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A result on single valued neutrosophic refined rough approximation operators[1]

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Abstract. Smarandache (1998) initiated neutrosophic sets as a new mathematical tool for dealing with problems involving incomplete, indeterminant and inconsistent knowledge. By simplifying neutrosophic sets, Smarandache (1998) and Wang et al. (2010) proposed the concept of single valued neutrosophic sets and studied some properties of single valued neutrosophic sets. Recently, Bao and Yang (2017) introduced n-dimension single valued neutrosophic refined rough sets by combining single valued neutrosophic refined sets with rough sets and further studied the hybrid model from two perspectives–constructive viewpoint and axiomatic viewpoint. A natural problem is: Can the supremum and infimum of n-dimension single valued neutrosophic refined rough approximation operators be given? Following the idea of Bao and Yang, in this paper, let X be a set, $H_n(X)$ and $L_n(X)$ denote the family of all n-dimension single valued neutrosophic refined upper and lower approximation operators in X, respectively. We can define appropriate order relation $\leqq$ on $H_n(X)$ (resp., $L_n(X)$) such that both $(H_n(X), \leqq)$ and $(L_n(X), \leqq)$ are complete lattices. In particular, both $(H, \leqq)$ and $(L, \leqq)$ are complete lattices, where H and L denote the family of single valued neutrosophic upper and lower approximation operators in X, respectively.

Keywords: Single valued neutrosophic refined rough relations, single valued neutrosophic refined lower approximation operators, single valued neutrosophic refined upper approximation operators, complete lattices

1. Introduction

In order to deal with imprecise information and knowledge. Smarandache [14, 15] first introduced the notion of neutrosophic set by fusing the non-standard analysis and a tri-component set. A neutrosophic set consists of three membership functions (truth-membership function, indeterminacy membership function and falsity-membership function), where every function value is a real standard or non-standard subset of the nonstandard unit interval $]0^-, 1^+[$. Since then, many authors have been studied various aspects of neutrosophic sets from different point of view, for example, in order to apply the neutrosophic idea to logics. Rivieccio [11] proposed neutrosophic logics which is a generalization of fuzzy logics and studied some basic properties. Guo and Cheng [4] and Guo and Sengur [5] obtained a good applications in image processing and cluster analysis by using neutrosophic sets. Salama and Broumi [13] and Broumi and Smarandache [3] first given a new hybrid mathematical structure called rough neutrosophic sets, handling incomplete and indeterminate information, and studied some operations and their properties.

Wang et al. [16] proposed single valued neutrosophic sets by simplifying neutrosophic sets. single valued neutrosophic sets can also be looked as an

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extension of intuitionistic fuzzy sets (Atanassov [1]), in which three membership functions are unrelated and their function values belong to the unit closed interval. Single valued neutrosophic sets results in a new hot research issue. Ye [19–21] proposed decision making based on correlation coefficients and weighted correlation coefficient of single valued neutrosophic sets, and illustrated the application of proposed methods. Majumdar and Samant [9] studied distance, similarity and entropy of single valued neutrosophic sets from a theoretical aspect.

Şahin and Küçük [12] proposed a subsethood measure of single valued neutrosophic sets based on distance and showed its effectiveness by an example. We known that there's a certain connection between fuzzy relations and fuzzy rough approximation operators (resp., fuzzy topologies, information systems [6–8]). Hence, Yang et al. [17] firstly proposed single valued neutrosophic relations and studied some kinds of kernels and closures of single valued neutrosophic relations, then they proposed single valued neutrosophic rough sets [18] by fusing single valued neutrosophic sets and rough sets (Pawlak, [10]), and explored a general framework of the study of single valued neutrosophic rough sets. Concretely, they studied the hybrid model by using constructive and axiomatic approaches. Recently, Bao and Yang [2] introduced n-dimension single valued neutrosophic refined rough sets by combining single valued neutrosophic refined sets with rough sets and further studied the hybrid model from two perspectives– constructive viewpoint and axiomatic viewpoint. However, the supremum and infimum of n-dimension single valued neutrosophic refined rough approximation operators were not given. Along this line, in the present paper, let X be a set, $H_n(X)$ and $L_n(X)$ denote the family of all n-dimension single valued neutrosophic refined upper and lower approximation operators in X, respectively. We can define appropriate order relation $\leqq$ on $H_n(X)$ (resp., $L_n(X)$) such that both $(H_n(X), \leqq)$ and $(L_n(X), \leqq)$ are complete lattices. In particular, both $(H, \leqq)$ and $(L, \leqq)$ are complete lattices, where H and L denote the family of single valued neutrosophic upper and lower approximation operators in X, respectively.

