\approx 1.419151582 \\ x_4 &\approx -0.5263885898 \end{aligned}$$

which are shown on the graph of Figure 1 obtained easily from Desmos online tool.[***]

Clearly x_3 and x_4 are not acceptable solutions because they do not belong to $[0, 1]$. If we take $x_1 \approx 0.7774780438$ then will get $y_1 = (a+b-x_1)/(1-x_1) = (0.9-x_1)/(1-x_1) \approx 0.5506061437$, so the pair $(x_1, y_1) \in [0, 1]^2$ is a solution of the decomposition problem of the BBA $m(\cdot)$. If we take $x_2 \approx 0.9297589637$ then will get $y_2 = (a+b-x_2)/(1-x_2) = (0.9-x_2)/(1-x_2) \approx -0.4236692006$,

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¶¶We did also obtain the same solutions with Maple, and also with Matlab.

***https://www.desmos.com/calculator

so we see that $y_2 \notin [0,1]$ and therefore the pair (x_2, y_2) cannot be a solution of the decomposition problem of the BBA $m(\cdot)$. Therefore the canonical masses $m_p(\cdot)$ and $m_c(\cdot)$ are given by

$$m_p(A) \approx 0.7774780438, \quad m_p(A \cup \bar{A}) \approx 0.2225219562$$
$$m_c(\bar{A}) \approx 0.5506061437, \quad m_c(A \cup \bar{A}) \approx 0.4493938563.$$

It can be verified that the PCR5 combination of the BBAs m_p and m_c, denoted $PCR5(m_p, m_c)$, is equal to the BBA $m(\cdot)$. The following important theorem holds.

Theorem 1. *Consider a dichotomous FoD $\Theta = \{A, \bar{A}\}$ with $A \neq \Theta$ and $A \neq \emptyset$ and a nondogmatic BBA $m(\cdot): 2^\Theta \to [0,1]$ defined on Θ by $m(A) = a$, $m(\bar{A}) = b$, and $m(A \cup \bar{A}) = 1 - a - b$, where $a, b \in [0,1]$ and $a + b < 1$. Then the BBA $m(\cdot)$ has a unique canonical decomposition using PCR5 rule of combination of the form $m = PCR5(m_p, m_c)$ with pro-evidence $m_p(A) = x$, $m_p(A \cup \bar{A}) = 1 - x$ and contra-evidence $m_c(\bar{A}) = y$, $m_c(A \cup \bar{A}) = 1 - y$, where $x, y \in [0,1]$.*

Proof. Based on (29) and (30), we have to prove that the following system $S_{a,b}$ of equations always admits one and only one solution $(x, y) \in [0,1] \times [0,1]$

$$S_{a,b}: \begin{cases} h(x,y) = a \\ h(y,x) = b \end{cases} \tag{43}$$

with $h(x,y) = \frac{x^2 + xy - xy^2}{x+y} = x - \frac{xy^2}{x+y}$. The h function can be prolonged in $(0,0)$ by continuity by setting $h(0,0) = 0$.

One has to prove the existence of a unique $x \in [a, a+b] \subset [0,1]$ and $y \in [b, a+b] \subset [0,1]$ solutions of $S_{a,b}$, or equivalently solutions of $y = \frac{a+b-x}{1-x}$ and of (41) $P(x) = 0$ with

$$P(x) \triangleq x^4 + (-a-2)x^3 + (2a+b)x^2 + (a+b)(1-b)x - a(a+b). \tag{44}$$

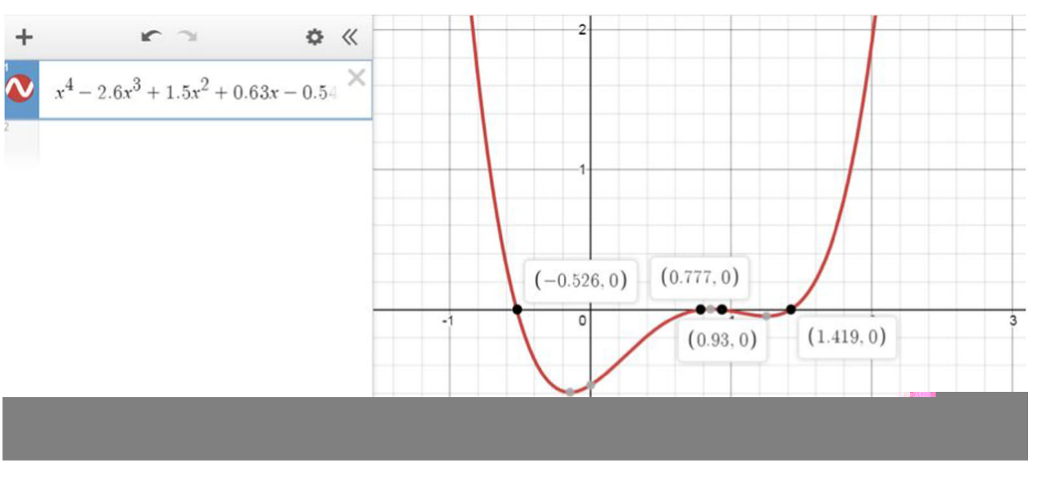

FIGURE 1 Plot of the quartic function [Color figure can be viewed at wileyonlinelibrary.com]

Because[†††] $\lim_{x\to-\infty} P(x) = +\infty$ and[‡‡‡] $P(a) < 0$, there exists $x_1 \in (-\infty, a)$ such that $P(x_1) = 0$. The solution x_1 is not acceptable because $x_1 \notin [a, a+b]$. Because[§§§] $P(1) < 0$ and $\lim_{x\to+\infty} P(x) = +\infty$, there exists also $x_4 \in (1, +\infty)$ such that $P(x_4) = 0$. The solution x_4 is not acceptable because $x_4 \notin [a, a+b]$. For $a + b \neq 1$, one has[¶¶¶] $P(a+b) > 0$ and $P(1) < 0$. Therefore there exists $x_3 \in (a+b, 1)$ such that $P(x_3) = 0$ but this solution x_3 is also not acceptable because $x_3 \notin [a, a+b]$. Because $P(a) < 0$ and $P(a+b) > 0$ there exists $x_2 \in [a, a+b]$ such that $P(x_2) = 0$ which is the only satisfactory solution. The value y_2 is given by $y_2 = \frac{a+b-x_2}{1-x_2}$, and one has $y_2 > 0$ because $x_2 < a + b$ and $y_2 < 1$ because $a + b < 1$. Moreover, from (33), $y_2 - b = \frac{x_2^2 y_2}{x_2 + y_2}$ which is always positive, therefore $y_2 > b$, and from (34) $y_2 - (a+b) = x_2(y_2 - 1)$ which is always negative, therefore $y_2 < a + b$. This completes the proof of Theorem 1. □

6 | PARTICULAR CASES OF DECOMPOSITIONS

Here we examine the canonical decomposition of particular cases, including dogmatic BBA.

6.1 | Dogmatic BBA: $a + b = 1$

Theorem 2. *Any dogmatic BBA defined by $m(A) = a$ and $m(\bar{A}) = b$, where $a, b \in [0, 1]$ and $a + b = 1$, has a canonical decomposition using PCR5 rule of combination of the form $m = PCR5(m_p, m_c)$ with $m_p(A) = x$, $m_p(A \cup \bar{A}) = 1 - x$ and $m_c(\bar{A}) = y$, $m_c(A \cup \bar{A}) = 1 - y$ where $x, y \in [0, 1]$.*

Proof. Any solution of $S_{a,b}$ verifies

$$x - a = \frac{xy^2}{x + y}, \tag{45}$$

$$y - b = \frac{x^2 y}{x + y}, \tag{46}$$

and therefore from (45) + (46) one has

$$(x - y) - (a - b) = \frac{xy(y - x)}{x + y} \tag{47}$$

[†††] $P(x)$ being polynomial, it is continuous and if $P(c)P(d) < 0$ there exist at least one solution between $[c, d]$. Therefore, we are not sure a priori there is only one solution between $[c, d]$. In our case, the signs of $P(x)$ for $x = -\infty, a, a+b, 1, +\infty$ are, respectively, +, −, +, −, and +. But because one has four intervals, into each interval it is not possible to have more than one solution (because otherwise will get five or more solutions, while this equation has only up to four real solutions). Therefore in each interval, there exists only one real solution.

[‡‡‡] Because $P(a) = a^2 b - ab(a+b) = -ab^2$.