2. Preliminaries

In this section, we briefly recall some basic definitions which will be used in the paper.

2.1. Single valued neutrosophic rough sets

Definition 2.1. (Smarandache 1998) Let X be a space of points (objects), with a generic element in X denoted by a. A neutrosophic set A in X consists of three membership functions (truth-membership function T_A, indeterminacy membership function I_A and falsity-membership function F_A, where every function value is a real standard or non-standard subset of the nonstandard unit interval $]0^-, 1^+[$.

There is no restriction on the sum of $T_A(a)$, $I_A(a)$ and $F_A(a)$, thus

$$0^- \leq \sup T_A(a) + \sup I_A(a) + \sup F_A(a) \leq 3^+.$$

In order to apply neutrosophic sets conveniently, Wang et al. proposed single neutrosophic sets as follows.

Definition 2.2. (Wang et al. 2010) Let X be a space of points (objects), with a generic element in X denoted by a. A single valued neutrosophic set A in X consists of three membership functions (truth-membership function T_A, indeterminacy membership function I_A and falsity-membership function F_A, where every function value is a real standard subset of the unit interval [0, 1].

There is no restriction on the sum of $T_A(a)$, $I_A(a)$ and $F_A(a)$, thus

$$0 \leq \sup T_A(a) + \sup I_A(a) + \sup F_A(a) \leq 3.$$

The family of all single valued neutrosophic sets in X will be denoted by $\mathbb{SVNS}(X)$.

Definition 2.3. (Ye 2014) Let A and B be two single valued neutrosophic sets in X, $T_A(a) \leq T_B(a)$, $I_A(a) \geq I_B(a)$ and $F_A(a) \geq F_B(a)$ for each $a \in X$, then we called A is contained in B, i.e., $A \Subset B$. If $A \Subset B$ and $B \Subset A$, then we called A is equal to B, denoted by $A = B$.

Definition 2.4. (Yang 2016) Let A and B be two single valued neutrosophic sets in X,

(1) The union of A and B is a single valued neutrosophic set C, denoted by $A \Cup B$, where $\forall x \in X$,

$$T_C(a) = max\{T_A(a), T_B(a)\},$$

$$I_C(a) = min\{I_A(a), I_B(a)\},$$

and

$$F_C(a) = min\{F_A(a), F_B(a)\}.$$

(2) The intersection of A and B is a single valued neutrosophic set D, denoted by $A \Cap B$, where $\forall x \in X$,

$$T_D(a) = min\{T_A(a), T_B(a)\},$$

$$I_D(a) = max\{I_A(a), I_B(a)\},$$

and

$$F_D(a) = max\{F_A(a), F_B(a)\}.$$

Definition 2.5. (Yang 2016) Let X be a set, and let $R = \{< (a, b), T_R(a, b), I_R(a, b), F_R(a, b) >| (a, b) \in X \times X\}$ be a single valued neutrosophic relation in X, where $T_R : X \times X \longrightarrow [0, 1], I_R : X \times X \longrightarrow [0, 1], F_R : X \times X \longrightarrow [0, 1]$ denote the truth-membership mapping, indeterminacy membership mapping and falsity-membership mapping of R, respectively. The family of all single valued neutrosophic relations in X will be denoted by $\boldsymbol{SVNR}(X)$.

Definition 2.6. (Yang 2017) Let R be a single valued neutrosophic relation in X, the pair (X, R) is called a single valued neutrosophic approximation space. $\forall A \in \mathbb{SVNS}(X)$, the lower and upper approximations of A with respect to (X, R), denoted by $\underline{R}(A)$ and $\overline{R}(A)$, are two single valued neutrosophic sets whose membership functions are defined as: $\forall a \in X$,

$$T_{\underline{R}(A)}(a) = \bigwedge_{b \in X} [F_R(a, b) \vee T_A(b)],$$

$$I_{\underline{R}(A)}(a) = \bigvee_{b \in X} [(1 - I_R(a, b)) \wedge I_A(b)],$$

$$F_{\underline{R}(A)}(a) = \bigvee_{b \in X} [T_R(a, b) \wedge F_A(b)],$$

and

$$T_{\overline{R}(A)}(a) = \bigvee_{b \in X} [T_R(a, b) \wedge T_A(b)],$$

$$I_{\overline{R}(A)}(a) = \bigwedge_{b \in X} [I_R(a, b) \vee I_A(b)],$$

$$F_{\overline{R}(A)}(a) = \bigwedge_{b \in X} [F_R(a, b) \vee F_A(b)].$$

The pair $(\underline{R}(A), \overline{R}(A))$ is called the single valued neutrosophic rough set of A with respect to (X, R). $\underline{R}$ and $\overline{R}$ are referred to as the single valued neutrosophic lower and upper approximation operators, respectively.