[§§§] Because $P(1) = -1 + a + b + (a+b)(1-b-a) = -(a+b-1)^2$.

[¶¶¶] Because from (40), $P(a+b)/(1-a-b)^2 = (a+b)^2 - a(a+b) \Rightarrow P(a+b) = b(a+b)(1-a-b)^2 > 0$.

which can be rewritten as

$$(x - y)\left[1 + \frac{xy}{x + y}\right] = (a - b). \tag{48}$$

This means that differences $(x - y)$ and $(a - b)$ have the same sign. Moreover from (34) with $a + b = 1$ one has $x + y - xy = 1$, or equivalently $(1 - x)(1 - y) = 0$ which is satisfied if $x = 1$, or if $y = 1$ or both equal one. We must distinguish three cases as follows:

- If $a < b$ then[****] $x < y$ therefore $y = 1$ and $h(x, 1) = a$. Solving $h(x, 1) = a$ is equivalent to solve $x^2 - ax - a = 0$ which admits only one positive solution $x \in \; a \; a \quad b$

Proof. We have the following system of equations to solve with $0 \leq x, y \leq 1$ and $1 - xy \neq 0$

$$\frac{x - xy}{1 - xy} = a, \tag{51}$$

$$\frac{y - xy}{1 - xy} = b. \tag{52}$$

After adding the two Equations (51) and (52) and because $a + b = 1$, we obtain $\frac{x - xy + y - xy}{1 - xy} = a + b = 1$, whence $x + y - 2xy = 1 - xy$, or $x + y - xy = 1$, or $x + y(1 - x) = 1$, or $y(1 - x) = 1 - x$, or $=\frac{1-x}{1-x} = 1$ when $x \neq 1$. From (52), one should have $\frac{y - xy}{1 - xy} = b$ with $y = 1$, that is $\frac{1 - x \cdot 1}{1 - x \cdot 1} = b$, or $1 = b$ which is false because if $0 < a < 1$ then $b = 1 - a \neq 1$. This completes the proof of Theorem 4. □

Lemma *The dogmatic BBAs* $m(A) = 1$, $m(\bar{A}) = 0$ (*case* $(a, b) = (1, 0)$), *or* $m(A) = 0$, $m(\bar{A}) = 1$ (*case* $(a, b) = (0, 1)$) *have infinitely many decompositions based on Dempster's rule of combination.*

Proof. For the case $(a, b) = (1, 0)$ one has to solve with $0 \leq x, y \leq 1$ and $1 - xy \neq 0$ the system of equations

$$\frac{x - xy}{1 - xy} = 1 \quad \text{and} \quad \frac{y - xy}{1 - xy} = 0. \tag{53}$$

This system is satisfied for $x = 1$ and $y \in [0, 1)$, that is any value in $[0, 1)$ can be chosen for

6.3 | Case when $a = 0$, or $b = 0$

In the case $a = 0$ and $0 < b \leq 1$, then for conjunctive rule, Yager's, Dubois-Prade's, Dempster's, and PCR5 rules one has $x = 0$ and $y = b$ (conflict between canonical masses is zero) and $m(\cdot)$ corresponds to the fusion of vacuous pro-evidence $m_p = m_v$ with the contra-evidence $m_c = m$. In the case $0 < a \leq 1$ and $b = 0$, then for conjunctive rule, Yager's, Dubois-Prade's, Dempster's, and PCR5 rules one has $x = a$ and $y = 0$ (conflict between canonical masses is zero) and $m(\cdot)$ corresponds to the fusion of the pro-evidence $m_p = m$ with the vacuous contra-evidence $m_c = m_v$. These cases have no particular interest because they can be seen just as the combination of pros (or cons) BBA with the vacuous BBA.

6.4 | Case when $a = b \in (0, 0.5)$

Theorem 5. *In the case $a = b \in (0, 0.5)$, the BBA $m(A) = m(\bar{A}) = a$ and $m(A \cup \bar{A}) = 1 - 2a$ can be canonically decomposed from PCR5 rule with the BBAs $m_p(A) = 1 - \sqrt{1-2a}$, $m_p(A \cup \bar{A}) = \sqrt{1-2a}$ and $m_c(\bar{A}) = 1 - \sqrt{1-2a}$, $m_c(A \cup \bar{A}) = \sqrt{1-2a}$.*