2.2. Single valued neutrosophic refined rough sets

Definition 2.7. (Ye and Ye 2014) Let X be a space of points(objects), with a generic element in X denoted by x. A single valued neutrosophic refined set B in X is characterized by three membership functions: a truth-membership function T_B, an indeterminacy membership function I_B and a falsity-membership function F_B as follows:

$$B = \{< x, T_B(x), I_B(x), F_B(x) >| x \in X\},$$

where

$$T_B(x) = \{T_{1B}(x), T_{2B}(x), \cdots, T_{nB}(x)\},$$

$$I_B(x) = \{I_{1B}(x), I_{2B}(x), \cdots, I_{nB}(x)\},$$

$$F_B(x) = \{F_{1B}(x), F_{2B}(x), \cdots, F_{nB}(x)\},$$

n is a positive integer, $T_{iB}(x), I_{iB}(x), F_{iB}(x) \in [0, 1]$ and $0 \leq T_{iB}(x) + I_{iB}(x) + F_{iB}(x) \leq 3$ for $i = 1, 2, \cdots, n$. Also, n is referred to as the dimension of B. For convenient, we take $SVNRS_n(X)$ to represent the family of all n-dimension single valued neutrosophic refined set. Obviously, when $n = 1$, the single valued neutrosophic refined set will degenerate into a single valued neutrosophic set. Moreover, $\forall B, C \in SVNRS_n(X)$,

• B is referred to as an empty single valued neutrosophic refined set iff $T_{iB}(x) = 0, I_{iB}(x) = F_{iB}(x) = 1 (i = 1, 2, \cdots, n)$ for all $x \in X$. The n-dimension empty single valued neutrosophic refined set is denoted by $\emptyset_n$.

• B is referred to as a full single valued neutrosophic refined set iff $T_{iB}(x) = 1, I_{iB}(x) = F_{iB}(x) = 0\ (i = 1, 2, \cdots, n)$ for all $x \in X$. The n-dimension full single valued neutrosophic refined set is denoted by X_n.

• The complement of B is denoted by B^c and defined as:

$$B^c = \{< x, T_{B^c}(x), I_{B^c}(x), F_{B^c}(x) >| x \in X\},$$

where

$$T_{B^c}(x) = F_B(x) = \{F_{1B}(x), F_{2B}(x), \cdots, F_{nB}(x)\},$$

$$I_{B^c}(x) = \sim I_B(x) = \{1 - I_{1B}(x), 1 - I_{2B}(x), \cdots, 1 - I_{nB}(x)\},$$

and

$$F_{B^c}(x) = T_B(x) = \{T_{1B}(x), T_{2B}(x), \cdots, T_{nB}(x)\}.$$

• The intersection of B and C is denoted by $B \sqcap C$ and defined as:

$$B \sqcap C = \{< x, T_{B\sqcap C}(x), I_{B\sqcap C}(x), F_{B\sqcap C}(x) >| x \in X\},$$

where

$$T_{B\sqcap C}(x) = T_B(x) \widetilde{\wedge} T_C(x) = \{T_{1B}(x) \wedge T_{1C}(x), T_{2B}(x) \wedge T_{2C}(x), \cdots, T_{nB}(x) \wedge T_{nC}(x)\},$$

$$I_{B\sqcap C}(x) = I_B(x) \widetilde{\vee} I_C(x) = \{I_{1B}(x) \vee I_{1C}(x), I_{2B}(x) \vee I_{2C}(x), \cdots, I_{nB}(x) \vee I_{nC}(x)\},$$

and

$$F_{B\sqcap C}(x) = F_B(x) \widetilde{\vee} F_C(x) = \{F_{1B}(x) \vee F_{1C}(x), F_{2B}(x) \vee F_{2C}(x) \cdots, F_{nB}(x) \vee F_{nC}(x)\}.$$