Proof. From (29) and (30), one has $\frac{x^2 + xy - xy^2}{x+y} = a$ and one has also in this case $\frac{y^2 + xy - x^2 y}{x+y} = b = a$. Therefore $x^2 + xy - xy^2 = y^2 + xy - x^2 y$, or $x^2 - xy^2 - y^2 + x^2 y = 0$, or $(x - y)(x + y + xy) = 0$. $x \geq 0$ and $y \geq 0$ because they represent the masses. Therefore $x + y + xy \geq 0$. The sum $x + y + xy = 0$ if and only if $x = y = 0$, but this produces the degenerate case, which is corresponding to $a = b = 0$ (ie, the vacuous BBA). Yet, in our theorem's hypothesis we assumed $a, b \in (0, 0.5)$, so $a > 0$, and $b > 0$. Therefore $x + y + xy > 0$. Hence $x = y$. Therefore the canonical BBAs must be of the form $m_p(A) = x$, $m_p(A \cup \bar{A}) = 1 - x$ and $m_c(\bar{A}) = x$, $m_c(A \cup \bar{A}) = 1 - x$. So one must solve the equation[††††] $x - x^2 + \frac{x^2}{2} = m(A) = a$, or equivalently $\frac{1}{2}x^2 - x + a = 0$, whose solutions are $x_1 = 1 + \sqrt{1-2a}$, and $x_2 = 1 - \sqrt{1-2a}$. For $0 < a < 0.5$, the solution $x_1 > 1$ is not admissible because $x_1 \notin [0, 1]$. The solution x_2 is acceptable because if $0 < a < 0.5$, then $0 < 2a < 1$, or $-1 < -1 + 2a < 0$, or (by multiplying by -1 the inequalities) $1 > 1 - 2a > 0$, or $0 < 1 - 2a < 1$, or $\sqrt{0} < \sqrt{1-2a} < \sqrt{1}$, or $0 > -\sqrt{1-2a} > -1$, or $1 > 1 - \sqrt{1-2a} > 0$ hence $x_2 \in (0, 1)$. This completes the proof of the Theorem 5. □

7 | EXAMPLES

We give in Tables 1 to 9 some numerical examples of PCR5-based canonical decompositions of BBA $m(\cdot)$ for different sampled values of a and b for convenience. These numerical examples may be useful for researchers working with BF and interested by this new type of decomposition in their own examples. The values have been approximated at the 10th digit. Figures 2 and 3 show the shapes of the pro-evidence $x = f(a, b)$ and the contra-evidence

[††††] In fact, we have also the second equation $x - x^2 + \frac{x^2}{2} = m(\bar{A}) = b = a$ to solve which is the same as the first one.

$y = g(a, b)$ surfaces proving graphically the existence of canonical decomposition based on PCR5 at the sampling rate of 0.025. The values (a, b) for which $a + b > 1$ are not acceptable and $f(a, b)$ and $g(a, b)$ have been set to zero in the figures.

8 | INTEREST OF CANONICAL DECOMPOSITION

The canonical decomposition based on PCR5 offers several practical interests and advantages that are briefly listed here (Figure 4).

1. From the theoretical standpoint, one has proved that the canonical decomposition based on PCR5 rule always exists in all the cases for nondogmatic or dogmatic BBAs contrariwise to other rules of combination that only work in some restrictive cases. Therefore this decomposition is more general and mathematically well justified.
2. This canonical decomposition of any dichotomous BBA $m(\cdot)$ into the pro-evidence $m_p(\cdot)$ and the contra-evidence $m_c(\cdot)$ allows to define now the notion of internal conflict of a (dichotomous) source of evidence, denoted $K_{\text{int}}(m)$, by