• The union of B and C is denoted by $B \sqcup C$ and defined as:

$$B \sqcup C = \{< x, T_{B\sqcup C}(x), I_{B\sqcup C}(x), F_{B\sqcup C}(x) >| x \in X\},$$

where

$$T_{B\sqcup C}(x) = T_B(x) \widetilde{\vee} T_C(x),$$

$$I_{B\sqcup C}(x) = I_B(x) \widetilde{\wedge} I_C(x),$$

and

$$F_{B\sqcup C}(x) = F_B(x) \widetilde{\wedge} F_C(x).$$

• B is contained in C is denoted by $B \sqsubseteq C$ and defined as: $B \sqsubseteq C$ if and only if $T_B(x) \preceq T_C(x)$, $I_C(x) \preceq I_B(x)$ and $F_C(x) \preceq F_B(x)$ for any $x \in X$, i.e., $T_{iB}(x) \leq T_{iC}(x)$, $I_{iC}(x) \leq I_{iB}(x)$ and $F_{iC}(x) \leq F_{iB}(x)$ for all $i = 1, 2, \cdots, n$.

Definition 2.8. (Bao and Yang 2017) Let $\mathcal{R}$ be a n-dimension single valued neutrosophic refined set in $X \times X$, then $\mathcal{R}$ is called a n-dimension single valued neutrosophic refined relation in X, and the pair $(X, \mathcal{R})$ is called a n-dimension single valued neutrosophic refined approximation space.

Definition 2.9. (Bao and Yang 2017) Let $(X, \mathcal{R})$ be a n-dimension single valued neutrosophic refined approximation space. $\forall A \in SVNRS_n(X)$, the lower and upper approximations of A with respect to $(X, \mathcal{R})$, denoted by $\underline{\mathcal{R}}(A)$ and $\overline{\mathcal{R}}(A)$, are two n-dimension single valued neutrosophic refined sets whose membership functions are defined as: $\forall a \in X$,

$$T_{\underline{\mathcal{R}}(A)}(a) = \widetilde{\bigwedge_{b\in X}} [F_{\mathcal{R}}(a, b) \vee T_A(b)],$$

$$I_{\underline{\mathcal{R}}(A)}(a) = \widetilde{\bigvee_{b\in X}} [(\sim I_{\mathcal{R}}(a, b)) \wedge I_A(b)],$$

$$F_{\underline{\mathcal{R}}(A)}(a) = \widetilde{\bigvee_{b\in X}} [T_{\mathcal{R}}(a, b) \wedge F_A(b)];$$

$$T_{\overline{\mathcal{R}}(A)}(a) = \widetilde{\bigvee_{b\in X}} [T_{\mathcal{R}}(a, b) \wedge T_A(b)],$$

$$I_{\overline{\mathcal{R}}(A)}(a) = \widetilde{\bigwedge_{b\in X}} [I_{\mathcal{R}}(a, b) \vee I_A(b)],$$

$$F_{\overline{\mathcal{R}}(A)}(a) = \widetilde{\bigwedge_{b\in X}} [F_{\mathcal{R}}(a, b) \vee F_A(b)].$$

The pair $(\underline{\mathcal{R}}(A), \overline{\mathcal{R}}(A))$ is called the single valued neutrosophic refined rough set of A with respect to $(X, \mathcal{R})$. $\underline{R}$ and $\overline{R}$ are referred to as the single valued neutrosophic refined lower and upper approximation operators, respectively.

Lemma 2.10. *(Bao and Yang 2017) Let $\mathcal{R}_1$ and $\mathcal{R}_1$ be two n-dimension single valued neutrosophic refined relations in X, $\forall A \in SVNRS_n(X)$, we have*

(1)

$$\underline{\mathcal{R}_1 \sqcup \mathcal{R}_2}(A) = \underline{\mathcal{R}_1}(A) \sqcap \underline{\mathcal{R}_2}(A),$$

(2)

$$\overline{\mathcal{R}_1 \sqcup \mathcal{R}_2}(A) = \overline{\mathcal{R}_1}(A) \sqcup \overline{\mathcal{R}_2}(A),$$

(3)

$$\underline{\mathcal{R}_1 \sqcap \mathcal{R}_2}(A) \sqsupseteq \underline{\mathcal{R}_1}(A) \sqcup \underline{\mathcal{R}_2}(A) \sqsupseteq \underline{\mathcal{R}_1}(A) \sqcap \underline{\mathcal{R}_2}(A),$$

(4)

$$\overline{\mathcal{R}_1 \sqcap \mathcal{R}_2}(A) \sqsubseteq \overline{\mathcal{R}_1}(A) \sqcap \overline{\mathcal{R}_2}(A).$$

By lemma 2.10, we can easily obtain the following corollary:

Corollary 2.11. *(Bao and Yang 2017) Let $\mathcal{R}_1$ and $\mathcal{R}_1$ be two n-dimension single valued neutrosophic refined relations in X. If $\mathcal{R}_1 \sqsubseteq R_2$, then $\forall A \in SVNRS_n(X)$, $\overline{\mathcal{R}_1}(A) \sqsubseteq \overline{\mathcal{R}_2}(A)$ and $\underline{\mathcal{R}_2}(A) \sqsubseteq \underline{\mathcal{R}_1}(A)$.*

3. The lattice structure of *n*-dimension single valued neutrosophic refined rough approximation operators.

In this section, we take $SVNRR_n(X)$ to represent the family of all n-dimension single valued neutrosophic refined relations in X.