$$K_{\text{int}}(m) \triangleq m_p(A)m_c(\bar{A}), \tag{55}$$

where $m_p(A) = x$ and $m_c(\bar{A}) = y$ are the canonical factors of the BBA $m(\cdot)$ based on PCR5 rule of combination. It is worth noting that the BBA $m(\cdot)$ has no internal conflict, if and only if at least one of its factor is the vacuous belief mass, that is, if $x = 0$ or $y = 0$, or both, which makes sense. For instance the BBA $m(A) = 0.3$ and $m(A \cup \bar{A}) = 0.7$ does not carry internal conflict because $m_p = m$ and $m_c = m_v$ (the vacuous BBA) so that its internal conflict $K_{\text{int}}(m) \triangleq m_p(A)m_c(\bar{A}) = 0.3 \cdot 0 = 0$. In fact, in this example, the BBA $m(\cdot)$ carries only uncertain pro-evidence, and vacuous contra-evidence. This internal conflict measure should contribute somehow in the definition of the information content carried by a (dichotomous) source of evidence. This aspect however is not detailed in this paper and is left for future research works. It is clear that the maximum of internal conflict $K_{\text{int}}(m) = 1$ is obtained for the dogmatic BBA $m(A) = m(\bar{A}) = 0.5$ whose canonical decomposition by PCR5 is

TABLE 1 Decomposition of basic belief assignment when $m(A) = 0.1$

(a,b)	*x*	*y*
(0.1,0.1)	0.1055728059	0.1055728059
(0.1,0.2)	0.1155063468	0.2085867463
(0.1,0.3)	0.1283308324	0.3116654549
(0.1,0.4)	0.1445620975	0.4155040377
(0.1,0.5)	0.1653570911	0.5207531320
(0.1,0.6)	0.1926613985	0.6284087006
(0.1,0.7)	0.2298437881	0.7403124237
(0.1,0.8)	0.2834628414	0.8604398965
(0.1,0.9)	0.3701562119	1

TABLE 2 Decomposition of basic belief assignment when $m(A) = 0.2$

(a,b)	x	y
(0.2,0.1)	0.2085867463	0.1155063468
(0.2,0.2)	0.2254033308	0.2254033308
(0.2,0.3)	0.2477759456	0.3353044255
(0.2,0.4)	0.2763932022	0.4472135955
(0.2,0.5)	0.3133633342	0.5630877072
(0.2,0.6)	0.3628331876	0.6861104563
(0.2,0.7)	0.4339764332	0.8233289109
(0.2,0.8)	0.5582575695	1

TABLE 3 Decomposition of basic belief assignment when $m(A) = 0.3$

(a,b)	x	y
(0.3,0.1)	0.3116654549	0.1283308324
(0.3,0.2)	0.3353044255	0.2477759456
(0.3,0.3)	0.3675444680	0.3675444680
(0.3,0.4)	0.4098895428	0.4916206002
(0.3,0.5)	0.4669657064	0.6247896197
(0.3,0.6)	0.5506061437	0.7774780438
(0.3,0.7)	0.7178908346	1

$m_p(A) = 1$ and $m_c(\bar{A}) = 1$ which shows the full conflict between the pro-evidence $m_p(\cdot)$ and the contra-evidence $m_c(\cdot)$ of the source. Of course, there is no internal conflict for the vacuous BBA. More precisely, $K_{\text{int}}(m_v) = 0$ because if $a = b = 0$ then one has $x = y = 0$ calculated from PCR5-based decomposition.

3. This canonical decomposition allows also to define the notion of level of uncertainty $U(m)$ of a dichotomous source of evidence $m(\cdot)$ as the conjunction of the uncertainties of pro and contra evidence, that is

TABLE 4 Decomposition of basic belief assignment when $m(A) = 0.4$

(a,b)	x	y
(0.4,0.1)	0.4155040377	0.1445620975
(0.4,0.2)	0.4472135955	0.2763932022
(0.4,0.3)	0.4916206002	0.4098895428
(0.4,0.4)	0.5527864045	0.5527864045
(0.4,0.5)	0.6442577571	0.7188975951
(0.4,0.6)	0.8633249581	1

TABLE 5 Decomposition of basic belief assignment when $m(A) = 0.5$

(a,b)	***x***	***y***
(0.5,0.1)	0.5207531320	0.1653570911
(0.5,0.2)	0.5630877072	0.3133633342
(0.5,0.3)	0.6247896197	0.4669657064
(0.5,0.4)	0.7188975951	0.6442577571
(0.5,0.5)	1	1

TABLE 6 Decomposition of basic belief assignment when $m(A) = 0.6$

(a,b)	***x***	***y***
(0.6,0.1)	0.6284087006	0.1926613985
(0.6,0.2)	0.6861104563	0.3628331876
(0.6,0.3)	0.7774780438	0.5506061437
(0.6,0.4)	1	0.8633249581

TABLE 7 Decomposition of basic belief assignment when $m(A) = 0.7$

(a,b)	***x***	***y***
(0.7,0.1)	0.7403124237	0.2298437881
(0.7,0.2)	0.8233289109	0.4339764332
(0.7,0.3)	1	0.7178908346

TABLE 8 Decomposition of basic belief assignment when $m(A) = 0.8$

(a,b)	***x***	***y***
(0.8,0.1)	0.8604398965	0.2834628414
(0.8,0.2)	1	0.5582575695

TABLE 9 Decomposition of basic belief assignment when $m(A) = 0.9$

(a,b)	***x***	***y***
(0.9,0.1)	1	0.3701562119

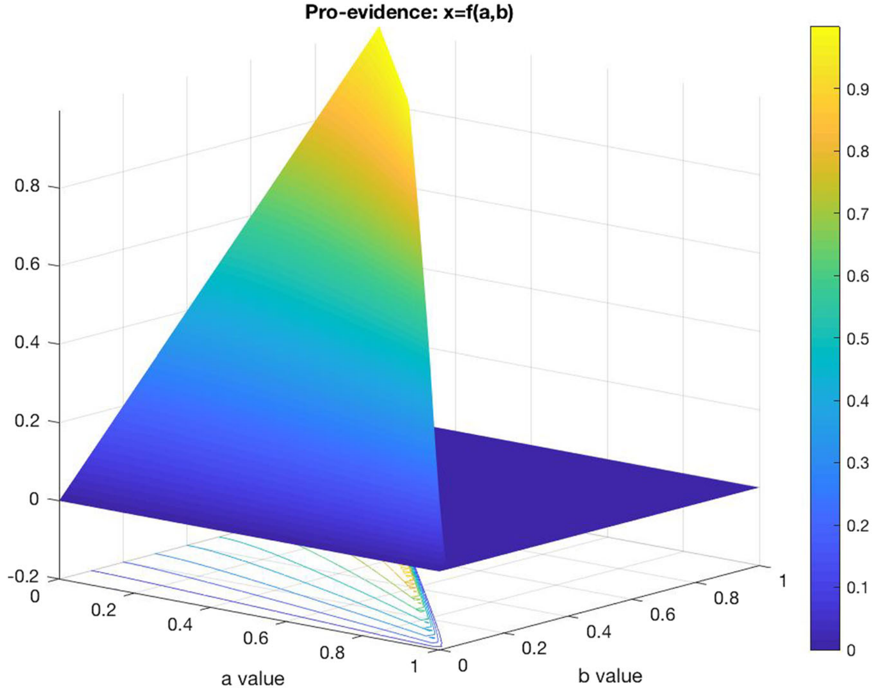

FIGURE 2 Plot of $x = f(a, b)$ pro-evidence surface [Color figure can be viewed at wileyonlinelibrary.com]