Let

$$H_n(X) = \{\overline{\mathcal{R}} \mid \mathcal{R} \in SVNRR_n(X)\}$$

and

$$L_n(X) = \{\underline{\mathcal{R}} \mid \mathcal{R} \in SVNRR_n(X)\}$$

be the family of n-dimension single valued neutrosophic refined upper and lower approximation operators in X, respectively. We will study the lattice structure of n-dimension single valued neutrosophic refined rough approximation operators.

Theorem 3.1.

(1) Defined a relation $\sqsubseteq$ on $SVNRR_n(X)$ as follows: $\mathcal{R}_1 \sqsubseteq \mathcal{R}_2$ if and only if for any $(a, b) \in X \times X$,

$$T_{i\mathcal{R}_1}(a, b) \leq T_{i\mathcal{R}_2}(a, b),$$

$$I_{i\mathcal{R}_2}(a, b) \leq I_{i\mathcal{R}_1}(a, b)$$

and

$$F_{i\mathcal{R}_2}(a, b) \leq F_{i\mathcal{R}_1}(a, b)$$

for all $i = 1, 2, \ldots, n$. Then $(SVNRR_n(X), \sqsubseteq)$ is a poset.

(2) Defined a relation $\leqq$ on $H_n(X)$ as follows: $\overline{\mathcal{R}_1} \leqq \overline{\mathcal{R}_2}$ if and only if $\overline{\mathcal{R}_1}(A) \sqsubseteq \overline{\mathcal{R}_2}(A)$ for each $A \in SVNRS_n(X)$. Then $(H_n(X), \leqq)$ is a poset.

(3) Defined a relation $\leqq$ on $L_n(X)$ as follows: $\underline{\mathcal{R}_1} \leqq \underline{\mathcal{R}_2}$ if and only if $\underline{\mathcal{R}_2}(A) \sqsubseteq \underline{\mathcal{R}_1}(A)$ for each $A \in SVNRS_n(X)$. Then $(L_n(X), \leqq)$ is a poset.

Proof. The proof is straightforward and we omit it.

$\forall \{\mathcal{R}_i\}_{i\in I} \subseteq SVNRR_n(X)$ and I be a index set, we can define union and intersection of $\mathcal{R}_i$ as follows:

$$T_{\sqcup_{i\in I}\mathcal{R}_i}(a, b) = \widetilde{\bigvee_{i\in I}} T_{\mathcal{R}_i}(a, b),$$

$$I_{\sqcup_{i\in I}\mathcal{R}_i}(a, b) = \widetilde{\bigwedge_{i\in I}} I_{\mathcal{R}_i}(a, b),$$

$$F_{\sqcup_{i\in I}\mathcal{R}_i}(a, b) = \widetilde{\bigwedge_{i\in I}} F_{\mathcal{R}_i}(a, b),$$

and

$$T_{\sqcap_{i\in I}\mathcal{R}_i}(a, b) = \widetilde{\bigwedge_{i\in I}} T_{\mathcal{R}_i}(a, b),$$

$$I_{\sqcap_{i\in I}\mathcal{R}_i}(a, b) = \widetilde{\bigvee_{i\in I}} I_{\mathcal{R}_i}(a, b),$$

$$F_{\sqcap_{i\in I}\mathcal{R}_i}(a, b) = \widetilde{\bigvee_{i\in I}} F_{\mathcal{R}_i}(a, b).$$

Then $\sqcup_{i\in I}\mathcal{R}_i$ and $\sqcap_{i\in i}\mathcal{R}_i$ are two n-dimension single valued neutrosophic refined relations, and we easily show that $\sqcup_{i\in I}\mathcal{R}_i$ and $\sqcap_{i\in I}\mathcal{R}_i$ are infimum and supremum of $\{\mathcal{R}_i\}_{i\in I}$, respectively. Hence we can easily obtain the following theorem:

Theorem 3.2. $(SVNRR_n(X), \sqsubseteq, \sqcup, \sqcap)$ is a complete lattice, X_n and $\emptyset_n$ are its top element and bottom element, respectively, where X_n and $\emptyset_n$ are two n-dimension single valued neutrosophic refined relations in X and defined as follows: $\forall (a, b) \in X \times X$,

$$T_{X_n}(a, b) = (1, 1, \cdots, 1),$$

$$I_{X_n}(a, b) = (0, 0, \cdots, 0),$$

$$F_{X_n}(a, b) = (0, 0, \cdots, 0),$$

and

$$T_{\emptyset_n}(a, b) = (0, 0, \cdots, 0),$$

$$I_{\emptyset_n}(a, b) = (1, 1, \cdots, 1),$$

$$F_{\emptyset_n}(a, b) = (1, 1, \cdots, 1).$$

Remark 3.3. By Definition 2.9, $\forall A \in SVNRS_n(X)$, $\forall a \in X$, we have

$$T_{\underline{\emptyset_n}(A)}(a) = \widetilde{\bigwedge_{b\in X}} \left[F_{\emptyset_n}(a,b) \vee T_A(b)\right] = (1,1,\cdots,1),$$

$$I_{\underline{\emptyset_n}(A)}(a) = \widetilde{\bigvee_{b\in X}} \left[(\sim I_{\emptyset_n}(a,b)) \wedge I_A(b)\right] = (0,0,\cdots,0),$$

$$F_{\underline{\emptyset_n}(A)}(a) = \widetilde{\bigvee_{b\in X}} \left[T_{\emptyset_n}(a,b) \wedge F_A(b)\right] = (0,0,\cdots,0),$$

and

$$T_{\overline{\emptyset_n}(A)}(a) = \widetilde{\bigvee_{b\in X}} \left[T_{\emptyset_n}(a,b) \wedge T_A(b)\right] = (0,0,\cdots,0),$$

$$I_{\overline{\emptyset_n}(A)}(a) = \widetilde{\bigwedge_{b\in X}} \left[I_{\emptyset_n}(a,b) \vee I_A(b)\right] = (1,1,\cdots,1),$$

$$F_{\overline{\emptyset_n}(A)}(a) = \widetilde{\bigwedge_{b\in X}} \left[F_{\emptyset_n}(a,b) \vee F_A(b)\right] = (1,1,\cdots,1).$$

Thus $\forall \mathcal{R} \in SVNRR_n(X)$, $\overline{\emptyset_n}(A) \sqsubseteq \overline{\mathcal{R}}(A)$ and $\underline{\mathcal{R}}(A) \sqsubseteq \underline{\emptyset_n}(A)$. This shows $\overline{\emptyset_n} \leqq \overline{\mathcal{R}}$ and $\underline{\emptyset_n} \leqq \underline{\mathcal{R}}$. Hence, $\overline{\emptyset_n}$ is the bottom element of $(H_n(X), \leqq)$ and $\underline{\emptyset_n}$ is the bottom element of $(L_n(X), \leqq)$.

According to the result of Remark 3.3, we have the following two theorems:

Theorem 3.4. $\forall\{\overline{\mathcal{R}_i}\}_{i\in I} \subseteq (H_n(X), \leqq)$ and I be a index set, we can define union and intersection of $\overline{\mathcal{R}_i}$ as follows:

$$\widehat{\bigvee_{i\in I}}\overline{\mathcal{R}_i} = \overline{\sqcup_{i\in I}\mathcal{R}_i}, \quad \widehat{\bigwedge_{i\in I}}\overline{\mathcal{R}_i} = \overline{[\sqcap_{i\in I}\mathcal{R}_i]},$$

where $[\sqcap_{i\in I}\mathcal{R}_i] = \sqcup\{\mathcal{R} \in SVNRR_n(X) \mid \forall A \in SVNRS_n(X), \overline{\mathcal{R}}(A) \sqsubseteq \sqcap_{i\in I}\overline{\mathcal{R}_i}(A)\}$. Then $\widehat{\bigvee_{i\in I}}\overline{\mathcal{R}_i}$ and $\widehat{\bigwedge_{i\in I}}\overline{\mathcal{R}_i}$ are supremum and infimum of $\{\overline{\mathcal{R}_i}\}_{i\in I}$, respectively.

Proof.

Let $\mathcal{R} = \sqcup_{i\in I}\mathcal{R}_i$, then $\mathcal{R}_i \sqsubseteq \mathcal{R}$ for each $i \in I$. By Corollary 2.11 and Theorem 3.1, we have $\overline{\mathcal{R}_i} \leqq \overline{\mathcal{R}}$. If $\mathcal{R}'$ is a n-dimension single valued neutrosophic refined relation such that $\overline{\mathcal{R}_i} \leqq \overline{\mathcal{R}'}$ for each $i \in I$, then

$$\forall A \in SVNRS_n(X),\ \overline{\mathcal{R}_i}(A) \sqsubseteq \overline{\mathcal{R}'}(A).$$

Hence,

$$\overline{\mathcal{R}}(A) = \overline{\sqcup_{i\in I}\mathcal{R}_i}(A) = \sqcup_{i\in I}\overline{\mathcal{R}_i}(A) \sqsubseteq \overline{\mathcal{R}'}(A).$$

Thus $\overline{\mathcal{R}} \leqq \overline{\mathcal{R}'}$. So

$$\widehat{\bigvee_{i\in I}}\overline{\mathcal{R}_i} = \overline{\mathcal{R}} = \overline{\sqcup_{i\in I}\mathcal{R}_i}$$

is the supremum of $\{\overline{\mathcal{R}_i}\}_{i\in I}$.

Let $\mathcal{R}^* = [\sqcap_{i\in I}\mathcal{R}_i]$, then $\forall B \in SVNRS_n(X)$, we have

$$\overline{\mathcal{R}^*}(B) = \overline{[\sqcap_{i\in I}\mathcal{R}_i]}(B) \sqsubseteq \sqcap_{i\in I}\overline{\mathcal{R}_i}(B) \sqsubseteq \overline{\mathcal{R}_i}(B).$$

By Theorem 3.1, we have $\overline{\mathcal{R}^\star} \leqq \overline{\mathcal{R}_i}$ for each $i \in I$. If $\mathcal{R}'$ is a n-dimension single valued neutrosophic refined relation such that $\overline{\mathcal{R}'} \leqq \overline{\mathcal{R}_i}$ for each $i \in I$, then

$$\forall A \in SVNRS_n(X),\ \overline{\mathcal{R}'}(A) \sqsubseteq \overline{\mathcal{R}_i}(A).$$

Thus

$$\overline{\mathcal{R}'}(A) \sqsubseteq \sqcap_{i\in I}\overline{\mathcal{R}_i}(A).$$

By the construction of $[\sqcap_{j\in J}\mathcal{R}_j]$, we can easily obtain $\mathcal{R}' \sqsubseteq [\sqcap_{i\in I}\mathcal{R}_i]$. Hence,

$$\overline{\mathcal{R}'} \leqq \overline{[\sqcap_{i\in I}\mathcal{R}_i]} = \overline{\mathcal{R}^*}.$$

So

$$\widehat{\bigwedge_{i\in I}}\overline{\mathcal{R}_i} = \overline{\mathcal{R}^*} = \overline{[\sqcap_{i\in I}\mathcal{R}_i]}$$

is the infimum of $\{\overline{\mathcal{R}_i}\}_{i\in I}$.

Theorem 3.5. $\forall\{\underline{\mathcal{R}_j}\}_{j\in J} \subseteq (L_n(X), \leqq)$ and J be a index set, we can define union and intersection of $\underline{\mathcal{R}_j}$ as follows:

$$\widehat{\bigvee_{j\in J}}\underline{\mathcal{R}_j} = \underline{\sqcup_{j\in J}\mathcal{R}_j}, \ \widehat{\bigwedge_{j\in J}}\underline{\mathcal{R}_j} = \underline{< \sqcup_{j\in J}\mathcal{R}_j >},$$

where $< \sqcup_{j\in J}\mathcal{R}_j >= \sqcup\{\mathcal{R} \in SVNRR_n(X) \mid \forall A \in SVNRS_n(X), \sqcup_{j\in J}\underline{\mathcal{R}_j}(A) \sqsubseteq \underline{\mathcal{R}}(A)\}$. Then $\widehat{\bigvee_{j\in J}}\underline{\mathcal{R}_j}$ and

$\widehat{\bigwedge\limits_{j\in J}}\underline{\mathcal{R}_j}$ are supremum and infimum of $\{\underline{\mathcal{R}_j}\}_{j\in J}$, respectively.

Proof.