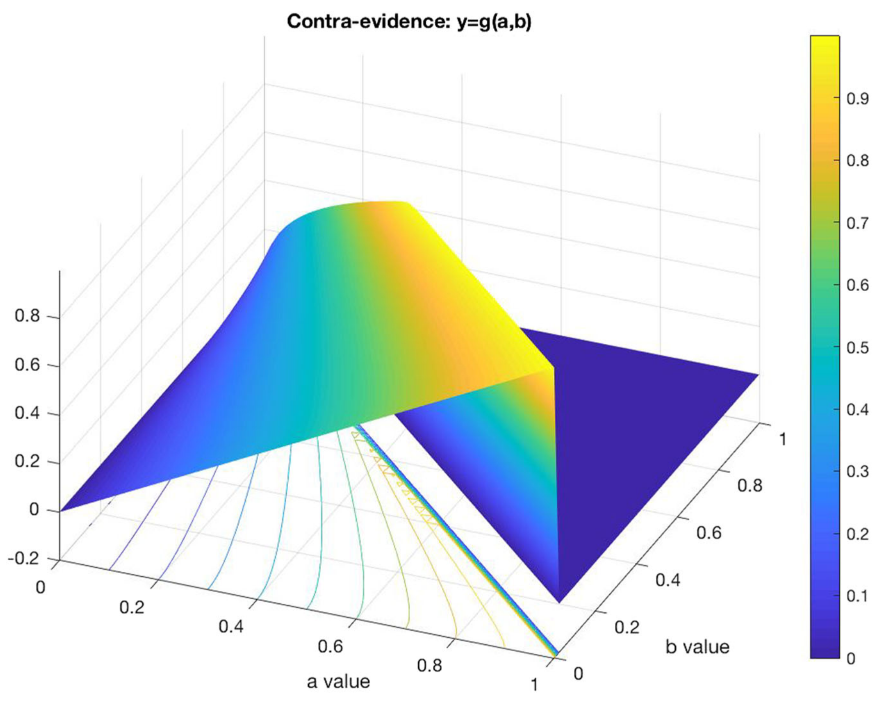

FIGURE 3 Plot of $y = g(a, b)$ contra-evidence surface [Color figure can be viewed at wileyonlinelibrary.com]

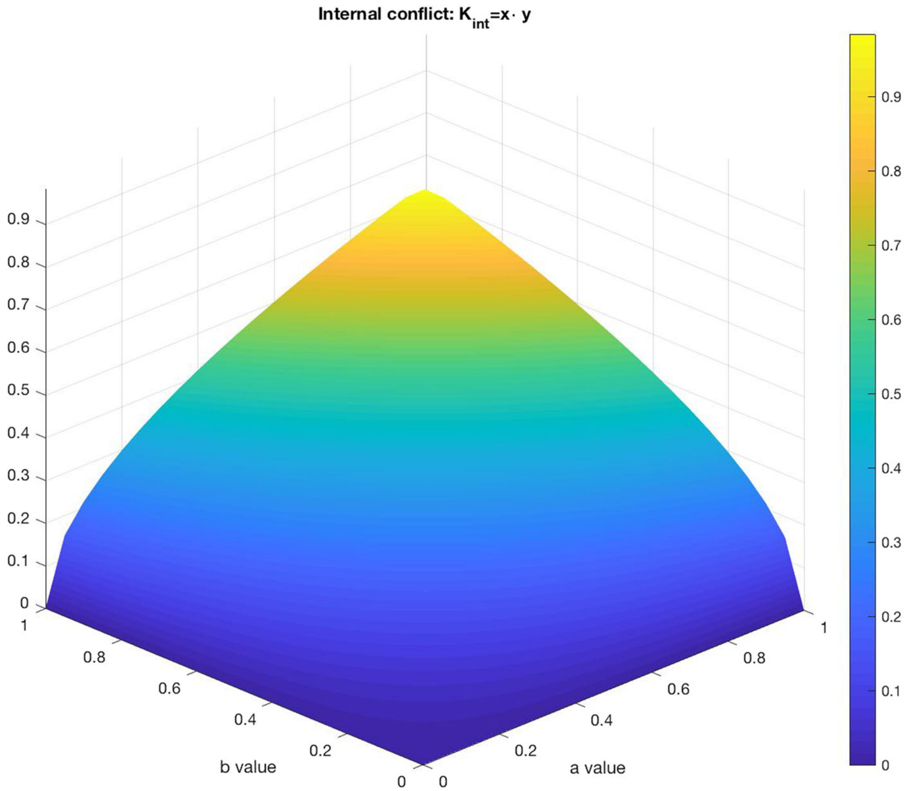

FIGURE 4 Internal conflict $K_{\text{int}}(m)$ [Color figure can be viewed at wileyonlinelibrary.com]

$$
\begin{aligned}
U(m) &\triangleq m_p(A \cup \bar{A}) m_c(A \cup \bar{A}) \\
&= (1-x)(1-y) = 1 - x - y + xy \\
&= 1 - x - y + K_{\text{int}}(m).
\end{aligned}
\tag{56}
$$

Because of PCR5-based decomposition one gets (as already shown in (31)) $U(m) = 1 - a - b$ which always belongs to $[0, 1]$. The formula (56) is interesting because it clearly shows the link between the pro-evidence value x, the contra-evidence value y and the internal conflict $K_{\text{int}}(m) = xy$. Clearly, if $x = 0$ and $y = 0$, then $K_{\text{int}}(m) = 0$ and the uncertainty is maximal (ie, $U(m) = 1$) because the dichotomous BBA m is the vacuous BBA $m(A \cup \bar{A}) = 1$. It can be verified that a dichotomous BBA m has no uncertainty ($U(m) = 0$) if and only if $x = 1$, or $y = 1$, or both which means that $m(\cdot)$ is a Bayesian dichotomous BBA.