Let $\mathcal{R} = \sqcup_{j\in J}\mathcal{R}_j$, then $\mathcal{R}_j \sqsubseteq \mathcal{R}$ for each $j \in J$. By Corollary 2.11 and Theorem 3.1, we have $\underline{\mathcal{R}_j} \leqq \underline{\mathcal{R}}$. If $\mathcal{R}'$ is a n-dimension single valued neutrosophic refined relation such that $\underline{\mathcal{R}_j} \leqq \underline{\mathcal{R}'}$ for each $j \in J$, then

$$\forall A \in SVNRS_n(X),\ \underline{\mathcal{R}'}(A) \sqsubseteq \underline{\mathcal{R}_j}(A).$$

Hence,

$$\underline{\mathcal{R}}(A) = \underline{\sqcup_{j\in J}\mathcal{R}_j}(A) = \sqcap_{j\in J}\underline{\mathcal{R}_j}(A) \sqsupseteq \underline{\mathcal{R}'}(A).$$

Thus $\underline{\mathcal{R}} \leqq \underline{\mathcal{R}'}$. So

$$\widehat{\bigvee_{j\in J}}\underline{\mathcal{R}_j} = \underline{\mathcal{R}} = \underline{\sqcup_{j\in J}\mathcal{R}_j}$$

is the supremum of $\{\underline{\mathcal{R}_j}\}_{j\in J}$.

Let $\mathcal{R}^\star =< \sqcup_{j\in J}\mathcal{R}_j >$, then $\forall B \in SVNRS_n(X)$, we have

$$\underline{\mathcal{R}^\star}(B) = \underline{< \sqcup_{j\in J}\mathcal{R}_j >}(B) \sqsupseteq \sqcup_{j\in J}\underline{\mathcal{R}_j}(B) \sqsupseteq \underline{\mathcal{R}_j}(B).$$

By Theorem 3.1, we have $\underline{\mathcal{R}^\star} \leqq \underline{\mathcal{R}_j}$ for each $j \in J$. If $\mathcal{R}'$ is a n-dimension single valued neutrosophic refined relation such that $\underline{\mathcal{R}'} \leqq \underline{\mathcal{R}_j}$ for each $j \in J$, then

$$\forall A \in SVNRS_n(X),\ \underline{\mathcal{R}_j}(A) \sqsubseteq \underline{\mathcal{R}'}(A).$$

Thus

$$\sqcup_{j\in J}\underline{\mathcal{R}_j}(A) \sqsubseteq \underline{\mathcal{R}'}(A).$$

By the construction of $< \sqcup_{j\in J}\mathcal{R}_j >$, we can easily obtain $< \sqcup_{j\in J}\mathcal{R}_j > \sqsupseteq \mathcal{R}'$. By Corollary 2.11 and Theorem 3.1, we have

$$\underline{\mathcal{R}'} \leqq \underline{< \sqcup_{j\in J}\mathcal{R}_j >} = \underline{\mathcal{R}^\star}.$$

So

$$\widehat{\bigwedge_{j\in J}}\underline{\mathcal{R}_j} = \underline{\mathcal{R}^\star} = \underline{< \sqcup_{j\in J}\mathcal{R}_j >}$$

is the infimum of $\{\underline{\mathcal{R}_j}\}_{j\in J}$.

By Remark 3.3 and Theorems 3.4 and 3.5, we have the following theorem:

Theorem 3.6. Both $(H_n(X), \leqq, \widehat{\vee}, \widehat{\wedge})$ and $(L_n(X), \leqq, \widehat{\vee}, \widehat{\wedge})$ are complete lattices.

Obviously, when $n = 1$, both $(H_1(X), \leqq, \widehat{\vee}, \widehat{\wedge})$ and $(L_1(X), \leqq, \widehat{\vee}, \widehat{\wedge})$ are complete lattices. Notice that, when $n = 1$, the n-dimension single valued neutrosophic refined relation will degenerate into a single valued neutrosophic relation. Hence we have the following corollary:

Corollary 3.7. If H (resp., L) denote the family of single valued neutrosophic upper (resp., lower) approximation operators in X, then both $(H, \leqq)$ and $(L, \leqq)$ are all complete lattices.

4. Conclusion

Following the notion of single valued neutrosophic refined rough approximation operators as introduced by Bao and Yang(2017), we proved that both $(H_n(X), \leqq, \widehat{\vee}, \widehat{\wedge})$ and $(L_n(X), \leqq, \widehat{\vee}, \widehat{\wedge})$ are complete lattices. In particular, both $(H, \leqq)$ and $(L, \leqq)$ are all complete lattices. To study the category of single valued neutrosophic refined rough sets in our forthcoming research and there are still exists a problem to be solved as follows: What is the relationship between three complete lattices $(SVNRR_n(X), \sqsubseteq, \sqcup, \sqcap)$, $(H_n(X), \leqq, \widehat{\vee}, \widehat{\wedge})$ and $(L_n(X), \leqq, \widehat{\vee}, \widehat{\wedge})$.

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