4. The canonical decomposition allows also to adjust/revise easily a dichotomous source of evidence (if needed) according to the knowledge one has on it. For instance, suppose one knows that the source which provides the BBA $m(\cdot)$ usually over estimates with a reinforcement factor of $\beta_p = 20\%$ the belief mass committed to hypothesis A but is always fair (unbiased) when committing its mass to $\bar{A}$. Under this condition, we make the canonical decomposition of $m(\cdot)$ to get $m_p(\cdot)$ and $m_c(\cdot)$ and we have to

discount[‡‡‡‡] the pro-evidence $m_p(\cdot)$ with the discounting rate of $\alpha_p = 1/(1+\beta_p)$ to get the new unbiased BBA $m'_p(\cdot)$ and keep the contra-evidence $m_c(\cdot)$ unchanged, so that the corrected (unbiased) BBA $m'(\cdot)$ will be obtained by the PCR5 combination of $m'_p(\cdot)$ with $m_c(\cdot)$. Of course, similar principles can be applied to discount (or reinforce) $m_c(\cdot)$ as we prefer (and when necessary) by choosing the adequate discounting (or reinforcing) factors.

5. This canonical decomposition opens the door to new rules of combination for the fusion of $S \geq 2$ (dichotomous) distinct[§§§§] BBAs $m_s(\cdot)$, $s = 1, 2..., S$. After making their canonical decompositions to get S pro-evidence $m_{p,s} = (m_{p,s}(A), m_{p,s}(\bar{A}), m_{p,s}(A \cup \bar{A})) = (x_s, 0, 1 - x_s)$ and S contra-evidence $m_{c,s} = (m_{c,s}(A), m_{c,s}(\bar{A}), m_{c,s}(A \cup \bar{A})) = (0, y_s, 1 - y_s)$ for $s = 1, 2, ..., S$, one can for instance combine the S informative nonconflicting pro-evidence $m_{p,s}$ altogether by the conjunctive rule (or any rule one prefers) to get the combined pro-evidence $m_p(\cdot)$, and do similarly to combine altogether the nonconflicting contra-evidence $m_{c,s}$ to get the combined contra-evidence $m_c(\cdot)$. Once $m_p(\cdot)$ and $m_c(\cdot)$ are calculated, we combine them with PCR5 to get the final resulting BBA. Processing this way will greatly simplify the combination of many dichotomous BBAs. Once the decomposition of each dichotomous BBA is done, we could also consider to apply some importance discounting[13] with rates β_s to combine separately the set of BBAs $\{m_{p,s}, s = 1, ..., S\}$ and the set of BBAs $\{m_{c,s}, s = 1, ..., S\}$ before making their PCR5 combination.

9 | CONCLUSIONS

In this study, we have proved that any dichotomous BBA (nondogmatic, or dogmatic) can be decomposed into two simpler proper belief assignments called the pro-evidence and contra-evidence that can be combined with PCR5 rule to retrieve the original BBA. This canonical decomposition is unique and is always possible. No simple explicit form of the expression of the solution exists but the solution can be found quite easily with numerical solvers (Matlab, Maple, etc). We have also shown that the decomposition of any dichotomous BBA cannot be done in all the cases with other well-known rules of combination, which reinforce the interest of the PCR5 principle for BF combination. This PCR5-based canonical decomposition allows also to establish the notion of the internal conflict of a dichotomous source of evidence which could be helpful in some applications. It offers the possibility to combine several dichotomous sources of evidence based on the fusion of their canonical components. This will be presented in detail in a forthcoming publication. The open challenging question is how to extend this notion of canonical decomposition for working with more general BBAs to make their combination more effective (if possible), and how could we define a measure of (uncertain) information thanks to this canonical decomposition.

ACKNOWLEDGMENTS

Jean Dezert is very grateful to Professor Christine Marois of the Faculté de Droit d'Économie et de Gestion of Orléans University, France for her inspiration and valuable help for the establishment of the concise proof of the Theorem 1.

[‡‡‡‡]We use classical Shafer's discounting method.[1]

[§§§§]That is, cognitively independent.

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How to cite this article: Dezert J, Smarandache F. Canonical decomposition of dichotomous basic belief assignment. *Int J Intell Syst*. 2020;1–21.
https://doi.org/10.1002/int.22236